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SOLVABILITY OF NONLINEAR INTEGRAL EQUATIONS OF PRODUCT TYPE

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Abstract. This article concerns nonlinear functional integral equations of product type. The first two equations set on a the positive half-axis encompass different classes of nonlinear integral equations and may involve the product of finitely many integral functions. The existence of integrable solutions is based on improved versions of Krasnoselskii's fixed point theorem combined with techniques of measure of weak noncompactness and some elements from functional analysis. The third one is an integro-differential equation set on a bounded interval, for which the existence of absolutely continuous solutions is provided. Examples show the applicability of our results.

1. Introduction

Nonlinear integral equations appear in several mathematical problems modeling nonlinear phenomena. As special cases, integral equations of product type arise, e.g., in the study of the spread of an infectious disease that does not induce permanent immunity (see, e.g., [3, 12, 14, 15, 29] and references therein). For instance, Gripenberg [14] studied the existence of periodic solutions to the following integral equation of product type:

$$x(t) = k\Big(P - \int_{-\infty}^{t} A(t-s)x(s)ds\Big)\Big(\int_{-\infty}^{t} a(t-s)x(s)ds\Big), \quad t \in \mathbb{R}.$$

This equation is related to models of disease spread that does not induce permanent immunity and the function x stands for the infection rate, i.e., the rate at which individuals susceptible to catch the disease become infected. Then $\int_{-\infty}^{t} a(t-s)x(s)ds$ is approximately proportional to the total infectivity if the average infectivity of an individual infected at time s is proportional to $a(t-s)$ at time $t > s$. P is the size of population and $P - \int_{-\infty}^{t} A(t-s)x(s)ds$ is approximately the number of susceptibles provided that the cumulative probability for the loss of immunity of an individual infected at time s is $1 - A(t-s)$ (see [14, 15]).

Gripenberg [15] also studied the existence and the uniqueness of a bounded, continuous, and nonnegative solution to the following integral equation of product

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type:

$$x(t) = k\Big(p(t) + \int_0^t A(t-s)x(s)ds\Big)\Big(q(t) + \int_0^t B(t-s)x(s)ds\Big), \tag{1.1}$$

for $t > 0$, under appropriate assumptions on functions A and B. The functions p, q are related to the past-time infection. Gripenberg also obtained sufficient conditions that guarantee the convergence of the solution as $t \to \infty$.

Pachpatte [27] provided a new integral inequality that he used to study the boundedness, asymptotic behavior, and growth of solutions of equation (1.1).

Abdeldaim [1], and Li et al. [21] generalized Pachpatte's inequality to some integral inequalities in order to study the boundedness and the asymptotic behavior of continuous solutions to equation (1.1).

Olaru [25] generalized (1.1) and showed the existence and uniqueness of a continuous solution to the following integral equation:

$$x(t) = \prod_{i=1}^m A_i(x)(t), \quad a < t < b,$$

where $A_i(x)(t) = g_i(t) + \int_a^t K_i(t,s,x(s))ds$, $t \in [a,b]$, and K_i is continuous Lipschitzian for $i = 1, \dots, m$.

Later Olaru [26] generalized (1.1) by studying the existence of a continuous solution to the integral equation

$$x(t) = \Big(g_1(t) + \int_0^t K_1(t,s,x(s))ds\Big)\Big(g_2(t) + \int_0^t K_2(t,s,x(s))ds\Big), \tag{1.2}$$

for $t > 0$. He employed the weakly Picard technique operators in a gauge space.

Finally we mention Bellour et al. [8] who studied the existence of an integrable solution to the following integral equation on the interval $[0,1]$,

$$\begin{aligned} x(t) = u(t,x(t)) &+ \Big(p(t) + \int_0^t k_1(t,s)f_1(s,x(s))ds\Big) \\ &\times \Big(q(t) + \int_0^t k_2(t,s)f_2(s,x(s))ds\Big). \end{aligned} \tag{1.3}$$

In this paper, we consider the more general nonlinear integral equation

$$x(t) = f(t,x(t)) + f_1\Big(t, \int_0^t v_1(t,s,x(s))ds\Big) f_2\Big(t, \int_0^t v_2(t,s,x(s))ds\Big), \tag{1.4}$$

for $t > 0$. This equation encompasses many important integral and functional equations that arise in nonlinear analysis and its applications, in particular integral equations (1.1), (1.2), and (1.3) (see also [13, 20] for some other special cases). When considering continuous solutions, we refer to [9] and some references therein.

However, many models of the spread of infectious diseases include data functions, which are discontinuous. For this reason, we devote our investigations to extend the theory developed for (1.1) and (1.2) to discuss the existence of a solution to (1.4) in the space of integrable real functions on $\mathbb{R}_+$ when f_1, f_2 obey linear growths in the second argument, which ensures continuity of the superposition operators. The product term involves two nonlinear operators acting from L^1 to L^1 and to L^∞, respectively. An example is included to illustrate the applicability of our first existence result. This is the content of Section 3.

Section 4 is devoted to a generalization of equation (1.4) to m product terms ($m \geq 2$), each transforming L^1 into L^{p_i} with conjugate exponents $p_i > 1$ ($1 \leq i \leq m$), which we call (L^p, L^q) product integrals:

$$x(t) = f(t, x(t)) + \prod_{i=1}^{m} f_i\Big(t, \int_0^t v_i(t, s, x(s))ds\Big), \quad t > 0.$$

Theorem 4.1 providing existence of integrable solution is proved via a fixed point argument.

The main tools used in our considerations rely on conjunction of some techniques of measures of noncompactness together with compactness criteria and an improved version of Krasnosel'skii fixed point theorem proved in [24].

The third nonlinear integral equation discussed in this work is also of product type and is set a bounded interval $[a, b]$:

$$x(t) = x_0 + \int_0^t f(s, x(s))ds + \int_0^t \big(\alpha(s) + V_1 x(s)\big)\big(\beta(s) + V_2 x(s)\big)ds,$$

for $a < t < b$. The existence of absolutely continuous solutions is obtained in Theorem 5.1, Section 5, extending results from [25].

Some elements from functional analysis including Dunford-Pettis weak compactness criterion and fixed point theorems are collected in next Section 2.

2. Preliminary results

We denote by $L^p = L^p(\mathbb{R}_+)$ ($1 \leq p < \infty$) the Banach space of equivalence classes of measurable functions on $\mathbb{R}_+$ such that $\int_0^{+\infty} |x(t)|^p dt < \infty$. It is equipped with the norm $\|x\|_p = \big(\int_0^{+\infty} |x(t)|^p dt\big)^{1/p}$. $L^\infty = L^\infty(\mathbb{R}_+)$ will refer to the Banach space of classes of measurable functions that are essentially bounded. Its norm is referred to by $\|x\|_\infty = ess\sup_{t\geq 0} |x(t)|$. For the sake of clarity, we will shorten $\|x\|_1$ to $\|x\|$, unless specified otherwise.

Theorem 2.1 (Generalized Hölder's theorem [10])**.** *Assume that $f_1, f_2, \dots, f_n$ are functions such that*

$$f_i \in L^{p_i}(\mathbb{R}_+), \quad 1 \leq i \leq n \quad \text{with} \quad \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_n} = \frac{1}{p} \leq 1. \tag{2.1}$$

Then the product $f = f_1 f_2 \dots f_n$ belongs to $L^p(\mathbb{R}_+)$ and

$$\|f\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_n\|_{p_n} \tag{2.2}$$

The following result is a kind of converse to the Lebesgue dominated convergence theorem.

Theorem 2.2 ([10, Théorème IV.9])**.** *Let Ω be a measurable set of $\mathbb{R}^n$ and (f_n) a sequence in $L^p(\Omega)$. If $f_n \to f$ in $L^p(\Omega)$ with $p \geq 1$, then there exists a subsequence (f_{n_k}) of (f_n) and a function $g \in L^p(\Omega)$ such that:*

1. *$f_{n_k} \to f$, a.e. in Ω,*
2. *$|f_{n_k}(t)| \leq g(t)$, for all $k \geq 1$ and a.e $t \in \Omega$.*

Also we need the following result.

Lemma 2.3. *Let E be a topological space and $(x_n)_n$ a sequence in E. If there exists $x \in E$ such that any subsequence $(x_{n_k})_k$ of (x_n) has a new subsequence $(x_{n_{k_l}})_l$ such that $x_{n_{k_l}} \to x$ in E, as $l \to \infty$. Then $x_n \to x$ in E, as $n \to \infty$.*

This is a classical result in topology whose proof is sketched here for the sake of completeness (see, e.g., [4, Exercise 9, Section 3.4, p.80].

