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REGULARITY OF THE LOWER POSITIVE BRANCH FOR SINGULAR ELLIPTIC BIFURCATION PROBLEMS

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Abstract. We consider the problem

$$\begin{gathered} -\Delta u = au^{-\alpha} + f(\lambda,\cdot,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega, \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\lambda \geq 0$, $0 \leq a \in L^\infty(\Omega)$, and $0 < \alpha < 3$. It is known that, under suitable assumptions on f, there exists $\Lambda > 0$ such that this problem has at least one weak solution in $H_0^1(\Omega) \cap C(\overline{\Omega})$ if and only if $\lambda \in [0,\Lambda]$; and that, for $0 < \lambda < \Lambda$, at least two such solutions exist. Under additional hypothesis on a and f, we prove regularity properties of the branch formed by the minimal weak solutions of the above problem. As a byproduct of the method used, we obtain the uniqueness of the positive solution when $\lambda = \Lambda$.

1. Introduction and statement of main results

Let Ω be a bounded domain in $\mathbb{R}^n$, and let a, and f be functions defined on Ω and $[0,\infty) \times \overline{\Omega} \times [0,\infty)$ respectively. For $\lambda \geq 0$ and $\alpha > 0$, consider the singular semilinear elliptic problem:

$$\begin{gathered} -\Delta u = au^{-\alpha} + f(\lambda,\cdot,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega. \end{gathered} \tag{1.1}$$

Singular elliptic problems like (1.1) appear in the study of many nonlinear phenomena, for instance in models of heat conduction in electrical conductors, in the study of chemical catalysts reactions, and in models of non Newtonian flows (see e.g., [10, 6, 16, 20]).

Fulks and Maybee [20], Crandall, Rabinowitz and Tartar [11], Lazer and McKenna [35], Díaz, Morel and Oswald [16], Del Pino [14], and Bougherara, Giacomoni and Hernández [3], addressed, under different assumptions on a, the existence of solutions to problem (1.1) in the case $f \equiv 0$. The case when $f \equiv 0$, and a is a measure, was treated by Oliva and Petitta [38].

Problem (1.1) was studied by Shi and Yao [43], in the case when Ω and a are regular enough (with a that may change sign), and $f(\lambda, x, s) = \lambda s^p$, with $0 < \alpha < 1$,

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and $0 < p < 1$. Dávila and Montenegro [13] considered free boundary singular elliptic problems of the form $-\Delta u = \chi_{\{u>0\}}(-u^{-\alpha} + \lambda g(\cdot, u))$ in Ω, $u = 0$ on $\partial\Omega$, $u \geq 0$ in Ω, $u \not\equiv 0$ in Ω (that is: $|\{x \in \Omega : u(x) > 0\}| > 0$).

Singular problems of the form

$$
\begin{gathered}
-\Delta u = g(x,u) + h(x, \lambda u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega, \\
u > 0 \quad \text{in } \Omega,
\end{gathered}
\tag{1.2}
$$

were studied by Coclite and Palmieri in [9]. They proved that, if $g(x,u) = au^{-\alpha}$, $a \in C^1(\overline{\Omega})$, $a > 0$ in $\overline{\Omega}$, $h \in C^1(\overline{\Omega} \times [0,\infty))$, and $\inf_{\overline{\Omega}\times[0,\infty)} \frac{h(x,s)}{1+s} > 0$, then there exists $\lambda^* > 0$ such that, for any $\lambda \in [0, \lambda^*)$, (1.2) has a positive classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$; and, for $\lambda > \lambda^*$, (1.2) has no positive classical solution.

Papageorgiou and Rădulescu [39] investigated the existence and nonexistence of positive weak solutions to problems of the form

$$
\begin{gathered}
-\Delta u = -u^{-\gamma} + \lambda f(x,u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega, \\
u > 0 \quad \text{in } \Omega,
\end{gathered}
\tag{1.3}
$$

in the case where Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $\gamma > 0$, $\lambda > 0$, and f is a Carathéodory function satisfying some further assumptions. They proved that, if $0 < \gamma < 1$, then there exists $\lambda^* > 0$ such that (1.3) has a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ when $\lambda > \lambda^*$, and has no solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ for $\lambda < \lambda^*$. They also proved that, when $\gamma \geq 1$, (1.3) has no solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Godoy and Guerin ([28], [29] and [30]) obtained existence results for weak solutions in $H_0^1(\Omega)$ to problems of the form

$$
\begin{gathered}
-\Delta u = \chi_{\{u>0\}} g(\cdot, u) + f(\cdot, u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega, \\
u \geq 0 \quad u \not\equiv 0 \quad \text{in } \Omega,
\end{gathered}
$$

where $s \to g(x,s)$ is singular at the origin, and $f : \Omega \times [0,\infty) \to \mathbb{R}$ is sublinear at ∞. While in [28] and [29] the singular part g was of the form $au^{-\alpha}$, a more general singular term g was allowed in [30].

Ghergu and Rădulescu [25] proved existence and nonexistence theorems for positive classical solutions of singular biparametric bifurcation problems of the form $-\Delta u = g(u) + \lambda |\nabla u|^p + \mu h(\cdot, u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, in the case where Ω is a smooth bounded domain in $\mathbb{R}^n$, $0 < p \leq 2$, $\lambda, \mu \geq 0$, $h(x,s)$ is nondecreasing with respect to s, g is unbounded around the origin, and both are positive Hölder continuous functions. They also established the asymptotic behavior of the solution around the bifurcation point, provided that $g(s)$ behaves like $s^{-\alpha}$ around the origin, for some $\alpha \in (0,1)$.

Dupaigne, Ghergu and Rădulescu [19] studied Lane-Emden-Fowler equations with convection term and singular potential; and Rădulescu [40] investigated the existence of blow-up boundary solutions for logistic equations, and also for Lane-Emden-Fowler equations with a singular nonlinearity and a subquadratic convection term.

The existence of positive solutions to the problem $-\Delta u = ag(u) + \lambda h(u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω was considered by Cîrstea, Ghergu and Rădulescu [12] under the assumptions that Ω is a smooth bounded domain in $\mathbb{R}^n$, $0 \le a \in C^{\beta}(\overline{\Omega})$, $0 < h \in C^{0,\beta}[0,\infty)$ for some $\beta \in (0,1)$, h is nondecreasing on $[0,\infty)$, $h(s)/s$ is nonincreasing for $s > 0$, g is non-increasing on $(0,\infty)$, $\lim_{s\to 0^+} g(s) = \infty$; and $\sup_{s\in(0,\sigma_0)} s^{\alpha} g(s) < \infty$ for some $\alpha \in (0,1)$ and $\sigma_0 > 0$.

Ghergu and Rădulescu [22], studied the Lane-Emden-Fowler singular equation $-\Delta u = \lambda f(u) + a(x)g(u)$ in Ω, $u = 0$ on $\partial\Omega$, when Ω is a bounded and smooth domain in $\mathbb{R}^n$, λ is a positive parameter, f is a nondecreasing function such that $s^{-1}f(s)$ is nondecreasing, $a \in C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, and g is singular at the origin. Under suitable additional assumptions on a, f, and g, they proved that, for some explicitly characterized $\lambda^* > 0$:

(i) For any $\lambda \in [0,\lambda^*)$, there exists a unique solution $u_\lambda \in \mathcal{E}$ (whose behavior near $\partial\Omega$ was established), where

$$\mathcal{E} := \{u \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega}) \text{ such that } \Delta u \in L^1(\Omega)\}.$$

(ii) For $\lambda \ge \lambda^*$ the problem has no solution in $\mathcal{E}$.

Ghergu and Rădulescu [24], established the existence of a ground state solution of the following problem involving the singular Lane-Emden-Fowler equation with convection term:

$$\begin{gathered} -\Delta u = p(x)(g(u) + f(u) + |\nabla u|^{\alpha}) \quad \text{in } \mathbb{R}^n, \\ u > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0, \end{gathered}$$

where $n \ge 3$, $0 < \alpha < 1$, p positive in $\mathbb{R}^n$, f positive, nondecreasing, with sublinear growth, and g positive, decreasing and singular at the origin.

Ghergu and Rădulescu [23], proved existence and nonexistence results for the two parameter singular problem $-\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x)$ in Ω, $u = 0$ on $\partial\Omega$, when Ω is a smooth bounded domain in $\mathbb{R}^n$, λ and μ are positive parameters, h is a positive function, f has sublinear growth, K may change sign, and g is nonnegative and singular at the origin.

Aranda and Godoy [2] found a multiplicity result for positive solutions in the space $W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ to problems of the form $-\Delta_p u = g(u) + \lambda h(u)$ in Ω, $u = 0$ on $\partial\Omega$, when Ω is a C^2 bounded and strictly convex domain in $\mathbb{R}^n$, $1 < p \le 2$; and g, h are locally Lipschitz functions on $(0,\infty)$ and $[0,\infty)$ respectively, with g nonincreasing, possibly singular at the origin; and h nondecreasing, with subcritical growth, and such that $\inf_{s>0} s^{-p+1}h(s) > 0$.

Kaufmann and Medri [34] proved existence and nonexistence results for positive solutions to one dimensional singular problems of the form $-(|u'|^{p-2}u')' = m(x)u^{-\gamma}$ in Ω, $u = 0$ on $\partial\Omega$, in the case where $\Omega \subset \mathbb{R}$ is a bounded open interval, $p > 1$, $\gamma > 0$, and $m : \Omega \to \mathbb{R}$ is a function that may change sign in Ω.

Chhetri, Drábek and Shivaji [8] studied the problem $-\Delta_p u = K(x)f(u)u^{-\delta}$ in $\mathbb{R}^n \setminus \Omega$, $u = 0$ on $\partial\Omega$, $\lim_{|x|\to\infty} u(x) = 0$, under the assumptions that Ω is a simply connected bounded and smooth domain in $\mathbb{R}^n$, $0 \in \Omega$, $n \ge 2$, $1 < p < n$, and $0 \le \delta < 1$. Under a decay assumption on K at infinity, and a growth restriction on f, they proved the existence of a weak solution $u \in C^1(\overline{\mathbb{R}^n \setminus \Omega})$. Also, under an additional condition on K, the uniqueness of such a solution was proved. The existence of radial solutions in the case when Ω is a ball centered at the origin was also addressed.

Saoudi, Agarwal and Mursaleen [41] considered singular elliptic problems of the form $-\operatorname{div}(A(x)\nabla u) = u^{-\alpha} + \lambda u^p$ in Ω, $u = 0$ on $\partial\Omega$, with $0 < \alpha < 1 < p < \frac{n+2}{n-2}$, and A uniformly elliptic on $\overline{\Omega}$. They proved that, for λ positive and small enough, at least two positive weak solutions in $H_0^1(\Omega)$ exist.

Giacomoni, Schindler and Takac [26] studied the existence of weak solutions in $W_0^{1,p}(\Omega)$ of the problem $-\Delta_p u = \lambda u^{-\alpha} + u^q$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, in the case where $0 < \alpha < 1$, $1 < p < \infty$, $q < \infty$ and $p - 1 < q \le p^{\#} - 1$, with $p^{\#}$ defined by $\frac{1}{p^{\#}} = \frac{1}{p} - \frac{1}{n}$ if $p < n$, $p^{\#} = \infty$ if $p > n$, and where $p^{\#} \in (p, \infty)$ is arbitrarily large if $p = n$. They proved the existence of $\Lambda \in (0, \infty)$ such that: a solution exists if $\lambda \in (0, \Lambda]$, no solution exists if $\lambda > \Lambda$, and at least two solutions exist if $\lambda \in (0, \Lambda)$.

Additional references, and a comprehensive treatment of the subject, can be found in [21] and [40]; see also [15].

