Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 117, pp. 1–32.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

INITIAL VALUE PROBLEMS FOR CAPUTO FRACTIONAL EQUATIONS WITH SINGULAR NONLINEARITIES

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Abstract. We consider initial value problems for Caputo fractional equations of the form $D_C^\alpha u = f$ where f can have a singularity. We consider all orders and prove equivalences with Volterra integral equations in classical spaces such as $C^m[0,T]$. In particular for the case $1 < \alpha < 2$ we consider nonlinearities of the form $t^{-\gamma} f(t, u, D_C^\beta u)$ where $0 < \beta \le 1$ and $0 \le \gamma < 1$ with f continuous, and we prove results on existence of global C^1 solutions under linear growth assumptions on $f(t, u, p)$ in the u, p variables. With a Lipschitz condition we prove continuous dependence on the initial data and uniqueness. One tool we use is a Gronwall inequality for weakly singular problems with double singularities. We also prove some regularity results and discuss monotonicity and concavity properties.

1. Introduction

The study of fractional integrals and fractional differential equations has expanded dramatically in recent years, there are now literally thousands of research papers dealing with various versions of fractional derivatives.

In this paper we will discuss Initial Value Problems (IVPs) mainly for the Caputo fractional derivative, but also for the Riemann-Liouville fractional derivative, the two fractional derivative that are most commonly used, both are defined in terms of the Riemann-Liouville fractional integral. There are relatively few recent papers on IVPs, as compared with the number dealing with boundary value problems, since many results can be found in textbooks such as [7, 17, 24].

Our main goal, achieved in Section 8, is to prove a global existence theorem for initial value problems for Caputo fractional differential equations involving a nonlinear term with a singularity and depending on lower order fractional derivatives. In particular for fractional derivatives of order between 1 and 2, we treat in detail the following problem for Caputo fractional derivatives in the space $C^1[0,T]$

$$D^\alpha u(t) = t^{-\gamma} f(t, u(t), D^\beta u(t)), \quad u(0) = u_0, u'(0) = u_1, \tag{1.1}$$

for $0 \le \gamma < 1$, $1 < \alpha < 2$ and $0 < \beta \le 1$ when f is continuous. We will prove a global existence result under the assumption $|f(t, u, p)| \le a(t) + M(|u| + |p|)$ for some $a \in L^\infty$ and constant $M > 0$. Under a Lipschitz condition, with no restriction on the size of the Lipschitz constant, we also prove continuous dependence on the initial

2010 *Mathematics Subject Classification.* 34A08, 34A12, 26A33, 26D10.
Key words and phrases. Fractional derivatives; Volterra integral equation; weakly singular kernel; Gronwall inequality.

Submitted November 20, 2018. Published October 30, 2019.

data and uniqueness. An important tool we employ is a Gronwall inequality in a weakly singular case. For weakly singular Gronwall type inequalities the pioneering work was by Henry [14] who proved, by an iterative process, some L^1 bounds given by series related to the Mittag-Leffler function. References are often given to the paper [28] which used Henry's method to replace a constant by a nondecreasing function which can in fact be simply deduced from the original result in [14]. For a similar inequality involving an integral with a doubly singular kernel we proved in [27] some L^∞ bounds which involve the exponential function. Medved [21] proved some L^∞ inequalities of a different type by use of Hölder's inequality. With a similar method to that of Medved some other inequalities were given by Zhu [29].

For a real number $\alpha \in (0,1)$ the Riemann-Liouville fractional integral of order α is defined informally as an integral with a singular kernel by

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds.$$

When $u \in L^1$ the definition becomes precise if equality is understood to hold in the L^1 sense and so it holds for almost every (a.e.) t.

It is frequently claimed that finding solutions of a fractional differential equation is *equivalent* to finding solutions of a Volterra integral equation. For example, for the IVP for a Caputo fractional derivative of order α with $0 < \alpha < 1$ with f continuous,

$$D_C^\alpha u(t) = f(t, u(t)), \quad u(0) = u_0, \tag{1.2}$$

and the Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, ds \tag{1.3}$$

it is often claimed that u is a solution of (1.2) *if and only if* u is a solution of (1.3). Apart from the fact that 'solution' means different things for the two problems and is often not made precise, there is a more serious issue. There are, in fact, two commonly used definitions of Caputo derivative, recalled below, which we will denote by D_C^α and D_*^α, the second one we will call the *modified Caputo derivative*, and an equivalence has been proved for only the second of these definitions, a fact that has often been overlooked.

In this paper we will give precise definitions and prove equivalences for all order fractional derivative cases in a more general case, when the nonlinearity is of the form $t^{-\gamma} f$, with $0 \le \gamma < 1$ and f is continuous, which, of course, includes the previous case.

We believe our work that allows the singular term $t^{-\gamma}$ in the nonlinearity, especially the treatment of (1.1) is new.

We give some properties of the Riemann-Liouville integral in Section 3, some of which may have some novelty, they seem to be not as well known as they should be. We include proofs of some known results for completeness.

In Sections 4,5, and 6 we give some equivalences between solutions of IVPs and solutions of corresponding integral equations.

In Section 7 we discuss the relationship between an increasing function and the positivity of its Caputo fractional derivative of order $\alpha \in (0,1)$ with a singularity allowed, only one direction of implication is valid. We also discuss concavity properties for Caputo and Riemann-Liouville fractional derivatives of orders $\alpha \in (1,2)$, in particular we give counter-examples to some claims in recently published papers.

We turn to existence of solutions of the IVP in Section 8. Kosmatov [18] studied the solvability of integral equations associated with the initial value problem $D_C^{\alpha}u(t) = f(t, D_C^{\beta}u(t))$ (of all orders α) and depends on the fractional derivative of lower order, assuming that the nonlinear term f is continuously differentiable. The case studied by Kosmatov was continued in [6] which uses similar hypotheses. Our method uses the Gronwall type inequality of [27] to obtain a priori bounds with fewer restrictions. Also we allow $f = f(t, u, D^{\beta}u)$ to also depend explicitly on u and we have the extra singular term $t^{-\gamma}$.

A local existence theorem for fractional equations in the special case $\gamma = 0$ is given in Diethelm [7, Theorem 6.1] when f is continuous. A global existence result is proved in [7, Corollary 6.3] when it is assumed that $\gamma = 0$ and f is continuous and there exist constants $c_1 > 0$, $c_2 > 0$, $0 \le \mu < 1$ such that $|f(t, u)| \le c_1 + c_2|u|^{\mu}$ but that result does not allow $\mu = 1$. Since, for $0 < \mu < 1$, $|u|^{\mu} \le 1 + |u|$ our result includes that one and covers the case $\mu = 1$. Under a Lipschitz condition [7, Theorem 6.8] proves an existence and uniqueness result by a very different argument to ours.

Some existence results for the case $0 < \alpha < 1$ were given by this author in the previous paper [27]. Li and Sarwar [20] also considered the IVP of order $0 < \alpha < 1$ with nonlinearity $t^{-\gamma}f$, they first prove a local existence theorem, then a continuation result to get global existence under the same type of condition as in [27] but only the case $\gamma = 0$ is treated in the global result. Also they use the first definition of Caputo derivative so this is an example where the claimed equivalence with the Volterra integral equation is not valid.

Eloe and Masthay [12] consider an initial-value problem for the modified Caputo fractional derivative of order $\alpha \in (n-1, n]$ with a nonlinearity which depends on classical, not fractional, derivatives of order at most $n-1$. They establish a Peano type local existence theorem, a Picard type existence and uniqueness theorem, and some results related to maximal intervals of smooth solutions.

We prove some regularity results in Section 9, the solution can have more regularity when f is more regular, but there is a limit to what can be obtained, see Theorem 9.1 for the details.

We end the paper by making some remarks on the implications for boundary value problems of the equivalences between IVPs and Volterra integral equations.

2. Preliminaries

For simplicity we consider functions defined on an arbitrary finite interval $[0, T]$, which is, by a simple change of variable, equivalent to any finite interval. For this case we use simpler notations for fractional derivatives than are frequently used. In this paper all integrals are Lebesgue integrals and $L^1 = L^1[0, T]$ denotes the usual space of Lebesgue integrable functions.

In the study of fractional integrals and fractional derivatives the Gamma and Beta functions occur frequently. The Gamma function is, for $p > 0$, given by

$$\Gamma(p) := \int_0^{\infty} s^{p-1} \exp(-s)\, ds \tag{2.1}$$

which is an improper Riemann integral but is well defined as a Lebesgue integral, and is an extension of the factorial function: $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. The Beta function is defined by

$$B(p,q) := \int_0^1 (1-s)^{p-1} s^{q-1}\, ds \tag{2.2}$$

which is a well defined Lebesgue integral for $p > 0, q > 0$ and it is well known, and proved in calculus texts, that $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

The following simple lemma is elementary and classical. Since it is useful to us we sketch the proof for completeness.

Lemma 2.1. *Let $0 \le \tau < t$ and $p > 0$, $q > 0$. Then we have*

$$\int_\tau^t (t-s)^{p-1}(s-\tau)^{q-1}\, ds = (t-\tau)^{p+q-1} B(p,q). \tag{2.3}$$

Proof. Change the variable of integration from s to σ where $s = \tau + \sigma(t-\tau)$ and the integral becomes

$$\begin{aligned} &\int_0^1 \big((1-\sigma)(t-\tau)\big)^{p-1} \big(\sigma(t-\tau)\big)^{q-1} (t-\tau)\, d\sigma \\ &= (t-\tau)^{p+q-1} \int_0^1 (1-\sigma)^{p-1} \sigma^{q-1}\, d\sigma \\ &= (t-\tau)^{p+q-1} B(p,q). \end{aligned}$$

$\square$

The space of functions that are continuous on $[0,T]$ is denoted by $C[0,T]$ or sometimes simply C or C^0 and is endowed with the supremum norm $\|u\|_\infty := \max_{t\in[0,T]} |u(t)|$. For $n \in \mathbb{N}$ we will write $C^n = C^n[0,T]$ to denote those functions u whose n-th derivative $u^{(n)}$ is continuous on $[0,T]$.

We will also use the space of absolutely continuous functions which is denoted $AC = AC[0,T]$. For $n \in \mathbb{N}$, $AC^n = AC^n[0,T]$ will denote those functions u whose n-th derivative $u^{(n)}$ is in $AC[0,T]$, hence $u^{(n+1)}(t)$ exists for a.e. t and is an L^1 function. A note of caution: some authors denote this space as AC^{n+1}.

The space AC is the appropriate space for the fundamental theorem of the calculus for Lebesgue integrals. In fact, we have the following equivalence.

$$\begin{gathered} u \in AC[0,T] \quad \text{if and only if } u'(t) \text{ exists for a.e. } t \in [0,T] \\ \text{with } u' \in L^1[0,T] \text{ and } u(t) - u(0) = \int_0^t u'(s)\, ds \text{ for all } t \in [0,T]. \end{gathered} \tag{2.4}$$

If $f \in L^1$ and $If(t) := \int_0^t f(s)\, ds$ then $If \in AC$ and $(If)'(t) = f(t)$ for a.e. t. But if g is a continuous function and $g' \in L^1$ exists a.e. it does not follow that $g \in AC$, as shown for example by the well-known Lebesgue's singular function F (also known as the Cantor-Vitali function, or Devil's staircase) which is continuous on $[0,1]$ and has zero derivative a.e., but is not AC, in fact $F(0) = 0$, $F(1) = 1$ and thus $F(1) - F(0) \ne \int_0^1 F'(s)\, ds$.

We write $g \in \mathrm{Lip}$ and say that g is Lipschitz (or satisfies a Lipschitz condition) if there is a constant $L > 0$ such that $|g(u) - g(v)| \le L|u - v|$ for all $u, v \in \mathrm{dom}(g)$.

The following facts are well known (on a bounded interval).

$$C^1 \subset \mathrm{Lip} \subset AC \subset \text{ differentiable a.e.},$$

$$AC \subset \text{ uniformly continuous} \subset C^0.$$

It is also known that, on a bounded interval $[0,T]$, the sum and pointwise product of functions in AC belong to AC and if $u \in AC$ and $g \in \mathrm{Lip}$ then the composition $g \circ u \in AC$, but the composition of AC functions need not be AC.

We also have the following positive result, which may be known but we have not seen it in the literature, when v is assumed to be 'almost AC', $v' \in L^1$ exists a.e. so v' can only blow up at 0 in an integrable manner.

Proposition 2.2. *Let $v \in C[0,T]$ be such that $v'(t) = f(t)$ for a.e. $t \in [0,T]$ where $f \in L^1[0,T]$ and suppose that $v \in AC[\delta, T]$ for every $\delta > 0$. Then $v \in AC[0,T]$.*

Proof. Since $v' = f \in L^1$ we have to prove that $v(t) - v(0) = \int_0^t f(s)\,ds$ for every $t \in [0,T]$. This is obviously true for $t = 0$, so suppose $t > 0$. Let $\varepsilon > 0$ and, since $f \in L^1$, let $0 < \delta < t$ be chosen so that $\int_0^\delta |f(s)|\,ds < \varepsilon$. As v is continuous at 0, by choosing δ smaller if necessary, we can suppose that $|v(\delta) - v(0)| < \varepsilon$. Since $v \in AC[\delta, T]$ with $v' = f$ a.e. we have $v(t) - v(\delta) = \int_\delta^t f(s)\,ds$. We then have

$$\begin{aligned}
&\big|v(t) - v(0) - \int_0^t f(s)\,ds\big| \\
&= \big|v(t) - v(\delta) + v(\delta) - v(0) - \int_0^\delta f(s)\,ds - \int_\delta^t f(s)\,ds\big| \\
&= \big|v(\delta) - v(0) - \int_0^\delta f(s)\,ds\big| < 2\varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary this proves that $v(t) - v(0) = \int_0^t f(s)\,ds$. $\square$

Remark 2.3. The hypotheses of Proposition 2.2 hold if $v \in C[0,T] \cap C^1(0,T]$ and $v' \in L^1$. An example is, for $0 < \gamma < 1$,

$$v(t) = \begin{cases} t^\gamma \ln(t), & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$

When studying fractional integrals and derivatives, functions such as $t^{\alpha-1}$ arise where typically $0 < \alpha < 1$. This leads to consideration of a weighted space of functions that are continuous except at $t = 0$ and have an integrable singularity at $t = 0$. For $\gamma > -1$ we define the space denoted $C_\gamma = C_\gamma[0,T]$ by

$$C_\gamma[0,T] := \{u \in C(0,T] \text{ such that } \lim_{t \to 0+} t^{-\gamma} u(t) \text{ exists}\},$$

then $u \in C_\gamma$ if and only if $u(t) = t^\gamma v(t)$ for some function $v \in C[0,T]$ and we define $\|u\|_\gamma := \|v\|_\infty$. The spaces of functions with singularity at $t = 0$ are $C_{-\gamma}$ where $\gamma > 0$. The space C_0 coincides with the space $C^0 = C[0,T]$. Clearly, for $\gamma > 0$ the space C_γ is a subspace of $C[0,T]$. We also define the space

$$C_{1,\gamma}[0,T] := \{u \in C(0,T] \text{ such that } u(t) = t^\gamma v(t) \text{ for some } v \in C^1[0,T]\}.$$

Note that $u \in C_{1,\gamma}$ is AC if $\gamma \geq 0$ but need not be a C^1 function if $0 < \gamma < 1$.