Proof. On the contrary, there would exist some $\varepsilon_0 > 0$ such that for all $k = 1, 2, \dots$, there exists $n_k > k$ such that $|x_{n_k} - x| > \varepsilon$. Then the sequence (x_{n_k}) has no convergent subsequence, a contradiction. Another way to see this result is to let $\underline{x} = \liminf_{n\to\infty} x_n$ and $\overline{x} = \limsup_{n\to\infty} x_n$. Now consider two subsequences (x_{n_k}) and (x_{n_l}) that converge to $\underline{x}$ and $\overline{x}$ respectively. By Assumption these subsequences have subsequences that converge to x. As a consequence $x = \underline{x} = \overline{x}$. $\square$

Definition 2.4. Let $I \subset \mathbb{R}$ be an interval (bounded or unbounded) and $n \geq 1$ an integer. A function $f : I \times \mathbb{R}^n \to \mathbb{R}$ satisfies Carathéodory's conditions if

(i) for all $x \in \mathbb{R}^n$, the function $t \mapsto f(t, x)$ is Lebesgue measurable on I,
(ii) for almost every (a.e. for short) $t \in I$, the function $x \mapsto f(t, x)$ is continuous on $\mathbb{R}^n$.

One of the most important operators in nonlinear analysis is the superposition (or Nemytskii) operator generated by a time-space argument function f and defined by $(Fx)(t) = f(t, x(t))$, where $x : I \to \mathbb{R}$ is a measurable function. It is well known that Nx is also measurable and that if N is defined in L^p with values in L^q, then it is bounded and continuous. Moreover Krasnosel'skii [18] and Appell and Zabreiko [2] proven the following characterization.

Theorem 2.5. *Let $I \subset \mathbb{R}$ be an interval (bounded or unbounded) and $p, q \in [1, +\infty)$. Then the superposition operator generated by Carathéodory's function f maps continuously the space $L^p(I)$ into $L^q(I)$ if and only if $|f(t, x)| \leq a(t) + c|x|^{\frac{p}{q}}$, for a.e. $t \in I$ and all $x \in \mathbb{R}$, where c is a nonnegative constant and $a \in L^q(I, (0, +\infty))$.*

The Sorza Dragoni Theorem reads as follows.

Theorem 2.6 ([23])**.** *Let $I \subset \mathbb{R}$ be a bounded interval and let $f : I \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying Carathéodory's conditions. Then, for each $\varepsilon > 0$, there exists a closed subset $D_\varepsilon \subset I$ such that* $\operatorname{meas}(I \setminus D_\varepsilon) < \varepsilon$ *and the restriction of f on the set $D_\varepsilon \times \mathbb{R}^n$ is continuous.*

The Dunford-Pettis Theorem provides a useful characterization of weakly compact sets of L^1.

Theorem 2.7 ([10])**.** *A bounded subset $\mathcal{M}$ of the Banach space $L^1(\mathbb{R}_+)$ has compact closure in the weak topology if and only if the following two conditions are fulfilled:*

(a) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that*
$$\int_D |x(t)|dt \leq \varepsilon, \quad \forall D \subset \mathbb{R}_+,\ \operatorname{meas}(D) \leq \delta,\ \forall x \in \mathcal{M}$$
(b) *for each $\varepsilon > 0$ there exists $T > 0$ such that*
$$\int_T^{+\infty} |x(t)|dt \leq \varepsilon, \quad \forall x \in \mathcal{M}.$$

Given a Banach space E, let $\mathcal{B}(E)$ denote the family of all nonempty bounded subsets of E and $\mathcal{W}(E)$ the subset of $\mathcal{B}(E)$ consisting of all relatively weakly compact subsets of E. B_r will refer to the closed ball centered at 0 with radius r in E. The following concept of the measure of weak noncompactness was first introduced by [11]; see also [6]. It is recalled in its axiomatic form.

Definition 2.8. A function $\mu : \mathcal{B}(E) \to \mathbb{R}_+$ is called a measure of weak noncompactness if it satisfies the conditions:

(1) The set $\ker(\mu) = \{M \in \mathcal{B}(E) : \mu(M) = 0\}$ is nonempty and $\ker(\mu) \subset \mathcal{W}(E)$.
(2) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
(3) $\mu(co(M)) = \mu(M)$, where $co(M)$ is the closed convex hull of M.
(4) $\mu(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\mu(M_1) + (1-\lambda)\mu(M_2)$, for all $\lambda \in [0,1]$.
(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty, weakly closed subsets of E with M_1 bounded and $M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ such that $\lim_{n\to\infty} \mu(M_n) = 0$, then $M_\infty := \cap_{n=1}^{\infty} M_n$ is nonempty.

An important example of measure of weak noncompactness in $L^1(\mathbb{R}_+)$ has been constructed by Banas and Knap [7] in the following way: for a bounded subset X of $L^1(\mathbb{R}_+)$, let

$$\mu(X) = c(X) + d(X),$$

where

$$c(X) = \lim_{\varepsilon\to 0} \Big(\sup_{x\in X} \Big\{ \sup \Big\{ \int_D |x(t)|dt : D \subset \mathbb{R}_+, \operatorname{meas}(D) \leq \varepsilon,\ x \in X \Big\} \Big\} \Big),$$

$$d(X) = \lim_{T\to\infty} \Big(\sup \Big\{ \int_T^{+\infty} |x(t)|dt : x \in X \Big\} \Big).$$

Notice that the first term is related to integrability condition (a) in Theorem 2.7 while the second one treats the equiconvergence at positive infinity, namely condition (b) in Theorem 2.7. Moreover by Dunford-Pettis theorem 2.7, the kernel of the measure of weak noncompactness μ coincides with the collection of all weakly relatively compact subsets of the Banach space $L^1(\mathbb{R}_+)$.

The following two definitions are needed in Theorems 2.11 and 2.12. The first one extends the concept of nonlinear contraction.

Definition 2.9. [22] Let (X, d) be a metric space. we say that $T : X \to X$ is a separate contraction if there exist two functions $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

(1) $\psi(0) = 0$, ψ is strictly increasing,
(2) $d(Tx, Ty) \leq \varphi(d(x,y))$, for all $x, y \in X$,
(3) $\psi(r) + \varphi(r) \leq r$, for $r > 0$.

Definition 2.10. [16] Let M be a subset of a Banach space E. A continuous map $A : M \to E$ is said to be (ws)-compact if for every weakly convergent sequence $(x_n)_n$ in M, the sequence $(Ax_n)_n$ has a strongly convergent subsequence in E.

Our existence results are based on the following two fixed point theorems. The first one is a Krasnosels'kii type theorem under the weak topology.

Theorem 2.11 ([24]). *Let M be a nonempty, bounded, closed, and convex subset of a Banach space E. Suppose that $F : M \to E$ and $G : M \to E$ satisfy:*

(i) *F is a separate contraction,*
(ii) *G is (ws)-compact,*
(iii) *there exists $\gamma \in [0,1)$ such that $\mu(FS + GS) \leq \gamma\mu(S)$ for all $S \subset M$, where μ is an arbitrary measure of weak noncompactness on E,*
(iv) *$F(M) + G(M) \subseteq M$.*

Then there exists $x \in M$ such that $Fx + Gx = x$.

This is a generalization of the following result.

Theorem 2.12 ([20]). *Let $\mathcal{M}$ be a nonempty bounded closed convex subset of a Banach space E. Suppose that $A : \mathcal{M} \to \mathcal{M}$ satisfies:*

(i) *A is (ws)-compact.*
(ii) *$A(\mathcal{M})$ is relatively weakly compact.*

Then there is a $x \in \mathcal{M}$ such that $Ax = x$.

We finish this section with some reminders and properties of absolute continuous functions (see, e.g., [17, 28])

Definition 2.13. A function $\theta : [a, b] \to \mathbb{R}$ is absolutely continuous if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |\theta(x_i') - \theta(x_i)| < \epsilon,$$

for any finite collection $\{(x_i, x_i') : i = 1, ..., n\}$ of pairwise disjoint intervals in $[a, b]$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$.

Absolutely continuous functions enjoy important properties.

Theorem 2.14. *If θ is absolutely continuous on $[a, b]$, then θ has a derivative defined almost everywhere on $[a, b]$. Moreover $\theta'(t)$ is integrable on $[a, b]$ and*

$$\theta(t) = \theta(a) + \int_a^t \theta'(s) ds.$$

Theorem 2.15. *Let θ be an integrable function on $[a, b]$, then the function*

$$\vartheta(t) = \vartheta(a) + \int_a^t \theta(s) ds$$

is absolutely continuous. Moreover, ϑ is derivable almost everywhere on $[a, b]$ and $\vartheta'(t) = \theta(t)$ a.e. $t \in [a, b]$.

