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NEUTRAL STOCHASTIC PARTIAL FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS DRIVEN BY G-BROWNIAN MOTION

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Abstract. In this article, we define the Hilbert-valued stochastic calculus with respect to G-Brownian motion in G-framework. On that basis, we prove the existence and uniqueness of mild solution for a class of neutral stochastic partial functional integro-differential equations driven by G-Brownian motion with non-Lipschitz coefficients. Our results are established by means of the Picard approximation. Moreover, we establish the stability of mild solution. An example is given to illustrate the theory.

1. Introduction

Neutral stochastic partial functional differential equations driven by Browniam motion (or Lévy process) arise in many areas of applied mathematics. For this reason, the study of this type of equations has been receiving increased attention in the last few years (see [2] and references therein). In [2], Bao and Hou studied a stochastic neutral partial functional differential equation. Then Diop et al. [5] extended the results to stochastic neutral partial functional integro-differential equations. In this paper, we focus on the study of neutral stochastic partial functional integro-differential equations in G-expectation framework.

Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng systemically established a time-consistent fully expectation theory. As a typical and important case, Peng introduced the G-expectation theory(see[12, 13, 14, 15]), which provides a unified tool for stochastic analysis problems that involve non-dominated family of probability measures. In the G-expectation framework, the notion of G-Brownian motion and the corresponding stochastic calculus of Itô's type were established. On that basis, many results about the stochastic ordinary different equations driven by G-Brownian motion are studied(see[1, 6, 7, 9, 11, 17, 18, 19]). However, so far as we known, there is no result about stochastic partial differential equations driven by G-Brownian motion. The main difficulty is that the infinite dimension G-Brownian motion and infinite dimension stochastic calculus in G-framework are undefined.

Motivated by the above mentioned works, in this paper, we first define $\mathbb{H}$-valued stochastic calculus with respect to one-dimensional G-Brownian motion, where $\mathbb{H}$

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is a real separable Hilbert space. On that basis, we study the existence, uniqueness and stability of mild solution for the following neutral stochastic partial functional integro-differential equations driven by G-Brownian motion in the space of $L_G^2(\Omega_T, \mathbb{H})$,

$$\begin{aligned} &d(u(t) + g(t, u_t)) \\ &= A(t)(u(t) + g(t, u_t))dt + \int_0^t R(t-s)(u(s) + g(s, u_s))ds \\ &\quad + f(t, u_t)dt + h(t, u_t)d\langle B\rangle_t + \sigma(t, u_t)dB_t, \quad t \in [0, T], \\ &\qquad u_0(\cdot) = \varphi \in C([-r, 0]; L_G^2(\Omega_T, \mathbb{H})), \quad r > 0, \end{aligned} \tag{1.1}$$

where $A(t) : D(A(t)) \subset L_G^2(\Omega_T, \mathbb{H}) \to L_G^2(\Omega_T, \mathbb{H})$, $R(t) : D(R(t)) \subset L_G^2(\Omega_T, \mathbb{H}) \to L_G^2(\Omega_T, \mathbb{H})$ are linear, closed, and densely defined operator on the space $L_G^2(\Omega_T, \mathbb{H})$, $u_t(\theta) = u(t + \theta)$ for $\theta \in [-r, 0]$. $(\langle B\rangle_t)_{t\geq 0}$ is the quadratic variation process of the G-Brownian motion $(B_t)_{t\geq 0}$. The coefficients $f, g, h, \sigma : [0, T] \times L_G^2(\Omega_T, \mathbb{H}) \to L_G^2(\Omega_T, \mathbb{H})$ are jointly continuous functions will be proposed in section 3.

The rest of this article is organized as follows. In section 2, we introduce some preliminaries and define the Hilbert-valued stochastic calculus with respect to G-Brownian motion . In section 3, we prove the existence and uniqueness of mild solution. The stability of mild solution in the mean square is discussed in section 4. Finally, we give an example in section 5.

2. Preliminaries

In this section, we introduce notation and preliminary results in G-framework that are needed in the sequel. More details can be found in [4, 10, 14, 15].

Let $\Omega_T = C_0([0, T]; R)$, the space of real valued continuous functions on $[0, T]$ with $w_0 = 0$, be endowed with the distance

$$d(w^1, w^2) := \sum_{N=1}^{\infty} 2^{-N}\big((\max_{0\leq t\leq N} |w_t^1 - w_t^2|) \wedge 1\big),$$

and let $B_t(w) = w_t$ be the canonical process. Denote by $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ the natural filtration generated by B. Let $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : \forall n \geq 1, t_1, \dots, t_n \in [0, T], \forall \varphi \in C_{b,Lip}(R^n)\}$, where $C_{b,Lip}(R^n)$ denotes the set of bounded Lipschitz functions on R^n. A sublinear functional on $L_{ip}(\Omega_T)$ is defined as follows: for all $X, Y \in L_{ip}(\Omega_T)$,

- (i) Monotonicity: $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$.
- (ii) Constant preserving: $\mathbb{E}[C] = C$ for $C \in R$.
- (iii) Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
- (iv) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda\mathbb{E}[X]$ for $\lambda \geq 0$.

The triplete $(\Omega, L_{ip}(\Omega_T), \mathbb{E})$ is called a sublinear expectation space and $\mathbb{E}$ is called a sublinear expectation. $X \in L_{ip}(\Omega_T)$ is called a random variable in $(\Omega, L_{ip}(\Omega_T), \mathbb{E})$.

Definition 2.1. A random variable $X \in L_{ip}(\Omega_T)$ is G-normal distributed with parameters $(0, [\underline{\sigma}^2, \bar{\sigma}^2])$, i.e., $X \sim N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$, if for each $\varphi \in C_{b,Lip}(R)$, $u(t, x) := \mathbb{E}[\varphi(x + \sqrt{t}X)]$ is a viscosity solution to the following PDE on $R^+ \times R$:

$$\begin{gathered} \frac{\partial u}{\partial t} - G(\frac{\partial^2 u}{\partial x^2}) = 0, \\ u_{t_0} = \varphi(x), \end{gathered} \tag{2.1}$$

where $G(a) := \frac{1}{2}(a^+\bar{\sigma}^2 - a^-\underline{\sigma}^2), a \in R$.