3. Riemann-Liouville fractional integrals

Some authors 'define' $I^\alpha u$ by:

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \quad \text{provided the integral exists.}$$

This does not specify which functions are being considered and leaves open whether the integral is to exist for all t, or for all nonzero t, or for a.e. t. A precise definition for integrable functions is the following.

Definition 3.1. The Riemann-Liouville (R-L) fractional integral of order $\alpha > 0$ of a function $u \in L^1[0,T]$ is defined for a.e. t by

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds.$$

The integral $I^\alpha u$ is the convolution of the L^1 functions h, u where $h(t) = t^{\alpha-1}/\Gamma(\alpha)$, so by the well known results on convolutions $I^\alpha u$ is defined as an L^1 function, in particular $I^\alpha u(t)$ is finite for a.e. t. If $\alpha = 1$ this is the usual integration operator which we denote I. We set $I^0 u = u$.

The R-L fractional integral operator has the following properties, which do not seem to be as well known as they deserve to be, some seem to be new and perhaps some of our proofs are new. Some of these say that I^α (partially) removes singularities at $t = 0$. The trickiest cases are when $0 < \alpha < 1$ and the integrand is singular. References are given in the Remark following the proofs.

Proposition 3.2. *Let $\alpha > 0$ and $0 \le \gamma < 1$.*

(1) *I^α is a linear operator defined on L^1. For $1 \le p \le \infty$, I^α is a bounded operator from L^p into L^p and*
$$\|I^\alpha u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|u\|_{L^p}.$$
(2) *For $1 \le p < 1/\alpha$, I^α is a bounded operator from $L^p[0,T]$ into $L^r[0,T]$ for $1 \le r < p/(1-\alpha p)$. If $1 < p < 1/\alpha$, then I^α is a bounded operator from $L^p[0,T]$ into $L^r[0,T]$ for $r = p/(1-\alpha p)$.*
(3) *For $1/p < \alpha < 1 + 1/p$ or $p = 1$ and $1 \le \alpha < 2$, the fractional integral operator I^α is bounded from L^p into a Hölder space $C^{0,\alpha-1/p}$, hence, for $u \in L^p$, $I^\alpha u$ is Hölder continuous with exponent $\alpha - 1/p$, thus $I^\alpha u$ is continuous. Moreover, $I^\alpha u(t) \to 0$ as $t \to 0+$, that is $I^\alpha u(0) = 0$.*
(4) *I^α is a bounded operator from $C_{-\gamma}[0,T]$ into $C_{\alpha-\gamma}[0,T]$. Moreover we have $\|I^\alpha u\|_{\alpha-\gamma} \le \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \|u\|_{-\gamma}$.*
(5) *If $0 \le \gamma \le \alpha < 1$ then I^α is a bounded operator from $C_{-\gamma}[0,T]$ into $C[0,T]$. Moreover, if $u(t) = t^{-\gamma} v(t)$ where $v \in C[0,T]$ then $\lim_{t\to 0+} I^\alpha u(t) = 0$ if either $\gamma < \alpha$ or $v(0) = 0$.*
(6) *I^α maps $AC[0,T]$ into $AC[0,T]$.*
(7) *If $0 < \gamma \le \alpha < 1$ (or if $\gamma = 0$ and $0 < \alpha < 1$) and $u' \in C_{-\gamma}$ then $I^\alpha u \in C^1[0,T]$ if and only if $u(0) = 0$. However, I^α does not map $C^1[0,1]$ into $C^1[0,1]$ in general, in fact it does not map C^∞ into C^1.*
(8) *for $m \in \mathbb{N}$, and $0 \le \gamma \le \alpha < 1$, $I^{m+\alpha}$ maps $C_{-\gamma}[0,T]$ into $C^m[0,T]$.*
(9) *I^α maps $C_{1,-\gamma}$ into $C_{1,\alpha-\gamma}$ and maps $C_{1,-\gamma}$ into AC if $\alpha \ge \gamma$.*

(10) *If $u \in L^1$ and u is non-decreasing function of t then $I^\alpha u(t)$ is also a non-decreasing function of t.*

Proof. (1) It is clear that I^α acts linearly on u and, by known results on convolutions $I^\alpha u$ is defined as an L^1 function, in particular $I^\alpha u(t)$ is finite for a.e. t.

The proof uses Young's convolution theorem (this can be found in many texts, for example [10, Chapter 5, Theorem 1.2]):
If $1 \le p, q, r \le \infty$ and $1 + 1/r = 1/p + 1/q$, then for $h \in L^q$, $u \in L^p$, it follows that $h * u \in L^r$ and $\|h * u\|_r \le \|h\|_q \|u\|_p$.

We have $I^\alpha u = h * u$ for $h(t) = t^{\alpha-1}/\Gamma(\alpha)$ and $h \in L^1$ since $\alpha > 0$. Taking $r = p, q = 1$ gives

$$\|I^\alpha u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|u\|_{L^p}.$$

The case $p = \infty$ is simple: for $u \in L^\infty$ the integral for $I^\alpha u$ is well defined and we have

$$\begin{aligned} I^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u\|_\infty\, ds \\ &= \frac{1}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} \|u\|_\infty, \end{aligned}$$

hence $\|I^\alpha u\|_\infty \le \dfrac{T^\alpha}{\Gamma(\alpha+1)} \|u\|_\infty$.

(2) We only give the case $r < p/(1-\alpha p)$, see the remark below for the case of equality. Take $h(t) = t^{\alpha-1}/\Gamma(\alpha)$ as in (1), and again apply Young's convolution theorem. We have $h \in L^q$ if $q(\alpha-1) > -1$, that is $q < 1/(1-\alpha)$ and hence $1/r > 1/p - \alpha$, that is $r < p/(1-\alpha p)$.

(3). Since we do not consider Hölder continuity in this paper we do not give any proof concerning Hölder spaces here, see the references in the remark below. For completeness we give a short proof of the last part, which is known from [1]. We have

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds.$$

Let $q = p/(p-1)$ be the conjugate exponent of p. Note that for fixed t, $s \mapsto (t-s)^{\alpha-1} \in L^q[0,t]$ since $(\alpha-1)q > -1$. Therefore by Hölder's inequality we have

$$\begin{aligned} |I^\alpha u(t)| &\le \frac{1}{\Gamma(\alpha)} \Big(\int_0^t (t-s)^{(\alpha-1)q}\, ds \Big)^{1/q} \Big(\int_0^t |u|^p(s)\, ds \Big)^{1/p}, \\ &= \frac{1}{\Gamma(\alpha)} \Big(\frac{t^{(\alpha-1)q+1}}{(\alpha-1)q+1} \Big)^{1/q} \Big(\int_0^t |u|^p(s)\, ds \Big)^{1/p}, \end{aligned}$$

and both terms in the product have a zero limit as $t \to 0+$.

(4) For $u \in C_{-\gamma}$ we have $u(t) = t^{-\gamma} v(t)$ for some function $v \in C[0,T]$ and $\|u\|_{-\gamma} = \|v\|_\infty$. So we have

$$\begin{aligned} I^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} v(s)\, ds, \\ &= \frac{1}{\Gamma(\alpha)} t^{\alpha-\gamma} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{-\gamma} v(t\sigma)\, d\sigma. \end{aligned} \tag{3.1}$$

Since v is continuous on $[0,T]$ it is bounded, say $\|v\|_\infty = M$, and we have

$$\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{-\gamma} v(t\sigma)\,d\sigma \le MB(\alpha, 1-\gamma)$$

Therefore $\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{-\gamma} v(t\sigma)\,d\sigma$ is a continuous function of t by the dominated convergence theorem, thus $I^\alpha u \in C_{\alpha-\gamma}[0,T]$. Moreover,

$$\|I^\alpha u\|_{\alpha-\gamma} \le \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{-\gamma}\,d\sigma \|v\|_\infty = \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\|u\|_{-\gamma}.$$

(5) Let $0 \le \gamma \le \alpha$. By part (3), I^α maps $C_{-\gamma}$ into $C_{\alpha-\gamma} \subset C[0,T]$ and from (3.1) we obtain $\|I^\alpha u\|_\infty \le T^{\alpha-\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\|u\|_{-\gamma}$.

Also from (3.1) we obtain $\lim_{t\to 0+} I^\alpha u(t) = 0$ if $\gamma < \alpha$. By the dominated convergence theorem this also holds if $\gamma = \alpha$ and $v(0) = 0$.

(6) For $u \in AC$, $u' \in L^1$ exists a.e. and $u(t) - u(0) = Iu'(t)$ for all t. Then

$$I^\alpha u(t) = I^\alpha\, Iu'(t) + I^\alpha u(0) = I\, I^\alpha u'(t) + u(0)t^\alpha/\Gamma(\alpha+1) \tag{3.2}$$

where $I^\alpha u' \in L^1$ so the first term is in AC, and the second term is also in AC since $\alpha > 0$.

(7) When $u' \in C_{-\gamma}$, we have $I^\alpha u' \in C_{\alpha-\gamma} \subset C$ by part (4), hence $I(I^\alpha u') \in C^1$ and from (3.2) we see that $I^\alpha u \in C^1[0,T]$ if and only if $u(0) = 0$. For the last part, taking $v(t) \equiv 1$ we have $v \in C^\infty$ and $I^\alpha v = t^\alpha/\Gamma(\alpha+1)$ is an AC function but is not in C^1 for $\alpha < 1$.

(8) This follows at once since $I^{m+\alpha}u = I^m I^\alpha u$ where $I^\alpha u \in C$ by part (5).

(9) Let $u \in C_{1,-\gamma}[0,1]$ so that $u(t) = t^{-\gamma}v(t)$ for some $v \in C^1[0,1]$. Then we have as above

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-\gamma}\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{-\gamma} v(t\sigma)\,d\sigma,$$

where, by differentiation under the integral sign,

$$\frac{d}{dt}\Big(\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{-\gamma} v(t\sigma)\,d\sigma\Big) = \int_0^1 (1-\sigma)^{\alpha-1}\sigma^{1-\gamma} v'(t\sigma)\,d\sigma.$$

Since v' is continuous the integral on the right is a continuous function of t by the dominated convergence theorem. Hence

$$\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{-\gamma} v(t\sigma)\,d\sigma$$

is in $C^1[0,1]$,that is $I^\alpha u \in C_{1,\alpha-\gamma}$; the example $v(t) \equiv 1$ shows this is optimal. Also, if $\alpha \ge \gamma$, $I^\alpha u$ is $t^{\alpha-\gamma}$ multiplied by a C^1 function so is in AC.

(10) We have

$$\begin{aligned} I^\alpha u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds \\ &= \frac{1}{\Gamma(\alpha)} t^\alpha \int_0^1 (1-\sigma)^{\alpha-1}u(t\sigma)d\sigma, \end{aligned}$$

and this is a non-decreasing function of t when u is non-decreasing. $\square$

Remark 3.3. Part (1) is stated in [17, Lemma 2.1 a] and Theorem 2.6 of [24] states: “may be verified by simple operations using the generalized Minkowski's inequality”.

Part (2) The first part for $p \geq 1$ is stated in [17, Lemma 2.1 b] and is proved in [24, Theorem 3.5] by another method. The fact that I^α is bounded from L^p (with $p > 1$) into L^r with the equality $r = p/(1-\alpha p)$ was proved by Hardy and Littlewood [13, Theorem 4]. Hardy and Littlewood show that the result does not hold if $p = 1$ for any $\alpha \in (0,1)$, and they also show that if $0 < \alpha < 1$ and $p = 1/\alpha$ the potentially plausible result is false, that is, $I^\alpha u$ is not necessarily bounded.

Part (3) This was proved by Hardy and Littlewood in [13, Theorem 12]. The proof can be found in [7, Theorem 2.6] and in [24, Theorem 3.6]. The Hölder space $C^{0,\lambda}$ has other notations in these references. Hardy and Littlewood point out that the result is not true, in the cases $p > 1$, $\alpha = 1/p$, and $\alpha = 1 + 1/p$. The continuity of $I^\alpha u$ is proved in [1, Lemma 2.2] by a direct method using Hölder's inequality and the last part is also proved in [1, Lemma 2.1] as in the given proof. The result is also a consequence of Corollary to Theorem 3.6 of [24].

Part (4) is stated in [17, Lemma 2.8] with α, γ complex numbers, the proof is referred to [16], a paper in Russian.

Part (5), the first part is stated in [17, Lemma 2.8]; the last part may be novel.

Part (6), this is proved in [24, Lemma 2.1], with a different notation, by a longhand version of the same proof, and is proved in [19, Lemma 2.3] by using Proposition 3.6 below.

Part (7) seems to be new for $\gamma > 0$, the case $\gamma = 0$ is known, see [7, Theorem 6.26]. The last part is well known and is pointed out in [7, Example 6.4].

Part (8) should be known but we do not have a reference, it improves [12, Lemma 2.4] which has the case $\gamma = 0$ and a different longer proof.

Part (9) This seems to be new when $\gamma \neq 0$.

Part (10) is presumably well known but we do not know a reference.

Interchanging the order of integration, using Fubini's theorem, shows that these fractional integral operators satisfy a semigroup property as follows.

Lemma 3.4. *Let $\alpha, \beta > 0$ and $u \in L^1[0,T]$. Then $I^\alpha I^\beta(u) = I^{\alpha+\beta}(u)$ as L^1 functions, thus, $I^\alpha I^\beta(u)(t) = I^{\alpha+\beta}(u)(t)$ for a.e. $t \in [0,T]$. If u is continuous, or if $u \in C_{-\gamma}$ and $\alpha + \beta \geq \gamma$, this holds for all $t \in [0,T]$. If $u \in L^1$ and $\alpha + \beta \geq 1$ equality again holds for all $t \in [0,T]$.*

Proof. For $u \in L^1$ the fractional integrals $I^\beta u$ and $I^{\alpha+\beta} u$ exist as L^1 functions so are finite almost everywhere. For each t for which $I^{\alpha+\beta}|u|(t)$ exists (finite), that is a.e. t, we have

$$
\begin{aligned}
&\Gamma(\alpha)\Gamma(\beta) I^\alpha(I^\beta u)(t) \\
&= \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s (s-\tau)^{\beta-1} u(\tau)\, d\tau \Big)\, ds \\
&= \int_0^t u(\tau) \Big(\int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{\beta-1}\, ds \Big)\, d\tau, \quad \text{by Fubini's theorem,} \\
&= \int_0^t (t-\tau)^{\alpha+\beta-1} u(\tau) B(\alpha,\beta)\, d\tau, \quad \text{by Lemma 2.1,}
\end{aligned}
$$

which proves the first part by the relationship between the Beta and Gamma functions stated earlier. When u is continuous all terms are continuous, see Proposition 3.2 part (5), so equality holds for all t. For $u \in C_{-\gamma}$, by Proposition 3.2 part (4), $I^\beta u \in C_{\beta-\gamma}$ and $I^\alpha(I^\beta(u)) \in C_{\alpha+\beta-\gamma} \subset C^0$ when $\alpha+\beta \ge \gamma$, so both sides are continuous functions and equality holds for all t. For the last part if $\beta \ge 1$ or $\alpha \ge 1$ then the terms on both sides are continuous so the only case to consider is $0 < \alpha, \beta < 1$. When $\alpha + \beta \ge 1$ we have $I^{\alpha+\beta}(u)(t) = I(I^{\alpha+\beta-1}u)(t)$ and the right side of this equation is AC so $I^{\alpha+\beta}u(t)$ exists for every t and equals $I^\alpha I^\beta u(t)$ by the first part. □

Remark 3.5. Some of this is proved in [7, Theorem 2.2] and in [24, (2.21)]. The part concerning $u \in C_{-\gamma}$ seems to be new. Also we make the observation that when $\alpha+\beta > 1$ and $u \in L^1$ both sides are also Hölder continuous by using Proposition 3.2 parts (2) and (3). Note that there are functions that are Hölder continuous but not AC and AC functions that are not Hölder continuous.