3. (L^1, L^∞) PRODUCT TYPE INTEGRAL EQUATION

To investigate the existence of integrable solutions to equation (1.4), we adopt the following assumptions on the given nonlinearities. Notice that by Theorem 2.5, sublinear growth conditions are optimal to assure continuity of superposition operators in L^1.

(A1) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory's conditions and it is a separate contraction with respect the second variable; moreover there exist a function $\varphi \in L^1(\mathbb{R}_+)$ and a positive constant c such that

$$|f(t, x)| \leq \varphi(t) + c|x|,$$

for a.e, $t \in \mathbb{R}_+$ and all $x \in \mathbb{R}$.

(A2) The functions $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ satisfy Carathéodory's conditions and there exist two functions $\varphi_1 \in L^1(\mathbb{R}_+)$, $\varphi_2 \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, and positive constants c_i $(i = 1, 2)$ such that

$$|f_i(t, x)| \leq \varphi_i(t) + c_i|x|,$$

for a.e. $t, s \in \mathbb{R}_+$ and all $x \in \mathbb{R}$.

(A3) The functions $v_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ satisfy Carathéodory's conditions and there exist two functions $a_i \in L^1(\mathbb{R}_+)$ and positive constants b_i such that for a.e. $t, s \in \mathbb{R}_+$ and all $x \in \mathbb{R}$

$$|v_i(t, s, x)| \le k_i(t, s)(a_i(s) + b_i|x|),$$

where $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ satisfy Carathéodory's conditions.

(A4) The linear Volterra operator K_i $(i = 1, 2)$ transforms the space $L^1(\mathbb{R}_+)$ into itself and K_2 transforms continuously the space $L^1(\mathbb{R}_+)$ into $L^\infty(\mathbb{R}_+)$, where

$$K_i x(t) = \int_0^t k_i(t, s)x(s)ds, \quad t > 0.$$

Let $\|K_i\|$ be the norm of the bounded linear operator K_i $(i = 1, 2)$.

Remark 3.1. A sufficient condition for the linear operator

$$(Kx)(t) = \int_0^t k(t, s)x(s)ds, \quad t \in \mathbb{R}_+$$

map L^1 into itself is that the mapping

$$s \mapsto \int_s^{+\infty} |k(t, s)|dt$$

be $L^\infty(\mathbb{R})$ (see [5, Theorem 2]). This implies that K is continuous (see [30]). Clearly a sufficient condition for the linear operator K map L^1 into L^∞ is that $k \in L^\infty(\mathbb{R}^2)$.

Observe that solving (1.4) amounts to finding a fixed point of the operator

$$H := F + G : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+) \tag{3.1}$$

defined by the right side of equation (1.4). Furthermore the map H can be written as

$$Hx(t) = Fx(t) + G_1x(t) \times G_2x(t), \quad t \in \mathbb{R}_+, \tag{3.2}$$

where F is the Nemytskii operator generated by the function f, i.e.:

$$Fx(t) = f(t, x(t)),$$

$$G_ix(t) = f_i\Big(t, \int_0^t v_i(t, s, x(s))ds\Big), \quad t > 0, \quad (i = 1, 2).$$

Let $Gx(t) = G_1x(t) \times G_2x(t)$. $\mathbb{N} = \{1, 2, 3, \dots\}$ will denote the set of positive integers. To abbreviate notation, we put

$$\begin{gathered}\alpha = b_1b_2c_1c_2\|K_1\|\|K_2\|,\\ \beta = \|\varphi\| + \big(\|\varphi_1\| + c_1\|K_1\|\|a_1\|\big)\big(\|\varphi_2\|_\infty + c_2\|K_2\|\|a_2\|\big),\\ \delta = c + b_1c_1\|K_1\|\big(\|\varphi_2\|_\infty + c_2\|K_2\|\|a_2\|\big) + b_2c_2\|K_2\|\big(\|\varphi_1\| + c_1\|K_1\|\|a_1\|\big).\end{gathered} \tag{3.3}$$

We start our proof with a compactness result crucial for our subsequent arguments.

Lemma 3.2. *Under Assumptions* (A1)–(A4)*, operators G_1 and G_2 are* (ws)*-compact from $L^1(\mathbb{R}_+)$ into it self.*

Proof. Let $(y_n)_n$ be a weakly convergent sequence in $L^1(\mathbb{R}_+)$. Then the set $X = \{y_n : n \in \mathbb{N}\}$ is relatively weakly compact, hence bounded for the L^1-norm. Consequently some positive constant r exists and satisfies $\|y_n\| \le r$, for all integer n. Let $\varepsilon > 0$. Appealing to Dunford-Pettis theorem 2.7, Assumptions (A2)–(A4) guarantee the existence of some positive constant T and $\delta > 0$ such that for each closed subset $D \subset \mathbb{R}_+$ with $\operatorname{meas}(D) \le \delta$, we have for all integer $n \in \mathbb{N}$

$$\int_D |G_1 y_n(t)|dt + \int_T^\infty |(G_1 y_n)(t)|dt \le \frac{\varepsilon}{4}. \tag{3.4}$$

Theorem 2.6 ensures the existence of a closed subset D_ε of the interval $[0,T]$ satisfying $\operatorname{meas}([0,T] \setminus D_\varepsilon) \le \varepsilon$ and such that the functions φ_1, k_1, v_1, and f_1 are continuous on the sets D_ε, $D_\varepsilon \times [0,T], D_\varepsilon \times [0,T] \times \mathbb{R}$, and $D_\varepsilon \times \mathbb{R}$ respectively.

Claim 1. The set $G_1(X)$ is relatively compact in $L^1(\mathbb{R}_+)$. Let

$$\overline{\varphi_1} = \sup\big\{\varphi_1(t) : t \in D_\varepsilon\big\}, \quad \overline{k_1} = \sup\big\{k_1(t,s) : (t,s) \in D_\varepsilon \times [0,T]\big\}.$$

Then for $n \in \mathbb{N}$ and for each $t \in D_\varepsilon$, we have

$$\begin{aligned} \Big|\int_0^t v_1(t,s,y_n(s))ds\Big| &\le \int_0^t \Big(k_1(t,s)[a_1(s) + b_1|y_n(s)|\Big)ds \\ &\le \overline{k_1}\big(\|a_1\| + b_1 r\big) := \overline{K_1}(\varepsilon). \end{aligned} \tag{3.5}$$

Consequently,

$$\Big|f_1\Big(t, \int_0^t v_1(t,s,y_n(s))ds\Big)\Big| \le \overline{\varphi_1} + c_1\overline{k_1}\big(\|a_1\| + b_1 r\big) := \overline{G_1}(\varepsilon). \tag{3.6}$$

This proves that $G_1(X)$ is equibounded on the subset D_ε. To show that $G_1(X)$ is equicontinuous on D_ε, take t_1 and t_2 in D_ε. Without loss of generality we may assume that $t_1 < t_2$. Then for each $n \in \mathbb{N}$, we have the estimate

$$\begin{aligned} &\Big|\int_0^{t_2} v_1(t_2,s,y_n(s))ds - \int_0^{t_1} v_1(t_1,s,y_n(s))ds\Big| \\ &\le \int_0^{t_1} |v_1(t_2,s,y_n(s)) - v_1(t_1,s,y_n(s))|ds + \Big|\int_{t_1}^{t_2} v_1(t_2,s,y_n(s))ds\Big| \\ &\le \int_{D_\varepsilon} |v_1(t_2,s,y_n(s)) - v_1(t_1,s,y_n(s))|ds \\ &\quad + \int_{[0,t_1]\setminus D_\varepsilon} |v_1(t_2,s,y_n(s))|ds + \int_{[0,t_1]\setminus D_\varepsilon} |v_1(t_1,s,y_n(s))|ds \\ &\quad + \overline{k_1}\Big(\int_{t_1}^{t_2} a_1(s)ds + b_1\int_{t_1}^{t_2} |y_n(s)|ds\Big) \\ &\le \operatorname{meas}(D_\varepsilon)\,\omega^T(v_1, t_2 - t_1) \\ &\quad + 2\overline{k_1}\Big(\int_{[0,t_1]\setminus D_\varepsilon} a_1(s)ds + b_1\int_{[0,t_1]\setminus D_\varepsilon} |y_n(s)|ds\Big) \\ &\quad + \overline{k_1}\Big(\int_{t_1}^{t_2} a_1(s)ds + b_1\int_{t_1}^{t_2} |y_n(s)|ds\Big), \end{aligned}$$

where $w^T(v_1, t_2 - t_1)$ refers to the modulus of continuity of v_1 on the cartesian product $D_\varepsilon \times [0,T] \times [-\overline{K_1}(\varepsilon), \overline{K_1}(\varepsilon)]$. Since a single set of L^1 is weakly relatively compact, we deduce from Theorem 2.7 that the terms of the real sequence $\big(\int_{t_1}^{t_2} |y_n(s)|ds\big)_n$ as well as $\int_{t_1}^{t_2} a_1(s)ds$ are arbitrarily small provided that the number $t_2 - t_1$ is small enough. In addition the function f_1 is uniformly continuous on the product $D_\varepsilon \times [-\overline{K_1}(\varepsilon), \overline{K_1}(\varepsilon)]$, then the set $G_1(X)$ is equicontinuous and equibounded on D_ε. Ascoli-Arzela Theorem then implies that $G_1(X)$ is relatively strongly compact in $C(D_\varepsilon)$. Consequently, for each integer $p \in \mathbb{N}$ there exists a closed subset D_p of $[0,T]$ with $\operatorname{meas}([0,T]\backslash D_p) \le \frac{1}{p}$ such that $G_1(X)$ is relatively compact in $C(D_p)$.