Finally, in [31] and [33], existence and multiplicity results were obtained for positive solutions of problem (1.1) for $0 < \alpha < 3$, $0 \le a \in L^\infty(\Omega)$, $a \not\equiv 0$ in Ω, and for some nonlinearities f satisfying that $f(\lambda, x, .)$ is superlinear with subcritical growth at ∞.

Our aim in this work is to complement the results obtained in [31] and [33] (see also [32]). To do that, we assume, from now on, that α, a, and f satisfy the following conditions:

(H1) $0 < \alpha < 3$

(H2) $0 \le a \in L^\infty(\Omega)$, and there exists $\delta > 0$ such that $\inf_{A_\delta} a > 0$, where $A_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \delta\}$ and, for any measurable $E \subset \Omega$, $\inf_E$ means the essential infimum on E.

(H3) $0 \le f \in C([0, \infty) \times \overline{\Omega} \times [0, \infty))$, and $f(0, \cdot, \cdot) = 0$ on $\overline{\Omega} \times [0, \infty)$.

(H4) There exist numbers $\eta_0 > 0$, $q \ge 1$, and a nonnegative function $b \in L^\infty(\Omega)$, such that $b \not\equiv 0$ and $f(\lambda, \cdot, s) \ge \lambda b s^q$ a.e. in Ω whenever $\lambda \ge \eta_0$ and $s \ge 0$.

(H5) There exist $p \in (1, \frac{n+2}{n-2})$, and $h \in C((0, \infty) \times \overline{\Omega})$ that satisfy $\inf_{[\eta, \infty) \times \overline{\Omega}} h > 0$ for any $\eta > 0$, and such that, for every $\sigma > 0$,

$$\lim_{(\lambda, s) \to (\sigma, \infty)} s^{-p} f(\lambda, \cdot, s) = h(\sigma, \cdot) \quad \text{uniformly on } \overline{\Omega}.$$

(H6) For any $(\lambda, x) \in (0, \infty) \times \Omega$, the function $f(\lambda, x, \cdot)$ is nondecreasing on $[0, \infty)$ and, for any $(x, s) \in \Omega \times (0, \infty)$, the function $f(\cdot, x, s)$ is strictly increasing on $[0, \infty)$.

(H7) For any $(\lambda, x, s) \in (0, \infty) \times \Omega \times (0, \infty)$, $\frac{\partial f}{\partial \lambda}(\lambda, x, s)$ exists and it is finite and positive, and for any $(\lambda, x) \in (0, \infty) \times \Omega$, the function $\frac{\partial f}{\partial \lambda}(\lambda, x, \cdot)$ is nondecreasing on $(0, \infty)$.

(H8) $f(\cdot, x, \cdot) \in C^2([0, \infty) \times [0, \infty))$ for almost all $x \in \Omega$; and, for any $M > 0$, $\|\frac{\partial f}{\partial s}(\lambda, \cdot, M)\|_\infty < \infty$ whenever $\lambda \in [0, M]$, and both $\frac{\partial f}{\partial \lambda}|_{(0,M) \times \Omega \times (0,M)}$ and $\frac{\partial f}{\partial s}|_{(0,M) \times \Omega \times (0,M)}$ belong to $L^\infty((0, M) \times \Omega \times (0, M))$.

(H9) For almost all $x \in \Omega$, $\frac{\partial f}{\partial s}(\cdot, x, \cdot) > 0$ in $[0, \infty) \times [0, \infty)$, and $\frac{\partial^2 f}{\partial s^2}(\cdot, x, \cdot) > 0$ in $[0, \infty) \times [0, \infty)$.

Since our results rely heavily on those in [33], the next remark summarize the main results given there.

Remark 1.1 (See [33, Theorems 1.2 and 1.3])**.** Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and assume that (H1)–(H6) hold. Then there exists $\Lambda > 0$ such that:

(i) For $\lambda = 0$, (1.1) has a unique weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and it belongs to $C(\overline{\Omega})$,

(ii) For $\lambda = \Lambda$, (1.1) has at least one weak solution in $H_0^1(\Omega) \cap C(\overline{\Omega})$,

(iii) For $\lambda \in (0, \Lambda)$, problem (1.1) has at least two positive weak solutions in $H_0^1(\Omega) \cap C(\overline{\Omega})$,

(iv) For $\lambda > \Lambda$, there is no weak solution of (1.1) in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

(v) For any $\lambda \in [0, \Lambda]$, problem (1.1) has a minimal weak solution $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$, in the sense that $u_\lambda \le v$ for any weak solution $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.1). Also, $u_\lambda \in C(\overline{\Omega})$ and, if $0 \le \lambda_1 < \lambda_2 \le \Lambda$, then there exists a positive constant c such that $u_{\lambda_1} + c d_\Omega \le u_{\lambda_2}$ in Ω; where $d_\Omega := \operatorname{dist}(\cdot, \partial\Omega)$.

(vi) If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.1) for some $\lambda \in [0, \Lambda]$, then $u \in C(\overline{\Omega})$, and there exists a positive constant c', independent of λ and u, such that $u \ge c' d_\Omega^{\tau_\alpha}$ in Ω, with $\tau_\alpha := 1$ if $0 < \alpha < 1$, and $\tau_\alpha := \frac{2}{1+\alpha}$ if $1 \le \alpha < 3$.

In the previous remark and below, by a weak solution, we mean a weak solution in usual sense:

Definition 1.2. Let $h : \Omega \to \mathbb{R}$ be a measurable function such that $h\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$. We say that $u : \Omega \to \mathbb{R}$ is a weak solution of the problem

$$\begin{gathered} -\Delta u = h \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

if $u \in H_0^1(\Omega)$ and $\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega h\varphi$ for any $\varphi \in H_0^1(\Omega)$.

For $u \in H^1(\Omega)$, and h as above, we will write $-\Delta u \ge h$ in Ω (respectively $-\Delta u \le h$ in Ω) to mean that $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \ge \int_\Omega h\varphi$ (resp. $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \le \int_\Omega h\varphi$) for any nonnegative $\varphi \in H_0^1(\Omega)$.

Let us state our results.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and assume* (H1)–(H9)*. Let Λ be given by Remark 1.1 and, for $\lambda \in [0, \Lambda]$, let u_λ be the minimal solution given by Remark 1.1 (v). Then:*

(i) *The map $\lambda \to u_\lambda$ is continuous from $[0, \Lambda]$ to $C(\overline{\Omega})$.*

(ii) *The map $\lambda \to u_\lambda$ is continuously differentiable from $(0, \Lambda)$ to $C(\overline{\Omega})$.*

(iii) *The map $\lambda \to u_\lambda$ is continuously differentiable from $(0, \Lambda)$ to $H_0^1(\Omega)$.*

Theorem 1.4. *Assume the hypothesis of Theorem 1.3, and let Λ be given by Remark 1.1. Then for $\lambda = \Lambda$ there exists a unique weak solution u in $H_0^1(\Omega) \cap L^\infty(\Omega)$ to problem* (1.1)*, and (according to Remark 1.1) it belongs to $C(\overline{\Omega})$.*

To prove Theorems 1.3 and 1.4 we follow an implicit function theorem approach. We rewrite (1.1) as $T(\lambda, u) = 0$, where

$$T(\lambda, u) := u - (-\Delta)^{-1}(a u^{-\alpha} + f(\lambda, \cdot, u)).$$

In Section 2, we define a suitable Banach space X_α, and an open subset $D_\alpha \subset X_\alpha$, such that $u_\lambda \in D_\alpha$ for any $\lambda \in [0, \Lambda]$. We prove that $T((0, \infty) \times D_\alpha) \subset X_\alpha$ and that $T : (0, \infty) \times D_\alpha \to X_\alpha$ is a continuously Fréchet differentiable map.

In Section 3, we consider, for $u \in D_\alpha$, and for a nonnegative and not identically zero $m \in L^\infty(\Omega)$, the following principal eigenvalue problem with singular potential $\alpha a u^{-\alpha-1}$, and weight function m :

$$-\Delta w + \alpha a u^{-\alpha-1} w = \mu_{m,u} m w \quad \text{in } \Omega,$$

$$
\begin{gathered}
w=0 \quad \text{on } \partial\Omega, \\
w>0 \quad \text{in } \Omega.
\end{gathered}
$$

We prove that this problem has a positive principal eigenvalue $\mu_{m,u}$, with a positive associated eigenfunction $w \in D_\alpha$. A corresponding maximum principle with weight is also proved.

In Section 4, we prove that, if $\lambda \in (0,\Lambda)$ and $m_\lambda := \frac{\partial f}{\partial s}(\lambda,\cdot,u_\lambda)$, then $\mu_{m_\lambda,u_\lambda} > 1$, with u_λ given by Remark 1.1 (v); and that, if (1.1) had at least two solutions for $\lambda = \Lambda$, then the same assertion would hold for $\lambda = \Lambda$. Moreover, we also prove that, in both cases, T satisfies the hypothesis of the implicit function theorem for Banach spaces at (λ, u_λ). Finally, from these facts, and from some additional auxiliary results, Theorems 1.3 and 1.4, as well as two results concerning uniformity properties of the family $\{u_\lambda\}_{\lambda\in[0,\Lambda]}$, are proved in Section 6.

2. Preliminaries

From now on, we We assume, conditions (H1)–(H9). For $1 \le p \le \infty$, let p' be given by $\frac{1}{p} + \frac{1}{p'} = 1$, and let p^* be defined by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ if $p < n$ and by $p^* := \infty$ otherwise. For a measurable function $v : \Omega \to \mathbb{R}$ such that $v\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, S_v will denote the functional $S_v : H_0^1(\Omega) \to \mathbb{R}$ defined by $S_v(\varphi) := \int_\Omega v\varphi$; and we will write $v \in (H_0^1(\Omega))'$ to mean that $S_v \in (H_0^1(\Omega))'$.

Remark 2.1. Let us recall the Hardy inequality (see e.g., [4, p. 313]): There exists a positive constant c such that $\|\frac{\varphi}{d_\Omega}\|_{L^2(\Omega)} \le c\|\nabla\varphi\|_{L^2(\Omega)}$ for all $\varphi \in H_0^1(\Omega)$.

Lemma 2.2. *If either $v \in L^{(2^*)'}(\Omega)$ or $d_\Omega v \in L^2(\Omega)$, then:*

- (i) *The functional $S_v : H_0^1(\Omega) \to \mathbb{R}$ is well defined, belongs to $(H_0^1(\Omega))'$, and there exists a positive constant c, independent of v, such that $\|S_v\| \le c\|v\|_{(2^*)'}$ when $v \in L^{(2^*)'}(\Omega)$, and $\|S_v\| \le c\|d_\Omega v\|_2$ when $d_\Omega v \in L^2(\Omega)$.*
- (ii) *The problem $-\Delta z = v$ in Ω, $z = 0$ on $\partial\Omega$, has a unique weak solution $z \in H_0^1(\Omega)$, and it satisfies, for some positive constant c independent of v, $\|z\|_{H_0^1(\Omega)} \le c\|v\|_{(2^*)'}$ if $v \in L^{(2^*)'}(\Omega)$, and $\|z\|_{H_0^1(\Omega)} \le c\|d_\Omega v\|_2$ if $d_\Omega v \in L^2(\Omega)$.*

Proof. Let $\varphi \in H_0^1(\Omega)$. If $v \in L^{(2^*)'}(\Omega)$ then, from the Hölder and Poincaré inequalities, there exist positive constants c and c', independent of v and φ, such that

$$
|S_v\varphi| \le \int_\Omega |v\varphi| \le c'\|v\|_{(2^*)'}\|\varphi\|_{2^*} \le c\|v\|_{(2^*)'}\|\nabla\varphi\|_2.
$$

If $d_\Omega v \in L^2(\Omega)$ then, applying the Hölder and the Hardy inequalities, we obtain

$$
|S_v\varphi| \le \int_\Omega |v\varphi| \le c'\|d_\Omega v\|_2\|d_\Omega^{-1}\varphi\|_2 \le c\|d_\Omega v\|_2\|\nabla\varphi\|_2,
$$

with c' and c constants independent of v and φ. Thus (i) holds, and from *i),* the Riesz theorem gives (ii). □

Remark 2.3. Let $v : \Omega \to \mathbb{R}$ be a measurable function such that $v\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$. If $S_v \in (H_0^1(\Omega))'$, then, by the Riesz theorem, the problem

$$
-\Delta z = v \text{ in } \Omega, \quad z = 0 \text{ on } \partial\Omega,
$$

has a unique weak solution $z \in H_0^1(\Omega)$, and it satisfies $\|z\|_{H_0^1(\Omega)} = \|S_v\|_{(H_0^1(\Omega))'}$.