For $\alpha \in (0, 1)$ we will see that in discussing fractional differential equations via the corresponding Volterra integral equation it is necessary to have $I^{1-\alpha}u \in AC$. The following result gives conditions for this to hold; the result is known, it is contained in the proof of [24, Theorem 2.1].

Proposition 3.6. *Let $u \in L^1[0, T]$ and $\alpha \in (0, 1)$. Then $I^{1-\alpha}u \in AC$ and $I^{1-\alpha}u(0) = 0$ if and only if there exists $f \in L^1$ such that $u = I^\alpha f$.*

Proof. Suppose that there exists $f \in L^1$ such that $u = I^\alpha f$. Then, by Lemma 3.4

$$I^{1-\alpha}u = I^{1-\alpha}I^\alpha f = If \in AC,$$

and $I^{1-\alpha}u(0) = \lim_{t\to 0+} \int_0^t f(s)\,ds = 0$, since $f \in L^1$. Conversely suppose that $I^{1-\alpha}u \in AC$ and $I^{1-\alpha}u(0) = 0$. Let $F(t) := I^{1-\alpha}u(t)$ so that $F \in AC$ and $F(0) = 0$. Then $f := F'$ exists for a.e. t with $f \in L^1$, and $F(t) = If = \int_0^t f(s)\,ds$. From $F(t) := I^{1-\alpha}u(t)$ we have $I^\alpha F = Iu$ that is $Iu = I^\alpha\, If = I\, I^\alpha f$. By the definition (see Proposition 3.2 (1), (2) we have $I^\alpha f \in L^1$ and since $u \in L^1$ both Iu and $I\, I^\alpha f$ are absolutely continuous so their derivatives exist a.e. as L^1 functions and are equal, that is $u = I^\alpha f$. □

4. Fractional derivatives of order $\alpha \in (0, 1)$

Let D denote the usual differentiation operator. The Riemann-Liouville (R-L) fractional derivative of order $\alpha \in (0, 1)$ is defined as follows.

Definition 4.1. For $\alpha \in (0, 1)$ and $u \in L^1$ the R-L fractional derivative $D^\alpha u$ is defined when $I^{1-\alpha}u \in AC$ by

$$D^\alpha u(t) := D\, I^{1-\alpha}u(t), \text{ a.e. } t \in [0, T].$$

For $D\, I^{1-\alpha}u(t)$ to be defined for a.e. t, it is necessary that $I^{1-\alpha}u$ should be differentiable a.e., but we do not believe that alone is sufficient, and our discussions below show that it is necessary to always have $I^{1-\alpha}u \in AC$ in considering IVPs for R-L fractional differential equations via a Volterra integral equation, thus we make this requirement. This has been noted in the monograph [24], see [24, Definition 2.4] and the related comments in the 'Notes to §2.6'. Clearly $DI^{1-\alpha}u$ exists a.e. if $u = I^\alpha f$ for some $f \in L^1$, using Lemma 3.4, but Proposition 3.6 already implies that $I^{1-\alpha}u \in AC$.

It follows using Lemma 3.4 that the R-L derivative D^α is the left inverse of I^α, as shown, for example, in [7, Theorem 2.14].

Lemma 4.2. *Let $0 < \alpha \le 1$. Then, for every $h \in L^1$, $D^\alpha I^\alpha h(t) = h(t)$ for almost every t.*

Proof. Since $Ih \in AC$ we have

$$D^\alpha I^\alpha h(t) = D\, I^{1-\alpha}\, I^\alpha h(t) = D\, Ih(t) = h(t), \quad \text{for a.e. } t. \qquad \square$$

In general fractional derivatives do not commute, see [7, Examples 2.6, 2.7].

The Caputo fractional derivative is defined with the derivative and fractional integral taken in the reverse order to that of the R-L derivative.

Definition 4.3. For $\alpha \in (0,1)$ and $u \in AC$ the Caputo fractional derivative $D_C^\alpha u$ is defined for a.e. t by

$$D_C^\alpha u(t) := I^{1-\alpha} Du(t).$$

For $u \in AC$, $Du \in L^1$ and so $D_C^\alpha u = I^{1-\alpha}(Du)$ is defined as an L^1 function. The modified Caputo derivative is defined by $D_*^\alpha u := D^\alpha(u - u(0))$ whenever this R-L derivative exists, that is when $u(0)$ exists and $I^{1-\alpha} u \in AC$.

There is a connection between the R-L and Caputo derivatives for functions with some regularity.

Proposition 4.4. *Let $u \in AC$ and let u_0 denote the constant function with value $u(0)$. For $0 < \alpha < 1$ we have*

$$D_*^\alpha u(t) = D^\alpha(u - u_0)(t) = D_C^\alpha u(t), \quad \text{for a.e. } t. \tag{4.1}$$

Proof. Since $u \in AC$, by Proposition 3.2 (6), $I^{1-\alpha} u \in AC$ so the R-L derivative exists and we have

$$\begin{aligned} D_*^\alpha u = D^\alpha(u - u_0) &= DI^{1-\alpha}(u - u_0) \\ &= DI^{1-\alpha} Iu', \quad \text{since } u \in AC, \\ &= DI\, I^{1-\alpha} u', \quad \text{by Lemma 3.4 as } u' \in L^1, \\ &= I^{1-\alpha} u', \quad \text{since } I^{1-\alpha} u' \in L^1, \\ &= D_C^\alpha u. \qquad \square \end{aligned}$$

The result is a special case of the result given for fractional derivatives of all orders in [7, Theorem 3.1], and which is proved below in Lemma 4.10.

The modified Caputo derivative D_*^α is the left inverse of I^α for **continuous** functions; for the higher order case see [7, Theorem 3.7].

Lemma 4.5. *Let $\alpha \in (0,1)$ and let u be continuous, then $D_*^\alpha I^\alpha u(t) = u(t)$ for $t \in [0,T]$.*

Proof. By Proposition 3.2 (5) with $\gamma = 0$, $I^\alpha u$ is continuous and $I^\alpha u(0) = 0$, thus

$$D_*^\alpha I^\alpha u(t) = D^\alpha(I^\alpha u - 0)(t) = DI^{1-\alpha}(I^\alpha u)(t) = D(Iu)(t) = u(t),$$

which is valid for every t since $u \in C$. $\square$

The Caputo derivatives do not commute in general but Diethelm has a positive result.

Lemma 4.6 ([7, Lemma 3.13]). *Let $f \in C^k[0,T]$ for some $k \in \mathbb{N}$. Moreover let $\alpha, \beta > 0$ be such that there exists some $\ell \in \mathbb{N}$ with $\ell \le k$ and $\alpha, \alpha+\beta \in [\ell-1, \ell]$. Then,*
$$D_*^\alpha D_*^\beta f = D_*^{\alpha+\beta} f.$$

Remark 4.7. Diethelm [7] notes that existence of ℓ is important, and gives the example $f(t) = t$, $\alpha = 7/10$, $\beta = 7/10$ to show that it can fail when this condition is not satisfied. Also he notes that such a result cannot be expected to hold in general for Riemann-Liouville derivatives.

4.1. **Fractional derivatives of higher order.** For higher order derivatives the definitions are as follows. For a positive integer k let D^k denote the ordinary derivative operator of order k, and for $n \in \mathbb{N}$ and a function u such that $D^k u(0)$ exists for $k = 0, \dots, n$ let $T_n u(t) := \sum_{k=0}^n \frac{t^k D^k u(0)}{k!}$ be the Taylor polynomial of degree n and define $T_0 u(t) = u(0)$.

Definition 4.8. Let $\beta \in \mathbb{R}_+$ and let $n = \lceil \beta \rceil$ be the smallest integer greater than or equal to β (the ceiling function acting on β). The Riemann-Liouville fractional differential operator of order β is defined when $D^{n-1}(I^{n-\beta} u) \in AC$, that is $I^{n-\beta} u \in AC^{n-1}$, by
$$D^\beta u := D^n I^{n-\beta} u.$$
The Caputo derivative is defined for $u \in AC^{n-1}$, by the equation
$$D_C^\beta u := I^{n-\beta} D^n u.$$
The modified Caputo derivative is defined when $I^{n-\beta} u \in AC^{n-1}$ and $T_{n-1} u$ exists by
$$D_*^\beta u = D^\beta (u - T_{n-1} u),$$
where $T_{n-1} u$ is the Taylor polynomial of degree $n-1$.
Under the given conditions each fractional derivative exists a.e.

Remark 4.9. Diethelm [7, Definition 3.2] calls D_*^β the *Caputo differential operator of order* β and thereafter uses that definition.

In the following results m always denotes a positive integer.

Lemma 4.10. *If $D^m u \in AC$ then for $0 < \alpha < 1$, $D_C^{m+\alpha} u$ exists, and we have*
$$D_C^{m+\alpha} u(t) = D_C^\alpha D^m u(t), \quad \text{for a.e. } t \in [0,T]. \tag{4.2}$$

Proof. By definition of Caputo derivative and the fact that $D^m u \in AC$ we have
$$D_C^{m+\alpha} u = I^{1-\alpha} D^{m+1} u = I^{1-\alpha} D(D^m u) = D_C^\alpha D^m u.$$
□

Lemma 4.11. *For $0 < \alpha < 1$ if $u \in AC[0,T]$ then $D_*^{m+\alpha} u(t) = D_*^{m+\alpha-1} u'(t)$ whenever both fractional derivatives exist.*

Proof. By definition,
$$\begin{aligned}
D_*^{m+\alpha} u(t) &= D^{m+1}(I^{1-\alpha}(u - T_m u))(t) \\
&= D^{m+\alpha}(I^{1-\alpha} I(u' - T_{m-1} u))(t) \\
&= D^{m+\alpha}(I I^{1-\alpha}(u' - T_{m-1} u))(t) \\
&= D^{m+\alpha-1}(I^{1-\alpha}(u' - T_{m-1} u))(t) \\
&= D_*^{m+\alpha-1} u'(t).
\end{aligned}$$
□

The following result is proved in Diethelm [7, Theorem 3.1] with a different proof using integration by parts.

Lemma 4.12. *Let $D^m u \in AC$ and $0 < \alpha < 1$. Then*

$$D_*^{m+\alpha} u = D_C^{m+\alpha} u. \tag{4.3}$$

Proof. Since $D^m u \in AC$, u and derivatives of order up to m are absolutely continuous and all required fractional derivatives exist. By repeated application of Lemma 4.11

$$\begin{aligned} D_*^{m+\alpha} u &= D^{m+\alpha-1}(u') = D_*^{m+\alpha-2} u'' = \dots \\ &= D_*^{\alpha}(D^m u) = D_C^{\alpha}(D^m u), \text{ by Proposition 4.4,} \\ &= D_C^{m+\alpha} u, \text{by Lemma 4.10.} \end{aligned}$$

□

5. IVP for Caputo derivative of all orders

We now turn to initial value problems for Caputo derivatives. For Caputo fractional differential equations with a singular nonlinearity we have the following relationships with a Volterra integral equation with a doubly singular kernel.

Theorem 5.1. *Let f be continuous on $[0,T] \times \mathbb{R}$, let $0 < \alpha < 1$ and let $0 \le \gamma < \alpha$. For $m \in \mathbb{N}$, if a function u with $D^m u \in AC$ satisfies the Caputo fractional initial value problem*

$$\begin{gathered} D_C^{m+\alpha} u(t) = t^{-\gamma} f(t, u(t)), \text{ a.e. } t \in (0,T], \\ u(0) = u_0,\ u'(0) = u_1, \dots, u^{(m)}(0) = u_m, \end{gathered} \tag{5.1}$$

then u satisfies the Volterra integral equation

$$u(t) = \sum_{k=0}^{m} \frac{t^k}{k!} u_k + \frac{1}{\Gamma(m+\alpha)} \int_0^t (t-s)^{m+\alpha-1} s^{-\gamma} f(s, u(s))\, ds,\ t \in [0,T]. \tag{5.2}$$

Secondly, if $u \in C[0,T]$ satisfies (5.2) then $u \in C^m[0,T]$, and $D_*^{m+\alpha} u$ exists a.e. and satisfies

$$\begin{gathered} D_*^{m+\alpha} u(t) = t^{-\gamma} f(t, u(t)) \quad \text{for a.e. } t, \\ u(0) = u_0, u'(0) = u_1, \dots, u^{(m)}(0) = u_m. \end{gathered} \tag{5.3}$$

Moreover, $I^{1-\alpha}(u - T_m(u)) \in AC^m$.

Thirdly, if $u \in C^m[0,T]$ and $I^{1-\alpha}(u - T_m(u)) \in AC^m$ and if u satisfies (5.3), then u satisfies (5.2).

Proof. Let $g(t) := t^{-\gamma} f(t, u(t))$, $t \in (0,T]$, and note that $g \in L^1$ since $0 \le \gamma \le \alpha < 1$. First we show the result for the special case $m = 0$. So suppose that $u \in AC$ and $D_C^{\alpha} u(t) = t^{-\gamma} f(t, u(t)) = g(t)$, for a.e. $t \in (0,T]$. Then, by the definition of Caputo derivative, we have $I^{1-\alpha}(Du) = g$ where $Du \in L^1$ since $u \in AC$. This yields $I^{\alpha} I^{1-\alpha}(Du) = I^{\alpha} g$, hence, by Lemma 3.4, $I(Du) = I^{\alpha} g$, that is $u(t) - u(0) = I^{\alpha} g$ since $u \in AC$.

Now we consider the case $m > 0$. Let u with $D^m u \in AC$ satisfy (5.1). Then using Lemma 4.10 we have a.e.,

$$\begin{aligned} D_C^{m+\alpha} u = g &\implies D_C^{\alpha}(D^m u) = g, \\ &\implies D^m u = u_m + I^{\alpha} g, \quad \text{by the special case just proved.} \end{aligned}$$

Integrating m times gives $u = \sum_{k=0}^{m} \frac{t^k}{k!} u_k + I^{m+\alpha} g$.