Moreover there exists $p_0 \ge 1$ such that $\operatorname{meas}([0,T]\backslash D_{p_0}) \le \delta$. Then the sequence $(G_1(y_n))_n$ has a convergent subsequence $(G_1(z_n))_n$ with respect to the standard norm of $C(D_{p_0})$. Therefore some integer $n_0 \in \mathbb{N}$ exists and satisfies that for all m, $n \ge n_0$ and for every $t \in D_{p_0}$, we have

$$|G_1(z_n)(t) - G_1(z_m)(t)| \le \frac{\varepsilon}{1 + 2\operatorname{meas}(D_{p_0})}. \tag{3.7}$$

From (3.4) and (3.7), we deduce the estimates:

$$\begin{aligned}
&\int_0^\infty |G_1(z_n)(t) - G_1(z_m)(t)|dt \\
&\le \int_{D_{p_0}} |G_1(z_n)(t) - G_1(z_m)(t)|dt + \int_{[0,T]\backslash D_{p_0}} |G_1(z_n)(t)|dt \\
&\quad + \int_{[0,T]\backslash D_{p_0}} |G_1(z_m)(t)|dt + \int_T^\infty |G_1(z_n)(t) - G_1(z_m)(t)|dt \le \varepsilon.
\end{aligned}$$

Finally, we have proven that $(G_1(z_n))_n$ is a Cauchy sequence in the Banach space $L^1(\mathbb{R}_+)$, proving that $G_1(X)$ is strongly relatively compact.

Claim 2. G_1 is continuous. For this aim, consider a sequence $(x_n)_n$ converging to some limit x in L^1. Theorem 2.2 yields some subsequence $(x_{n_k})_k$ of $(x_n)_n$ and an integrable function g such that $x_{n_k} \to x$, as $k \to \infty$ for a.e. $t \in \mathbb{R}_+$ and $|x_{n_k}(t)| \le g(t)$, for a.e. $t \in \mathbb{R}_+$ and all $k \in \mathbb{N}$. Since v_1 satisfies Carathéodory's condition (A3), then $v_1(t,s,x_{n_k}(s)) \to v_1(t,s,x(s))$, as $k \to \infty$ for a.e. $t > 0$. According to Assumptions (A2) and (A3), we infer that

$$\int_0^t |v_1(t,s,x_{n_k}(s))|ds \le \int_0^t k_1(t,s)\big[a_1(s) + b_1 g(s)\big]ds \in L^1(\mathbb{R}_+). \tag{3.8}$$

Lebesgue's Dominated Convergence Theorem guarantees that

$$\int_{\mathbb{R}_+} \Big| \int_0^t v_1(t,s,x_{n_k}(s))ds - \int_0^t v_1(t,s,x(s))ds \Big| dt \to 0, \quad \text{as } k \to \infty. \tag{3.9}$$

Using Theorem 2.5, we deduce that

$$\|(G_1 x_{n_k}) - (G_1 x)\| \to 0, \quad \text{as } k \to +\infty. \tag{3.10}$$

This together with Lemma 2.3 imply that $\big\|(G_1 x_n) - (G_1 x)\big\| \to 0$, proving that $G_1 : L^1 \to L^1$ is continuous. We conclude that G_1 is (ws)-compact. By an argument similar to the one above, we infer that the set $G_2(X)$ is relatively compact in $L^1(\mathbb{R}_+)$ and that G_2 is continuous, proving that $G_2 : L^1 \to L^1$ is (ws)-compact. $\square$

Theorem 3.3. *In addition to (A1)-(A4) assume that*

(A5) $\sqrt{\alpha\beta} < \frac{1-\delta}{2}$, *where* α, β, δ *are defined in* (3.3).

Then the nonlinear integral equation (1.4) *has at least one solution* $x \in L^1(\mathbb{R}_+)$.

Proof. We will show that operator H defined by (3.1) satisfies all conditions of Theorem 2.11. The proof is split into three steps.

Claim 1. There exists $r_0 > 0$ such that $F(B_{r_0}) + G(B_{r_0}) \subseteq B_{r_0}$. To see this, let $x, y \in B_r$ for some positive constant r to be determined. We have the estimates:

$$\begin{aligned}
&\|Fx + Gy\| \\
&\le \int_{\mathbb{R}_+} |f(t, x(t))|dt \\
&\quad + \int_{\mathbb{R}_+} \Big|f_1\Big(t, \int_0^t v_1(t, s, y(s))ds\Big)\Big|\Big|f_2\Big(t, \int_0^t v_2(t, s, y(s))ds\Big)\Big|dt \\
&\le \|\varphi\| + c\|x\| + \int_{\mathbb{R}_+} \Big[\varphi_1(t) + c_1 \int_0^t \Big(k_1(t, s)[a_1(s) + b_1|y(s)|\Big)ds\Big] \\
&\quad \times \Big[\varphi_2(t) + c_2 \int_0^t \Big(k_2(t, s)[a_2(s) + b_2|y(s)|\Big)ds\Big]dt \\
&\le \|\varphi\| + c\|x\| + \Big[\|\varphi_1\| + c_1\|K_1\|\Big(\|a_1\| + b_1\|y\|\Big)\Big] \\
&\quad \times \Big[\|\varphi_2\|_\infty + c_2\|K_2\|\Big(\|a_2\| + b_2\|y\|\Big)\Big] \\
&\le \|\varphi\| + cr + \Big[\|\varphi_1\| + c_1\|K_1\|\Big(\|a_1\| + b_1 r\Big)\Big] \\
&\quad \times \Big[\|\varphi_2\|_\infty + c_2\|K_2\|\Big(\|a_2\| + b_2 r\Big)\Big].
\end{aligned} \tag{3.11}$$

Define the quadratic function $\theta(r) = \alpha r^2 + (\delta - 1)r + \beta, \quad r > 0$, where α, β, δ are defined in (3.3). According to Assumption $(A5)$, the discriminant $\Delta = (\delta-1)^2 - 4\alpha\beta$ of the equation

$$\theta(r) = 0 \tag{3.12}$$

is a positive and $0 < \delta < 1$. If $0 < r_1 < r_2$ are the roots of this equation, then taking any $r_0 \in [r_1, r_2]$ gives $\|Fx + Gy\| \le r_0$, proving our claim.