For $\delta > 0$ let $\Omega_\delta := \{x \in \Omega : d_\Omega(x) > \delta\}$. We will need the following lemma, which is a variant of [33, Lemma 3.2].

Lemma 2.4. *Let $u \in W^{1,2}_{\rm loc}(\Omega) \cap C(\Omega)$ be a solution, in the sense of distributions, to the problem $-\Delta u = u^{-1}$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω (respectively to the problem $-\Delta u = d_\Omega^{-1}$ in Ω, $u = 0$ on $\partial\Omega$) such that, for some positive constants c_1, c_2 and γ, $c_1 d_\Omega \le u \le c_2 d_\Omega^\gamma$ a.e. in Ω. Then $u \in H^1_0(\Omega) \cap C^1(\Omega) \cap C(\overline{\Omega})$, and u is a weak solution of the respective problem.*

Proof. Note that $\int_\Omega \langle \nabla u, \nabla\varphi\rangle = \int_\Omega u^{-1}\varphi$ (respectively $\int_\Omega \langle \nabla u, \nabla\varphi\rangle = \int_\Omega d_\Omega^{-1}\varphi$) for any $\varphi \in H^1_0(\Omega)$ such that $supp(\varphi) \subset \Omega$. Indeed, let $\delta > 0$ be such that $supp(\varphi) \subset \Omega_\delta$, and let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence in $C_c^\infty(\Omega)$ satisfying $supp(\varphi_j) \subset \Omega_\delta$ for all j, and such that $\{\varphi_j\}_{j\in\mathbb{N}}$ converges to φ in $H^1_0(\Omega_\delta)$. Now, $\nabla u\big|_{\Omega_\delta} \in L^2(\Omega_\delta, \mathbb{R}^n)$ and $u \ge c_1\delta$ on Ω_δ. Also, from the Hardy inequality $\int_{\Omega_\delta} |d_\Omega^{-1}\varphi| \le c\|\varphi\|_{H^1_0(\Omega_\delta)}$, with c a positive constant independent of φ. Then the maps $\varphi \to \int_{\Omega_\delta} \langle \nabla u, \nabla\varphi\rangle$ and $\varphi \to \int_{\Omega_\delta} u^{-1}\varphi$ (resp. $\varphi \to \int_{\Omega_\delta} \langle \nabla u, \nabla\varphi\rangle$ and $\varphi \to \int_{\Omega_\delta} d_\Omega^{-1}\varphi$) are continuous on $H^1_0(\Omega_\delta)$. Also, $\int_\Omega \langle \nabla u, \nabla\varphi_j\rangle = \int_\Omega u^{-1}\varphi_j$ for all j (respectively $\int_\Omega \langle \nabla u, \nabla\varphi_j\rangle = \int_\Omega d_\Omega^{-1}\varphi_j$). Then $\int_\Omega \langle \nabla u, \nabla\varphi\rangle = \lim_{j\to\infty} \int_\Omega \langle \nabla u, \nabla\varphi_j\rangle = \lim_{j\to\infty} \int_\Omega u^{-1}\varphi_j = \int_\Omega u^{-1}\varphi$ (respectively $\int_\Omega \langle \nabla u, \nabla\varphi\rangle = \lim_{j\to\infty} \int_\Omega \langle \nabla u, \nabla\varphi_j\rangle = \lim_{j\to\infty} \int_\Omega d_\Omega^{-1}\varphi_j = \int_\Omega d_\Omega^{-1}\varphi$).
For each $j \in \mathbb{N}$, let $h_j : \mathbb{R} \to \mathbb{R}$ be defined by

$$h_j(s) := \begin{cases} 0 & \text{if } s \le \frac{1}{j}, \\ -3j^2s^3 + 14js^2 - 19s + \frac{8}{j} & \text{if } \frac{1}{j} < s < \frac{2}{j}, \\ s & \text{if } \frac{2}{j} \le s. \end{cases}$$

Then $h_j \in C^1(\mathbb{R})$, $h_j'(s) = 0$ for $s < \frac{1}{j}$, $h_j'(s) \ge 0$ for $\frac{1}{j} < s < \frac{2}{j}$ and $h_j'(s) = 1$ for $\frac{2}{j} < s$. Also, $0 < h_j(s) < s$ for all $s \in (0, 2/j)$.

Let $h_j(u) := h_j \circ u$. Then, for all j, $\nabla(h_j(u)) = h_j'(u)\nabla u$ in $D'(\Omega)$. Since $u \in W^{1,2}_{\rm loc}(\Omega)$, it follows that $h_j(u) \in W^{1,2}_{\rm loc}(\Omega)$. Since $h_j(u)$ has compact support, $h_j(u) \in H^1_0(\Omega)$. Therefore, for all j, $\int_\Omega \langle \nabla u, \nabla(h_j(u))\rangle = \int_\Omega u^{-1}h_j(u)$ (resp. $\int_\Omega \langle \nabla u, \nabla(h_j(u))\rangle = \int_\Omega d_\Omega^{-1}h_j(u)$), i.e.,

$$\int_{\{u>0\}} h_j'(u)|\nabla u|^2 = \int_\Omega u^{-1}h_j(u) \quad (\text{resp. } = \int_\Omega d_\Omega^{-1}h_j(u)). \tag{2.1}$$

Now, $h_j'(u)|\nabla u|^2$ is nonnegative and $\lim_{j\to\infty} h_j'(u)|\nabla u|^2 = |\nabla u|^2$ a.e. in Ω, and so, from (2.1) and Fatou's lemma, we have

$$\int_\Omega |\nabla u|^2 \le \underline{\lim}_{j\to\infty} \int_\Omega u^{-1}h_j(u) \quad (\text{resp. } \int_\Omega |\nabla u|^2 \le \liminf_{j\to\infty} \int_\Omega d_\Omega^{-1}h_j(u)).$$

Since $u \le c_2 d_\Omega^\gamma$, we have $d_\Omega^{-1}u \in L^1(\Omega)$. Now, $\lim_{j\to\infty} u^{-1}h_j(u) = 1$ a.e. in Ω (resp. $\lim_{j\to\infty} d_\Omega^{-1}h_j(u) = d_\Omega^{-1}u$ a.e. in Ω) and, for any $j \in \mathbb{N}$, $0 \le u^{-1}h_j(u) \le 1$ in Ω (resp. $0 \le d_\Omega^{-1}h_j(u) \le d_\Omega^{-1}u$ in Ω). Then, Lebesgue's dominated convergence theorem gives

$$\lim_{j\to\infty} \int_\Omega u^{-1}h_j(u) = \int_\Omega 1 < \infty \qquad (\text{resp. } = \int_\Omega d_\Omega^{-1}u < \infty).$$

Thus $\int_\Omega |\nabla u|^2 < \infty$, and so $u \in H^1(\Omega)$. Now, $-\Delta u = u^{-1}$ in $D'(\Omega)$ (resp. $-\Delta u = d_\Omega^{-1}$ in $D'(\Omega)$), also $u \in L^\infty(\Omega)$; and $u^{-1} \in L^\infty_{\rm loc}(\Omega)$ (resp. and $d_\Omega^{-1} \in L^\infty_{\rm loc}(\Omega)$). Now, the inner elliptic estimates in [27, Theorem 8.24] give that $u \in C^1(\Omega)$ and, from the assumptions of the lemma u is continuous at $\partial\Omega$, and so $u \in C(\overline{\Omega})$. Thus, since $u \in H^1(\Omega)$, $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, we conclude that $u \in H_0^1(\Omega)$.

Let $\varphi \in H_0^1(\Omega)$. By the Hardy inequality, $\|u^{-1}\varphi\|_1 \le c_1^{-1}\|d_\Omega^{-1}\varphi\|_1 \le c\|\varphi\|_{H_0^1(\Omega)}$ (resp. $\|d_\Omega^{-1}\varphi\|_1 \le c\|\varphi\|_{H_0^1(\Omega)}$) for some positive constant c independent of φ. Then $\varphi \to \int_\Omega u^{-1}\varphi$ (resp. $\varphi \to \int_\Omega d_\Omega^{-1}\varphi$) is continuous on $H_0^1(\Omega)$. Also, $u \in H_0^1(\Omega)$, and so $\varphi \to \int_\Omega \langle \nabla u, \nabla\varphi\rangle$ is continuous on $H_0^1(\Omega)$. Therefore, since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, and

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle = \int_\Omega u^{-1}\varphi \quad (\text{resp. } \int_\Omega \langle \nabla u, \nabla\varphi\rangle = \int_\Omega d_\Omega^{-1}\varphi) \tag{2.2}$$

for any $\varphi \in C_c^\infty(\Omega)$; we conclude that (2.2) holds for all $\varphi \in H_0^1(\Omega)$. $\square$

Remark 2.5. Problems of the form

$$\begin{gathered} -\Delta u = \widetilde{a}u^{\widetilde{\alpha}} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega \end{gathered} \tag{2.3}$$

were considered in [37] when $\widetilde{\alpha} < 1$, $\widetilde{a} \in C_{\rm loc}^\eta(\Omega)$ for some $\eta \in (0,1)$, and such that, for some constants $c > 0$, and $p \le 2$,

$$\frac{1}{c}L(d_\Omega(x)) \le d_\Omega^p(x)\widetilde{a}(x) \le cL(d_\Omega(x)) \quad \text{for all } x \in \Omega, \tag{2.4}$$

where $L(t) = \exp\big(\int_t^{\omega_0} \frac{z(s)}{s}ds\big)$, with $\omega_0 > \operatorname{diam}(\Omega)$, and $z \in C([0,\omega_0])$ such that $z(0) = 0$ and $\int_0^{\omega_0} t^{1-p}L(t)dt < \infty$. Under the stated assumptions, [37, Theorem 1] says that problem (2.3) has a unique classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ which, for some positive constant c', satisfies

$$\frac{1}{c'}\theta_p(d_\Omega(x)) \le u(x) \le c'\theta_p(d_\Omega(x)) \quad \text{for all } x \in \Omega,$$

where

$$\theta_p(t) := \begin{cases} \big(\int_0^{\omega_0} \frac{L(s)}{s}ds\big)^{\frac{1}{1-\widetilde{\alpha}}} & \text{if } p = 2, \\ t^{\frac{2-p}{1-\widetilde{\alpha}}}(L(t))^{\frac{1}{1-\widetilde{\alpha}}} & \text{if } 1+\widetilde{\alpha} < p < 2, \\ t\big(\int_t^{\omega_0} \frac{L(s)}{s}ds\big)^{\frac{1}{1-\widetilde{\alpha}}} & \text{if } p = 1+\widetilde{\alpha}, \\ t & \text{if } p < 1+\widetilde{\alpha}. \end{cases}$$

In particular, when $\widetilde{\alpha} = 0$, $z \equiv 0$ (i.e., $L \equiv 1$), and $p = 1$ in (2.4),

$$\begin{gathered} -\Delta v = d_\Omega^{-1} \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.5}$$

has a unique classical solution $v \in C^2(\Omega) \cap C(\overline{\Omega})$, and that there exists a positive constant c such that

$$c^{-1}\log(\frac{\omega_0}{d_\Omega})d_\Omega \le v \le c\log\big(\frac{\omega_0}{d_\Omega}\big)d_\Omega \quad \text{in } \Omega. \tag{2.6}$$

Moreover, since $d_\Omega^{-1} \in (H_0^1(\Omega))'$, from Lemma 2.4, $v \in H_0^1(\Omega)$, and v is a weak solution of (2.5).