Secondly, let u be continuous and suppose that $u(t) = \sum_{k=0}^{m} \frac{t^k}{k!} u_k + I^{m+\alpha} g(t)$ with $g(t) = t^{-\gamma} f(t, u(t))$. To verify the initial conditions we observe that, for every $\beta \geq \alpha > \gamma$, setting $s = t\sigma$,

$$\begin{aligned} I^\beta g(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{-\gamma} f(s, u(s))\, ds, \\ &= \frac{1}{\Gamma(\beta)} \int_0^1 t^{\beta-\gamma} (1-\sigma)^{\beta-1} \sigma^{-\gamma} f(t\sigma, u(t\sigma))\, d\sigma \end{aligned}$$

exists for every t since $\int_0^1 (1-\sigma)^{\beta-1} \sigma^{-\gamma}\, d\sigma = B(\beta, 1-\gamma)$, and $f(t\sigma, u(t\sigma))$ is bounded by continuity of u and f. We also see that $I^\beta g(t)$ is a continuous function of $t \in [0,T]$. In particular, $I^\alpha g(t)$ is continuous and therefore $I^{m+\alpha} g \in C^m$, hence also $u \in C^m$. Furthermore for $\beta > \gamma$ we have

$$I^\beta g(0) = \lim_{t \to 0+} I^\beta g(t) = \frac{1}{\Gamma(\beta)} \lim_{t \to 0+} t^{\beta-\gamma} \int_0^1 (1-\sigma)^{\beta-1} \sigma^{-\gamma} f(t\sigma, u(t\sigma))\, d\sigma = 0,$$

and then taking $\beta = m + \alpha$ in the equation $u(t) = \sum_{k=0}^{m} \frac{t^k}{k!} u_k + I^{m+\alpha} g(t)$ we first obtain $u(0) = u_0$. By differentiation we have

$$u'(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} u_{k+1} + I^{m-1+\alpha} g(t),$$

and taking $\beta = m - 1 + \alpha$ we obtain $u'(0) = u_1$. Similarly, differentiating and evaluating we obtain $D^n u(0) = u_n$ for $n = 1, \dots, m$. Then we have

$$D_*^{m+\alpha} u = D^{m+\alpha}(u - T_m u) = D^{m+\alpha}\big(u - \sum_{k=0}^{m} \frac{t^k}{k!} u_k\big) = D^{m+\alpha} I^{m+\alpha} g = g.$$

This shows that $D^{m+1}(I^{1-\alpha}(u - T_m(u))) = g$ a.e..

Since we have

$$I^{1-\alpha}(u - T_m(u)) = I^{1-\alpha} I^{m+\alpha} g = I^{m+1} g = I^m I g$$

it follows that $D^m(I^{1-\alpha}(u - T_m(u))) = Ig \in AC$, that is, $I^{1-\alpha}(u - T_m(u)) \in AC^m$.

Thirdly, for $g(t) = t^{-\gamma} f(t, u(t))$, $g \in L^1$, since $I^{1-\alpha}(u - T_m(u)) \in AC^m$ the expression $D^{m+1}(I^{1-\alpha}(u - T_m(u)))(t) = g(t)$ can be integrated $m+1$ times to get

$$I^{1-\alpha}(u - T_m(u))(t) = I^{m+1} g(t) + a_0 + a_1 t + \dots + a_m t^m, \text{ for constants } a_i.$$

Applying I^α yields

$$I(u - T_m(u)) = I^{m+1+\alpha} g + b_0 t^\alpha + b_1 t^{1+\alpha} + \dots + b_m t^{m+\alpha},$$

for constants b_i whose precise values are not important here. Since $u \in C^m$ we have $I(u - T_m(u)) \in C^{m+1}$ and $I^{m+1+\alpha} g = I^{m+\alpha} I g \in C^{m+1}$ and therefore we must have $b_i = 0$ for every i. Then, we may differentiate to get $u - T_m(u) = I^{m+\alpha} g$. □

Remark 5.2. It has often been asserted (when $\gamma = 0$) that (5.1) is equivalent to (5.2), but it seems that the absolute continuity of solutions u of (5.2) has never been shown when f is at best continuous. The proved equivalence is:

if $u \in C^m[0,T]$ and $I^{1-\alpha}(u - T_m(u)) \in AC^m$ then u satisfies (5.3) if and only if u satisfies (5.2).

Some of these issues are discussed in the paper [19] where some positive results for boundary value problems involving the fractional derivative $D_C^{1+\alpha}u$ are obtained under a Lipschitz condition on f. For some more positive results see §9 below. It is important to have $I^{1-\alpha}(u - T_m(u)) \in AC^m$ in the third part of the theorem otherwise the integration in the last part is not valid. This is often implicit in the, often undefined, notion of 'solution' for the problem, for if $D_*^{m+\alpha}u$ exists and $D_*^{m+\alpha}u(t) = t^{-\gamma}f(t, u(t))$ when u, f are continuous then $t^{-\gamma}f(t, u(t)) \in L^1$ so that $D^{m+1}I^{1-\alpha}(u - T_m(u)) \in L^1$ and therefore $D^m I^{1-\alpha}(u - T_m(u)) \in AC$. When the term $t^{-\gamma}$ is absent and (5.3) is satisfied then $D^{m+1}I^{1-\alpha}(u - T_m(u))$ is continuous so $I^{1-\alpha}(u - T_m(u)) \in C^{m+1}$.

Remark 5.3. The case $m = 0$ is proved in the recent paper [27]. When there is no singular term $t^{-\gamma}$ and f is continuous, Diethelm [7, Lemma 6.2] proves the equivalence in the form that $u \in C[0, h]$ is solution of $D_*^{m+\alpha}u(t) = f(t, u(t))$ with initial conditions $D^k u(0) = u_k, k = 0, 1, \dots, m$, if and only if u is solution of

$$u(t) = \sum_{k=0}^{m} \frac{t^k}{k!}u_k + I^{m+\alpha}f(t).$$

There it is implicit that a solution of the fractional differential equation is a function for which $D_*^{m+\alpha}u$ exists, which requires more than continuity of u.

6. Initial value problems for the Riemann-Liouville fractional derivative

It has often been stated imprecisely that the fractional differential equation with the Riemann-Liouville fractional derivative is equivalent to an integral equation.

One such statement is as follows. Assume that $\alpha > 0$. Then u satisfies $D^\alpha u = f$ if and only if

$$u(t) = I^\alpha f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R},\ i = 1, 2, \dots n.$$

where $n = \lceil\alpha\rceil$, the smallest integer greater than or equal to α.

The difficulty is that it is not stated in what class of functions the solution is sought, one can consider $f \in L^1$ and seek solutions $u \in L^1$ or seek solutions in the weighted space $C_{\alpha-n}$, where the space C_γ was defined in §3. Seeking solutions in the space $C[0, T]$ can only be done for special cases since $c_n = 0$ is then necessary irrespective of any condition on $u(0)$, and then only $u(0) = 0$ is a consistent value.

6.1. R-L derivative of order $0 < \alpha < 1$.

For $0 < \alpha < 1$ a well posed initial value problem for the equation $D^\alpha u = f$ with $f \in L^1$ is given by the following Proposition.

Proposition 6.1. *Let $f \in L^1[0, T]$. Then a function $u \in L^1$ such that $I^{1-\alpha}u \in AC$ satisfies $D^\alpha u(t) = f(t)$ a.e. and $I^{1-\alpha}u(0) = c\Gamma(\alpha)$ if and only if $u(t) = ct^{\alpha-1} + I^\alpha f(t)$ a.e., where $c = I^{1-\alpha}u(0)/\Gamma(\alpha)$.*

Proof. Let $u \in L^1$ and suppose that $I^{1-\alpha}u \in AC$ and $D^\alpha u(t) = f(t)$ a.e. Thus $D(I^{1-\alpha}u)(t) = f(t)$ a.e. and since $I^{1-\alpha}u \in AC$ and $f \in L^1$ we may integrate to get $I^{1-\alpha}u(t) = a + If$ where $a = I^{1-\alpha}u(0)$. Applying the operator I^α gives $Iu(t) = at^\alpha/\Gamma(1+\alpha) + I^{1+\alpha}f(t)$. Differentiating these AC functions gives $u(t) = ct^{\alpha-1} + I^\alpha f(t)$ a.e. where $c = I^{1-\alpha}u(0)/\Gamma(\alpha)$.

Conversely, if $u(t) = ct^{\alpha-1} + I^\alpha f(t)$ a.e. then $u \in L^1$ and $I^{1-\alpha}u(t) = c\Gamma(\alpha) + If(t)$ so $I^{1-\alpha}u \in AC$ and $I^{1-\alpha}u(0) = c\Gamma(\alpha)$. Moreover $D(I^{1-\alpha}u)(t) = f(t)$ a.e. $\square$

Proposition 6.1 is stated in an equivalent form, also for the higher order case, as [17, Lemma 2.5(b)] and was proved in [24, Theorem 2.4]. it is also proved, under some slightly different hypotheses, in [3, Theorems 4.10 and 5.1], see Proposition 6.4 below.

When f depends also on u the result takes the following form.

Theorem 6.2. *Let $u \in L^1$ be such that $I^{1-\alpha}u \in AC$ and suppose that $t \mapsto f(t,u(t)) \in L^1$. Then $D^\alpha u(t) = f(t,u(t))$ a.e. and $I^{1-\alpha}u(0) = c\Gamma(\alpha)$ if and only if $u \in L^1$ with $t \mapsto f(t,u(t)) \in L^1$ satisfies $u(t) = ct^{\alpha-1} + I^\alpha f(t,u(t))$ a.e., where $c = I^{1-\alpha}u(0)/\Gamma(\alpha)$.*

The 'initial condition' is often given in terms of $\lim_{t\to 0+} u(t)t^{1-\alpha}$, when this exists. The result is proved for $\alpha \in \mathbb{C}$ with $0 < \mathrm{Re}(\alpha) < 1$ in [17, Lemma 3.2].

Lemma 6.3. *Let $0 < \alpha < 1$ and suppose that $u \in L^1$. Then*

$$\lim_{t\to 0+} u(t)t^{1-\alpha} = c \text{ implies that } \lim_{t\to 0+} I^{1-\alpha}u(t) = c\Gamma(\alpha).$$

Proof. For $\varepsilon > 0$ there exists $\delta > 0$ such that $|u(t)t^{1-\alpha} - c| < \varepsilon$ for $|t| < \delta$. Then for $|t| < \delta$ we have

$$I^{1-\alpha}u(t) - c\Gamma(\alpha) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}(u(s) - s^{\alpha-1}c)\, ds,$$

hence

$$\begin{aligned} |I^{1-\alpha}u(t) - c\Gamma(\alpha)| &\le \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} \varepsilon\, ds \\ &= \frac{\varepsilon}{\Gamma(1-\alpha)} \int_0^1 (1-\sigma)^{-\alpha}\sigma^{\alpha-1}\, d\sigma = \Gamma(\alpha)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary this proves that $\lim_{t\to 0+} I^{1-\alpha}u(t) = c\Gamma(\alpha)$. □

The following Proposition is given in [3, Theorem 6.2].

Proposition 6.4. *Let $0 < \alpha < 1$, and let f be continuous on $(0,T] \times J$ where $J \subset \mathbb{R}$ is an unbounded interval. If u is continuous on $(0,T]$ and u, $t \mapsto f(t,u(t))$ belong to $L^1[0,T]$, then u satisfies the initial value problem,*

$$D^\alpha u(t) = f(t,u(t)), \quad t \in (0,T], \; \lim_{t\to 0+} t^{1-\alpha}u(t) = u^0, \tag{6.1}$$

if and only if it satisfies the Volterra integral equation

$$u(t) = u^0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))\, ds, \; t \in (0,T]. \tag{6.2}$$

Remark 6.5. This is somewhat different to Theorem 6.2 since in Proposition 6.4 it is assumed that $D^\alpha u(t)$ exists for every $t \in (0,T]$ and functions that are in $L^1 \cap C(0,T]$ are considered, as opposed to supposing that functions are in L^1 and $D^\alpha u(t)$ exists a.e. in Theorem 6.2.

The converse of Lemma 6.3 is claimed in [3, Theorem 6.1] but this is not clear as the proof uses L'Hôpital's rule which assumes the limit exists whose existence is to be shown. Note that if $\lim_{t\to 0+} I^{1-\alpha}u(t) = c\Gamma(\alpha)$ and $c \neq 0$ then it is necessary that $u \notin L^p$ for every $p > 1/(1-\alpha)$ by Proposition 3.2 part (3). However, in [3] the result needed for solutions of (6.2) does hold since $I^{1-\alpha}I^\alpha f(t) = If(t) \to 0$ as $t \to 0+$ for $f \in L^1$.

Remark 6.6. Comparing Proposition 6.4 with Theorem 6.2 implies that the hypotheses of Proposition 6.4 should imply that $I^{1-\alpha}u \in AC$. In fact, if $u \in L^1$ and is a solution of (6.2) then writing $g(t) = f(t, u(t))$ we have $g \in L^1$ and

$$I^{1-\alpha}u(t) = I^{1-\alpha}\big(u^0 t^{\alpha-1} + I^\alpha g(t)\big) = \Gamma(\alpha)u^0 + Ig(t) \in AC.$$

If u satisfies (6.1) then $D(I^{1-\alpha}u)$ is continuous on $(0, T]$, and by Lemma 6.3, $\lim_{t\to 0+} I^{1-\alpha}u(t) = u^0\Gamma(\alpha)$ so $I^{1-\alpha}u$ satisfies the hypotheses of Proposition 2.2 and is therefore AC.

Remark 6.7. An early paper on the Riemann-Liouville IVP $D^\alpha u = f(t, u)$ is that of Delbosco and Rodino [5] who studied continuous solutions. In some cases it is implicit, but not explicit, that $u(0) = 0$. They also study the problem when f is replaced by $t^{-\gamma} f$ under a Lipschitz condition on f.

6.2. **R-L derivative of order** $1 < \beta < 2$**.** It is obvious that if $\beta = 1 + \alpha$ with $0 < \alpha < 1$ and $D^\beta u = D^{1+\alpha}u$ exists then $D^{1+\alpha}u = D(D^\alpha u)$. We have the following result.

Theorem 6.8. *Let $f \in L^1[0, T]$. Then $u \in L^1$ such that $D^\alpha u = D(I^{1-\alpha}u) \in AC$ satisfies $D^{1+\alpha}u(t) = f(t)$ a.e. and $I^{1-\alpha}u(0) = c_1\Gamma(\alpha)$ and $D^\alpha u(0) = c_2$ if and only if $u(t) = c_1 t^{\alpha-1} + c_2 t^\alpha + I^{1+\alpha} f(t)$ a.e., where $c_1 = I^{1-\alpha}u(0)/\Gamma(\alpha)$ and $c_2 = D^\alpha u(0)/\Gamma(1 + \alpha)$.*

Proof. Let $u \in L^1$ and suppose that $D^\alpha u \in AC$ and $D^{1+\alpha}u(t) = f(t)$ a.e. Thus $D(D^\alpha u)(t) = f(t)$ a.e. and since $D^\alpha u \in AC$ and $f \in L^1$ we may integrate to get $D^\alpha u(t) = a_2 + If(t)$ for all t, where $a_2 = D^\alpha u(0)$. Integrating again gives

$$I^{1-\alpha}u(t) = a_1 + a_2 t + I^2 f(t), \text{ where } a_1 = I^{1-\alpha}u(0).$$

Applying the operator I^α gives

$$Iu(t) = a_1 t^\alpha/\Gamma(1 + \alpha) + a_2 t^{1+\alpha}/\Gamma(2 + \alpha) + I^{2+\alpha} f(t).$$

Differentiating these AC functions gives

$$u(t) = a_1 t^{\alpha-1}/\Gamma(\alpha) + a_2 t^\alpha/\Gamma(1 + \alpha) + I^{1+\alpha} f(t), \quad \text{for a.e. } t.$$

Conversely, if $u(t) = a_1 t^{\alpha-1}/\Gamma(\alpha) + a_2 t^\alpha/\Gamma(1+\alpha) + I^{1+\alpha} f(t)$ a.e. then $u \in L^1$ and $I^{1-\alpha}u(t) = a_1 + a_2 t + I^2 f(t)$ so $I^{1-\alpha}u(0) = a_1$ and $D^\alpha u = D(I^{1-\alpha}u) = a_2 + If \in AC$ so that $D(I^{1-\alpha}u)(0) = D^\alpha u(0) = a_2$. Moreover $D^{1+\alpha}u = D(D^\alpha u) = f$ a.e. □

When f depends on u the result is as follows.