Definition 2.2. We call a sublinear expectation $\hat{\mathbb{E}} : L_{ip}(\Omega_T) \to R$ a G-expectation if the canonical process B is a G-Brownian motion under $\hat{\mathbb{E}}[\cdot]$, that is, for each $0 \le s \le t \le T$, the increment $B_t - B_s \sim N(0, [\underline{\sigma}^2(t-s), \bar{\sigma}^2(t-s)])$ and for all $n > 0, 0 \le t_1 \le \dots \le t_n \le T$ and $\varphi \in L_{ip}(\Omega_T)$,

$$\hat{\mathbb{E}}[\varphi(B_{t_1}, \dots, B_{t_{n-1}}, B_{t_n} - B_{t_{n-1}})] = \hat{\mathbb{E}}[\psi(B_{t_1}, \dots, B_{t_{n-1}})],$$

where $\psi(x_1, \dots, x_{n-1}) := \hat{\mathbb{E}}[\varphi(x_1, \dots, x_{n-1}, \sqrt{t_n - t_{n-1}}B_1)]$.

Let us define the conditional G-expectation $\hat{\mathbb{E}}_t$ of $\xi \in L_{ip}(\Omega_T)$ knowing $L_{ip}(\Omega_t)$, for $t \in [0, T]$. Without loss of generality, we can assume that ξ has the representation $\xi = \varphi(B(t_1), B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1}))$ with $t = t_i$, for some $1 \le i \le n$, and we put

$$\begin{aligned} &\hat{\mathbb{E}}_{t_i}[\varphi(B(t_1), B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1}))] \\ &= \widetilde{\varphi}(B(t_1), B(t_2) - B(t_1), \dots, B(t_i) - B(t_{i-1})), \end{aligned}$$

where $\widetilde{\varphi}(x_1, .., x_i) = \hat{\mathbb{E}}[\varphi(x_1, \dots, x_i, B(t_{i+1}) - B(t_i), \dots, B(t_n) - B(t_{n-1}))]$.

Definition 2.3. Quadratic variation process of G-Brownian motion defined by

$$\langle B \rangle_t := B_t^2 - 2\int_0^t B_s dB_s$$

is a continuous, nondecreasing process.

For $p \ge 1$, we denote by $L_G^p(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under the natural norm $\|X\|_{p,G} := (\hat{\mathbb{E}}[|X|^p])^{1/p}$. For all $t \in [0, T]$, $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}_t$ are continuous mapping on $L_{ip}(\Omega_T)$ endowed with the norm $\|\cdot\|_{1,G}$. Therefore, they can be extended continuous to $L_G^1(\Omega_T)$ under the norm $\|X\|_{1,G}$.

Theorem 2.4 ([4]). *There exists a weakly compact subset $\mathcal{P} \subset \mathcal{M}(\Omega_T)$, where $\mathcal{M}(\Omega_T)$ is the set of probability measures on $(\Omega_T, \mathcal{F}_T)$, such that*

$$\hat{\mathbb{E}}[\xi] = \max_{P \in \mathcal{P}} E_P(\xi) \quad \forall \xi \in L_G^1(\Omega_T).$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{\mathbb{E}}$. For this $\mathcal{P}$, we define capacity

$$C(A) = \sup_{P \in \mathcal{P}} P(A), A \in \mathcal{F}_T.$$

A set $A \subset \Omega_T$ is a polar set if $C(A) = 0$. A property holds quasi-surely (q. s.) if it holds outside a polar set.

Let $\mathbb{H}$ be a real separable Hilbert space equip with the norm $\|\cdot\|$ and the inner product $(\cdot, \cdot)_H$. We consider the following space of random variables:

$$\mathrm{Lip}(\Omega_T, \mathbb{H}) := \mathrm{span}\{X \in \mathbb{H} | \|X(w)\| \in L_{ip}(\Omega_T)\}, \tag{2.2}$$

where span indicates that the space of all linear combinations of the corresponding random variables is considered. We denote by $L_G^p(\Omega_T, \mathbb{H})$ the completion of $\mathrm{Lip}(\Omega_T, \mathbb{H})$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[\|X\|^p])^{1/p}$. Then $(L_G^p(\Omega_T, \mathbb{H}), \|X\|_p)$ is a Banach space and the operator $\hat{\mathbb{E}}[\cdot]$ can be continuously extended to the Banach space $L_G^p(\Omega_T, \mathbb{H})$.

Next, we introduce $\mathbb{H}$-valued Itô integral of G-Brownian motion. We consider the following type of simple process: for a given partition $\pi_T = t_0, t_1, \dots, t_N$ of $[0,T]$, set

$$\eta(t) = \sum_{k=0}^{N-1} \xi_k(w) I_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in L_G^p(\Omega_{t_k}, \mathbb{H}), k = 0, 1, \dots, N-1$ are given. The collection of these processes is denoted by $M_G^{p,0}(0,T;\mathbb{H})$. We define the following norm on $M_G^{p,0}(0,T;\mathbb{H})$:

$$\|\eta\|_{M^p} = (\frac{1}{T}\int_0^T \hat{\mathbb{E}}[\|\eta(t)\|^p]dt)^{1/p} = (\frac{1}{T}\sum_{k=0}^{N-1} \hat{\mathbb{E}}\|\xi_k(w)\|^p(t_{k+1}-t_k))^{1/p}.$$

Finally, we denote by $M_G^p(0,T;\mathbb{H})$ the completion of $M_G^{p,0}(0,T;\mathbb{H})$ under the above norm. Moreover, $M_G^p(0,T;\mathbb{H}) \subseteq M_G^q(0,T;\mathbb{H})$ for $p \geq q$.

Definition 2.5. For $\eta \in M_G^{1,0}(0,T;\mathbb{H})$, the related Bochner integral is

$$\int_0^t \eta(t)dt := \sum_{k=0}^{N-1} \xi_k(w)(t_{k+1} - t_k).$$

Lemma 2.6. *Let $\eta \in M_G^1(0,T;\mathbb{H})$, then we have*

$$\hat{\mathbb{E}}\|\int_0^t \eta(s)ds\| \leq \int_0^t \hat{\mathbb{E}}\|\eta(s)\|ds.$$

Proof. According to the real-valued Bochner integral defined in [14] and the property of sublinear expectation, we have

$$\hat{\mathbb{E}}\|\int_0^t \eta(s)ds\| \leq \hat{\mathbb{E}}\int_0^t \|\eta(s)\|ds \leq \int_0^t \hat{\mathbb{E}}\|\eta(s)\|ds. \qquad \square$$

Definition 2.7. For $\eta \in M_G^{2,0}(0,T;\mathbb{H})$ of the form $\eta(t) = \sum_{k=0}^{N-1} \xi_k(w) I_{[t_k,t_{k+1})}(t)$, define

$$I(\eta) := \int_0^T \eta(s)dB(s) := \sum_{k=0}^{N-1} \xi_k(B(t_{k+1}) - B(t_k)).$$

The mapping $I : M_G^{2,0}(0,T;\mathbb{H}) \to L_G^2(\Omega_T, \mathbb{H})$ can be continuously extended to $I : M_G^2(0,T;\mathbb{H}) \to L_G^2(\Omega_T, \mathbb{H})$. Moreover, we have the following Lemma.