Claim 2. There exists $\gamma \in [0, 1)$ such that $\mu(FX + GX) \le \gamma\mu(X)$ for all $X \subseteq B_{r_0}$. Let X be a nonempty subset of B_{r_0}, $\varepsilon > 0$, and D a nonempty measurable subset of $\mathbb{R}_+$ with $\operatorname{meas}(D) \le \varepsilon$. Then for all $x, y \in X$, we have the estimate

$$\begin{aligned}
&\int_D |Fx(t) + Gy(t)|dt \\
&\le \int_D |f(t, x(t))|dt + \int_D \Big|f_1\Big(t, \int_0^t v_1(t, s, y(s))ds\Big)\Big|\Big|f_2\Big(t, \int_0^t v_2(t, s, y(s))ds\Big)\Big|dt \\
&\le \|\varphi\|_{L^1(D)} + c\|x\|_{L^1(D)} + \int_D \Big[\varphi_1(t) + c_1 \int_0^t k_1(t, s)[a_1(s) + b_1|y(s)|]ds\Big] \\
&\times \Big[\varphi_2(t) + c_2 \int_0^t \Big(k_2(t, s)[a_2(s) + b_2|y(s)|\Big)ds\Big]dt \\
&\le \int_D \varphi(t)\,dt + c\|x\|_{L^1(D)} + \Big[\int_D \varphi_1(t)\,dt + c_1\|K_1\|\Big(\int_D a_1(t)\,dt + b_1\|y\|_{L^1(D)}\Big)\Big] \\
&\quad \times [\|\varphi_2\|_\infty + c_2\|K_2\|(\|a_2\| + b_2\, r_0)].
\end{aligned}$$

Using Theorem 2.7, we obtain

$$\lim_{\varepsilon \to 0} \Big(\sup\Big\{\int_D \xi(t)dt : D \subset \mathbb{R}_+, \operatorname{meas}(D) \le \varepsilon\Big\}\Big) = 0, \tag{3.13}$$

where ξ is any one of the functions φ, φ_1, a_1. Hence

$$c(FX + GX) \leq \gamma\, c(X), \tag{3.14}$$

where

$$\gamma := c + b_1 c_1 \|K_1\| \left[\|\varphi_2\|_\infty + c_2 \|K_2\| \left(\|a_2\| + b_2 r_0 \right) \right]. \tag{3.15}$$

Let us fix an arbitrary positive number T. Then for any functions $x, y \in X$, we have

$$\begin{aligned}
&\int_T^\infty |Fx(t) + Gy(t)| dt \\
&\leq \int_T^\infty |f(t, x(t))| dt + \int_T^\infty \Big| f_1\Big(t, \int_0^t v_1(t,s,y(s)) ds\Big) \Big| \\
&\quad \times \Big| f_2\Big(t, \int_0^t v_2(t,s,y(s)) ds\Big) \Big| dt \\
&\leq \int_T^\infty \varphi(t) dt + c \int_T^\infty |x(t)| dt \\
&\quad + \int_T^\infty \Big| \Big(\varphi_1(t) + c_1 \int_0^t (k_1(t,s)[a_1(s) + b_1|y(s)|]) ds\Big) \\
&\quad \times \Big(\varphi_2(t) + c_2 \int_0^t \big(k_2(t,s)[a_2(s) + b_2(s)|y(s)|]\big) ds\Big) \Big| dt \\
&\leq \int_T^\infty \varphi(t)\, dt + c \int_T^\infty |x(t)| dt \; + \Big(\int_T^\infty \varphi_1(t)\, dt + c_1 \|K_1\| \Big(\int_T^\infty a_1(t) dt \\
&\quad + b_1 \int_T^\infty |y(t)| dt \Big) \Big) \Big(\|\varphi_2\|_\infty + c_2 \|K_2\| (\|a_2\| + b_2 r_0) \Big).
\end{aligned} \tag{3.16}$$

A single set of L^1 being weakly relatively compact, by applying Dunford-Pettis theorem 2.7 with ξ any one of the functions $\varphi(t), \varphi_1(t)$, and $a_1(t)$, we find that

$$\lim_{T \to \infty} \int_T^{+\infty} \xi(t) dt = 0.$$

Hence

$$d(FX + GX) \leq \gamma\, d(X), \quad \text{for all } X \subset B_{r_0}. \tag{3.17}$$

Finally, adding (3.14) and (3.17) leads to

$$\mu(FX + GX) \leq \gamma\, \mu(X), \quad \text{for all } X \subset B_{r_0}. \tag{3.18}$$

Let

$$\eta = c + b_1 c_1 \|K_1\| \|\varphi_2\|_\infty + b_1 c_1 c_2 \|K_1\| \|K_2\| \|a_2\|.$$

Using notation (3.3), the constant γ in (3.15) may be rewritten as

$$\gamma = \eta + \alpha r_0 = \eta + \big(1 - \delta - \frac{\beta}{r_0}\big),$$

where r_0 is any root of the quadratic equation (3.12). Since $0 < \eta < \delta$, we deduce that $0 < \gamma < 1$, showing that $F + G$ is a strict γ-set contraction, as claimed.

Claim 3. Operator G is (ws)-compact. Let $(x_n)_n$ be a weakly convergent sequence in B_{r_0}. From Lemma 3.2, there exists a subsequence $(x_{n_k})_k$ and two functions $g_1, g_2 \in L^1(\mathbb{R}_+)$ such that the sequences $(G_1 x_{n_k})_k$ and $(G_2 x_{n_k})_k$ converge to g_1 and g_2 respectively for the L^1 norm. By Theorem 2.2, we can find a subsequence $(x_{n_{k'}})_{k'}$ of $(x_{n_k})_k$ such that $(G_2 x_{n_{k'}})_{k'}$ converges to g_2, as $k' \to \infty$, for a.e. $t \in \mathbb{R}_+$.

By straightforward computations, we obtain that g_2 is essentially bounded. Indeed for all integer k' and a.e. $t \in \mathbb{R}_+$ we have

$$\begin{aligned} \big|(G_2 x_{n_{k'}})(t)\big| &\le \varphi_2(t) + c_2 \int_0^t \Big(k_2(t,s)[a_2(s) + b_2|x_{n_{k'}}(s)|\Big)ds \\ &\le \|\varphi_2\|_\infty + c_2\|K_2\|\big(\|a_2\| + b_2 r_0\big) := M. \end{aligned} \tag{3.19}$$

Hence $\|G_2 x_{n_{k'}}\|_\infty \le M$. With the triangle and Hölder's inequalities, we deduce the following estimates:

$$\begin{aligned} &\|Gx_{n_{k'}} - g_1 g_2\| \\ &\le \|(G_1 x_{n_{k'}})(G_2 x_{n_{k'}}) - (G_2 x_{n_{k'}})g_1\| + \|(G_2 x_{n_{k'}})g_1 - g_1 g_2\|_1 \\ &\le \|G_2 x_{n_{k'}}\|_\infty \|G_1 x_{n_{k'}} - g_1\| + \|(G_2 x_{n_{k'}})g_1 - g_1 g_2\| \\ &\le M\|G_1 x_{n_{k'}} - g_1\| + \|(G_2 x_{n_{k'}})g_1 - g_1 g_2\|. \end{aligned} \tag{3.20}$$

Since for a.e. $t \in \mathbb{R}_+$, we have

$$\big|(G_2 x_{n_{k'}})(t) g_1(t) - g_1(t) g_2(t)\big| \le 2M|g_1(t)| \in L^1(\mathbb{R}_+),$$

and an application of Lebesgue's Dominated Convergence Theorem implies

$$\|(G_2 x_{n_{k'}})g_1 - g_1 g_2\| \to 0, \quad \text{as } k' \to +\infty. \tag{3.21}$$

Hence

$$\|Gx_{n_{k'}} - g_1 g_2\| \to 0, \quad \text{as } k' \to +\infty. \tag{3.22}$$

Then G is (ws)-compact. Finally Assumption (A1) guarantees that F is a separate contraction mapping and Theorem 2.11 completes the proof of Theorem 3.3. $\square$

Example 3.4. Consider the nonlinear integral equation of product type

$$\begin{aligned} x(t) &= \frac{1}{\pi(1+t^2)} + \frac{x^2(t)}{10(1+|x(t)|)} \\ &\quad + \Big(\frac{\exp(-t)}{10(1+t)} + \int_0^t \frac{1}{ts + \lambda + x^2(s)} \ln(1 + x^2(s))ds\Big) \\ &\quad \times \Big(\frac{\cos(t)}{1+t^2} + \int_0^t \exp(-(t+s))\Big(\frac{1}{\pi(1+s^2)} + \sin(x(s))\Big)ds\Big), \end{aligned} \tag{3.23}$$

for $t > 0$. Note that (3.23) is a special case of (1.4) where we have set

$$\begin{gathered} f(t,x) = \frac{1}{\pi(1+t^2)} + \frac{x^2}{10(1+|x|)}, \\ f_1(t,x) = \frac{\exp(-t)}{10(1+t)} + x, \quad f_2(t,x) = \frac{\cos(t)}{1+t^2} + x, \\ v_1(t,s,x) = \frac{1}{ts + \lambda + x^2} \ln(1+x^2), \\ v_2(t,s,x) = \exp(-(t+s))\Big(\frac{1}{\pi(1+s^2)} + \sin(x)\Big), \\ k_1(t,s) = \frac{1}{ts + \lambda}, \quad k_2(t,s) = \exp(-(t+s)). \end{gathered}$$

By simple calculations, we can check that all of Assumptions (A1)–(A5) are fulfilled for every $\lambda > \lambda_0 = \big(\frac{5\pi}{128}\big)^2(\sqrt{13} + \sqrt{37})^4$. As a consequence, by Theorem 3.3, Equation (3.23) has at least one integrable solution, for all $\lambda > \lambda_0$.