Theorem 6.9. *Let $u \in L^1$ be such $D^\alpha u \in AC$ and that $t \mapsto f(t, u(t)) \in L^1$. Then $D^{1+\alpha}u(t) = f(t, u(t))$ a.e., $I^{1-\alpha}u(0) = c_1\Gamma(\alpha)$ and $D^\alpha u(0) = c_2$ if and only if $u(t) = c_1 t^{\alpha-1} + c_2 t^\alpha + I^{1+\alpha} f(t, u(t))$ a.e., where $c_1 = I^{1-\alpha}u(0)/\Gamma(\alpha)$ and $c_2 = D^{1-\alpha}u(0)$.*

We note that to prove this equivalence it is required that $D^\alpha u \in AC$, equivalently $I^{1-\alpha}u \in AC^1$. Also if we ask that $t \mapsto f(t, u(t)) \in L^1$ for *every* $u \in L^1$ then it is known that a necessary and sufficient condition is $|f(t, u)| \le a(t) + b|u|$ for all (t, u) for some $a \in L^1$ and some constant $b \ge 0$.

6.3. **RL derivative of higher order.** For the RL derivative of order $m+\alpha$ where $m \in \mathbb{N}$ and $0 < \alpha < 1$ the result corresponding to Theorem 6.9 is the following and can be proved in the same way.

Theorem 6.10. *Let $u \in L^1$ be such $D^{m-1+\alpha}u = D^m(I^{1-\alpha}u) \in AC$ (that is $I^{1-\alpha}u \in AC^m$) and suppose that $t \mapsto f(t,u(t)) \in L^1$. Then u satisfies*

$$D^{m+\alpha}u(t) = f(t,u(t)) \text{ a.e.},$$
$$I^{1-\alpha}u(0) = b_{m+1}\Gamma(\alpha), \quad \text{and} \quad D^{m+\alpha-k}u(0) = b_k,\ k = 1,\dots,m,$$

if and only if

$$u(t) = \sum_{k=1}^{m} b_k t^{m+\alpha-k} + I^{m+\alpha}f(t,u(t)) \text{ a.e.},$$

where $b_{m+1} = I^{1-\alpha}u(0)/\Gamma(\alpha)$ and $b_k = D^{m+\alpha-k}u(0)$.

This result is essentially given in Theorem 3.1 of [17] where it is assumed that f is continuous on $(0,T) \times G$ (G an open set in $\mathbb{R}$) satisfying $f(t,u) \in L^1$ for every $u \in G$. The proof claims that the necessary AC property holds but the results cross referenced seem to be the wrong ones as they do not seem to prove this. Under a Lipschitz condition on f it is also essentially given in [7, Lemma 5.2].

7. Monotonicity and concavity properties of fractional derivatives

The Caputo and R-L derivatives are nonlocal, that is, the fractional derivative at a point t depends on the values of $u(s)$ for all $s \in [0,t]$. However, the Caputo derivative, in particular, has some similarities with derivatives of integer powers. For example if u is non-decreasing on $[0,T]$ then for any one $\alpha \in (0,1)$, both $D_*^\alpha u(t)$ and $D^\alpha u(t)$ are nonnegative for a.e. t but the converse is not true as we show below, though it is falsely claimed in the recent paper [23]. The correct result implying monotonicity is given in [9], see Remark 7.3 below. Similarly if $u \in C^2$ and $u''(t) \le 0$, that is u is concave, then $D_*^{1+\alpha}u(t)$ is non-positive but the converse does not hold.

We first note the following simple fact, we include the proof for completeness.

Lemma 7.1. *If $u \in AC[0,T]$ then u is non-decreasing if and only if $u'(t) \ge 0$ for a.e. $t \in [0,T]$.*

Proof. Since $u \in AC$ the derivative u' exists for a.e. t. Suppose that u is non-decreasing and that $u'(\tau)$ exists at a point $\tau \in (0,T)$. For $h \ne 0$ sufficiently small we have $\dfrac{u(\tau+h)-u(\tau)}{h} \ge 0$. Taking the limit as $h \to 0$ shows that $u'(\tau) \ge 0$. Conversely, suppose that $u'(t) \ge 0$ for a.e. t. Since $u \in AC$ we have $u(t) - u(0) = \int_0^t u'(s)\,ds$. Then for $t > \tau$, $u(t) - u(\tau) = \int_\tau^t u'(s)\,ds \ge 0$, that is u is non-decreasing. □

Proposition 7.2. *Let $0 < \alpha < 1$ and let $0 \le \gamma < \alpha$. Suppose that u, v are continuous and that $I^{1-\alpha}u \in AC$, $I^{1-\alpha}v \in AC$ so that $D_*^\alpha u$ and $D_*^\alpha v$ exist a.e.*

1. *If u is non-decreasing then $D_*^\alpha u(t) \ge 0$ and $D^\alpha u(t) \ge 0$ for a.e. $t \in [0,T]$.*
2. *If $D_*^\alpha u \in C_{-\gamma}$ and if $D_*^\alpha u(t) \ge 0$ for $t > 0$ then $u(t) \ge u(0)$ for every $t \in [0,T]$. If $D_*^\alpha u(t) > 0$ for $t > 0$ then $u(t) > u(0)$ for every $t \in (0,T]$.*

(3) *If* $D_*^\alpha u \in C_{-\gamma}$, $D_*^\alpha v \in C_{-\gamma}$ *and* $D_*^\alpha u(t) \ge D_*^\alpha v(t)$ *for* $t > 0$ *and* $u(0) \ge v(0)$ *then it follows that* $u(t) \ge v(t)$ *for* $t \in [0,T]$. *If* $D_*^\alpha u(t) > D_*^\alpha v(t)$ *for* $t > 0$ *and* $u(0) \ge v(0)$ *then* $u(t) > v(t)$ *for all* $t \in (0,T]$.

(4) *If, for some* $\alpha \in (0,1)$, *we have* $D_*^\alpha u(t) \ge 0$ *for* $t \in [0,T]$ *(or if* $D^\alpha u(t) \ge 0$*) it does not follow that* u *is non-decreasing even if* $D_*^\alpha u$ *or* u *is a* C^∞ *function.*

Proof. (1) Let u_0 be the constant function taking the value $u(0)$. By definition, $D_*^\alpha u(t) = DI^{1-\alpha}(u-u_0)(t)$ and $I^{1-\alpha}(u-u_0)(t)$ is a non-decreasing function of t by Proposition 3.2 (10). Thus, by Lemma 7.1, its derivative $D_*^\alpha u$ is ≥ 0 a.e.

(2) Let $g(t) := D_*^\alpha u(t)$, then, by assumption, $g(t) := t^{-\gamma} f(t)$ where f is continuous. It follows from Theorem 5.1 part 3 that we have

$$u(t) = u(0) + \int_0^t (t-s)^{\alpha-1} g(s)\,ds$$

therefore $u(t) \ge u(0)$ since $g(s) \ge 0$ for a.e. $s > 0$. When the inequality is strict, we have $(t-s)^{\alpha-1} g(s) > 0$ for a.e. $s \in (0,t)$, so the integral is positive for $t > 0$, thus $u(t) > u(0)$ for $t > 0$.

(3) We have $D_*^\alpha(u-v) \ge 0$ so $u(t) - v(t) \ge u(0) - v(0) \ge 0$ by part (2). The strict inequality case is proved in the same way.

(4) We give two simple counter-examples, it is easy to give many other similar ones. We note that for for a function $u \in AC$ with $u(0) = 0$ the following equality holds: $D^\alpha u(t) = D_*^\alpha u(t) = D_C^\alpha u(t)$. For $p > 0$, the function $h_p(t) := t^p$ satisfies $h_p \in AC$ and $h_p(0) = 0$ and we will use the well-known and easily verified fact that for $0 < \alpha < 1$

$$D_C^\alpha h_p(t) = \frac{t^{p-\alpha}\Gamma(p+1)}{\Gamma(p+1-\alpha)}.$$

Let $\alpha = 1/2$ and $T = 1$ and firstly let $u(t) = 5t^{3/2} - 4t^{5/2}$. Then $u \in C^1$ and we obtain

$$D_C^{1/2} u(t) = 5\Gamma(5/2)(t-t^2) = \frac{15\sqrt{\pi}}{4}(t-t^2) \ge 0, \quad \text{for } t \in [0,1],$$

and $D_C^{1/2} u$ is a C^∞ function, but $u(t)$ is increasing on $[0,3/4]$ and decreasing on $[3/4,1]$.

Secondly, for $\alpha = 1/2$ and $T = 1$ take $v(t) = 6t^2 - 5t^3$ so $v \in C^\infty$. By the above formulas we obtain $D_C^{1/2} v(t) = 16(t^{3/2} - t^{5/2})/\sqrt{\pi} \ge 0$ for $t \in [0,1]$ but v is increasing on $[0,4/5]$ and decreasing on $[4/5,1]$. $\square$

Remark 7.3. The results with $\gamma \ne 0$ may be new. Diethelm [9] discusses monotonicity properties for $C^1[0,T]$ functions in terms of $D_C^\alpha u$. He proves part (1) for C^1 functions in his Theorem 2.1; the proof is also simple with the definition $D_C^\alpha u$ available. He further shows that monotonicity of $u \in C^1$ is equivalent to requiring that there is $\alpha_0 \in (0,1)$ such that the Caputo derivatives $D_C^\alpha u$ of orders $\alpha \in (\alpha_0, 1)$ do not change sign, and this is equivalent to the Caputo derivatives $D_C^\alpha u$ of all orders $\alpha \in (0,1)$ do not change sign. For part (4) he gives a different example to ours where $D_*^\alpha u(t)$ does not change sign for some, but not all, α and u is not monotone.

Theorem 2.4 in [4] partially proves part (3) when $\gamma = 0$ for the fractional derivative D_C^α but omits to state that f must be continuous and that u should be in AC.

The proof also claims the fractional differential and integral equations are *equivalent* but the correct equivalence is given in Theorem 5.1, however the implication of part 1 of Theorem 5.1 is sufficient there.

We now discuss concavity properties, for simplicity we consider functions defined on $[0,1]$. If $0<\alpha<1$ is given and $D_*^{1+\alpha}u(t)=g(t)$ we might hope that $g\le 0$ implies that u is concave but this is not valid.

Proposition 7.4. *If $0<\alpha<1$, $u\in AC[0,1]$, $I^{1-\alpha}u'\in AC$, and u is concave, that is $u'(t)$ is non-increasing, then $D_*^{1+\alpha}u(t)\le 0$. But $D_*^{1+\alpha}u(t)\le 0$ (for a fixed α) does not imply that u is concave even if $u\in C^2[0,1]$ or $D_*^{1+\alpha}u\in C^2[0,1]$ (or more regular).*

Proof. The assumption $I^{1-\alpha}u'\in AC$ ensures that the fractional derivative $D_*^{1+\alpha}u$ exists. Then by Lemma 4.11 we have

$$D_*^{1+\alpha}u(t)=D^{\alpha}(u'-u'(0))(t)=D(I^{1-\alpha}(u'-u'(0)))(t)\le 0$$

by Proposition 3.2 (10) since $u'(t)-u'(0)$ is non-increasing.

For the second part, we take $\alpha=1/2$ and note that for $h_p(t)=t^p$ we have

$$D_C^{3/2}h_p(t)=\begin{cases}0, & \text{if } p=0 \text{ or if } p=1,\\ \frac{\Gamma(p+1)}{\Gamma(p-1/2)}t^{p-3/2}, & \text{if } p>1.\end{cases}$$

Firstly we take the following C^∞ function $u(t)=a+bt+t^3-2t^2$ where a,b can be arbitrary constants. Then by the above formula $D_C^{3/2}u(t)=\frac{8}{\sqrt{\pi}}(t^{3/2}-t^{1/2})\le 0$ but $u''=6t-4$ changes sign at $t=2/3$ so u is not concave, also not convex, on $[0,1]$.

Secondly take, $u(t)=a+bt-\dfrac{8t^{5/2}}{15\sqrt{\pi}}+\dfrac{32t^{7/2}}{105\sqrt{\pi}}$ so that $D_C^{3/2}u(t)=-t+t^2\le 0$. Then $u''(t)=-2t^{1/2}/\sqrt{\pi}+8t^{3/2}/(3\sqrt{\pi})$ which changes sign at $t=3/4$ so u is neither convex nor concave. $\square$

Remark 7.5. Using Diethelm's result on monotonicity [9] mentioned above applied to u', it follows that if $u\in C^2[0,1]$ then u is concave if and only if for **every** $\alpha\in(0,1)$ the fractional derivative $D_C^{1+\alpha}u(t)$ is non-positive for $t\in[0,1]$. The application of this result could be limited by the fact that a solution of a problem $D_*^{1+\alpha}u(t)=f(t,u(t))$ with f continuous is not a $C^2[0,1]$ function in general, see [7, Theorem 6.25] and Theorem 9.1 below.

There have been some papers which make incorrect claims regarding concavity properties. Ntouyas and Pourhadi [22] discuss the following BVP with $0<\alpha<1$ (we have changed to the interval $[0,1]$):

$$D_C^{1+\alpha}u(t)+g(t)=0,\quad u(0)=0, u(1)=\beta u(\eta), \tag{7.1}$$

where $\beta\eta\ne 1$. They claim the following.

> [22, Lemma 2.6] Suppose that $g\in C^2([0,1];\mathbb{R})$ and $g(0)\ge 0$.
> (a) If g is convex, then the unique solution of (7.1) is concave.
> (b) If g is concave, then the unique solution of (7.1) is convex.

Unfortunately this is not correct. We can take the same example as above with $\alpha=1/2$, that is consider $D_C^{3/2}u$ for $u(t)=a+bt-\dfrac{8t^{5/2}}{15\sqrt{\pi}}+\dfrac{32t^{7/2}}{105\sqrt{\pi}}$, taking $a=0$ and

b chosen to make $u(1) = \beta u(\eta)$, then u satisfies $D_C^{3/2}u(t) + t - t^2 = 0$, and we have $g(t) = t - t^2$ which is concave and non-negative but the solution is neither convex nor concave. For example, if $\eta = 1/2$ and we also ask for $u(t) \geq 0$ for $t \in [0,1]$ (the paper [22] is interested in existence of positive solutions and assumes $\beta\eta < 1$ in some Lemmas) then we need $b \gtrapprox 0.129$ and then for any for any $\beta \in [0.0016, 1.944]$ such b exists, for example when $\beta = 0.01, \eta = 1/2$ we can take $b \approx 0.1292$, for $\beta = 1, \eta = 1/2$ we can take $b \approx 0.182$, or for $\beta = 1.944, \eta = 1/2$ we can take $b \approx 1.968$.

Eloe and Neugebauer [11] discuss concavity properties of the R-L fractional derivative. One of their results, [11, Theorem 3.7], claims that if $\alpha \in (0,1)$, and $D^{1+\alpha}u$ is continuous with $u(0) = 0$ and $u(1) \geq 0$, then, if $u \in C^2(0,1]$ (not necessarily at 0), $D^{1+\alpha}u(t) \leq 0$ implies that $u''(t) \leq 0$ for $t \in (0,1]$.