Lemma 2.8. *Suppose $\eta \in M_G^2(0,T;\mathbb{H})$, then for $t \in [0,T]$, we have*

$$\hat{\mathbb{E}}[\|\int_0^t \eta(s)dB(s)\|^2] \leq \bar{\sigma}^2 \int_0^t \hat{\mathbb{E}}\|\eta(s)\|^2 ds.$$

Proof. Let us consider the case

$$\eta(t) = \sum_{k=0}^{N-1} \xi_k(w) I_{[t_k,t_{k+1})}(t) \in M_G^{2,0}(0,T;\mathbb{H}).$$

Then

$$
\begin{aligned}
&\hat{\mathbb{E}}[\| \int_0^t \eta(s) dB(s) \|^2] \\
&= \hat{\mathbb{E}}[\| \sum_{k=0}^{N-1} \xi_k (B(t_{k+1} \wedge t) - B(t_k \wedge t)) \|^2] \\
&= \hat{\mathbb{E}}[\sum_{k=0}^{N-1} \| \xi_k (B(t_{k+1} \wedge t) - B(t_k \wedge t)) \|^2 \\
&\quad + 2 \sum_{j<k, j=0}^{N-1} (\xi_j (B(t_{j+1} \wedge t) - B(t_j \wedge t)), \xi_k (B(t_{k+1} \wedge t) - B(t_k \wedge t)))_H] \\
&\le \hat{\mathbb{E}}[\sum_{k=0}^{N-1} \| \xi_k (B(t_{k+1} \wedge t) - B(t_k \wedge t)) \|^2] \\
&+ 2\hat{\mathbb{E}}[\sum_{j<k, j=0}^{N-1} (\xi_j (B(t_{j+1} \wedge t) - B(t_j \wedge t)), \xi_k (B(t_{k+1} \wedge t) - B(t_k \wedge t)))_H].
\end{aligned} \tag{2.3}
$$

However, since $\hat{\mathbb{E}}[(B(t) - B(s))^2] = \bar{\sigma}^2 (t - s)$ and ξ_k is independent from $B(t_{k+1} \wedge t) - B(t_k \wedge t)$, we have

$$
\begin{aligned}
&\hat{\mathbb{E}}[\sum_{k=0}^{N-1} \| \xi_k (B(t_{k+1} \wedge t) - B(t_k \wedge t)) \|^2] \\
&= \hat{\mathbb{E}}[\sum_{k=0}^{N-1} \sum_l (\xi_k (B(t_{k+1} \wedge t) - B(t_k \wedge t)), e_l)_H^2] \\
&\le \sum_{k=0}^{N-1} \hat{\mathbb{E}}[\sum_l (\xi_k, e_l)_H^2 (B(t_{k+1} \wedge t) - B(t_k \wedge t))^2] \\
&\le \sum_{k=0}^{N-1} [\bar{\sigma}^2 \hat{\mathbb{E}}[\sum_l (\xi_k, e_l)_H^2] (t_{k+1} \wedge t - t_k \wedge t)] \\
&= \bar{\sigma}^2 \int_0^t \hat{\mathbb{E}} \| \eta(s) \|^2 ds.
\end{aligned} \tag{2.4}
$$

Here, $\{e_l\}$ is the standard orthonormal basis of space $\mathbb{H}$.

On the other hand, by the property of sublinear conditional expectation, we have

$$
\begin{aligned}
&\hat{\mathbb{E}}[\sum_{j<k, j=0}^{N-1} (\xi_j (B(t_{j+1} \wedge t) - B(t_j \wedge t)), \xi_k (B(t_{k+1} \wedge t) - B(t_k \wedge t)))_H] \\
&\le \sum_{j<k, j=0}^{N-1} \hat{\mathbb{E}}[(\xi_j (B(t_{j+1} \wedge t) - B(t_j \wedge t)), \xi_k (B(t_{k+1} \wedge t) - B(t_k \wedge t)))_H] \\
&= \sum_{j<k, j=0}^{N-1} \hat{\mathbb{E}}[(\xi_j, \xi_k)_H (B(t_{j+1} \wedge t) - B(t_j \wedge t)) \cdot (B(t_{k+1} \wedge t) - B(t_k \wedge t))]
\end{aligned} \tag{2.5}
$$

$$\begin{aligned}
&= \sum_{j<k,j=0}^{N-1} \hat{\mathbb{E}}[\hat{\mathbb{E}}_{t_k}[(\xi_j,\xi_k)_H(B(t_{j+1}\wedge t)-B(t_j\wedge t))\cdot(B(t_{k+1}\wedge t)-B(t_k\wedge t))]] \\
&= \sum_{j<k,j=0}^{N-1} \hat{\mathbb{E}}[(\xi_j,\xi_k)_H(B(t_{j+1}\wedge t)-B(t_j\wedge t))\cdot\hat{\mathbb{E}}_{t_k}[(B(t_{k+1}\wedge t)-B(t_k\wedge t))]] \\
&= 0.
\end{aligned}$$

Then by (2.3)–(2.5), we have

$$\hat{\mathbb{E}}[\|\int_0^t \eta(s)dB(s)\|^2] \le \bar{\sigma}^2 \int_0^t \hat{\mathbb{E}}\|\eta(s)\|^2 ds.$$