Similarly, taking $\widetilde{\alpha}=-1$, $L\equiv 1$, and $p=0$ in (2.4), the problem

$$\begin{gathered}-\Delta w=w^{-1} \quad \text{in } \Omega,\\ w=0 \quad \text{on } \partial\Omega\\ w>0 \quad \text{in } \Omega\end{gathered} \tag{2.7}$$

has a unique classical solution $w\in C^2(\Omega)\cap C(\overline{\Omega})$, and (2.6) holds with w instead of v. Also notice that $\log(\frac{\omega_0}{d_\Omega})d_\Omega\in (H_0^1(\Omega))'$; indeed, for $\varphi\in H_0^1(\Omega)$,

$$\int_\Omega |\log(\frac{\omega_0}{d_\Omega})d_\Omega\varphi| \le \|\log(\frac{\omega_0}{d_\Omega})d_\Omega^2\|_\infty \int_\Omega |d_\Omega^{-1}\varphi| \le c\|\varphi\|_{H_0^1(\Omega)}$$

for some positive constant c independent of φ. Then, from Lemma 2.4, $w\in H_0^1(\Omega)$, and w is a weak solution of (2.7).

Remark 2.6. The following result is a particular case of [12, Theorem 1]. Let $\beta\in(0,1)$ and let $\mathcal{E}:=\{u\in C^2(\Omega)\cap C^{1,1-\beta}(\overline{\Omega}):\Delta u\in L^1(\Omega)\}$. Then the problem

$$\begin{gathered}-\Delta u=u^{-\beta} \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega,\\ u>0 \quad \text{in } \Omega\end{gathered} \tag{2.8}$$

has a unique classical solution $u\in\mathcal{E}$ and there exist a positive constant c such that $c^{-1}d_\Omega\le u\le cd_\Omega$ in Ω.

Let us observe that, since u is a solution of (2.8) in the sense of distributions, and since $u\in H_0^1(\Omega)$, and the map $\varphi\to\int_\Omega u^{-\beta}\varphi$ belongs to $(H_0^1(\Omega))'$ (because $u\ge c^{-1}d_\Omega$ in Ω), a standard density argument shows that u is a weak solution of (2.8).

For $\xi\in(H_0^1(\Omega))'$, $(-\Delta)^{-1}\zeta$ will denote, as usual, the unique solution $u\in H_0^1(\Omega)$ (given by the Riesz theorem) to the problem $-\Delta u=\zeta$ in Ω, $u=0$ on $\partial\Omega$.

Lemma 2.7. *If $0\le\beta<1$, then $(-\Delta)^{-1}(d_\Omega^{-\beta})\in H_0^1(\Omega)$, and $d_\Omega^{-1}(-\Delta)^{-1}(d_\Omega^{-\beta})\in L^\infty(\Omega)$.*

Proof. The lemma clearly holds when $\beta=0$, because $(-\Delta)^{-1}(\mathbf{1})\in C^1(\overline{\Omega})$ and $(-\Delta)^{-1}(\mathbf{1})=0$ on $\partial\Omega$. If $\beta\in(0,1)$, let $\zeta\in H_0^1(\Omega)$ be the weak solution to (2.8) given by Remark 2.6. Note that, according to Remark 2.6, $c''d_\Omega\le\zeta\le c'd_\Omega$ in Ω for some positive constants c' and c''; and thus $d_\Omega^{-\beta}\le(c')^\beta\zeta^{-\beta}$ in Ω. Therefore, for $\varphi\in H_0^1(\Omega)$, and some constant c independent of φ, we have

$$\begin{aligned}|\int_\Omega d_\Omega^{-\beta}\varphi| &\le (c')^\beta\int_\Omega\zeta^{-\beta}|\varphi| = (c')^\beta\int_\Omega d_\Omega\zeta^{-\beta}|\frac{\varphi}{d_\Omega}|\\ &\le (c')^\beta(c'')^{-\beta}\int_\Omega d_\Omega^{1-\beta}|\frac{\varphi}{d_\Omega}| \le c\|\varphi\|_{H_0^1(\Omega)},\end{aligned}$$

the above inequality holds, by the Hölder and the Hardy inequalities. Thus $d_\Omega^{-\beta}\in(H_0^1(\Omega))'$, and so $(-\Delta)^{-1}(d_\Omega^{-\beta})$ is a well defined element in $H_0^1(\Omega)$. Also, from the weak maximum principle, and since $c''d_\Omega\le\zeta\le c'd_\Omega$ in Ω, we obtain

$$0\le(-\Delta)^{-1}(d_\Omega^{-\beta})\le(c')^{-\beta}(-\Delta)^{-1}(\zeta^{-\beta})=\zeta\le c'd_\Omega \quad \text{in } \Omega.$$

which completes the proof. ☐

Definition 2.8. For $\alpha \in (0,3)$, let τ_α be as in Remark 1.1 (vi), let ω_0 be as in Remark 2.5, and let $\vartheta_\alpha : \Omega \to \mathbb{R}$ be defined by $\vartheta_\alpha := d_\Omega^{\tau_\alpha}$ if $\alpha \neq 1$, and by $\vartheta_1 := \log(\frac{\omega_0}{d_\Omega})d_\Omega$.

Lemma 2.9. *Let $\alpha \in (0,3)$. If $g \in L^\infty(\Omega)$, then $\vartheta_\alpha^{-\alpha} g \in (H_0^1(\Omega))'$ (and so $(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha} g) \in H_0^1(\Omega)$), and there exists a positive constant c, independent of g, such that:*

(i) $\|(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha} g)\|_{H_0^1(\Omega)} \le c\|g\|_\infty$,
(ii) $\|\vartheta_\alpha^{-1}(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha} g)\|_\infty \le c\|g\|_\infty$.

Proof. Note that $\int_\Omega |\vartheta_\alpha^{-\alpha} g\varphi| \le \|g\|_\infty \int_\Omega \vartheta_\alpha^{-\alpha} d_\Omega |d_\Omega^{-1}\varphi|$ for any $\varphi \in H_0^1(\Omega)$. If $\alpha \neq 1$, we have $\vartheta_\alpha^{-\alpha} d_\Omega \in L^2(\Omega)$ (because $2(1-\alpha\tau_a) > -1$), and then, by the Hölder and the Hardy inequalities, $S_{\vartheta_\alpha^{-\alpha} g}$ is well defined on $H_0^1(\Omega)$, and belongs to $(H_0^1(\Omega))'$. Moreover, $\|S_{\vartheta_\alpha^{-\alpha} g}\|_{(H_0^1(\Omega))'} \le c\|g\|_\infty$ with c a constant independent of g, and so, by Remark 2.3, $\|(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha} g)\|_{H_0^1(\Omega)} \le c\|g\|_\infty$. Thus (i) holds when $\alpha \neq 1$. If $\alpha = 1$ then $\vartheta_\alpha^{-\alpha} d_\Omega \in L^\infty(\Omega)$ and so, again now, we obtain (i).

To see (ii), consider the function $z := (-\Delta)^{-1}(\vartheta_\alpha^{-\alpha} g)$. We have, in the weak sense,

$$-\vartheta_\alpha^{-\alpha}\|g\|_\infty \le -\Delta z \le \vartheta_\alpha^{-\alpha}\|g\|_\infty \quad \text{in } \Omega, \tag{2.9}$$

and then, by the weak maximum principle,

$$-\|g\|_\infty(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha}) \le z \le \|g\|_\infty(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha}) \quad \text{a.e. in } \Omega,$$

i.e., $|z| \le \|g\|_\infty(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha})$ a.e. in Ω.

If $0 < \alpha < 1$ then $\vartheta_\alpha^{-\alpha} = d_\Omega^{-\alpha}$, and so, by Lemma 2.7, $\|\vartheta_\alpha^{-\alpha} z\|_\infty \le c\|g\|_\infty$, with c independent of g. Thus (ii) holds when $0 < \alpha < 1$

If $1 < \alpha < 3$, consider the weak solution $w \in H_0^1(\Omega)$ to the problem

$$-\Delta w = w^{-\alpha} \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega,\ w > 0 \text{ in } \Omega. \tag{2.10}$$

(such a solution exists and it is unique, for instance, by Remark 1.1, taking there $a = 1$ and $\lambda = 0$). By [31, Lemmas 2.9 and 2.11], there exist positive constants c_1 and c_2 such that $c_1\vartheta_\alpha \le w \le c_2\vartheta_\alpha$ a.e. in Ω, and then $c_2^{-\alpha}\vartheta_\alpha^{--\alpha} \le w^{-\alpha} \le c_1^{-\alpha}\vartheta_\alpha^{-\alpha}$ a.e. in Ω; thus, from (2.9), we have

$$\begin{aligned} -c_2^\alpha\|g\|_\infty(-\Delta w) &= -c_2^\alpha\|g\|_\infty w^{-\alpha} \le -\Delta z \\ &\le c_2^\alpha\|g\|_\infty w^{-\alpha} = c_2^\alpha\|g\|_\infty(-\Delta w) \quad \text{in } \Omega, \end{aligned}$$

and then, from the weak maximum principle, $-c_2^\alpha\|g\|_\infty w \le z \le c_2^\alpha\|g\|_\infty w$ a.e. in Ω, i.e., $|z| \le c_2^\alpha\|g\|_\infty w$ a.e. in Ω. Since $w \le c_2\vartheta_\alpha$ a.e. in Ω, we obtain that (ii) holds also when $1 < \alpha < 3$. Consider now the case $\alpha = 1$. Let $w \in H_0^1(\Omega)$ be the weak solution to the problem

$$-\Delta w = w^{-1} \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega, \quad w > 0 \text{ in } \Omega. \tag{2.11}$$

From Remark 2.5, there exist positive constants c_1 and c_2 such that $c_1\vartheta_1 \le w \le c_2\vartheta_1$ a.e. in Ω, and then $c_2^{-1}\vartheta_1^{-1} \le w^{-1} \le c_1^{-1}\vartheta_1^{-1}$ a.e. in Ω; thus, from (2.11), in the weak sense, we have

$$\begin{aligned} -c_2\|g\|_\infty(-\Delta w) &= -c_2\|g\|_\infty w^{-1} \le -\Delta z \\ &\le c_2\|g\|_\infty w^{-1} = c_2\|g\|_\infty(-\Delta w) \quad \text{in } \Omega, \end{aligned}$$

and then, from the weak maximum principle, $-c_2\|g\|_\infty w \le z \le c_2\|g\|_\infty w$ a.e. in Ω, i.e., $|z| \le c_2\|g\|_\infty w$ a.e. in Ω. Since $w \le c_2\vartheta_1$ a.e. in Ω, we obtain (ii) also when $\alpha = 1$. $\square$

3. Towards an application of the implicit function theorem

Definition 3.1. Let X_α, $\|\cdot\|_{X_\alpha} : X_\alpha \to [0,\infty)$, and D_α, be defined by

$$\begin{gathered} X_\alpha := \{u \in H_0^1(\Omega) : \vartheta_\alpha^{-1}u \in L^\infty(\Omega)\}, \\ \|u\|_{X_\alpha} := \|\nabla u\|_2 + \|\vartheta_\alpha^{-1}u\|_\infty, \\ D_\alpha := \{u \in X_\alpha : \inf_\Omega \vartheta_\alpha^{-1}u > 0\}. \end{gathered}$$

Note that X_α and $\mathbb{R}\times X_\alpha$, equipped with the norms $\|\cdot\|_{X_\alpha}$ and $|\cdot| + \|cdot\|_{X_\alpha}$ respectively, are Banach spaces.