4. (L^p, L^q) PRODUCT TYPE INTEGRAL EQUATION

In what follows, let $m \geq 2$ be an integer and $p_i \in (1, +\infty)$ $(i = 1, \dots, m)$ satisfy $\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1$. Consider the product functional integral equation

$$x(t) = f(t, x(t)) + \prod_{i=1}^{m} f_i\Big(t, \int_0^t v_i(t, s, x(s))ds\Big), \quad t \in \mathbb{R}_+ \tag{4.1}$$

and set

(A2') For $i = 1, \dots, m$, the functions $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ $(i = \overline{1, m})$ satisfy Carathéodory's conditions and there exist a function $\varphi_i \in L^{p_i}(\mathbb{R}_+)$ and positive constants c_i such that

$$|f_i(t, x)| \leq \varphi_i(t) + c_i|x|^{1/p_i}, \quad \text{for a.e. } t, s \in \mathbb{R}_+ \text{ and all } x \in \mathbb{R}.$$

(A4') The linear Volterra operator K_i $(i = 1, \dots, m)$ maps continuously the space $L^1(\mathbb{R}_+)$ into itself. $\|K_i\|$ denotes the norm of the linear operator K_i.

(A5') $c + \prod_{i=1}^m c_i(b_i\|K_i\|)^{1/p_i} < 1$,

Theorem 4.1. *Under Assumptions* (A1), (A2'), (A3), (A4'), (A5'), *equation* (4.1) *has at least one integrable solution on* $\mathbb{R}_+$.

Proof. Note that, in view of our assumptions, Theorem 2.5 assures that the Nemytskii operator F is continuous from L^1 into L^1 while G_i $(i = 1, \dots m)$ is continuous from L^1 into L^{p_i} $(i = 1, \dots, m)$. In addition the generalized Hölder inequality implies that the operator $G := \prod_{i=1}^m G_i : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$ is well defined and thus the operator $F + G$ is also well defined from $L^1(\mathbb{R}_+)$ into itself. Observe further that for any measurable set $\Omega \subseteq \mathbb{R}_+$ and for $x, y \in L^1(\mathbb{R}_+)$, by Hölder's inequality (2.1) we have the estimates:

$$\begin{aligned}
&\|Fx + Gy\|_{L^1(\Omega)} \\
&\leq \int_\Omega |f(t, x(t))|dt + \int_\Omega \prod_{i=1}^m \Big|f_i\Big(t, \int_0^t v_i(t, s, y(s))ds\Big)\Big|dt \\
&\leq \|\varphi\|_{L^1(\Omega)} + c\|x\|_{L^1(\Omega)} \\
&\quad + \prod_{i=1}^m \Big(\int_\Omega \Big|f_i\Big(t, \int_0^t v_i(t, s, y(s))ds\Big)\Big|^{p_i} dt\Big)^{1/p_i} \\
&\leq \|\varphi\|_{L^1(\Omega)} + c\|x\|_{L^1(\Omega)} \\
&\quad + \prod_{i=1}^m \Big(\|\varphi_i\|_{L^{p_i}(\Omega)} + \Big(\int_\Omega \int_0^t \Big(k_i(t, s)[a_i(s) + b_i|y(s)|]\Big)ds\, dt\Big)^{1/p_i}\Big) \\
&\leq \|\varphi\|_{L^1(\Omega)} + c\|x\|_{L^1(\Omega)} \\
&\quad + \prod_{i=1}^m \Big[\|\varphi_i\|_{L^{p_i}(\Omega)} + c_i\|K_i\|^{1/p_i}\Big(\|a_i\|_{L^1(\Omega)} + b_i\|y\|_{L^1(\Omega)}\Big)^{1/p_i}\Big].
\end{aligned} \tag{4.2}$$

Claim 1. There exists $r_0 > 0$ such that $F(B_{r_0}) + G(B_{r_0}) \subseteq B_{r_0}$. From (4.2), for $x, y \in B_r$ we have

$$\begin{aligned}
&\|Fx + Gy\| \\
&\leq \|\varphi\| + cr + \prod_{i=1}^m \Big(\|\varphi_i\|_{p_i} + c_i\|K_i\|^{1/p_i}\Big(\|a_i\|_{L^1(\mathbb{R}_+)} + b_i r\Big)^{1/p_i}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \|\varphi\| + cr + \prod_{i=1}^{m} r^{1/p_i}\Big(\frac{\|\varphi_i\|_{p_i}}{r^{1/p_i}} + c_i\|K_i\|^{1/p_i} b_i^{1/p_i}\Big(\frac{\|a_i\|}{b_i r} + 1\Big)^{1/p_i}\Big)\\
&= \|\varphi\| + cr + r\prod_{i=1}^{m}\Big(\frac{\|\varphi_i\|_{p_i}}{r^{1/p_i}} + c_i\|K_i\|^{1/p_i} b_i^{1/p_i}\Big(\frac{\|a_i\|}{b_i r} + 1\Big)^{1/p_i}\Big).
\end{aligned}
$$

By Assumption (A5'), we conclude that

$$
\lim_{r\to+\infty} \|\varphi\| + c\,r + r\prod_{i=1}^{m}\Big(\frac{\|\varphi_i\|_{p_i}}{r^{1/p_i}} + c_i\|K_i\|^{1/p_i} b_i^{1/p_i}\Big(\frac{\|a_i\|}{b_i r} + 1\Big)^{1/p_i}\Big) - r = -\infty.
$$

Consequently some positive number r_0 exists and satisfies $\|Fx + Gy\| \le r_0$, for all $x, y \in B_{r_0}$.

Claim 2. There exists $\gamma \in [0,1)$ such that $\mu(FX+GX) \le \gamma\mu(X)$, for all $X \subseteq B_{r_0}$. Let X be a nonempty subset of B_{r_0}, $\varepsilon > 0$, and D a nonempty measurable subset of $\mathbb{R}_+$ with $\operatorname{meas}(D) \le \varepsilon$. Using (4.2), we obtain for $x, y \in X$

$$
\begin{aligned}
\|Fx + Gy\|_{L^1(D)} \le \|\varphi\|_{L^1(D)} + c\|x\|_{L^1(D)} + \prod_{i=1}^{m}\Big[\|\varphi_i\|_{L^{p_i}(D)}\\
+ c_i\|K_i\|^{1/p_i}\Big(\|a_i\|_{L^1(D)} + b_i\|y\|_{L^1(D)}\Big)^{1/p_i}\Big].
\end{aligned} \tag{4.3}
$$

Letting $\varepsilon \to 0$ and taking into account the fact that the single sets $\{\varphi\}, \{|\varphi_i|^{p_i}\}$, and $\{a_i\}$ are weakly relatively compact in L^1, we obtain that

$$
c(FX + GX) \le \gamma\, c(X),
$$

where

$$
\gamma := c + \prod_{i=1}^{m} c_i(b_i\|K_i\|)^{1/p_i} < 1. \tag{4.4}
$$

Similarly, for each $T > 0$, we have

$$
\begin{aligned}
&\|Fx + Gy\|_{L^1([T,+\infty[)}\\
&\le \|\varphi\|_{L^1([T,+\infty[)} + c\|x\|_{L^1([T,+\infty[)} + \prod_{i=1}^{m}\Big[\Big(\int_T^{+\infty} |\varphi_i|^{p_i}(t)dt\Big)^{1/p_i}\\
&\quad + c_i\|K_i\|^{1/p_i}\Big(\int_T^{+\infty} |a_i(t)|dt + b_i\int_T^{+\infty} |y(t)|dt\Big)^{1/p_i}\Big].
\end{aligned} \tag{4.5}
$$

Letting $T \to +\infty$, we obtain $c(FX + GX) \le \gamma c(X)$. Hence

$$
\mu(FX + GX) \le \gamma\,\mu(X), \quad \forall X \subseteq B_{r_0}.
$$

Claim 3. Operator $G : L^1 \to L^1$ is (ws)-compact. To see that G is continuous take a sequence $(x_n)_n$ converging to some limit $x \in L^1$. Since $G_i : L^1 \to L^{p_i}$ are continuous, we conclude that for each $1 \le i \le m$,

$$
\lim_{n\to\infty} \|G_i x_n - G_i x\|_{p_i} = 0. \tag{4.6}
$$

Moreover, by Hölder's inequality, we infer that the sequence $\big(\prod_{i=2}^{m} G_i x_n\big)_n$ converges to $\prod_{i=2}^{m} G_i x$ in L^r-norm with $\frac{1}{r} = \frac{1}{p_2} + \frac{1}{p_3} + \dots \frac{1}{p_m}$. Hence there exists some

$M > 0$ with $\Big\| \prod_{i=2}^m G_i x_n \Big\|_r \leq M$, for all integer n. As a consequence

$$\|Gx_n - Gx\| \leq M\|G_1x_n - G_1x\|_{p_1} + \|G_1x\|_{p_1}\Big\|\prod_{i=2}^m G_i x_n - \prod_{i=2}^m G_i x\Big\|_r, \tag{4.7}$$

showing that G is continuous.