For a general $\eta \in M_G^2(0,T;\mathbb{H})$, choose $\{\eta^n, n\ge 1\} \in M_G^{2,0}(0,T;\mathbb{H})$ such that

$$\begin{gathered}
\int_0^T \hat{\mathbb{E}}[\|\eta(s)-\eta^n(s)\|^2]ds \to 0, \\
\hat{\mathbb{E}}[\|\int_0^T \eta^n(s)dB(s) - \int_0^T \eta dB(s)\|^2] \to 0,
\end{gathered}$$

as $n\to\infty$. Then we have

$$\begin{aligned}
&\hat{\mathbb{E}}[\|\int_0^T \eta(s)dB(s)\|^2] \\
&\le \hat{\mathbb{E}}[\|\int_0^T (\eta(s)-\eta^n(s))dB(s)\|^2] + \hat{\mathbb{E}}[\|\int_0^T \eta^n(s)dB(s)\|^2].
\end{aligned} \tag{2.6}$$

It is easy to get the result when $n\to\infty$ on (2.6). □

Definition 2.9. For $\eta \in M_G^{1,0}(0,T;\mathbb{H})$ of the form $\eta(t) = \sum_{k=0}^{N-1} \xi_k(w) I_{[t_k,t_{k+1})}(t)$, define

$$Q(\eta) := \int_0^T \eta(s)d\langle B\rangle_s := \sum_{k=0}^{N-1} \xi_k(\langle B\rangle_{t_{k+1}} - \langle B\rangle_{t_k}).$$

The mapping $Q: M_G^{1,0}(0,T;\mathbb{H}) \to L_G^1(\Omega_T,\mathbb{H})$ can be continuously extended to $Q: M_G^1(0,T;\mathbb{H}) \to L_G^1(\Omega_T,\mathbb{H})$. Using a similar discussion as in Lemma 2.8 and the fact that $\langle B\rangle_t - \langle B\rangle_s \le \overline{\sigma}^2(t-s)$ in $L_G^1(\Omega_T)$ for $0\le s\le t<\infty$, it is easy to get the following lemma.

Lemma 2.10. *Let $\eta \in M_G^2(0,T;\mathbb{H})$, then there exists a constant $C_1>0$ such that*

$$\hat{\mathbb{E}}[\|\int_0^t \eta(s)d\langle B\rangle_s\|^2] \le C_1 \hat{\mathbb{E}}[\int_0^T \|\eta(t)\|^2 dt].$$

Lemma 2.11 (Bihari's inequality [3])**.** *Let $H(y)$ be a continuous, nondecreasing, and concave function in $y\in R^+$ such that $H(0)=0$. Let $y(\cdot)$ be a Borel measurable bounded nonnegative function on $[0,T]$, and let $v(\cdot)$ be a nonnegative integrable function on [0,T]. If*

$$y(t) \le c + \int_0^t v(s)H(y(s))ds, \quad t\in[0,T].$$

Then

$$y(t) \le J^{-1}(J(c) + \int_0^t v(s)ds)$$

holds for $t \in [0,T]$ *such that* $J(c)+\int_0^t v(s)ds \in Dom(J^{-1})$, *where* $J(\tau) = \int_0^\tau \frac{1}{H(s)}ds$, *on* $\tau > 0$, *and* J^{-1} *is the inverse function of* J. *In particular, if* $c = 0$ *and* $\int_{0+} \frac{1}{H(y)}dy = +\infty$, *then* $y(t) = 0$ *for* $t \in [0,T]$.

Lemma 2.12 ([16])**.** *Let the assumptions of Lemma 2.11 hold. If for* $\forall \varepsilon > 0$, *there exists* $t_1 \geq 0$ *such that for* $0 \leq y_0 < \varepsilon$, $\int_{t_1}^T v(s)ds \leq \int_{y_0}^\varepsilon \frac{1}{H(y)}dy$. *Then for* $t \in [t_1,T]$, *the estimate* $y(t) < \varepsilon$ *holds.*

3. Existence and uniqueness of the solution

In this section, we aim to study the existence, uniqueness of mild solution for equation (1.1) in the space of $L_G^2(\Omega_T,\mathbb{H})$, where $A(t) : D(A(t)) \subset L_G^2(\Omega_T,\mathbb{H}) \to L_G^2(\Omega_T,\mathbb{H})$, $R(t) : D(R(t)) \subset L_G^2(\Omega_T,\mathbb{H}) \to L_G^2(\Omega_T,\mathbb{H})$ are linear, closed, and densely defined operator on the space $L_G^2(\Omega_T,\mathbb{H})$, $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r,0]$. The coefficients $f,g,h,\sigma : [0,T] \times L_G^2(\Omega_T,\mathbb{H}) \to L_G^2(\Omega_T,\mathbb{H})$ are jointly continuous functions. In other words, we have $\sup_{0\leq t\leq T} \hat{\mathbb{E}}[\|\phi(t)\|^2] < \infty$, which implies that $\phi \in M_G^2(0,T;\mathbb{H})$ for $\phi = f,g,h,\sigma$. Thus the stochastic calculuses appeared in (1.1) are meaningful.

Let $C([-r,0];L_G^2(\Omega_T,\mathbb{H}))$ be the family of all continuous functions in $L_G^2(\Omega_T,\mathbb{H})$. Let $\|\varphi\|_C^2 := \sup_{-r\leq\theta\leq 0} \hat{\mathbb{E}}[\|\varphi(\theta)\|^2]$. Let A be the infinitesimal generator of a strongly continuous semigroup on $L_G^2(\Omega_T,\mathbb{H})$ and $(S_t)_{t\geq 0}$ be the corresponding resolvent operator on $L_G^2(\Omega_T,\mathbb{H})$. Regarding the theory of of partial integro-differential equations and resolvent operator we refer the readers to [8], we omit it in this paper. Now, we give the definition of mild solution for equation (1.1).

Definition 3.1. A process $\{u(t), -r \leq t \leq T\}$ is said to be a mild solution of (1.1), if $u(t) = \varphi(t)$ on $[-r,0]$, and the following conditions hold:

- (i) $u(t) \in L_G^2(\Omega_T,\mathbb{H})$ is $\mathcal{F}_t$-adapted and continuous in $t \in [0,T]$ q.s.;
- (ii) For $t \in [0,T]$, we have q.s.

$$\begin{aligned} u(t) + g(t,u_t) &= S(t)(\varphi(0)+g(0,\varphi)) + \int_0^t S(t-s)f(s,u_s)ds \\ &\quad + \int_0^t S(t-s)h(s,u_s)d\langle B\rangle_s + \int_0^t S(t-s)\sigma(s,u_s)dB_s. \end{aligned} \tag{3.1}$$

To prove the required results, we assume the following conditions.