Unfortunately this is not correct. We again take $\alpha = 1/2$ and consider

$$u(t) = bt^{1/2} - \frac{8t^{5/2}}{15\sqrt{\pi}} + \frac{32t^{7/2}}{105\sqrt{\pi}}$$

and we can choose, for example, $b \geq 0.129$ to make $u(1) \geq 0$; a simple choice is $b = 1/6$ which we now make. Note that, by properties of the R-L derivative, $D^{3/2}t^{1/2} = 0$. Then we obtain $D^{3/2}u(t) = -t + t^2 \leq 0$ but by calculation $u''(t)$ changes sign when $t \approx 0.794$, thus u is neither concave nor convex. An error is that the 'if' part of their Lemma 2.1 which is used in the proof of their Theorem 3.7 is not correct, a counter example to that implication on $[0,1]$ is $u(t) = t/(1 + 4t^2)$ where u'' changes sign at $t = \sqrt{3}/2$. We note that for the 'only if' part of their Lemma 2.1 (for which there is a simpler proof directly using the definition of concavity) the condition $u(0) \geq 0$ is necessary for its validity; in the paper [11] the main results have $u(0) = 0$.

8. Existence of Caputo IVPs with nonlinearity depending on a fractional derivative

We will now consider the Caputo IVP:

$$D_*^{1+\alpha}u(t) = t^{-\gamma}f(t, u(t), D_C^{\beta}u(t)) \quad \text{for } t > 0,\ u(0) = u_0, u'(0) = u_1,$$

where $0 \leq \gamma < \alpha < 1$ and $0 < \beta \leq 1$. We believe our results here are new even for the case $\beta = 1$ since we include the singular term $t^{-\gamma}$.

For this problem it is natural to work in the space $C^1[0,T]$ in which we will use the norm $\|u\|_1 := \|u\|_\infty + \|u'\|_\infty$. Since $u \in C^1$ we have $D_*^{\beta}u(t) = D_C^{\beta}u(t)$ by Proposition 4.4, but we do not necessarily have $D_*^{1+\alpha}u(t) = D_C^{1+\alpha}u(t)$. We write $D_C^{\beta}u$ since this is the form we use in the proof below. First we have the following equivalence.

Theorem 8.1. *Let $0 \leq \gamma < \alpha < 1$ and $0 < \beta \leq 1$. Let f be continuous on $[0,T] \times \mathbb{R} \times \mathbb{R}$. If a function $u \in C^1[0,T]$ is such that $D_*^{1+\alpha}u$ exists a.e. and satisfies the Caputo fractional initial value problem*

$$D_*^{1+\alpha}u(t) = t^{-\gamma}f(t, u(t), D_C^{\beta}u(t)), \quad \text{for a.e. } t > 0, \quad u(0) = u_0, \quad u'(0) = u_1. \tag{8.1}$$

then u satisfies the Volterra integral equation

$$u(t) = u_0 + u_1\, t + I^{1+\alpha}(t^{-\gamma}f(t, u(t), D_C^{\beta}u(t))). \tag{8.2}$$

Conversely, if $u \in C^1[0,T]$ satisfies (8.2) *then $D_*^{1+\alpha}u$ exists a.e. and u satisfies* (8.1)*.*

Proof. Let $g(t) := f(t, u(t), D_C^{\beta}u(t))$. First we note that for $u \in C^1$ the Caputo fractional derivative $D_C^{\beta}u$ is continuous. In fact,

$$D_C^{\beta}u = I^{1-\beta}u' \in C_{1-\beta} \subset C, \quad \text{by Proposition 3.2 (5)}.$$

Hence for $u \in C^1$, g is continuous. The proof is now almost identical with that of Theorem 5.1 so is omitted. □

We will prove a global existence theorem under the growth assumption

$$|f(t,u,p)| \le a(t) + M(|u| + |p|) \quad \text{for some } a \in L^{\infty} \text{ and constant } M > 0.$$

Under an extra Lipschitz condition we will also prove continuous dependence on initial data, and a uniqueness result.

The problem

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} f(s, u(s))\, ds$$

when $0 \le \gamma < \alpha < 1$ and f is continuous was recently studied by this author in [27] by employing a new L^{∞} Gronwall inequality which established an a priori bound for solutions of a problem with weakly singular kernel in the space $C[0,T]$. For problem (8.1) the appropriate space is $C^1[0,T]$ and we must obtain a priori bounds on both the function u and its derivative u'. We will use the Gronwall inequality from [27] which we now recall in a simplified form.

Theorem 8.2. *Let $a \ge 0$ and $b > 0$ be constants and suppose that $\eta > 0$, $\gamma \ge 0$ and $\eta + \gamma < 1$. Suppose that $u \in L_+^{\infty}[0,T]$ satisfies the inequality*

$$u(t) \le a + b \int_0^t (t-s)^{-\eta} s^{-\gamma} u(s)\, ds, \quad \text{for a.e. } t \in [0,T]. \tag{8.3}$$

Then there is an explicit constant $B = B(b, \eta, \gamma)$ (see [27] *for details) such that*

$$u(t) \le \frac{a(1-\gamma)}{1-\eta-\gamma} \exp\Big(Bt^{1-\gamma}\Big), \quad \text{for a.e. } t \in [0,T]. \tag{8.4}$$

In particular, there is an explicit constant $C = C(b, \eta, \gamma, T)$ such that $u(t) \le aC$ for a.e. $t \in [0,T]$.

When u is continuous the inequality holds for all t not only a.e. t.

We will now discuss solutions in the space $C^1[0,T]$ of the Volterra integral equation

$$u(t) = u_0 + u_1 t + I^{1+\alpha}(t^{-\gamma} f(t, u(t), D_C^{\beta}u(t))), \tag{8.5}$$

where $0 < \alpha < 1$ and $0 < \beta \le 1$. If $u \in C^1$ is a solution then its derivative satisfies

$$u'(t) = u_1 + I^{\alpha}(t^{-\gamma} f(t, u(t), D_C^{\beta}u(t))). \tag{8.6}$$

Our result reads as follows.

Theorem 8.3. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, let $0 < \alpha < 1$, $0 \le \gamma < \alpha$, and $0 < \beta \le 1$, and let $u_0, u_1 \in \mathbb{R}$. Suppose there are $a \in L^{\infty}$ and a constant*

$M > 0$ such that $|f(t,u,p| \le a(t) + M(|u| + |p|)$ for all $t \in [0,T]$ and $u, p \in \mathbb{R}$. Then the integral operator

$$Nu(t) := u_0 + tu_1 + \frac{1}{\Gamma(1+\alpha)} \int_0^t (t-s)^\alpha s^{-\gamma} f(s, u(s), D_C^\beta u(s))\, ds$$

has a fixed point in $C^1[0,T]$. If, in addition, there exists $L > 0$ such that f satisfies the Lipschitz condition

$$|f(t,u,p) - f(t,v,q)| \le L(|u-v| + |p-q|), \quad \text{for all } t \in [0,T],\ u,v,p,q \in \mathbb{R},$$

then fixed points depend continuously on the initial data and, in particular, for a given u_0, u_1 there is a unique fixed point.

Proof. The proof of existence consists of showing that $N : C^1 \to C^1$ is completely continuous, that is N is continuous and maps bounded subsets of C^1 into relatively compact subsets of C^1, and that there is a bounded open set U containing 0 such that $Nu \neq \lambda u$ for all $u \in \partial U$ and all $\lambda \ge 1$. This will prove the Leray-Schauder degree $\deg(N, U, 0) = 1$ which proves that N has a fixed point in U. To show that N is completely continuous we will show N is continuous and will apply the Arzelà-Ascoli criterion to show that $N(U)$ is bounded and that $(Nu)(t)$ and $(Nu)'(t)$ are equicontinuous for $u \in U$.

We will use the simple fact that

$$s^{-\gamma}|D_C^\beta u(s)| \le I^{1-\beta}(s^{-\gamma}|u'(s)|) \quad \text{for a.e. } s. \tag{8.7}$$

In fact, since u' is continuous $s \mapsto s^{-\gamma}|u'(s)|$ is an L^1 function and we have

$$\begin{aligned} s^{-\gamma}|D_C^\beta u(s)| &= s^{-\gamma}|I^{1-\beta}u'(s)| \\ &\le \frac{1}{\Gamma(1-\beta)} s^{-\gamma} \int_0^s (s-\sigma)^{-\beta}|u'(\sigma)|\, d\sigma \\ &\le \frac{1}{\Gamma(1-\beta)} \int_0^s \sigma^{-\gamma}(s-\sigma)^{-\beta}|u'(\sigma)|\, d\sigma \\ &= I^{1-\beta}(s^{-\gamma}|u'(s)|). \end{aligned}$$

We now show the existence of a suitable set U. In fact, if there exists $\lambda \ge 1$ and $u \neq 0$ such that $\lambda u = Nu$ then $\lambda u(t) = Nu(t)$ and $\lambda u'(t) = (Nu)'(t)$, and we have

$$\begin{aligned} |u(t)| \le \lambda|u(t)| = |Nu(t)| &\le |u_0| + t|u_1| + I^{1+\alpha}s^{-\gamma}|f(s,u(s),D_C^\beta u(s))| \\ &\le |u_0| + T|u_1| + I^{1+\alpha}(s^{-\gamma}(a(s) + M(|u(s)| + |D_C^\beta u(s)|) \\ &\le C_0(a,u_0,u_1,T) + MI^{1+\alpha}(s^{-\gamma}|u(s)|) + MI^{1+\alpha}I^{1-\beta}|s^{-\gamma}u'(s)| \\ &= C_0(a,u_0,u_1,T) + MI^{1+\alpha}s^{-\gamma}|u(s)| + MI^{2+\alpha-\beta}|s^{-\gamma}u'(s)|. \end{aligned}$$

The constant C_0 can be given explicitly but since it is not important to us we omit the explicit value; we also do this with other constants below. Similarly we obtain

$$\begin{aligned} |u'(t)| &\le |u_1| + I^\alpha|f(s,u(s),D_C^\beta u(s))| \\ &\le C_1(a,u_1,T) + MI^\alpha(s^{-\gamma}|u(s)|) + MI^{1+\alpha-\beta}|s^{-\gamma}u'(s)|. \end{aligned}$$

Now we note that

$$I^{\alpha_1}v \le C_2(\alpha_1,\alpha_2,T)I^{\alpha_2}v \quad \text{whenever } \alpha_1 \ge \alpha_2. \tag{8.8}$$

In fact,

$$\begin{aligned}
I^{\alpha_1}v(t) &= \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}v(s)\,ds\\
&= \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_2-1}(t-s)^{\alpha_1-\alpha_2}v(s)\,ds\\
&\le \frac{1}{\Gamma(\alpha_1)}T^{\alpha_1-\alpha_2}\int_0^t (t-s)^{\alpha_2-1}v(s)\,ds\\
&\le \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)}T^{\alpha_1-\alpha_2}I^{\alpha_2}v.
\end{aligned}$$

Adding the above inequalities and using (8.8) several times (noting that α is the smallest exponent) gives

$$\begin{aligned}
|u(t)|+|u'(t)| &\le C_3(a,u_0,u_1,T) + C_4(a,u_0,u_1,T)I^{\alpha}s^{-\gamma}(|u(s)|+|u'(s)|)\\
&= C_3 + C_4(a,u_0,u_1,T)\int_0^t (t-s)^{\alpha-1}s^{-\gamma}(|u(s)|+|u'(s)|)\,ds.
\end{aligned}$$

Since $1-\alpha+\gamma<1$, by Theorem 8.2 there is a constant C_5, independent of u, such that $|u(t)|+|u'(t)|\le C_5$ for all $t\in[0,T]$. Choose $R>C_5$, and let U_R, be the open ball in C^1 of radius R centred at 0. We have shown that $Nu\neq\lambda u$ for all $u\in\partial U_R$ and all $\lambda\ge 1$.

Now we show that $N(\overline{U}_R)$ is bounded. We have $\|u\|_1\le R$ so that $|u(t)|+|u'(t)|\le R$ for all $t\in[0,T]$, and we obtain

$$|D^{\beta}u(t)| = |I^{1-\beta}u'| \le \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}|u'|(s)\,ds \le R_1 := \frac{RT^{1-\beta}}{\Gamma(2-\beta)}.$$

As f is uniformly continuous on $[0,T]\times[0,R]\times[0,R_1]$, there exists $M=M(R,\beta,T)<\infty$ such that $|f(t,u,p)|\le M$ for all $t\in[0,T]$, $u\in[0,R]$, $p\in[0,R_1]$. Thus for $u\in\overline{U}_R$ we have

$$\begin{aligned}
|(Nu)'(t)| &\le |u_1| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\gamma}|f(s,u(s),D^{\beta}u(s))|\,ds\\
&\le |u_1| + \frac{1}{\Gamma(\alpha)}M\int_0^t (t-s)^{\alpha-1}s^{-\gamma}\,ds\\
&= |u_1| + \frac{1}{\Gamma(\alpha)}Mt^{\alpha-\gamma}B(\alpha,1-\gamma)\\
&\le |u_1| + \frac{1}{\Gamma(\alpha)}MT^{\alpha-\gamma}B(\alpha,1-\gamma).
\end{aligned}$$

A similar calculation is valid for $Nu(t)$. This proves that the set $N(\overline{U}_R)$ is bounded.

The proof of equicontinuity of $(Nu)'(t)$ is essentially identical with the proof in [27] since the property of f used is the uniform bound, hence we omit this. The proof of equicontinuity for $\{Nu(y) : u\in\overline{U}_R\}$ is similar but easier. The equicontinuity shows that N maps bounded sets into relatively compact sets by the Arzelà-Ascoli theorem. Also it shows that $Nu(t)$ and $(Nu)'(t)$ are continuous in t so $N(\overline{U}_R)\subset C^1$.

Finally we show that N is continuous on C^1, that is, $u_n(s)\to u(s)$ and $u_n'(s)\to u'(s)$ uniformly in s implies $Nu_n(t)\to Nu(t)$ and $(Nu)_n'(t)\to(Nu)'(t)$ uniformly in t. For $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|u_n'(s) - u'(s)| < \varepsilon$ for $n > n_0$ and all $s \in [0,T]$. Then we have

$$\begin{aligned}|D^\beta u_n(t) - D^\beta u(t)| &\le \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} |u_n'(s) - u'(s)|\, ds \\ &< \frac{1}{\Gamma(1-\beta)} \varepsilon \int_0^t (t-s)^{-\beta}\, ds \le \frac{\varepsilon T^{1-\beta}}{\Gamma(1-\beta)}.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary this shows that $D^\beta u_n(t) \to D^\beta u(t)$ uniformly in t. Hence, since $\{u_n\}$ is bounded, say $\|u_n\|_1 \le R$, and f is uniformly continuous on $[0,T] \times [0,R] \times [0,R_1]$, for $\varepsilon > 0$ there exists $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that

$$|f(s, u_n(s), D^\beta u_n(s)(s)) - f(s, u(s), D^\beta u(s)| < \varepsilon,$$

for all $n > n_1$ and all $s \in [0,T]$. Then we have, for every $t \in [0,T]$,

$$\begin{aligned}&|(Nu_n)'(t) - (Nu)'(t)| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} |f(s, u_n(s), D^\beta u_n(s)) - f(s, u(s), D^\beta u(s))|\, ds \\ &< \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma}\, ds \\ &\le \frac{\varepsilon}{\Gamma(\alpha)} T^{\alpha-\gamma} B(\alpha, 1-\gamma), \quad \text{for } n > n_1.\end{aligned}$$

This proves that $(Nu_n)'(t) \to (Nu)'(t)$ uniformly in t and a similar calculation shows that $(Nu_n)(t) \to (Nu)(t)$ uniformly in t. This completes the proof for existence of a solution.