Recall that $\lambda \in \mathbb{R}$ is called a principal eigenvalue for $-\Delta$ in Ω, with homogeneous Dirichlet boundary condition, if the problem $-\Delta\phi = \lambda\phi$ in Ω, $\phi = 0$ on $\partial\Omega$ has a solution ϕ (called a principal eigenfunction) such that $\phi > 0$ in Ω. It is a well known fact that this problem has a unique positive principal eigenvalue, noted $\lambda_1(b)$ (see e.g., [17]).

Lemma 3.2. *D_α is a nonempty open set in X_α.*

Proof. Let φ_1 be the positive principal eigenfunction for $-\Delta$ in Ω with homogeneous Dirichlet boundary condition, normalized by $\|\varphi_1\|_\infty = 1$. Then (see e.g. [17]), $\varphi_1 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for any $p \in [1,\infty)$ (in particular $\varphi_1 \in C^1(\overline{\Omega})$), and there exist positive constants c_1, c_2 such that

$$c_1 d_\Omega \le \varphi_1 \le c_2 d_\Omega \quad \text{in } \Omega. \tag{3.1}$$

Therefore $\varphi_1 \in D_\alpha$ for each $\alpha \in (0,1)$. If $\alpha \in (1,3)$ then $\frac{2}{1+\alpha} > 1/2$ and so $\varphi_1^{\frac{2}{1+\alpha}} \in H_0^1(\Omega)$. Also, $c_1^{\frac{2}{1+\alpha}} d_\Omega^{\widetilde{\tau}_\alpha} \le \varphi_1^{\frac{2}{1+\alpha}} \le c_2^{\frac{2}{1+\alpha}} d_\Omega^{\widetilde{\tau}_\alpha}$, which gives $\varphi_1^{\frac{2}{1+\alpha}} \in D_\alpha$. If $\alpha = 1$ note that $\log(\frac{\omega_0}{\varphi_1})\varphi_1 \in H_0^1(\Omega)$ and that, for some positive constant c, $c^{-1}\vartheta_1 \le \log(\frac{\omega_0}{\varphi_1})\varphi_1 \le c\vartheta_1$ in Ω, and so $\log(\frac{\omega_0}{\varphi_1})\varphi_1 \in D_1$.

To see that D_α is open in X_a, observe that if $u_0 \in D_\alpha$, then, for some positive constant c, $u_0 \ge c\vartheta_\alpha$ in Ω. Let $\varepsilon := \frac{c}{2}$ and let $u \in X_\alpha$ such that $\|u - u_0\|_{X_\alpha} \le \varepsilon$. Then $\|\vartheta_\alpha^{-1}(u - u_0)\|_\infty \le \varepsilon$, and so $-\varepsilon\vartheta_\alpha \le u - u_0 \le \varepsilon\vartheta_\alpha$ a.e. in Ω. Then $u \ge u_0 - \varepsilon\vartheta_\alpha \ge \frac{c}{2}\vartheta_\alpha$ a.e. in Ω, therefore $u \in D_\alpha$. $\square$

Lemma 3.3. *For any $(\lambda, u) \in (0,\infty)\times D_\alpha$, $au^{-\alpha} + f(\lambda,\cdot,u) \in (H_0^1(\Omega))'$, and $(-\Delta)^{-1}(au^{-\alpha} + f(\lambda,\cdot,u)) \in X_\alpha$.*

Proof. Let $(\lambda,u) \in (0,\infty)\times D_\alpha$. Since $u \in D_\alpha \subset X_\alpha$, and $\vartheta_\alpha \in L^\infty(\Omega)$, we have $u \in L^\infty(\Omega)$. Then $f(\lambda,\cdot,u) \in L^\infty(\Omega)$. Also, since $u \in D_\alpha$, we have $u \ge c\vartheta_\alpha$ a.e. in Ω for some positive constant c. Therefore $au^{-\alpha} + f(\lambda,\cdot,u) \le c^{-\alpha}\|a\|_\infty\vartheta_\alpha^{-\alpha} + \|f(\lambda,\cdot,u)\|_\infty$ a.e. in Ω. Also, for some constant c',

$$c^{-\alpha}\|a\|_\infty\vartheta_\alpha^{-\alpha} + \|f(\lambda,\cdot,u)\|_\infty \le c'\vartheta_\alpha^{-\alpha} \quad \text{in } \Omega. \tag{3.2}$$

Then $0 \le au^{-\alpha} + f(\lambda,\cdot,u) \le c'\vartheta_\alpha^{-\alpha}$; and the lemma follows from Lemma 2.9. $\square$

Lemma 3.4. *If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.1), then there exists a positive constant c such that $u \ge c\vartheta_\alpha$ in Ω.*

Proof. When $\alpha \neq 1$, Remark 1.1 proves the lemma. Consider the case $\alpha = 1$. Let δ be as in (H2). From Remark 1.1 we know that $u \in C(\overline{\Omega})$, and that, for some positive constant c_1, $u \geq c_1 d_\Omega$ in Ω. Thus there exists a positive constant c_2 such that

$$u \geq c'' \vartheta_1 \quad \text{a.e. in } \Omega \setminus A_{\delta/8}. \tag{3.3}$$

Let U be a $C^{1,1}$ domain such that $A_{3\delta/4} \subset U \subset A_\delta$. As shown in [32], we have $\partial U \setminus \partial\Omega \subset \Omega \setminus A_{\delta/2}$ and

$$d_U = d_\Omega \quad \text{in } A_{\delta/8}, \tag{3.4}$$

where $d_U := \operatorname{dist}(\cdot, \partial U)$.

Since $U \subset A_\delta$, from (H2) we have $\underline{a} := \inf_U a > 0$. Let $v \in H_0^1(\Omega) \cap C^2(\Omega) \cap C(\overline{\Omega})$ be the weak solution, given by Remark 2.5, to the problem

$$\begin{gathered} -\Delta v = v^{-1} \quad \text{in } U, \\ v = 0 \quad \text{on } \partial U, \\ v > 0 \quad \text{in } U. \end{gathered}$$

Let ω_0 be as in Remark 2.5, and let $\widetilde{\vartheta}_1 : U \to \mathbb{R}$ be defined by $\widetilde{\vartheta}_1 := \log(\frac{\omega_0}{d_U}) d_U$. Thus, by Remark 2.5, there exists a positive constant c_3 such that $v \geq c_3 \widetilde{\vartheta}_1$ in U. Observe that $-\Delta((\underline{a})^{1/2} v) = \underline{a}((\underline{a})^{1/2} v)^{-1} \leq a((\underline{a})^{1/2} v)^{-1}$ in U, and that, in weak sense, $-\Delta u \geq a u^{-1}$ in Ω. Then $-\Delta(u - (\underline{a})^{1/2} v) \geq a(u^{-1} - ((\underline{a})^{1/2} v)^{-1})$ in $(H_0^1(U))'$; and clearly $u \geq (\underline{a})^{1/2} v$ in ∂U. Then, taking $(u - (\underline{a})^{1/2} v)^-$ as a test function, we obtain $u \geq (\underline{a})^{1/2} v$ in U. Thus there exists a positive constant c_4 such that $u \geq c_4 \widetilde{\vartheta}_1$ in U. Moreover, since $d_U = d_\Omega$ in $A_{\delta/8}$, we have $\widetilde{\vartheta}_1 = \vartheta_1$ in $A_{\delta/8}$, and so

$$u \geq c_4 \vartheta_1 \quad \text{in } A_{\delta/8}\,. \tag{3.5}$$

From (3.3), (3.5), and (3.4), we obtain $u \geq c \vartheta_1$ in Ω, with $c := \min\{c_2, c_4\}$, which completes the proof. $\square$

Lemma 3.5. *If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of* (1.1)*, then $u \in D_\alpha$.*

Proof. From Lemma 3.4 there exists a positive constant c such that $u \geq c\vartheta_\alpha$ in Ω. Since $u \in L^\infty(\Omega)$ we have $f(\lambda, \cdot, u) \in L^\infty(\Omega)$. Thus, for some constant c', $0 \leq a u^{-\alpha} + f(\lambda, \cdot, u) \leq c^{-\alpha} \|a\|_\infty \vartheta_\alpha^{-\alpha} + \|f(\lambda, \cdot, u)\|_\infty \leq c' \vartheta_\alpha^{-\alpha}$. Since

$$u = (-\Delta)^{-1}(a u^{-\alpha} + f(\lambda, \cdot, u)),$$

from Lemma 2.9 we obtain $u \leq c'' \vartheta_\alpha$, and so $u \in X_\alpha$. Let $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be the weak solution of $-\Delta w = a w^{-\alpha}$ in Ω, $w = 0$ on $\partial\Omega$ (given by Remark 1.1). Then, by Remark 1.1, $w \geq c\vartheta_\alpha$. Also we have

$$-\Delta(u - w) \geq a(u^{-\alpha} - w^{-\alpha}) \text{ in } \Omega, \quad \text{and} \quad u - w = 0 \text{ on } \partial\Omega\,. \tag{3.6}$$

Now, taking $(u - w)^-$ as a test function in (3.6), we obtain $u \geq w$. Thus $u \geq c\vartheta_\alpha$ and so $u \in D_\alpha$ $\square$

Definition 3.6. Let $T : (0, \infty) \times D_\alpha \to X_\alpha$ be the operator defined by

$$T(\lambda, u) = u - (-\Delta)^{-1}(a u^{-\alpha} + f(\lambda, \cdot, u)) \tag{3.7}$$

Lemma 3.7. *$T : (0, \infty) \times D_\alpha \to X_\alpha$ is Fréchet differentiable, and its differential at $(\lambda, u) \in (0, \infty) \times D_\alpha$, noted $DT_{(\lambda,u)}$, is given by*

$$DT_{(\lambda,u)}(\tau, \psi) = \psi - (-\Delta)^{-1}(-\alpha a \psi u^{-\alpha-1} + \tau \frac{\partial f}{\partial \lambda}(\lambda, \cdot, u) + \psi \frac{\partial f}{\partial s}(\lambda, \cdot, u)), \tag{3.8}$$

for any $(\tau,\psi)\in\mathbb{R}\times X_\alpha$.