- (H1) A is the infinitesimal generator of a strongly continuous semigroup on $L_G^2(\Omega_T,\mathbb{H})$. $(S(t))_{t\geq 0}$ is the corresponding resolvent operator which satisfies $S(0) = I$ and $\|S(t)\|_{\mathcal{L}} \leq Ne^{\beta t}$ for some constants N and β, where $\|\cdot\|_{\mathcal{L}}$ denote the operator norm.
- (H2) $f,g,h,\sigma : [0,T] \times L_G^2(\Omega_T,\mathbb{H}) \to L_G^2(\Omega_T,\mathbb{H})$ are jointly continuous functions and satisfy the following non-Lipschitz condition: for any $\xi,\eta \in C([-r,0];L_G^2(\Omega_T,\mathbb{H}))$,

$$\hat{\mathbb{E}}\|f(t,\xi)-f(t,\eta)\|^2 \vee \hat{\mathbb{E}}\|h(t,\xi)-h(t,\eta)\|^2 \vee \hat{\mathbb{E}}\|\sigma(t,\xi)-\sigma(t,\eta)\|^2 \leq H(\|\xi-\eta\|_C^2),$$

  where $H(y)$ is continuous, nondecreasing, and concave in $y \in R^+$ such that $H(0) = 0$ and $\int_{0+} \frac{1}{H(y)}dy = +\infty$.
- (H3) For each $t \in [0,T]$, there exists a positive constant K' such that

$$\|f(t,0)\|^2 \vee \|h(t,0)\|^2 \vee \|\sigma(t,0)\|^2 \leq K'.$$

(H4) For $\xi,\eta \in C([-r,0];L^2_G(\Omega_T,\mathbb{H}))$, there exists a positive constant $K<1$, for $\forall t\in[0,T]$, we have

$$\hat{\mathbb{E}}\|g(t,\xi)-g(t,\eta)\|^2 \le K^2\|\xi-\eta\|_C^2.$$

Moreover, $g(t,0)=0$ for $t\in[0,T]$.

We construct a sequence of successive approximations defined as follows:

$$u^0(t)=\begin{cases}\varphi(t), & t\in[-r,0],\\ S(t)\varphi(0), & t\in[0,T]\end{cases} \tag{3.2}$$

and for $n\ge 1$,

$$u^n(t)=\begin{cases}\varphi(t), \quad t\in[-r,0],\\ S(t)(\varphi(0)+g(0,\varphi))-g(t,u_t^n)+\int_0^t S(t-s)f(s,u_s^{n-1})ds\\ +\int_0^t S(t-s)h(s,u_s^{n-1})d\langle B\rangle_s\\ +\int_0^t S(t-s)\sigma(s,u_s^{n-1})dB(s), \quad t\in[0,T].\end{cases} \tag{3.3}$$

Lemma 3.2. *If* (H1)–(H4) *hold, then there exists a constant $C>0$ which is independent of $n\ge 1$, such that*

$$\sup_{0\le t\le T}\hat{\mathbb{E}}\|u^n(t)\|^2\le C.$$

Proof. Form the elementary inequality $(a+b+c+d)^2\le 4(a^2+b^2+c^2+d^2)$, we have

$$\begin{aligned}
&\sup_{0\le t\le T}\hat{\mathbb{E}}\|u^n(t)+g(t,u_t^n)\|^2\\
&\le 4\sup_{0\le t\le T}\hat{\mathbb{E}}\|S(t)(\varphi(0)+g(0,\varphi))\|^2+4\sup_{0\le t\le T}\hat{\mathbb{E}}\|\int_0^t S(t-s)f(s,u_s^{n-1})ds\|^2\\
&\quad+4\sup_{0\le t\le T}\hat{\mathbb{E}}\|\int_0^t S(t-s)h(s,u_s^{n-1})d\langle B\rangle_s\|^2\\
&\quad+4\sup_{0\le t\le T}\hat{\mathbb{E}}\|\int_0^t S(t-s)\sigma(s,u_s^{n-1})dB_s\|^2=:4\sum_{i=1}^4 I_i.
\end{aligned} \tag{3.4}$$

From (H1) and (H4), we have

$$I_1\le 2M(1+K^2)\|\varphi\|_C^2, \tag{3.5}$$

where $M=\sup_{0\le t\le T}\|S(t)\|_{\mathcal{L}}^2$. On the other hand, in view of (H2), we obtain from the Hölder inequality that

$$\begin{aligned}
I_2&\le T\sup_{0\le t\le T}\hat{\mathbb{E}}\int_0^t\|S(t-s)f(s,u_s^{n-1})\|^2ds\\
&\le 2TM[K'T+\int_0^T H(\|u_s^{n-1}\|_C^2)ds].
\end{aligned} \tag{3.6}$$

According to Lemmas 2.8 and 2.10, and (H2) and (H3), we have

$$\begin{aligned} I_3 &\le C_1 \sup_{0\le t\le T} \int_0^t \hat{\mathbb{E}}\|S(t-s)h(s,u_s^{n-1})\|^2 ds \\ &\le C_1 M \int_0^T \hat{\mathbb{E}}\|h(s,u_s^{n-1}) - h(s,0) + h(s,0)\|^2 ds \\ &\le 2C_1 M[K'T + \int_0^T H(\|u_s^{n-1}\|_C^2) ds], \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} I_4 &\le \sup_{0\le t\le T} \bar{\sigma}^2 M \int_0^t \hat{\mathbb{E}}\|\sigma(s,u_s^{n-1})\|^2 ds \le \bar{\sigma}^2 M \int_0^T \hat{\mathbb{E}}\|\sigma(s,u_s^{n-1})\|^2 ds \\ &\le \bar{\sigma}^2 M \int_0^T \hat{\mathbb{E}}\|\sigma(s,u_s^{n-1}) - \sigma(s,0) + \sigma(s,0)\|^2 ds \\ &\le 2\bar{\sigma}^2 M[K'T + \int_0^T H(\|u_s^{n-1}\|_C^2) ds]. \end{aligned} \tag{3.8}$$

Notice that $H(u)$ is a concave function on $u \ge 0$, thus there exists a pair of positive constants a, b such that $H(u) \le a + bu$. Putting (3.5)—(3.8) into (3.4) yields for some positive constants C_2 and C_3, such that

$$\sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t) + g(t,u_t^n)\|^2 \le C_3 + C_2 \int_0^T \|u_s^{n-1}\|_C^2 ds. \tag{3.9}$$