Let $(y_n)_n$ be a weakly convergent sequence in $L^1(\mathbb{R}_+)$. Then the set $X = \{y_n : n \in \mathbb{N}\}$ is relatively weakly compact, hence bounded for the L^1-norm. As a result, some positive constant r exists and satisfies $\|y_n\| \leq r$, for all integer n. Let $\varepsilon > 0$. Since $G(X)$ is weakly relatively compact, Dunford-Pettis Theorem 2.7 guarantees the existence of some positive constants T and δ such that for each closed subset $D \subset \mathbb{R}_+$ with $\operatorname{meas}(D) \leq \delta$ and all integer $n \in \mathbb{N}$, we have

$$\int_D |Gy_n(t)|dt + \int_T^\infty |(Gy_n)(t)|dt \leq \frac{\varepsilon}{4}. \tag{4.8}$$

Theorem 2.6 implies the existence of a closed subset D_ε of the interval $[0,T]$ satisfying $\operatorname{meas}([0,T] \setminus D_\varepsilon) \leq \varepsilon$ and such that the functions φ_i, k_i, v_i, and f_i for $(i = 1, \dots, m)$ are continuous on the sets D_ε, $D_\varepsilon \times [0,T], D_\varepsilon \times [0,T] \times \mathbb{R}$, and $D_\varepsilon \times \mathbb{R}$ respectively.

We show that the set $G(X)$ is relatively compact in $L^1(\mathbb{R}_+)$. From (3.5) and Assumption (A2'), we deduce that for each $n \in \mathbb{N}$ and for each $t \in D_\varepsilon$, we have

$$\begin{aligned} \Big|\int_0^t v_i(t,s,y_n(s))ds\Big| &\leq \int_0^t \Big(k_i(t,s)[a_i(s) + b_i|y_n(s)|]\Big)ds \\ &\leq \overline{k_i}\big(\|a_i\| + b_i r\big) := \overline{K_i}(\varepsilon). \end{aligned} \tag{4.9}$$

Hence

$$\Big|f_i\Big(t, \int_0^t v_i(t,s,y_n(s))ds\Big)\Big| \leq \overline{\varphi_i} + c_i\big(\overline{K_i}(\varepsilon)\big)^{1/p_i} := \overline{G_i}(\varepsilon). \tag{4.10}$$

This proves that for each $i = 1, \dots m$, the set $G_i(X)$ is equibounded on D_ε. Arguing as in Lemma 3.2, we can see that the sequences $\big(\int_0^t v_i(t,s,y_n(s))ds\big)_n$ is equicontinuous on D_ε. Since the function $g := \prod_{i=1}^m f_i : \mathbb{R}^{2m} \to \mathbb{R}$ defined by

$$g(x_1, \dots, x_{2m}) = \prod_{i=1}^m f_i(x_{2i-1}, x_{2i})$$

is uniformly continuous on the product $\prod_{i=1}^m D_\varepsilon \times [-\overline{K_i}(\varepsilon), \overline{K_i}(\varepsilon)]]$, then the set $G(X)$ is equicontinuous and equibounded on D_ε. By Ascoli-Arzela Theorem, the set $G(X)$ is relatively strongly compact in $C(D_\varepsilon)$. Consequently, for each integer $p \in \mathbb{N}$, there exists a closed subset D_p of $[0,T]$ with $\operatorname{meas}([0,T]\backslash D_p) \leq \frac{1}{p}$ such that $G(X)$ is relatively compact in $C(D_p)$. Moreover there exists $p_0 \geq 1$ such that $\operatorname{meas}([0,T]\backslash D_{p_0}) \leq \delta$. Therefore the sequence $(G(y_n))_n$ has a subsequence, still denoted $(G(y_n))_n$, which converges with respect to the standard norm of $C(D_{p_0})$. Then some integer $n_0 \in \mathbb{N}$ exists and satisfies that for all $m, n \geq n_0$ and for every $t \in D_{p_0}$:

$$|G(y_n)(t) - G(y_m)(t)| \leq \frac{\varepsilon}{1 + 2\operatorname{meas}(D_{p_0})}. \tag{4.11}$$

From (4.8) and (4.11), we deduce the estimates:

$$\int_0^\infty |G(y_n)(t) - G(y_m)(t)|dt$$

$$\begin{aligned} &\le \int_{D_{p_0}} |G(y_n)(t) - G(y_m)(t)|dt + \int_{[0,T]\setminus D_{p_0}} |G(y_n)(t)|dt \\ &\quad + \int_{[0,T]\setminus D_{p_0}} |G(y_m)(t)|dt + \int_T^\infty |G(y_n)(t) - G(y_m)(t)|dt \le \varepsilon. \end{aligned}$$

We conclude that $(G(y_n))_n$ is a Cauchy sequence in the Banach space $L^1(\mathbb{R}_+)$, proving that $G(X)$ is strongly relatively compact. Finally G is (ws)-compact, which completes the proof. □

Remark 4.2. A comparison between conditions (A5) and (A5') shows that the first one derived from an algebraic quadratic equation is optimal for existence of solution in case of (L^1, L^∞) product operators. However, the second condition, derived from a first-order inequality is a sufficient condition for existence. In this respect, it is to point out that Theorem 4.1 does not encompass Theorem 3.3.

5. Absolutely continuous solutions for a nonlinear integro-differential equation of product type

In this section, we study the nonlinear integro-differential equation of product type in the space $AC([a, b])$ $(a < b)$:

$$\begin{gathered} x'(t) = f(t, x(t)) + \Big(\alpha(t) + \int_0^t v_1(t, s, x(s))ds\Big) \\ \times \Big(\beta(t) + \int_0^t v_2(t, s, x(s))ds\Big), \\ x(0) = x_0. \end{gathered} \tag{5.1}$$

Consider the following assumptions:

- (A6) The function $\alpha \in L^1([a, b])$ and $\beta \in L^\infty([a, b])$.
- (A7) The function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory's conditions and there exist a function $\phi \in L^1([a, b])$ and a positive constant c such that
$$|f(t, x)| \le \phi(t) + c|x|,$$
for a.e. $t \in [a, b]$ and for all $x \in \mathbb{R}$.
- (A8) The functions $v_1,\ v_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory's conditions and there exist a constant $b_i > 0$ and two functions $a_i \in L^1([a, b])$ $(i = 1, 2)$ such that
$$|v_i(t, s, x)| \le k_i(t, s)\big(a_i(s) + b_i|x|\big),$$
for a.e. $t, s \in [a, b]$, where $k_i : [a, b] \times [a, b] \to \mathbb{R}$, $(i = 1, 2)$ satisfy Carathéodory's conditions.
- (A9) The linear Volterra operator K_1 transforms the space $L^1([a, b])$ into itself and K_2 transforms continuously the space $L^1([a, b])$ into $L^\infty([a, b])$, where
$$K_i x(t) = \int_0^t k_i(t, s)x(s)ds, \quad t \in [a, b] \quad (i = 1, 2).$$
Let $\|K_i\|$ be a norm of the linear operator K_i.

Solving (5.1) is equivalent to finding a fixed point of the operator Q defined on the space $L^1([a, b])$ into itself by

$$Qx(t) = x_0 + \int_0^t f(s, x(s))ds + \int_0^t \big(\alpha(s) + V_1x(s)\big)\big(\beta(s) + V_2x(s)\big)ds, \tag{5.2}$$

where

$$V_i x(s) = \int_0^s v_i(s,\tau,x(\tau))d\tau \quad (i=1,2). \tag{5.3}$$

Theorem 5.1. *Assume* (A6)-(A9) *and that*

(A10)
$$\begin{aligned} &2\Big(b_1 b_2\|K_1\|\|K_2\|\Big[|x_0| + \|\phi\| + \big(\|\alpha\| + \|K_1\|\|a_1\|\big)\big(\overline{\beta} \\ &+ \|K_2\|\|a_2\|\big)\Big]\Big)^{1/2} + c + b_1\|K_1\|\big(\overline{\beta} + \|K_2\|\|a_2\|\big) \\ &+ b_2\|K_2\|\big(\|\alpha\| + \|K_1\|\|a_1\|\big) \\ &< \frac{1}{b-a}, \end{aligned} \tag{5.4}$$

where $\overline{\beta} = \operatorname{ess\,sup}_{t\in[a,b]} \beta(t)$*.*

Then the nonlinear integro-differential equation (5.1) *has a solution x in the space $AC([a,b])$.*

Proof. We show that $Q : L^1([a,b]) \to L^1([a,b])$ satisfies all hypotheses of Theorem 2.12.