Proof. Let $(\lambda,u)\in(0,\infty)\times D_\alpha$ and let $r>0$ be such that $\lambda>4r$ and $\vartheta_\alpha^{-1}u>4r$ a.e. in Ω. Note that

$$N:=\{(\tau,\psi)\in\mathbb{R}\times X_\alpha : |\tau|<r \text{ and } \|\psi\|_{X_\alpha}<r\}$$

is an open neighborhood of $(0,0)$ in $\mathbb{R}\times X_\alpha$, and that $(\lambda,u)+N\subset(0,\infty)\times D_\alpha$. For $(\tau,\psi)\in N$ we have

$$T(\lambda+\tau,u+\psi)=T(\lambda,u)+\psi-(-\Delta)^{-1}(h_{\tau,\psi}), \tag{3.9}$$

where $h_{\tau,\psi}:=a(u+\psi)^{-\alpha}+f(\lambda+\tau,\cdot,u+\psi)-au^{-\alpha}-f(\lambda,\cdot,u)$. A computation gives

$$\begin{aligned} h_{\tau,\psi}&=a\int_0^1\frac{d}{dt}(u+t\psi)^{-\alpha}\,dt\\ &\quad+\int_0^1\big[\tau\frac{\partial f}{\partial\lambda}(\lambda+t\tau,\cdot,u+t\psi)+\psi\frac{\partial f}{\partial s}(\lambda+t\tau,\cdot,u+t\psi)\big]dt. \end{aligned}\tag{3.10}$$

Also,

$$\begin{aligned} &a\int_0^1\frac{d}{dt}(u+t\psi)^{-\alpha}\,dt\\ &=-\alpha a\psi\int_0^1[u^{-\alpha-1}+\int_0^t\frac{d}{d\sigma}(u+\sigma\psi)^{-\alpha-1}d\sigma]dt\\ &=-\alpha a\psi u^{-\alpha-1}+R_1(\psi), \end{aligned}\tag{3.11}$$

where

$$R_1(\psi):=\alpha(\alpha+1)a\psi^2\int_0^1\int_0^t(u+\sigma\psi)^{-\alpha-2}\,d\sigma\,dt. \tag{3.12}$$

Also,

$$\begin{aligned} &\int_0^1\big[\tau\frac{\partial f}{\partial\lambda}(\lambda+t\tau,\cdot,u+t\psi)+\psi\frac{\partial f}{\partial s}(\lambda+t\tau,\cdot,u+t\psi)\big]dt\\ &=\int_0^1\int_0^t\frac{d}{d\sigma}\big[\tau\frac{\partial f}{\partial\lambda}(\lambda+\sigma\tau,\cdot,u+\sigma\psi)+\psi\frac{\partial f}{\partial s}(\lambda+\sigma\tau,\cdot,u+\sigma\psi)\big]\,d\sigma\,dt\\ &\quad+\tau\frac{\partial f}{\partial\lambda}(\lambda,\cdot,u)+\psi\frac{\partial f}{\partial s}(\lambda,\cdot,u)\\ &=\tau\frac{\partial f}{\partial\lambda}(\lambda,\cdot,u)+\psi\frac{\partial f}{\partial s}(\lambda,\cdot,u)+R_2(\tau,\psi), \end{aligned}\tag{3.13}$$

where

$$\begin{aligned} R_2(\tau,\psi):=\tau^2&\int_0^1\int_0^t\frac{\partial^2 f}{\partial\lambda^2}(\lambda+\sigma\tau,\cdot,u+\sigma\psi)\,d\sigma\,dt\\ &+2\tau\psi\int_0^1\int_0^t\frac{\partial^2 f}{\partial\lambda\partial s}(\lambda+\sigma\tau,\cdot,u+\sigma\psi)\,d\sigma\,dt\\ &+\psi^2\int_0^1\int_0^t\frac{\partial^2 f}{\partial s^2}(\lambda+\sigma\tau,\cdot,u+\sigma\psi)\,d\sigma\,dt. \end{aligned}\tag{3.14}$$

Thus $T(\lambda+\tau,u+\psi)=T(\lambda,u)+L_{\lambda,u}(\tau,\psi)-(-\Delta)^{-1}(R_1(\psi)+R_2(\tau,\psi))$, where

$$L_{\lambda,u}(\tau,\psi):=\psi-(-\Delta)^{-1}(-\alpha a\psi u^{-\alpha-1}+\tau\frac{\partial f}{\partial\lambda}(\lambda,\cdot,u)+\psi\frac{\partial f}{\partial s}(\lambda,\cdot,u)).$$

Then, to conclude the proof of the lemma, it is sufficient to prove the following two assertions: (a) $L_{\lambda,u}(\mathbb{R}\times X_\alpha)\subset X_\alpha$ and $L_{\lambda,u}:\mathbb{R}\times X_\alpha\to X_\alpha$ is continuous; (b)

$$\|(-\Delta)^{-1}(R_1(\psi))\|_{X_\alpha}+\|(-\Delta)^{-1}(R_2(\tau,\psi))\|_{X_\alpha}\le c\|(\tau,\psi)\|_{X_\alpha}^2$$

for some positive constant c, independent of τ and ψ.

Let us prove (a). Since $u\in D_\alpha$, we have, for some positive constant c', $u\ge c'\vartheta_\alpha$ in Ω; then, taking into account that $|\vartheta_\alpha^{-1}\psi|\le\|\psi\|_{X_\alpha}$, for some positive constant c independent of (τ,ψ), we have $|\alpha a u^{-\alpha-1}\psi|\le c\vartheta_\alpha^{-\alpha}\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}$ in Ω. Also, $u\in X_\alpha$ implies $u\in L^\infty(\Omega)$, and so, by (H8)–(H9), $\frac{\partial f}{\partial\lambda}(\lambda,\cdot,u)$ and $\frac{\partial f}{\partial s}(\lambda,\cdot,u)$ belong to $L^\infty(\Omega)$. Then, for some positive constants c'' and c''' independent of (τ,ψ), we have

$$\begin{aligned}
|\tau\frac{\partial f}{\partial\lambda}(\lambda,\cdot,u)+\psi\frac{\partial f}{\partial s}(\lambda,\cdot,u)| &\le c''(|\tau|+|\psi|)\\
&= c''(|\tau|+\vartheta_\alpha|\frac{\psi}{\vartheta_\alpha}|)\\
&\le c'''\vartheta_\alpha^{-\alpha}\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}\ a.e.\text{ in }\Omega.
\end{aligned}\tag{3.15}$$

Then, for some positive constant c independent of (τ,ψ), it holds

$$|-\alpha a\psi u^{-\alpha-1}+\tau\frac{\partial f}{\partial\lambda}(\lambda,\cdot,u)+\psi\frac{\partial f}{\partial s}(\lambda,\cdot,u)|\le c\vartheta_\alpha^{-\alpha}\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}a.e.\text{ in }\Omega,$$

Lemma 2.9 now implies that (a) holds.

Let us prove (b). Let $\rho:=\max\{\lambda+r,(\|u\|_{X_\alpha}+r)\|\vartheta_\alpha\|_\infty\}$, and let

$$M:=\||\frac{\partial^2 f}{\partial\lambda^2}|+|\frac{\partial^2 f}{\partial\lambda\partial s}|+|\frac{\partial^2 f}{\partial s^2}|\|_{L^\infty((0,\rho)\times\Omega\times(0,\rho))}.$$

Note that, for any $(\tau,\psi)\in N$ and $\sigma\in[0,1]$, we have $0<\lambda+\sigma\tau<\lambda+r\le\rho$ and

$$0\le u+\sigma\psi\le(\|\vartheta_\alpha^{-1}u\|_\infty+\|\vartheta_\alpha^{-1}\psi\|_\infty)\vartheta_\alpha\le(\|u\|_{X_\alpha}+r)\|\vartheta_\alpha\|_\infty\le\rho.$$

Then, for such a (τ,ψ), and for some positive constant c independent of (τ,ψ), the following inequalities hold:

$$\begin{aligned}
\big|\tau^2\int_0^1\int_0^t\frac{\partial^2 f}{\partial\lambda^2}(\lambda+\sigma\tau,\cdot,u+\sigma\psi)\,d\sigma\,dt\big| &\le\frac12 M\tau^2\le c\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}^2,\\
\big|2\tau\psi\int_0^1\int_0^t\frac{\partial^2 f}{\partial\lambda\partial s}(\lambda+\sigma\tau,\cdot,u+\sigma\psi)\,d\sigma\,dt\big| &\le M|\tau||\psi|\le M\vartheta_\alpha|\tau||\frac{\psi}{\vartheta_\alpha}|\\
&\le c\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}^2,\\
\big|\psi^2\int_0^1\int_0^t\frac{\partial^2 f}{\partial s^2}(\lambda+\sigma\tau,\cdot,u+\sigma\psi)\,d\sigma\,dt\big| &\le\frac12 M\psi^2\le c\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}^2.
\end{aligned}$$

Therefore, from (3.14), $|R_2(\tau,\psi)|\le c\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}^2\le c'\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}^2\vartheta_\alpha^{-\alpha}$, with c and c' constants independent of (τ,ψ). Then, by Lemma 2.9, $(-\Delta)^{-1}(R_2(\tau,\psi))\in X_\alpha$ and $\|(-\Delta)^{-1}(R_2(\tau,\psi))\|_{X_\alpha}\le c\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}^2$, where c is a constant independent of (τ,ψ).

Consider now $R_1(\psi)$. Since $u\ge c\vartheta_\alpha$ a.e. in Ω, from (3.12), for a constant c' independent of (τ,ψ), we have

$$|R_1(\psi)|\le c'\vartheta_\alpha^{-(\alpha+2)}\psi^2=c'\vartheta_\alpha^{-\alpha}(\frac{\psi}{\vartheta_\alpha})^2\le c'\vartheta_\alpha^{-\alpha}\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}^2$$

and so, by Lemma 2.9, we have $(-\Delta)^{-1}(R_1(\psi)) \in X_\alpha$ and $\|(-\Delta)^{-1}(R_1(\psi))\|_{X_\alpha} \le c\|(\tau,\psi)\|^2_{\mathbb{R}\times X_\alpha}$, with c a positive constant independent of (τ,ψ). Thus (b) holds.

Then T is Fréchet differentiable at (λ,u), and its differential is given by (3.8). □

Corollary 3.8. *For $(\lambda,u)\in\mathbb{R}\times D_\alpha$, the partial derivative $D_2T_{(\lambda,u)}$ at (λ,u) (i.e. the Fréchet differential, at u, of $v\to T(\lambda,v)$) is given by*

$$(D_2T_{(\lambda,u)})(\psi)=\psi-(-\Delta)^{-1}\Big(\big(-\alpha a u^{-\alpha-1}+\frac{\partial f}{\partial s}(\lambda,x,u)\big)\psi\Big)\quad \textit{for any } \psi\in X_\alpha.$$

Lemma 3.9. *$T:(0,\infty)\times D_\alpha\to X_\alpha$ is continuously Fréchet differentiable.*

Proof. Let $(\lambda,u)\in(0,\infty)\times D_\alpha$, and let $\{(\lambda_j,u_j)\}_{j\in\mathbb{N}}\subset(0,\infty)\times D_\alpha$ be a sequence that converges to (λ,u) in $\mathbb{R}\times X_\alpha$. Thus $\{u_j\}_{j\in\mathbb{N}}$ converges to u in $H_0^1(\Omega)$, and $\{\vartheta_\alpha^{-1}u_j\}_{j\in\mathbb{N}}$ converges to $\vartheta_\alpha^{-1}u$ in $L^\infty(\Omega)$. In particular, $\{u_j\}_{j\in\mathbb{N}}$ converges to u in $L^\infty(\Omega)$. For $(\tau,\psi)\in\mathbb{R}\times X_\alpha$ and $j\in\mathbb{N}$, we have

$$\begin{aligned}
&DT_{(\lambda_j,u_j)}(\tau,\psi)-DT_{(\lambda,u)}(\tau,\psi)\\
&=(-\Delta)^{-1}(\alpha a\psi(u_j^{-\alpha-1}-u^{-\alpha-1}))\\
&\quad+\tau(-\Delta)^{-1}\big(\frac{\partial f}{\partial\lambda}(\lambda,\cdot,u)-\frac{\partial f}{\partial\lambda}(\lambda_j,\cdot,u_j)\big)\\
&\quad+(-\Delta)^{-1}\Big(\psi\big(\frac{\partial f}{\partial s}(\lambda,\cdot,u)-\frac{\partial f}{\partial s}(\lambda_j,\cdot,u_j)\big)\Big).
\end{aligned}\tag{3.16}$$