By (H4), for $K < 1$, it follows that

$$\begin{aligned} &\sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t)\|^2 \\ &\le \frac{1}{1-K} \sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t) + g(t,u_t^n)\|^2 + \frac{1}{K} \sup_{0\le t\le T} \hat{\mathbb{E}}\|g(t,u_t^n)\|^2 \\ &\le \frac{1}{1-K} \sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t) + g(t,u_t^n)\|^2 + K \sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t)\|^2 + K\|\varphi\|_C^2, \end{aligned} \tag{3.10}$$

which implies

$$\sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t)\|^2 \le \frac{1}{(1-K)^2} \sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t) + g(t,u_t^n)\|^2 + \frac{K}{1-K}\|\varphi\|_C^2. \tag{3.11}$$

Thus we have

$$\begin{aligned} \sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t)\|^2 &\le \frac{C_3}{(1-K)^2} + [\frac{C_2 T}{(1-K)^2} + \frac{K}{1-K}]\|\varphi\|_C^2 \\ &\quad + \frac{C_2}{(1-K)^2} \int_0^T \sup_{0\le\theta\le s} \hat{\mathbb{E}}\|u^{n-1}(\theta)\|^2 ds. \end{aligned} \tag{3.12}$$

Observing that

$$\max_{1\le n\le k} \sup_{0\le t\le T} \hat{\mathbb{E}}\|u^{n-1}(t)\|^2 \le \|\varphi\|_C^2 + \max_{1\le n\le k} \sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t)\|^2. \tag{3.13}$$

Then there exists positive constants C_4, C_5 such that

$$\max_{1\le n\le k} \sup_{0\le t\le T} \hat{\mathbb{E}}\|u^n(t)\|^2 \le C_5 + C_4 \int_0^T \max_{1\le n\le k} \sup_{0\le\theta\le s} \hat{\mathbb{E}}\|u^n(\theta)\|^2 ds. \tag{3.14}$$

By Gronwall's inequality, we derive that

$$\max_{1\le n\le k}\sup_{0\le t\le T}\hat{\mathbb{E}}\|u^n(t)\|^2 \le C_5\exp(C_4T). \tag{3.15}$$

This completes the proof since k is arbitrary. □

Theorem 3.3. *Let the assumptions of Lemma 3.2 be satisfied, then there exists a unique mild solution to* (1.1).

Proof. Applying (3.3), (H1)–(H4) as in the proof of Lemma 3.2, for all $t\in[0,T]$ and $m,n\ge 1$, we can show that there exists a positive constant C_6 such that

$$\begin{aligned}&\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{n+m}(s)-u^n(s)+g(s,u_s^{n+m})-g(s,u_s^n)\|^2\\&\le C_6\int_0^t H(\sup_{0\le\theta\le s}\hat{\mathbb{E}}\|u^{n+m-1}(\theta)-u^{n-1}(\theta)\|^2)ds.\end{aligned} \tag{3.16}$$

By (H4), we have

$$\begin{aligned}&\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{n+m}(s)-u^n(s)\|^2\\&\le\frac{1}{1-K}\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{n+m}(s)-u^n(s)+g(s,u_s^{n+m})-g(s,u_s^n)\|^2\\&\quad+K\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{n+m}(s)-u^n(s)\|^2\\&\le\frac{C_6}{1-K}\int_0^t H(\sup_{0\le\theta\le s}\hat{\mathbb{E}}\|u^{n+m-1}(\theta)-u^{n-1}(\theta)\|^2)ds\\&\quad+K\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{n+m}(s)-u^n(s)\|^2.\end{aligned} \tag{3.17}$$

Then we have

$$\begin{aligned}&\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{n+m}(s)-u^n(s)\|^2\\&\le\frac{C_6}{(1-K)^2}\int_0^t H(\sup_{0\le\theta\le s}\hat{\mathbb{E}}(\|u^{n+m-1}(\theta)-u^{n-1}(\theta)\|^2))ds.\end{aligned} \tag{3.18}$$

Let

$$Z(t)=\limsup_{m,n\to\infty}\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{n+m}(s)-u^n(s)\|^2.$$

From Lemma 3.2, (H2) and Fatou's lemma, we get

$$Z(t)\le\frac{C_6}{(1-K)^2}\int_0^t H(s,Z(s))ds.$$

By Lemma 2.11, we get

$$Z(t)=0. \tag{3.19}$$

So there exist a subsequence still denoted by $\{u^n\}_{n\in N}$ such that for any $n>1$

$$\sup_{0\le s\le t}\hat{\mathbb{E}}[\|u^{n+1}(s)-u^n(s)\|^2]\le\frac{1}{2^n}.$$

Then

$$\hat{\mathbb{E}}[\sum_{n=1}^\infty\|u^{n+1}(t)-u^n(t)\|^2]\le\sum_{n=1}^\infty\hat{\mathbb{E}}[\|u^{n+1}(t)-u^n(t)\|^2]\le 1.$$

By the Chebyshev inequality in G-framework, it is easy to see that

$$\sum_{n=1}^{\infty} \|u^{n+1}(t) - u^n(t)\| < \infty, \quad q.s.$$

Set $u(t) = u^0(t) + \sum_{n=0}^{\infty}(u^{n+1}(t) - u^n(t))$. From (3.19), $\{u^n\}_{n\in N}$ is a Cauchy sequence in $L_G^2(\Omega_T, \mathbb{H})$. Consequently, we can conclude that $\sup_{0\le s\le t} \hat{\mathbb{E}}(\|u^n(s) - u(s)\|^2) \to 0$ as $n \to \infty$. Hence in what follows, we claim that $u(t)$ is a mild solution to equation (1.1). In fact, by (H2), the Hölder inequality, Lemmas 2.8 and 2.10 and letting $n \to \infty$, we obtain

$$\begin{gathered}
\sup_{0<t\le T} \hat{\mathbb{E}}\| \int_0^t S(t-s)[f(s,u_s^n) - f(s,u_s)]ds\|^2 \to 0, \\
\sup_{0<t\le T} \hat{\mathbb{E}}\| \int_0^t S(t-s)[h(s,u_s^n) - h(s,u_s)]d\langle B\rangle_s\|^2 \to 0, \\
\sup_{0<t\le T} \hat{\mathbb{E}}\| \int_0^t S(t-s)[\sigma(s,u_s^n) - \sigma(s,u_s)]dB_s\|^2 \to 0.
\end{gathered}$$

On the other hand, by applying (H4), we can also claim, for $t \in [0,T]$, that

$$\hat{\mathbb{E}}\|g(t,u_t^n) - g(t,u_t)\|^2 \le K^2 \sup_{0\le s\le t} \hat{\mathbb{E}}\|u_s^n - u_s\|^2 \to 0,$$

as $n \to \infty$.