Now suppose that f satisfies the Lipschitz condition and let u, v be fixed points with $u(0) = u_0, u'(0) = u_1$, $v(0) = v_0, v'(0) = v_1$. Then we have

$$\begin{aligned}u(t) - v(t) = u_0 - v_0 + t(u_1 - v_1) + \frac{1}{\Gamma(1+\alpha)} \int_0^t (t-s)^\alpha s^{-\gamma} \big(f(s, u(s), D^\beta u(s)) \\ - f(s, v(s), D^\beta v(s)\big)\, ds,\end{aligned}$$

and

$$\begin{aligned}&u'(t) - v'(t) \\ &= u_1 - v_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} \big(f(s, u(s), D^\beta u(s)) - f(s, v(s), D^\beta v(s)\big)\, ds.\end{aligned}$$

By the arguments used above this gives

$$\begin{aligned}&|u(t) - v(t)| + |u'(t) - v'(t)| \\ &\le |u_0 - v_0| + (1+T)|u_1 - v_1| + C_6 L\, I^\alpha \big(s^{-\gamma}(|u(s) - v(s)| + |u'(s) - v'(s)|\big).\end{aligned}$$

By Theorem 8.2 we obtain, for $t \in [0,T]$,

$$|u(t) - v(t)| + |u'(t) - v'(t)| \le (|u_0 - v_0| + (1+T)|u_1 - v_1|) C_7 \exp\big(Bt^{1-\gamma}\big).$$

This proves the continuous dependence on initial data and taking $u_0 = v_0, u_1 = v_1$ proves uniqueness. $\square$

Remark 8.4. We can obtain existence of nonnegative solutions if we suppose that $f(t,u,p) \ge 0$ and that $u_0 \ge 0$ and $u_0 + Tu_1 \ge 0$. The proof is almost the same using fixed point index theory in place of degree theory.

Remark 8.5. Some previous work with nonlinearities involving fractional derivatives was done by Kosmatov [18] and continued by Deng and Deng [6]. They consider $D_C^{\alpha}u(t) = f(t, D_C^{\beta}u(t))$ (of any order α) when the nonlinear term f is assumed to be continuously differentiable. It is assumed in [18] that $|f(t,p)| \le a_1(t)+a_2(t)|p|$ and that a certain integral involving a_1 is finite and another involving a_2 is smaller than one in order to prove appropriate a priori bounds. Deng and Deng [6] use some of the results from Kosmatov [18] and generalize the main existence theorem by imposing the condition $|f(t,p)| \le a|p| + b$ for $t \in [0,1]$ with a, b positive constants but no restriction on the size.

For $1 < \alpha < 2$ and $0 < \beta < 1$ (a particular case) Kosmatov [18, Lemma 3.1] claims the equivalence of $D_C^{\alpha}u(t) = f(t, D_C^{\beta}u(t))$ and $u(0) = u_0, u'(0) = u_1$ with an integral equation for $C^2[0,T]$ functions. However, solutions of the integral equation do not have enough regularity for this in general, see Theorem 9.3, so this is not clear. Deng and Deng [6] use the claimed result without comment.

Our method uses Theorem 8.2, the Gronwall inequality of [27], to obtain a priori bounds with fewer restrictions. We work in the space C^1 and we do not need $f \in C^1$ or a condition such as $f(0,0) = 0$, also we allow $f = f(t,u,D^{\beta}u)$ to depend on u too and we have the extra singular term $t^{-\gamma}$.

The same methods as above apply to higher order problems. We only give the case for fractional derivatives of order between 2 and 3. We will consider the Caputo IVP:

$$\begin{gathered} D_*^{2+\alpha}u(t) = t^{-\gamma}f(t, u(t), D_C^{\beta_1}u(t), D_C^{1+\beta_2}u(t)), \quad \text{for } t > 0, \\ u(0) = u_0, \quad u'(0) = u_1, \quad u''(0) = u_2, \end{gathered}$$

where $0 \le \gamma < \alpha < 1$ and $0 < \beta_1, \beta_2 \le 1$.

For this problem we work in the space $C^2[0,T]$ in which we will use the norm $\|u\|_2 := \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$. As previously in Theorem 8.1 this is equivalent to the Volterra integral equation

$$u(t) = u_0 + u_1 t + u_2 t^2/2 + I^{2+\alpha}(t^{-\gamma}f(t, u(t), D_C^{\beta_1}u(t), D_C^{1+\beta_2}u(t))). \tag{8.9}$$

We note that $D_C^{1+\beta_2}u = I^{1-\beta_2}u'' \in C[0,T]$ and, as before, that

$$|s^{-\gamma}D_C^{1+\beta_2}u(s)| \le I^{1-\beta_2}|s^{-\gamma}u''(s)|.$$

We now have the following result.

Theorem 8.6. *Let $f : [0,T] \times \mathbb{R}^3$ be continuous, let $0 < \alpha < 1$, $0 \le \gamma < \alpha$, and $0 < \beta_i \le 1$ $(i = 1,2)$, and let $u_0, u_1, u_2 \in \mathbb{R}$. Suppose there are $a \in L^\infty$ and a constant $M > 0$ such that $|f(t,u,p,q)| \le a(t) + M(|u| + |p| + |q|)$ for all $t \in [0,T]$ and $u, p, q \in \mathbb{R}$. Then the integral operator*

$$\begin{aligned} Nu(t) :&= u_0 + tu_1 + (t^2/2)u_2 \\ &\quad + \frac{1}{\Gamma(2+\alpha)}\int_0^t (t-s)^{1+\alpha}s^{-\gamma}f(s, u(s), D_C^{\beta_1}u(s), D_C^{1+\beta_2}u(s)))\,ds \end{aligned}$$

has a fixed point in $C^2[0,T]$. If, in addition, there exists $L > 0$ such that f satisfies the Lipschitz condition

$$\begin{gathered} |f(t,u_1,p_1,q_1) - f(t,u_2,p_2,q_2)| \le L(|u_1 - u_2| + |p_1 - p_2|| + |q_1 - q_2|), \\ \textit{for all } t \in [0,T] \textit{ and all } u_i, p_i, q_i \in \mathbb{R}, \end{gathered}$$

then fixed points depend continuously on the initial data and in particular, for a given u_0, u_1, u_2 there is a unique fixed point.

Proof. If $u = \lambda Nu$ for $\lambda \geq 1$ then we have:

$$|u(t)| \leq |u_0| + t|u_1| + (t^2/2)|u_2| + I^{2+\alpha}|s^{-\gamma} f(s, u(s), D_C^{\beta_1} u(s), D_C^{1+\beta_2} u(s)|,$$
$$|u'(t)| \leq |u_1| + t|u_2| + I^{1+\alpha}|s^{-\gamma} f(s, u(s), D_C^{\beta_1} u(s), D_C^{1+\beta_2} u(s))|,$$
$$|u''(t)| \leq |u_2| + I^{\alpha}|s^{-\gamma} f(s, u(s), D_C^{\beta_1} u(s), D_C^{1+\beta_2} u(s))|.$$

These are estimated exactly as in the proof of Theorem 8.3 and the Gronwall inequality Theorem 8.2 is used to get the a priori bound on $|u| + |u'| + |u''|$. The rest of the proof is exactly similar to that of Theorem 8.3 hence is omitted. $\square$

A similar result can be given for higher order problems, since this requires no new ideas we do not state or prove this.

9. Regularity results for Caputo IVP

For the case of fractional derivatives D_*^{β} with $1 < \beta < 2$ we can give some regularity results when f has more regularity.

Theorem 9.1. (1) *Let f be continuous on $[0, T]$, let $0 < \gamma \leq \alpha < 1$ and let $g(t) := t^{-\gamma} f(t)$. If $u \in C[0, T]$ is such that $D_*^{1+\alpha} u$ exists and satisfies the Caputo fractional initial value problem*

$$D_*^{1+\alpha} u(t) = g(t) \quad \text{for a.e. } t, \quad u(0) = u_0, \quad u'(0) = u_1. \tag{9.1}$$

then $u \in C^1[0, T]$ and satisfies the Volterra integral equation $u(t) = u_0 + u_1 t + I^{1+\alpha} g(t)$, that is,

$$u(t) = u_0 + u_1 t + \frac{1}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha} s^{-\gamma} f(s)\, ds, \quad t \in [0, T]. \tag{9.2}$$

Conversely, if $u \in C[0, T]$ satisfies (9.2) then $u \in C^1[0, T]$ and satisfies (9.1).

(2) *If $f \in AC$ and $u \in C[0, T]$ satisfies*

$$D_*^{1+\alpha} u(t) = f(t) \quad \text{for } t \in [0, T], \quad u(0) = u_0, \quad u'(0) = u_1. \tag{9.3}$$

then $u(t) = u_0 + tu_1 + v(t) + \frac{f(0) t^{\alpha+1}}{\Gamma(\alpha+1)}$ where $v \in AC^1$, thus $u' \in AC$ and hence $D_*^{1+\alpha} u = D_C^{1+\alpha} u$.

(3) *If $f \in C^1$ and $u \in C[0, T]$ satisfies (9.3) then $u' \in AC$ and $u(t) = u_0 + tu_1 + v(t) + \frac{f(0)}{\Gamma(\alpha+1)} t^{1+\alpha}$ where $v \in C^2[0, T]$. In particular, $u \in C^2(0, T]$, moreover $u \in C^2[0, T]$ if and only if $f(0) = 0$.*

Proof. (1) By definition $D_*^{1+\alpha} u(t) = g(t)$ means that $DI^{1-\alpha}(u - T_1(u)) \in AC$ and $T_1(u)$ exists, and $D^2 I^{1-\alpha}(u - T_1(u)) = g \in L^1$. Integrating twice gives

$$I^{1-\alpha}(u - T_1(u))(t) = I^2 g(t) + a + bt.$$

By Proposition 3.2 (5), $I^{1-\alpha}(u - T_1(u))(0) = 0$ so we must have $a = 0$. Applying I^{α} gives

$$I(u - T_1(u)) = I^{2+\alpha} g + b t^{\alpha+1}/\Gamma(\alpha+2),$$

hence

$$(u - T_1(u))(t) = I^{1+\alpha} g(t) + b t^{\alpha}/\Gamma(\alpha+1).$$

Since $u'(0)$ exists we must have $b = 0$. Thus (9.2) holds and $u - T_1(u) = I^{1+\alpha} g = I(I^\alpha g) \in C^1$ since $I^\alpha g \in C_{\alpha-\gamma} \subset C$ by Proposition 3.2 (5).

Conversely, if $u \in C[0,T]$ satisfies (9.2) then since $I^{1+\alpha} g = I(I^\alpha g) \in C^1$ we obtain $u \in C^1$ and $D_*^{1+\alpha} u = D^{1+\alpha} I^{1+\alpha} g = g$ a.e.

(2) By part (1) $u - T_1 u = I^{1+\alpha} f$ and $u \in C^1$. For $f \in AC$ we have $f' \in L^1$ and $f(t) - f(0) = If'(t)$. Therefore

$$u - T_1 u = I^{1+\alpha}(If' + f(0)) = v(t) + \frac{f(0)t^{\alpha+1}}{\Gamma(\alpha+1)},$$

where $v(t) = I\, I^\alpha(If') \in AC^1$ (using Proposition 3.2) and $\frac{f(0)t^{\alpha+1}}{\Gamma(\alpha+1)} \in AC^1$ too. Thus $u \in AC^1$ and $D_*^{1+\alpha} u = D_C^{1+\alpha} u$ by Lemma 4.10.

(3) For $f \in C^1$, we have $f' \in C[0,T]$ and $f(t) - f(0) = If'(t)$ so

$$u - T_1 u = I^{1+\alpha}(If' + f(0)) = I^2 I^\alpha f' + f(0)t^{1+\alpha}/\Gamma(1+\alpha),$$

where $v(t) := I^2 I^\alpha f' \in C^2[0,T]$. This equation shows that $u \in C^2[0,T]$ if and only if $f(0) = 0$. □

Remark 9.2. Nothing much better than $u \in C^1$ is expected in the analogue to parts (2), (3) when there is the singular term $t^{-\gamma}$ since we can only prove $I^\alpha(t^{-\gamma} f(t)) \in C_{\alpha-\gamma}$.

For fractional differential equations of order $1+\alpha$ with $0 < \alpha < 1$ and when f depends on u but not on derivatives of u we have the following regularity results.

Theorem 9.3. *Let $0 < \gamma \le \alpha < 1$.*

(1) *Let f be continuous on $[0,T] \times \mathbb{R}$. If a function $u \in C[0,T]$ is such that $D_*^{1+\alpha} u$ exists and satisfies the Caputo fractional initial value problem*
$$D_*^{1+\alpha} u(t) = t^{-\gamma} f(t, u(t)) \quad \text{for a.e. } t, \quad ; u(0) = u_0, \quad u'(0) = u_1. \tag{9.4}$$
then $u \in C^1[0,T]$ and satisfies the Volterra integral equation
$$u(t) = u_0 + u_1\, t + I^{1+\alpha}(t^{-\gamma} f(t, u(t))). \tag{9.5}$$
Conversely, if $u \in C[0,T]$ satisfies (9.5) then $u \in C^1[0,T]$ and satisfies (9.4).

(2) *Let f be continuous on $[0,T] \times \mathbb{R}$ and such that*
$$t \mapsto f(t, u(t)) \in AC \quad \text{for every } u \in C^1[0,T]. \tag{9.6}$$
Then if $u \in C[0,T]$ satisfies
$$D_*^{1+\alpha} u(t) = f(t, u(t)) \quad \text{for a.e. } t, \quad u(0) = u_0, \quad u'(0) = u_1. \tag{9.7}$$
we have $u' \in AC[0,T]$ and $D_C^{1+\alpha} u(t) = f(t, u(t))$.

(3) *If $f \in C^1$, that is $\partial_t f$ and $\partial_u f$ are continuous functions, and $u \in C[0,T]$ satisfies (9.7) then $u(t) = v(t) + ct^{1+\alpha}$ where $v \in C^2[0,T]$ and c is a constant, in particular, $u \in C^2(0,T] \cap AC^1[0,T]$ and satisfies $D_C^{1+\alpha} u(t) = f(t, u(t))$. Moreover $u \in C^2[0,T]$ if and only if $f(0, u_0) = 0$.*

Proof. (1) Define $h(t) := f(t, u(t))$. If $u \in C[0,T]$ satisfies (9.4) then $D_*^{1+\alpha} u(t) = t^{-\gamma} h(t)$. By Theorem 9.1 (1) the result follows.

(2) Let $h(t) := f(t, u(t))$. By part (1), $u \in C^1[0,T]$ hence by assumption (9.6) we obtain $h \in AC$ and Theorem 9.1 (2) applies.

(3) In this case $h \in C^1$ so this follows from Theorem 9.1 (3). □

Remark 9.4. The regularity in (3) on all of $[0,T]$ cannot be improved in general even if $f \in C^\infty$. For $u(t) = t^{1+\alpha}$ satisfies $D_*^{1+\alpha} u = \text{constant} \in C^\infty$ but we only have $u \in AC^1[0,T]$. In [7, Theorem 6.25] it is proved (also there are results for the higher order case) that $u \in C^1[0,T]$ as in (1). In [7, Theorem 6.26] it is shown (also there is a higher order case result), as in (3), that $u \in C^2(0,T]$ and that $u \in C^2[0,T]$ if and only if $f(0,u_0) = 0$, by similar arguments but without the explicit functions. We have not seen the exact forms of our cases (2) and (3) in the literature so they may be novel. Although smooth solutions cannot exist in general, Diethelm [8] give a full characterization of the situations where smooth solutions exist, even analytic.