Let $c_0>0$ be such that $\vartheta_\alpha^{-1}u\ge c_0$ a.e. in Ω. Since $\{\vartheta_\alpha^{-1}u_j\}_{j\in\mathbb{N}}$ converges to $\vartheta_\alpha^{-1}u$ in $L^\infty(\Omega)$, there exists $j_0\in\mathbb{N}$ such that $\vartheta_\alpha^{-1}u_j\ge\frac12 c_0$ a.e. in Ω for any $j\ge j_0$. Then there exists a positive constant c_1 such that, for $j\ge j_0$, and for a.a. $x\in\Omega$, $t^{-\alpha-2}\le c_1\vartheta_\alpha^{-(\alpha+2)}$ whenever t lies on the line segment with endpoints $u(x)$ and $u_j(x)$. Thus, for $j\ge j_0$, and for some positive constant c, independent of (τ,ψ) and j, we have

$$\begin{aligned}
|a(u_j^{-\alpha-1}-u^{-\alpha-1})\psi| &=(\alpha+1)\big|a\psi\int_u^{u_j}t^{-\alpha-2}dt\big|\\
&\le c|u_j-u||\psi|\vartheta_\alpha^{-(\alpha+2)}\\
&\le c|\vartheta_\alpha^{-1}(u_j-u)||\vartheta_\alpha^{-1}\psi|\vartheta_\alpha^{-\alpha}\\
&\le c\|u_j-u\|_{X_\alpha}\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}\vartheta_\alpha^{-\alpha},
\end{aligned}\tag{3.17}$$

Then, by Lemma 2.9, $(-\Delta)^{-1}(a(u_j^{-\alpha-1}-u^{-\alpha-1})\psi)\in X_\alpha$, and, for some constant c independent of (τ,ψ) and j, we have

$$\|(-\Delta)^{-1}(\alpha a(u_j^{-\alpha-1}-u^{-\alpha-1})\psi)\|_{X_\alpha}\le c\|u_j-u\|_{X_\alpha}\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha},$$

which gives

$$\|(-\Delta)^{-1}(\alpha a(u_j^{-\alpha-1}-u^{-\alpha-1}))\|_{\mathcal{L}(\mathbb{R}\times X_\alpha,X_\alpha)}\le c\|u_j-u\|_{X_\alpha}.\tag{3.18}$$

Consider now the second term of the sum in the right-hand side of (3.16), i.e., the term $\tau(-\Delta)^{-1}(\frac{\partial f}{\partial\lambda}(\lambda,\cdot,u)-\frac{\partial f}{\partial\lambda}(\lambda_j,\cdot,u_j))$. As $\vartheta_\alpha^{-1}u\in L^\infty(\Omega)$, $\{\vartheta_\alpha^{-1}u_j\}_{j\in\mathbb{N}}$ converges to $\vartheta_\alpha^{-1}u$ in $L^\infty(\Omega)$, and $\{\lambda_j\}_{j\in\mathbb{N}}$ converges to λ in $\mathbb{R}$, there exists $\rho>0$ such that $\vartheta_\alpha\le\rho$ in Ω, $\|u_j\|_\infty\le\rho$, $|\lambda_j|\le\rho$ for any $j\in\mathbb{N}$, $\|u\|_\infty\le\rho$, and $|\lambda|\le\rho$. Then, by the mean value theorem, and by (H8), (H9), there exists $M>0$ such that, for all

j,

$$\begin{aligned}
&\big|\tau(\frac{\partial f}{\partial \lambda}(\lambda,\cdot,u)-\frac{\partial f}{\partial \lambda}(\lambda_j,\cdot,u_j))\big| \\
&\le M|\tau|(|u_j-u|+|\lambda_j-\lambda|) \\
&\le M|\tau|(\|\vartheta_\alpha\|_\infty\|\vartheta_\alpha^{-1}u_j-\vartheta_\alpha^{-1}u\|_\infty+|\lambda_j-\lambda|) \\
&\le c\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}\|(\lambda_j-\lambda,u_j-u)\|_{\mathbb{R}\times X_\alpha} \\
&\le c'\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}\|(\lambda_j-\lambda,u_j-u)\|_{\mathbb{R}\times X_\alpha}\vartheta_\alpha^{-\alpha} \quad \text{a.e. in } \Omega,
\end{aligned} \tag{3.19}$$

where c and c' are positive constants independent of (τ,ψ) and j. Then, from Lemma 2.9, we have

$$\begin{aligned}
&\|\tau(-\Delta)^{-1}(\frac{\partial f}{\partial \lambda}(\lambda,\cdot,u)-\frac{\partial f}{\partial \lambda}(\lambda_j,\cdot,u_j))\|_{X_\alpha} \\
&\le c''\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}\|(\lambda_j-\lambda,u_j-u)\|_{\mathbb{R}\times X_\alpha},
\end{aligned}$$

where c'' is a positive constant independent of (τ,ψ) and j. Thus we obtain, for a constant c independent of j,

$$\begin{aligned}
&\|\tau(-\Delta)^{-1}(\frac{\partial f}{\partial \lambda}(\lambda,\cdot,u)-\frac{\partial f}{\partial \lambda}(\lambda_j,\cdot,u_j))\|_{\mathcal{L}(\mathbb{R}\times X_\alpha,X_\alpha)} \\
&\le c\|(\lambda_j-\lambda,u_j-u)\|_{\mathbb{R}\times X_\alpha}.
\end{aligned} \tag{3.20}$$

Consider now the third term of the sum in the right-hand side of (3.16), i.e., the term $(-\Delta)^{-1}(\psi(\frac{\partial f}{\partial s}(\lambda,\cdot,u)-\frac{\partial f}{\partial s}(\lambda_j,\cdot,u_j)))$. We now have the following inequality, which is analogous to (3.19),

$$\begin{aligned}
&\big|\psi(\frac{\partial f}{\partial s}(\lambda,\cdot,u)-\frac{\partial f}{\partial s}(\lambda_j,\cdot,u_j))\big| \\
&\le M|\psi|(\|u_j-u\|_\infty+|\lambda_j-\lambda|) \\
&\le M\|\vartheta_\alpha\|_\infty|\vartheta_\alpha^{-1}\psi|(\|\vartheta_\alpha\|_\infty\|\vartheta_\alpha^{-1}u_j-\vartheta_\alpha^{-1}u\|_\infty+|\lambda_j-\lambda|) \\
&\le c\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}\|(\lambda_j-\lambda,u_j-u)\|_{\mathbb{R}\times X_\alpha} \\
&\le c'\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}\|(\lambda_j-\lambda,u_j-u)\|_{\mathbb{R}\times X_\alpha}\vartheta_\alpha^{-\alpha} \ a.e. \text{ in } \Omega.
\end{aligned} \tag{3.21}$$

where c and c' are positive constants independent of (τ,ψ) and j. Thus, from Lemma 2.9, we obtain

$$\begin{aligned}
&\big\|(-\Delta)^{-1}\Big(\psi\big(\frac{\partial f}{\partial s}(\lambda,\cdot,u)-\frac{\partial f}{\partial s}(\lambda_j,\cdot,u_j)\big)\Big)\big\|_{X_\alpha} \\
&\le c''\|(\tau,\psi)\|_{\mathbb{R}\times X_\alpha}\|(\lambda_j-\lambda,u_j-u)\|_{\mathbb{R}\times X_\alpha}
\end{aligned}$$

which implies that, for some constant c independent of j,

$$\begin{aligned}
&\big\|(\tau,\psi)\to(-\Delta)^{-1}\Big(\psi\big(\frac{\partial f}{\partial s}(\lambda,\cdot,u)-\frac{\partial f}{\partial s}(\lambda_j,\cdot,u_j)\big)\Big)\|_{\mathcal{L}(\mathbb{R}\times X_\alpha,X_\alpha)} \\
&\le c\big\|(\lambda_j-\lambda,u_j-u)\big\|_{\mathbb{R}\times X_\alpha}.
\end{aligned} \tag{3.22}$$

Then, from (3.18), (3.20) and (3.22), it follows that $\{DT_{(\lambda_j,u_j)}\}_{j\in\mathbb{N}}$ converges to $DT_{(\lambda,u)}$ in $\mathcal{L}(\mathbb{R}\times X_\alpha,X_\alpha)$, which completes the proof. $\square$

4. An operator with singular potential and a related principal eigenvalue problem with weight

Lemma 4.1. *For any $u \in D_\alpha$, it holds that $\alpha a u^{-\alpha-1} w\varphi \in L^1(\Omega)$ whenever w and φ belong to $H_0^1(\Omega)$. Moreover, there exists a positive constant, independent of w and φ, such that*

$$\|\alpha a u^{-\alpha-1} w\varphi\|_1 \le c\|w\|_{H_0^1(\Omega)}\|\varphi\|_{H_0^1(\Omega)}. \tag{4.1}$$

Proof. If $0 < \alpha \le 1$, then either $\vartheta_\alpha = d_\Omega$ or $\vartheta_\alpha = \log(\frac{\omega_0}{d_\Omega})d_\Omega$. In both cases there exists a positive constant c such that $u \ge c d_\Omega$ in Ω. Thus, for some positive constant c', independent of w and φ,

$$0 \le \alpha a u^{-\alpha-1} w\varphi = \alpha a d_\Omega^2 u^{-\alpha-1} \frac{w}{d_\Omega}\frac{\varphi}{d_\Omega} \le c' d_\Omega^{1-\alpha} \frac{w}{d_\Omega}\frac{\varphi}{d_\Omega} \text{ in } \Omega.$$

If $1 < \alpha < 3$, then $\vartheta_\alpha = d_\Omega^{\frac{2}{1+\alpha}}$ in Ω, and so there exists a positive constant c such that $u \ge c d_\Omega^{\frac{2}{1+\alpha}}$ in Ω. Thus, for some positive constant c' independent of w and φ, we have

$$0 \le \alpha a u^{-\alpha-1} w\varphi = \alpha a d_\Omega^2 u^{-\alpha-1} \frac{w}{d_\Omega}\frac{\varphi}{d_\Omega} \le c' d_\Omega^{2-\frac{2(\alpha-1)}{\alpha+1}} \frac{w}{d_\Omega}\frac{\varphi}{d_\Omega} = c' d_\Omega^{\frac{4}{\alpha+1}} \frac{w}{d_\Omega}\frac{\varphi}{d_\Omega},$$

and the lemma follows from the Hölder and the Hardy inequalities. □

Definition 4.2. For any $u \in D_\alpha$, and for any nonnegative $m \in L^\infty(\Omega)$ such that $m \not\equiv 0$ in Ω, we define

$$\mu_{m,u} := \inf_{\{w \in H_0^1(\Omega): \int_\Omega m w^2 > 0\}} \frac{\int_\Omega (|\nabla w|^2 + \alpha a u^{-\alpha-1} w^2)}{\int_\Omega m w^2} \tag{4.2}$$

or, equivalently,

$$\mu_{m,u} := \inf_{w \in W_m} \int_\Omega (|\nabla w|^2 + \alpha a u^{-\alpha-1} w^2), \tag{4.3}$$

where $W_m := \{w \in H_0^1(\Omega) : \int_\Omega m w^2 = 1\}$.

From Lemma 4.1, $\alpha a u^{-\alpha-1} w^2 \in L^1(\Omega)$ for any $w \in H_0^1(\Omega)$. Thus $\mu_{m,u}$ is well defined and finite.