Hence taking limit in both side of (3.2), we obtain that $u(t) \in L_G^2(\Omega_T, \mathbb{H}), t \in [0,T]$ satisfy equation (3.1). Moreover, $u(t)$ in continuous in $t \in [0,T]$ q.s., since the stochastic calculuses appeared in (3.1) have quasi-continuous modification. Thus $u(t)$ is a mild solution to (1.1). This shows the existence.

Denote by $u_1(t)$ and $u_2(t)$ the mild solutions to (1.1). In the same way as (3.18) was proved, we can show that for some constant $D > 0$,

$$\sup_{0\le s\le t} \hat{\mathbb{E}}\|u_1(s) - u_2(s)\|^2 \le D \int_0^t H(\sup_{0\le\theta\le s} \hat{\mathbb{E}}\|u_1(\theta) - u_2(\theta)\|^2)ds,$$

this leads to $\sup_{0\le s\le t} \hat{\mathbb{E}}\|u_1(s) - u_2(s)\|^2 = 0$, which also implies $u_1(t) = u_2(t)$ q.s. for any $0 \le t \le T$. This shows the uniqueness. □

4. Stability of the solution

In this section, we study the stability through the continuous dependence of mild solutions on the initial value by means of Lemma 2.12.

Definition 4.1. A mild solution $u^{\varphi_1}(t)$ of equation (1.1) with initial value φ_1 is said to be stable in the mean square if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{0\le s\le t} \hat{\mathbb{E}}\|u^{\varphi_1}(s) - u^{\varphi_2}(s)\|^2 < \varepsilon,$$

whenever $\|\varphi_1 - \varphi_2\|_C^2 < \delta$, for all $t \in [0,T]$. Here $u^{\varphi_2}(t)$ is another mild solution of equation (1.1) with initial value φ_2.

Theorem 4.2. *Let $u^{\varphi_1}(t)$ and $u^{\varphi_2}(t)$ be mild solutions of (1.1) with initial values φ_1 and φ_2, respectively. If the assumptions of Theorem 3.3 are satisfied and $K < \frac{1}{\sqrt{5}}$, then the mild solution of equation (1.1) is stable in the mean square.*

Proof. Estimating as before, we can show that

$$
\begin{aligned}
&\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{\varphi_1}(s)-u^{\varphi_2}(s)\|^2\\
&\le 5\sup_{0\le s\le t}\hat{\mathbb{E}}\|S(s)(\varphi_1(0)-\varphi_2(0)+g(0,\varphi_1)-g(0,\varphi_2))\|^2\\
&\quad+5\sup_{0\le s\le t}\hat{\mathbb{E}}\|\int_0^s S(s-l)[f(l,u_l^{\varphi_1})-f(l,u_l^{\varphi_2})]dl\|^2\\
&\quad+5\sup_{0\le s\le t}\hat{\mathbb{E}}\|g(s,u_s^{\varphi_1})-g(s,u_s^{\varphi_2})\|^2\\
&\quad+5\sup_{0\le s\le t}\hat{\mathbb{E}}|\int_0^s S(s-l)[h(l,u_l^{\varphi_1})-h(l,u_l^{\varphi_2})]d\langle B\rangle_l\|^2\\
&\quad+5\sup_{0\le s\le t}\hat{\mathbb{E}}\|\int_0^s S(s-l)[\sigma(l,u_l^{\varphi_1})-\sigma(r,u_l^{\varphi_2})]dB_l\|^2\\
&\le(10M+10MK^2+5K^2)\|\varphi_1-\varphi_2\|_C^2+5K^2\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{\varphi_1}(s)-u^{\varphi_2}(s)\|^2\\
&\quad+5(Mt+C_1M+M\bar{\sigma}^2)\int_0^t H(\sup_{0\le\theta\le l}\hat{\mathbb{E}}\|u^{\varphi_1}(\theta)-u^{\varphi_2}(\theta)\|^2)dl.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{\varphi_1}(s)-u^{\varphi_2}(s)\|^2\\
&\le\frac{(10M+10MK^2+5K^2)}{1-5K^2}\|\varphi_1-\varphi_2\|_C^2\\
&\quad+\frac{5(Mt+C_1M+M\bar{\sigma}^2)}{1-5K^2}\int_0^t H(\sup_{0\le\theta\le l}\hat{\mathbb{E}}\|u^{\varphi_1}(\theta)-u^{\varphi_2}(\theta)\|^2)dl.
\end{aligned}\tag{4.1}
$$

Let

$$
\begin{gathered}
y_0=\frac{(10M+10MK^2+5K^2)}{1-5K^2}\|\varphi_1-\varphi_2\|_C^2,\\
y(t)=\sup_{0\le s\le t}\hat{\mathbb{E}}\|u^{\varphi_1}(s)-u^{\varphi_2}(s)\|^2,\\
\tilde{H}(y):=\frac{5(Mt+C_1M+M\bar{\sigma}^2)}{1-5K^2}H(y),
\end{gathered}
$$

and $v(t)=1$. Then (4.1)can be rewritten as $y(t)\le y_0+\int_0^t v(l)\tilde{H}(y(l))dl$. Moreover, by (H2), $\tilde{H}(y)$ is continuous, nondecreasing and concave in $y\in R^+$ such that $\tilde{H}(0)=0$ and $\int_{0^+}\frac{1}{\tilde{H}(y)}dy=+\infty$. So, for $\varepsilon>0$, we have $\lim_{s\to0^+}\int_s^\varepsilon\frac{1}{\tilde{H}(y)}dy=+\infty$. Thus, there exists a positive constant $\delta<\varepsilon$, such that $\int_\delta^\varepsilon\frac{1}{\tilde{H}(y)}dy\ge T$. Then for $y_0\le\delta\le\varepsilon$, we have

$$
\int_{y_0}^\varepsilon\frac{1}{\tilde{H}(y)}dy\ge\int_\delta^\varepsilon\frac{1}{\tilde{H}(y)}dy\ge T=\int_0^T v(t)dt.
$$

Hence, by Lemma 2.12, for all $t\in[0,T]$, we have $y(t)<\varepsilon$. This shows the mild solution of (1.1) is stable in the mean square. ☐