Remark 9.5. Of course it is important to have some explicit conditions that imply (9.6). Clearly $f \in C^1$ is sufficient. The sum and pointwise product of functions in $AC[0,T]$ belong to $AC[0,T]$ and if $u \in AC$ then the composition $g \circ u \in AC$ if g satisfies a Lipschitz condition. Hence we have the following examples:

1. f is Lipschitz in both variables, that is $|f(t,u) - f(s,v)| \le L(|t-s| + |u-v|)$ for all $t,s \in [0,T]$ and all $u,v \in \mathbb{R}$. Then for $u \in C^1$, $t \mapsto f(t,u(t)) \in AC$.
2. $f(t,u) = h_1(t) + h_2(t) g(u)$ where $h_i \in AC[0,T]$ and $g \in \text{Lip}$.
3. Sums of terms of the above types.

The case (1) where f is Lipschitz in both variables is discussed in detail for several boundary value problems in [19] with somewhat different arguments.

For the problem $D_*^{1+\alpha} u(t) = t^{-\gamma} f(t, u(t), D^\beta u(t))$ which was studied above in the space C^1 via the Volterra integral equation

$$u(t) = u_0 + u_1\, t + I^{1+\alpha}(t^{-\gamma} f(t, u(t), D_C^\beta u(t))),$$

even if $f \in C^1$ we only have $t \mapsto f(t, u(t), D_C^\beta u(t)) \in C^0$ so we can not prove more than $u \in C^1$.

10. Comments on Boundary Value problems

For the Caputo fractional differential equation $D_*^\alpha u(t) = f(t,u(t))$, $t \in [0,T]$, of order $\alpha \in (0,1)$ it is appropriate to impose exactly one boundary condition. A natural type of condition is $au(0) + bu(T) = c$. When f is continuous it is shown in Lemma 6.40 of [7] that the Caputo problem is equivalent to a Fredholm integral equation. Tisdell [25] studies systems of such equations with this type of boundary condition. We prove a new equivalence result exactly similar to [7] for the single equation when the nonlinearity is allowed to be singular.

Theorem 10.1. *Let $0 \le \gamma < \alpha < 1$, let $a,b,c \in \mathbb{R}$ with $a + b \neq 0$ and let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Then, the function $u \in C[0,T]$ with $I^{1-\alpha} u \in AC$ is a solution of the boundary value problem*

$$D_*^\alpha u(t) = t^{-\gamma} f(t,u(t)), \quad t \in (0,T], \quad au(0) + bu(T) = c, \tag{10.1}$$

if and only if $u \in C[0,T]$ is a solution of the integral equation

$$\begin{aligned} u(t) = \frac{c}{a+b} &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} f(s,u(s))\, ds \\ &- \frac{b}{a+b} \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} s^{-\gamma} f(s,u(s))\, ds. \end{aligned} \tag{10.2}$$

Proof. Let $u \in C[0,T]$ satisfy (10.2), let $g(t) := t^{-\gamma} f(t,u(t))$ so $g \in L^1$. Then u satisfies

$$u(t) = \frac{c}{a+b} + I^\alpha g(t) - \frac{b}{a+b} I^\alpha g(T).$$

Thus $u(0) = \frac{c}{a+b} - \frac{b}{a+b} I^\alpha g(T)$ and $u(T) = \frac{c}{a+b} + \frac{a}{a+b} I^\alpha g(T)$, hence $au(0) + bu(t) = c$. Furthermore, $D_*^\alpha u = D^\alpha(u - u(0)) = D^\alpha I^\alpha g = g$ a.e. so $D_*^\alpha u = t^{-\gamma} f(t,u(t)$ for $t \in (0,T]$. Moreover it follow that $I^{1-\alpha} u \in AC$ as in Theorem 5.1. Conversely, let $u \in C[0,T]$ with $I^{1-\alpha} u \in AC$ satisfy (10.1). By Theorem 5.1, $u(t) = u_0 + I^\alpha g(t)$ and so $u_0 = u(0), u(T) = u_0 + I^\alpha g(T)$. The boundary conditions give $au_0 + bu_0 + bI^\alpha g(T) = c$ thus $u_0 = \frac{c - bI^\alpha g(T)}{a+b}$, which proves that u satisfies (10.2). □

Remark 10.2. When $\gamma = 0$, $D_*^\alpha u = f$ means that $D(I^{1-\alpha}(u - u_0)) = f$ so $I^{1-\alpha}(u - u_0) \in C^1$ and is automatically AC; this recovers the result of [7, Lemma 6.40].

For the Caputo fractional differential equation $D_*^{1+\alpha} u(t) = f(t,u(t))$, $t \in [0,T]$, where $\alpha \in (0,1)$ it is natural to impose two boundary conditions. The case with $D_C^{1+\alpha} u(t) = f(t,u(t))$ and general separated boundary conditions (also sometimes called Sturm-Liouville boundary conditions) was extensively studied in [19]. Lemma 6.43 of [7] proves an equivalence when f is continuous and $D_*^{1+\alpha} u(t) = f(t,u(t))$ together with boundary conditions of this type. We can allow a singularity as follows.

Theorem 10.3. *Let $0 \le \gamma < \alpha \in (0,1)$, let $a,b,c,d \in \mathbb{R}$ with $ad + bc + acT \neq 0$ and let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Then, the function $u \in C[0,T]$ with $I^{1-\alpha} u \in AC^1$ is a solution of the boundary value problem*

$$\begin{gathered} D_*^{1+\alpha} u(t) = t^{-\gamma} f(t,u(t)), \quad t \in (0,T], \\ au(0) - bu'(0) = 0, \quad cu(T) + du'(T) = 0, \end{gathered} \tag{10.3}$$

if and only if $u \in C[0,T]$ is a solution of an integral equation equation

$$u(t) = \int_0^T G(t,s) s^{-\gamma} f(s,u(s))\, ds,$$

where G is called the Green's function.

Proof. By Theorem 5.1 for $u \in C[0,T]$ with $I^{1-\alpha} u \in AC^1$ satisfying $D_*^{1+\alpha} u(t) = t^{-\gamma} f(t,u(t))$ we have $u(t) = u_0 + tu_1 + I^{1+\alpha} g(t)$ where $g(t) = t^{-\gamma} f(t,u(t))$. The boundary conditions are satisfied if

$$au_0 - bu_1 = 0,\ cu_0 + cTu_1 + cI^{1+\alpha} g(T) + du_1 + dI^\alpha g(T) = 0.$$

These equations have unique solution u_0, u_1 if and only if

$$\det \begin{bmatrix} a & -b \\ c & cT + d \end{bmatrix} = ad + bc + acT \neq 0.$$

These values are then substituted into $u_0 + tu_1 + I^{1+\alpha} g(t)$ to get the formula for the Green's function. The converse is similar to previous arguments. Since we are not pursuing any existence theory for boundary value problems here we omit the formulas for the Green's function which can be readily found and are given in [19]. □

Remark 10.4. In [19] the authors assume a Lipschitz condition on f in order that the solutions of the integral equation satisfy $D_C^{1+\alpha} u(t) = f(t,u(t))$.

For Riemann-Liouville fractional derivatives the type of boundary conditions that can be considered depend on the function space in which the solution is to be found. We will only discuss the case $D^{1+\alpha}$ for $0 < \alpha < 1$.

The equation $D^{1+\alpha}u = f$ is equivalent to $D^2(I^{1-\alpha})u = f$ where we should suppose explicitly that $I^{1-\alpha}u \in AC^1$. By writing $v = I^{1-\alpha}u$ we see that $v' \in AC$ and v satisfies the second order ordinary differential equation $v'' = f$. Therefore natural BVPs would have boundary condition defined in terms of $I^{1-\alpha}u$ and its derivative $D(I^{1-\alpha})u = D^\alpha u$ evaluated at $t = 0, t = T$. The physical interpretation of such boundary conditions is not obvious, but we recall that $I^{1-\alpha}u(0)$ is related to $\lim_{t\to 0+} t^{1-\alpha}u(t)$, see Lemma 6.3.

If, as is often the case, solutions are to be found in the space $C[0,T]$ as fixed points of a Fredholm integral operator, with kernel the Green's function, then certain boundary conditions are not allowed. The Green's function would be found by using the equivalence from Theorem 6.8. When f is continuous the procedure would begin by claiming the solution is of the form

$$u(t) = c_1 t^{\alpha-1} + c_2 t^\alpha + I^{1+\alpha} f(t, u(t)) \tag{10.4}$$

and then use the boundary conditions to determine unique values for c_1, c_2 as functions of u, f. We note that if $c_1 \neq 0$ then u is not defined at 0 and if $c_2 \neq 0$ then u' is not defined at 0. If we seek a solution $u \in C[0,T]$ of (10.4) then it is necessary that $c_1 = 0$ irrespective of the value $u(0)$, but when $c_1 = 0$ in (10.4) it is necessary that $u(0) = 0$ so that is the only compatible boundary condition.

If we seek a solution $u \in C[0,T]$ of (10.4) and try to impose a boundary condition at 0 involving $u'(0)$ then we must have $c_2 = 0$ in addition to $c_1 = 0$ so the only problem that can be considered is $u(t) = I^{1+\alpha}f(t, u(t))$ which is a Volterra equation, not a Fredholm equation, and any extra BC at $t = T$ would give an over-determined problem, in other words a BC at 0 involving $u'(0)$ is not allowed with any other BC at $t = T$ for the R-L derivative case if solutions in $C[0,T]$ are sought.

If one BC is $u(0) = 0$ then one can impose many possible BCs at $t = T$ and solutions of the integral equation will give solutions of the BVP. This has been done in many papers, too large a number to be given here.

However, it should be noted that one can never get periodic solutions (or anti-periodic ones) for fractional equations, it has been shown by Kaslik and Sivasundaram [15] in 2012 and by, Wang, Feckan, and Zhou [26] in 2013, and by Area, Losada, and Nieto [2] in 2014 that the fractional derivative or integral of a periodic function u defined on the whole real line cannot be periodic, except, in the Riemann-Liouville case for the trivial example of $u = 0$, in the Caputo case for u constant. Boundary value problems on a finite interval with boundary conditions of periodic type can be, and have been, considered.

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Addendum posted on August 5, 2020

Some further information and corrections.

(1) Proposition 3.2 (10) needs the extra condition that u is non-negative. Signs are important, the product of two non-decreasing functions is non-decreasing if both are non-negative but not necessarily if they have opposite signs.

Proposition 3.2 (10) is usually false when u is increasing but is negative on some interval $(0,\delta)$. Then $I^\alpha f(t)$ is negative for $t\in(0,\delta)$. For example if $f\in L^p$ for $p>1/\alpha$ then $I^\alpha f(0)=0$ (see Proposition 3.2 (3)), hence $I^\alpha f$ is not increasing for t near 0. A simple example, for $t\geq 0$, is $u(t)=3kt-1$ and $\alpha=1/2$, where $k>0$ can be arbitrarily large. Then u is increasing, negative for $t<1/(3k)$ and positive for $t>1/(3k)$. By a small calculation, $I^{1/2}u(t)=(4kt^{3/2}-2t^{1/2})/\sqrt{\pi}$ which is zero at 0, is then negative and decreasing, has a negative minimum at $t=1/(6k)$, and then increases, crossing zero at $t=1/(2k)$.

For $0<\gamma<1$ the $L^1[0,T]$ function $u(t)=t^{-\gamma}$ has

$$I^\alpha u(t)=\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}t^{\alpha-\gamma}$$

so gives an example of a positive decreasing L^1 function whose fractional derivative is increasing for $\alpha>\gamma$ and is decreasing for $\alpha<\gamma$.

By applying the result to $-u$ there is the obvious **dual result** that *if $u(t)\leq 0$ and non-increasing for $t\geq 0$ then $I^\alpha u(t)$ is non-increasing.*

Proposition 3.2 (10) is correctly used in two places, in Proposition 7.2 where it is applied to $v(t)=u(t)-u(0)$ with u nondecreasing so v is also non-negative, and in Proposition 7.4 where the dual is applied to $u'(t)-u'(0)$ with u' non-increasing.

(2) Proposition 3.2 (3) can be extended.

Lemma A1. *For $0<\alpha<1$ and $p>1/\alpha$, I^α is a compact operator from $L^p[0,T]$ to $C[0,T]$. Hence I^α is also compact from $C[0,T]$ to $C[0,T]$.*

This is because I^α maps into a Hölder space and the fact that for $0<\beta<1$ the Hölder space $C^{0,\beta}[0,T]$ is compactly embedded in $C[0,T]$.

The proof of this compact embedding is simple, so we give it here. The norm in $C^{0,\beta}[0,T]$ is

$$\|u\|_{0,\beta}:=\sup_{x\in[0,T]}|u(x)|+\sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\beta}.$$

Lemma A2. *If $\{u_n\}$ is a bounded sequence in $C^{0,\beta}[0,T]$, then $\{u_n\}$ is relatively compact in $C[0,T]$.*

Proof. Let $\|u_n\|_{0,\beta}\leq M$. For $x\neq y$ we have

$$|u_n(x)-u_n(y)|=\frac{|u_n(x)-u_n(y)|}{|x-y|^\beta}|x-y|^\beta\leq M|x-y|^\beta$$

so $\{u_n\}$ is bounded and equicontinuous hence relatively compact in $C[0,T]$ by the Ascoli-Arzelà theorem. □

(3) With a closer reading of the Hardy-Littlewood paper [13] I discovered that they have answered the following question which arises from Remark 5.2:

Question. Does there exist a function $w\in C[0,T]$ such that $I^\alpha w\notin AC$?

From the known results (see Proposition 3.2 (6)) w must not be AC.

The answer is yes. In §5.5 of [13] they give the following example. For $a > 1$ and any $k \in (0, 1)$, let w_k be the Weierstrass function

$$w_k(t) = \sum_{n=0}^{\infty} a^{-nk} \cos(a^n t).$$

Then state (and have proved in other papers cited there) that w_k is Hölder continuous with exponent k but $I^{1-k} w_k$ is not differentiable at any point so is not AC.

This implies that, for $0 < \alpha < 1$, if all we know is that u and f are continuous and $u(t) = u_0 + I^\alpha f(t)$ then it cannot be inferred that D_C^α exists; D_*^α does exist since $I^{1-\alpha}(u - u_0) = If \in C^1$. Thus, in Theorem 5.1 the often supposed equivalence between (5.1) and (5.2) is false in general. Therefore to study a Caputo fractional differential equation via the corresponding fractional integral equation the definition D_*^α should be used, as is done in the well-known books [7] and [17].

This supports the maxim 'Read the masters', Hardy and Littlewood answered the question nearly a century before the answer was wanted by those studying Caputo fractional derivatives.

(4) Some typos:

page 11, In the sentence 'The result is a special case of the result given for fractional derivatives of all orders in [7, Theorem 3.1], and which is proved below in Lemma 4.10.', Lemma 4.10 here should be replaced by Lemma 4.12.

page 18, In Theorem 6.10, the sum should go from $k = 1$ to $k = m + 1$.

page 19, $1/\Gamma(\alpha)$ is omitted in front of the integral sign in the proof of Proposition 7.2 step (2).

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