Claim 1. There exists a ball $B_{r_0} = B(0,r_0)$ in $L^1([a,b])$ such that $Q(B_{r_0}) \subseteq B_{r_0}$. To see this, pick an arbitrary $x \in B_r$ for some positive constant r and observe that:

$$\begin{aligned} &\|Qx\|_{L^1([a,b])} \\ &\le \int_a^b |x_0|dt + \int_a^b\int_a^t |f(s,x(s))|ds\,dt + \int_a^b\int_a^t \Big|\alpha(s) + \int_a^s v_1(s,\tau,x(\tau))d\tau\Big| \\ &\quad\times \Big|\beta(s) + \int_a^s v_2(s,\tau,x(\tau))d\tau\Big|ds\,dt \\ &\le (b-a)\left(|x_0| + \|\phi\| + cr\right) + (b-a)\big[\overline{\beta} + \|K_2\|\big(\|a_2\| + b_2\,r\big)\big] \\ &\quad\times \big[\|\alpha\| + \|K_1\|\big(\|a_1\| + b_1\,r\big)\big]. \end{aligned} \tag{5.5}$$

Hence $\big\|Qx\big\|_{L^1([a,b])} \le r$ whenever $\varsigma(r) \le 0$, where

$$\begin{aligned} \varsigma(r) &= b_1 b_2\|K_1\|\|K_2\|\,r^2 + |x_0| + \|\phi\| + \big(\|\alpha\| + \|K_1\|\|a_1\|\big)\big(\overline{\beta} + \|K_2\|\|a_2\|\big) \\ &\quad + \Big[c + b_1\|K_1\|\big(\overline{\beta} + \|K_2\|\|a_2\|\big) + b_2\|K_2\|\big(\|\alpha\| + \|K_1\|\|a_1\|\big) - \frac{1}{b-a}\Big]r. \end{aligned}$$

From Assumption (A10), it suffices to choose

$$0 < r_0 = \frac{\frac{1}{b-a} - \big[c + b_1\|K_1\|\big(\overline{\beta} + \|K_2\|\|a_2\|\big) + b_2\|K_2\|\big(\|\alpha\| + \|K_1\|\|a_1\|\big)\big] + \sqrt{\Delta}}{2b_1 b_2\|K_1\|\|K_2\|},$$

where $\Delta > 0$ is the discriminant of the quadratic equation $\varsigma(r) = 0$.

Claim 2. The set $Q(B_{r_0})$ is relatively weakly compact. Take an arbitrary $\varepsilon > 0$ and a measurable subset D of $[a,b]$ such that $\operatorname{meas}(D) \le \varepsilon$. For each $x \in B_{r_0}$, arguing as in Claim 1, we obtain

$$\begin{aligned} \int_D |Qx(t)|dt &\le \operatorname{meas}(D)[|x_0| + (\|\phi\| + cr_0)] + \operatorname{meas}(D)\big[\overline{\beta} + \|K_2\|\big(\|a_2\| + b_2 r_0\big)\big] \\ &\quad\times \big[\|\alpha\| + \|K_1\|\big(\|a_1\| + b_1 r_0\big)\big]. \end{aligned}$$

Hence
$$\lim_{\varepsilon\to 0}\Big(\sup\Big\{\int_D |Qx(t)|dt : D\subset [a,b], \operatorname{meas}(D)\le\varepsilon\Big\}\Big)=0.$$
Consequently $Q(B_{r_0})$ is a weakly relatively compact subset of $L^1[a,b]$.

Claim 3. Operator $Q : L^1[a,b]\to L^1[a,b]$ is (ws)-compact. In view of assumptions and Theorem 2.5, Q is a continuous operator. Consider an arbitrary weakly convergence sequence (x_n) in $L^1[a,b]$; then there exists $r>0$ such that $\|x_n\|\le r$, for all $n\in\mathbb{N}$. Without loss of generality, let $t_1,t_2\in[a,b]$ be such that $t_1<t_2$. Then for each integer n, we have the estimate
$$\begin{aligned}
&|Qx_n(t_2)-Qx_n(t_1)|\\
&\le \Big|\int_a^{t_2} f(s,x_n(s))ds-\int_a^{t_1} f(s,x_n(s))ds\Big|\\
&\quad+\Big|\int_a^{t_2}\Big(\alpha(s)+\int_a^s v_1(s,\tau,x_n(\tau))d\tau\Big)\Big(\beta(s)+\int_a^s v_2(s,\tau,x_n(\tau))d\tau\Big)ds\\
&\quad-\int_a^{t_1}\Big(\alpha(s)+\int_a^s v_1(s,\tau,x_n(\tau))d\tau\Big)\Big(\beta(s)+\int_a^s v_2(s,\tau,x_n(\tau))d\tau\Big)ds\Big|\\
&\le \int_{t_1}^{t_2}|f(s,x_n(s))|ds+\int_{t_1}^{t_2}\Big|\Big(\alpha(s)+\int_a^s v_1(s,\tau,x_n(\tau))d\tau\Big)\\
&\quad\times\Big(\beta(s)+\int_a^s v_2(s,\tau,x_n(\tau))d\tau\Big)\Big|ds\\
&\le \int_{t_1}^{t_2}\phi(s)ds+c\int_{t_1}^{t_2}|x_n(s)|ds\\
&\quad+\int_{t_1}^{t_2}\Big[\alpha(s)+\int_a^s\Big(k_1(s,r)[a_1(\tau)+b_1|x_n(\tau)|\Big)d\tau\Big]\\
&\quad\times\Big[\beta(s)+\int_0^t\Big(k_2(s,\tau)[a_2(\tau)+b_2|x_n(\tau)|\Big)d\tau\Big]ds\\
&\le \int_{t_1}^{t_2}\phi(s)ds+c\int_{t_1}^{t_2}|x_n(s)|ds\\
&\quad+\big[\overline{\beta}+\|K_2\|\big(\|a_2\|+b_2r_0\big)\big]\int_{t_1}^{t_2}\big[\alpha(s)+\|K_1\|\big(a_1(s)+b_1|x_n(s)|\big)ds\big].
\end{aligned}$$

Since a single set of $L^1[a,b]$ is weakly relatively compact and from the relative weakly compactness of the set $\{x_n : n\in\mathbb{N}\}$, we conclude that the terms $|Qx_n(t_2)-Qx_n(t_1)|$ are arbitrarily small provided that the number t_2-t_1 is small enough. Hence the sequence $(Qx_n)_n$ is equicontinuous on $[a,b]$. Moreover for each $t\in[a,b]$ and for all $n\in\mathbb{N}$, we have
$$\begin{aligned}
|Qx_n(t)| &\le |x_0|+\int_a^t|f(s,x_n(s))|ds+\int_a^t\Big|\alpha(s)+\int_0^s v_1(s,\tau,x_n(\tau))d\tau\Big|\\
&\quad\times\Big|\beta(s)+\int_a^s v_2(s,\tau,x_n(\tau))d\tau\Big|ds\\
&\le |x_0|+\int_a^b\left(\phi(s)+c\,|x_n(s)|\right)ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_a^b\Big[\alpha(s)+\int_a^s\Big(k_1(s,r)[a_1(\tau)+b_1|x_n(\tau)|\Big)d\tau\Big]\\
&\times\big[\beta(s)+\int_a^t\big(k_2(s,\tau)[a_2(\tau)+b_2|x_n(\tau)|\big)d\tau\big]ds\\
\le{}& |x_0|+\|\phi\|+c\,r_0+\big[\overline{\beta}+\|K_2\|\big(\|a_2\|+b_2r_0\big)\big]\\
&\times\big[\|\alpha\|+\|K_1\|\big(\|a_1\|+b_1r_0\big)\big].
\end{aligned}
$$

This proves that the sequence $(Qx_n)_n$ is equibounded on $[a,b]$. By Ascoli Arzela Theorem, the sequence $(Qx_n)_n$ has a convergent subsequence with respect to the sup-norm. Therefore it is convergent in $L^1[a,b]$. This implies that operator Q is (ws)-compact. Finally, all conditions Theorem 2.12 are fulfilled. Hence equation (5.2) has at lest one solution $x\in L^1[a,b]$. Since the functions $f(\cdot,x(\cdot))$ and $(\alpha(\cdot)+V_1x(\cdot))(\beta(\cdot)+V_2x(\cdot))$ are integrable on $[a,b]$, we infer that the solution x is absolutely continuous on $[a,b]$. □

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