5. An application

In this section, we provide an example that illustrates the obtained theory. We consider the following neutral stochastic integro-differential equation driven by G-Brownian motion.

$$
\begin{aligned}
& d[v(t,x)+\int_{-r}^{0}\tilde{g}(t,v(t+\theta,x))d\theta] \\
& =\Big(\frac{\partial^2}{\partial^2 x}[v(t,x)+\int_{-r}^{0}\tilde{g}(t,v(t+\theta,x))d\theta]\Big)dt \\
& \quad +\Big(\int_0^t b(t-s)\frac{\partial^2}{\partial^2 x}[v(t,x)+\int_{-r}^{0}\tilde{g}(t,v(t+\theta,x))d\theta]ds\Big)dt \\
& \quad +\Big(\int_{-r}^{0}\tilde{f}(t,v(t+\theta,x))d\theta\Big)dt+\tilde{h}(t,v(t+\theta,x))d\langle B\rangle_t \\
& \quad +\tilde{\sigma}(t,v(t+\theta,x))dB_t, \quad \text{for } t\geq 0,\ \theta\in[-r,0],\ x\in[0,\pi], \\
& \qquad v(t,0)+\int_{-r}^{0}\tilde{g}(t,v(t+\theta,0))d\theta=0, \quad t\geq 0, \\
& \qquad v(t,\pi)+\int_{-r}^{0}\tilde{g}(t,v(t+\theta,\pi))d\theta=0, \quad t\geq 0, \\
& \qquad v(\theta,x)=v_0(\theta,x)\in C([-r,0];L_G^2(\Omega_T,L^2([0,\pi]))), \\
& \qquad\qquad \text{for } \theta\in[-r,0],\ x\in[0,\pi],
\end{aligned}
\tag{5.1}
$$

where $\tilde{f},\tilde{g},\tilde{h},\tilde{\sigma}: R^+\times L_G^2(\Omega_T,L^2([0,\pi]))\longrightarrow L_G^2(\Omega_T,L^2([0,\pi]))$ are jointly continuous functions, $b: R^+\to R$ is bounded and C^1 function such that b' is bounded and uniformly continuous. Let $Av(x)=v''(x)$, $x\in[0,\pi]$, $v\in D(A)$, where $D(A)=\{v\in L_G^2(\Omega_T,L^2([0,\pi]))|\ v'(x)$ is absolutely continuous on $[0,\pi], v''\in L^2([0,\pi]), v(0)=v(\pi)=0\}$. Then, A generates a strongly continuous semigroup. Let $R(t)z:=b(t)Az$ for $t\geq 0$ and $z\in D(A)$. By [5, Theorem 2.2], there exists a resolvent operator $(S(t))_{t\geq 0}$ on $L_G^2(\Omega_T,L^2([0,\pi]))$.

Moreover, for $t\geq 0,\xi,\eta\in L_G^2(\Omega_T,L^2([0,\pi]))$, with $\|\cdot\|$ the natural norm with respect to space $L^2([0,\pi])$, we suppose that:

(i) there exists a positive constant $L_g, rL_g<1$, such that
$$\|\tilde{g}(t,\xi)-\tilde{g}(t,\eta)\|\leq L_g\|\xi-\eta\|,$$

(ii) there exists a positive constant L_f, such that
$$\hat{\mathbb{E}}\|\tilde{f}(t,\xi)-\tilde{f}(t,\eta)\|^2\leq L_f H(\hat{\mathbb{E}}\|\xi-\eta\|^2),$$

(iii)
$$
\begin{aligned}
\hat{\mathbb{E}}\|\tilde{h}(t,\xi)-\tilde{h}(t,\eta)\|^2 &\leq H(\hat{\mathbb{E}}\|\xi-\eta\|^2),\\
\hat{\mathbb{E}}\|\tilde{\sigma}(t,\xi)-\tilde{\sigma}(t,\eta)\|^2 &\leq H(\hat{\mathbb{E}}\|\xi-\eta\|^2),
\end{aligned}
$$
where $H(\cdot)$ is defined in (H2).

For $x\in[0,\pi]$ and $\zeta\in C([-r,0];L_G^2(\Omega_T,L^2([0,\pi])))$, we define the operators $f,g,h,\sigma: R^+\times C([-r,0];L_G^2(\Omega_T,L^2([0,\pi])))\to L_G^2(\Omega_T,L^2([0,\pi]))$ as follows

$$g(t,\zeta)(x)=\int_{-r}^{0}\tilde{g}(t,\zeta(\theta)(x))d\theta,\quad f(t,\zeta)(x)=\int_{-r}^{0}\tilde{f}(t,\zeta(\theta)(x))d\theta,$$

$$h(t,\zeta)(x)=\tilde{h}(t,\zeta(\theta)(x)),\quad \sigma(t,\zeta)(x)=\tilde{\sigma}(t,\zeta(\theta)(x)).$$

If we put

$$\begin{gathered}u(t)=v(t,x),\quad t\in[0,T],\ x\in[0,\pi],\\ \varphi(\theta)(x)=v_0(\theta,x),\quad \theta\in[-r,0],\ x\in[0,\pi].\end{gathered}\tag{5.2}$$

Then we can rewrite equation (5.1) in the abstract form

$$\begin{aligned}&d(u(t)+g(t,u_t))\\ &=A(t)(u(t)+g(t,u_t))dt+\int_0^t R(t-s)(u(s)+g(s,u_s))ds\\ &\quad+f(t,u_t)dt+h(t,u_t)d\langle B\rangle_t+\sigma(t,u_t)dB_t,\quad t\in[0,T],\\ &\quad u_0(\cdot)=\varphi\in C([-r,0];L_G^2(\Omega_T,L^2([0,\pi]))),\quad r>0.\end{aligned}\tag{5.3}$$

By the continuity of $\tilde{f},\tilde{g},\tilde{h},\tilde{\sigma}$, it is clear that f,h,σ are jointly continuous on $R^+\times C([-r,0];L_G^2(\Omega_T,\ L^2([0,\pi])))$ with values in $L_G^2(\Omega_T,L^2([0,\pi]))$. On the other hand, by (i)-(iii), it is easy to show that all the assumptions of Theorem 3.3 are satisfied. Therefore, there exists a unique mild solution of (5.1). Moreover, this solution depends on the initial value by Theorem 4.2.

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