Lemma 4.3. *For any $u \in D_\alpha$, and for any nonnegative $m \in L^\infty(\Omega)$ such that $m \not\equiv 0$ in Ω, the following statements hold:*

(i) *The infimum in (4.3) is achieved at some nonnegative $w \in W_m$.*
(ii) $\mu_{m,u} > 0$.
(iii) *If $w \in W_m$ is a nonnegative minimizer for (4.3) then $\alpha a u^{-\alpha-1} w\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, and w satisfies, in weak sense,*

$$\begin{gathered} -\Delta w + \alpha a u^{-\alpha-1} w = \mu_{m,u} m w \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.4}$$

(iv) *If $w \in W_m$ is a nonnegative minimizer for (4.3) then, for any positive δ such that $\Omega_\delta \neq \emptyset$, there exists a positive constant c such that $w \ge c d_{\Omega_\delta}$ a.e. in Ω_δ. In particular, $w > 0$ a.e. in Ω.*

Proof. To prove (i), consider a minimizing sequence $\{w_j\}_{j\in\mathbb{N}} \subset W_m$ for (4.3). Note that $\{w_j\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, and so, taking a subsequence if necessary, we can assume that, for some $w \in H_0^1(\Omega)$, $\{w_j\}_{j\in\mathbb{N}}$ converges, strongly in $L^2(\Omega)$, and a.e. in Ω, to w; and that $\{\nabla w_j\}_{j\in\mathbb{N}}$ converges to ∇w weakly in $L^2(\Omega,\mathbb{R}^n)$. Now, $|\int_\Omega mw^2 - 1| = |\int_\Omega m(w^2 - w_j^2)| \le \|m\|_\infty \|w - w_j\|_2 \|w + w_j\|_2$. Since the last expression converges to zero as j tends to ∞, we obtain that $w \in W_m$. Then $\mu_{m,u} \le \int_\Omega (|\nabla w|^2 + \alpha a u^{-\alpha-1} w^2)$. On the other hand, from Fatou's lemma, and from the fact that $\{w_j\}_{j\in\mathbb{N}}$ converges to w weakly in $H_0^1(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega |\nabla w|^2 + \int_\Omega \alpha a u^{-\alpha-1} w^2 \\
&= \int_\Omega |\nabla w|^2 + \int_\Omega \liminf_{j\to\infty} \alpha a u^{-\alpha-1} w_j^2 \\
&\le \liminf_{j\to\infty} \int_\Omega |\nabla w_j|^2 + \int_\Omega \liminf_{j\to\infty} \alpha a u^{-\alpha-1} w_j^2 \\
&\le \liminf_{j\to\infty} \int_\Omega |\nabla w_j|^2 + \liminf_{j\to\infty} \int_\Omega \alpha a u^{-\alpha-1} w_j^2 \\
&\le \liminf_{j\to\infty} \int_\Omega \big(|\nabla w_j|^2 + \alpha a u^{-\alpha-1} w_j^2\big) = \mu_{m,u}.
\end{aligned}$$

Then $\int_\Omega |\nabla w|^2 + \int_\Omega \alpha a u^{-\alpha-1} w^2 \le \mu_{m,u}$. Since $w \in W_m$, the reverse inequality also holds. Thus the infimum in (4.3) is achieved at w. Since the infimum in (4.3) is also achieved at $|w|$, (i) holds.

To prove (ii), observe that, since $\mu_{m,u} = \int_\Omega (|\nabla w|^2 + \alpha a u^{-\alpha-1} w^2)$ for some $w \in W_m$, then $\mu_{m,u} \ge 0$. If $\mu_{m,u} = 0$, we would have $\int_\Omega |\nabla w|^2 = 0$, and so $w = 0$ a.e. in Ω, which would contradict $\int_\Omega mw^2 = 1$.

To prove (iii), consider a minimizer $w \in W_m$ for (4.3). Let $\varphi \in H_0^1(\Omega)$, and let $t \in \mathbb{R}$. Note that, by Lemma 4.1, $\alpha a u^{-\alpha-1}(w + t\varphi)^2 \in L^1(\Omega)$. Also observe

$$\mu_{m,u} \int_\Omega m(w + t\varphi)^2 \le \int_\Omega (|\nabla (w + t\varphi)|^2 + \alpha a u^{-\alpha-1}(w + t\varphi)^2). \tag{4.5}$$

Indeed, since, by *ii)*, $\mu_{m,u} \ge 0$, (4.5) clearly holds when $\int_\Omega m(w + t\varphi)^2 \le 0$. If $\int_\Omega m(w + t\varphi)^2 > 0$, (4.5) follows from (4.2). Now, since $w \in W_m$ and $\mu_{m,u} = \int_\Omega (|\nabla w|^2 + \alpha a u^{-\alpha-1} w^2)$, from (4.5) we obtain

$$\begin{aligned}
&\mu_{m,u} \int_\Omega m(2tw\varphi + t^2\varphi^2) \\
&\le \int_\Omega \big(t^2 |\nabla \varphi|^2 + 2t\langle \nabla w, \nabla \varphi\rangle + a u^{-\alpha-1}(2tw\varphi + t^2\varphi^2)\big).
\end{aligned}$$

Suppose $t > 0$, divide by t both sides of the last inequality, and take the limit as $t \to 0^+$; using that $a u^{-\alpha-1} w\varphi$ and $a u^{-\alpha-1}\varphi^2$ belong to $L^1(\Omega)$, Lebesgue's dominated convergence theorem gives $\mu_{m,u} \int_\Omega mw\varphi \le \int_\Omega (\langle \nabla w, \nabla \varphi\rangle + a u^{-\alpha-1} w\varphi)$. When $t < 0$, a similar procedure gives the reverse inequality. Then

$$\mu_{m,u} \int_\Omega mw\varphi = \int_\Omega (\langle \nabla w, \nabla \varphi\rangle + a u^{-\alpha-1} w\varphi), \tag{4.6}$$

whenever $\varphi \in H_0^1(\Omega)$, i.e., w is a weak solution of problem (4.4) .

Finally, note that it is enough to prove *iv)* when δ is positive and small enough. Let Ω^δ be a domain with C^2 boundary such that $\Omega_\delta \subset \Omega^\delta \subset \Omega_{\delta/2}$. Since $u \in D_\alpha$

we have $0 \leq \alpha a u^{-\alpha-1} \in L^\infty(\Omega^\delta)$. Since $m \geq 0$, $w \geq 0$ and $\int_\Omega m w^2 = 1$, we have $mw \geq 0$ and $|\{mw > 0\}| > 0$. Then, for δ and ε small enough, there exists a measurable set $E \subset \Omega^\delta$, with $|E| > 0$, such that $\mu_{m,u} m w \geq \varepsilon \chi_E$ in Ω^δ. Let $\zeta \in \cap_{1 \leq p < \infty} W^{2,p}(\Omega^\delta)$ be the solution to the problem $-\Delta\zeta + \alpha a u^{-\alpha-1}\zeta = \varepsilon\chi_E$ in Ω^δ, $\zeta = 0$ on $\partial\Omega^\delta$. Then $\zeta \in C^1(\overline{\Omega^\delta})$, and the Hopf boundary lemma (as stated in [42, Theorem 1.1]) gives $\frac{\partial\zeta}{\partial\nu} < 0$ on $\partial\Omega^\delta$, where ν denotes the outward unit normal at $\partial\Omega^\delta$. Moreover, the strong maximum principle (see [27, Theorem 9.6]) gives $w \geq \zeta$ and also $\zeta > 0$ in Ω^δ. Thus, for some positive constant c, $\zeta \geq c d_{\Omega^\delta}$ in Ω^δ, and *iv)* holds. $\square$

For $\zeta \in (H_0^1(\Omega))'$, we write $\zeta \geq 0$ to mean that $\zeta(\varphi) \geq 0$ for any nonnegative $\varphi \in H_0^1(\Omega)$.

Lemma 4.4. *Let $u \in D_\alpha$, let $m \in L^\infty(\Omega)$ be nonnegative and nonidentically zero, and let $\mu \in [0, \mu_{u,m})$. Then, for any $\zeta \in (H_0^1(\Omega))'$:*

(i) *There exists a unique weak solution $z \in H_0^1(\Omega)$ of the problem*
$$-\Delta z + \alpha a u^{-\alpha-1} z = \mu m z + \zeta, \tag{4.7}$$
i.e., z satisfies $mz\varphi \in L^1(\Omega)$ and $\int_\Omega \langle \nabla z, \nabla\varphi\rangle = \mu\int_\Omega m z\varphi + \zeta(\varphi)$ for any $\varphi \in H_0^1(\Omega)$.

(ii) *If, in addition, $\zeta \geq 0$, then $z \geq 0$.*

(iii) *If $\zeta = S_v$ for some measurable $v : \Omega \to \mathbb{R}$ such that $|v| \leq c\vartheta_\alpha^{-\alpha}$, then $z \in X_\alpha$.*

Proof. To prove (i), consider the symmetric bilinear form $A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by $A(u,v) := \int_\Omega (\langle \nabla u, \nabla v\rangle + \alpha a u^{-\alpha-1} uv - \mu m uv)$. By Lemma 4.1, we have $au^{-\alpha-1}uv \in L^1(\Omega)$, and $|\int_\Omega \alpha a u^{-\alpha-1} uv| \leq c\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}$ for some positive constant c independent of u and v; clearly a similar estimate holds for $|\int_\Omega \mu m uv|$. Thus, A is well defined and continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$. Taking into account (4.2), we have
$$\begin{aligned} A(v,v) &= \int_\Omega (|\nabla v|^2 + \alpha a u^{-\alpha-1} v^2 - \mu m v^2) \\ &\geq \big(1 - \frac{\mu}{\mu_{u,m}}\big) \int_\Omega (|\nabla v|^2 + \alpha a u^{-\alpha-1} v^2); \end{aligned}$$
then, since $\mu_{u,m} > \mu$, A is coercive on $H_0^1(\Omega)$. Thus, by the Lax Milgram Theorem (as stated in [4, Corollary 5.8]), there exists a unique weak solution $z \in H_0^1(\Omega)$ to problem (4.7). Moreover, z minimizes the functional $J : H_0^1(\Omega) \to \mathbb{R}$ given by
$$J(v) := \frac{1}{2}\int_\Omega (|\nabla v|^2 + \alpha a u^{-\alpha-1} v^2 - \mu m v^2) - \zeta(v).$$
Thus (i) holds.

To prove (ii), observe that, if $\zeta \geq 0$, taking $-z^-$ as a test function in (4.7), we obtain $\int_\Omega (|\nabla(z^-)|^2 + \alpha a u^{-\alpha-1}(z^-)^2) = \mu\int_\Omega m(z^-)^2 - \zeta(z^-)$, and thus
$$\begin{aligned} \int_\Omega (|\nabla(z^-)|^2 + \alpha a u^{-\alpha-1}(z^-)^2) &= \mu\int_\Omega m(z^-)^2 - \zeta(z^-) \\ &\leq \frac{\mu}{\mu_{u,m}} \int_\Omega (|\nabla(z^-)|^2 + \alpha a u^{-\alpha-1}(z^-)^2); \end{aligned}$$
which (since $\mu < \mu_{u,m}$), implies $z^- = 0$ in Ω. Thus (ii) holds.

To prove (iii), we consider first the case when $\zeta = S_v$, with $v \geq 0$ a.e. in Ω. We claim that, for any $k \in \mathbb{N} \cup \{0\}$, there exists a positive constant c_k such that

$$z \leq q^{2^k} (-\Delta)^{-2^k} z + c_k \vartheta_\alpha \quad \text{in } \Omega, \tag{4.8}$$

where $(-\Delta)^{-2^j} := ((-\Delta)^{-1})^{2^j}$. We prove (4.8) by induction on k. As $m \in L^\infty(\Omega)$, there exists $q \in (0,\infty)$ such that

$$-\Delta z \leq qz + c\vartheta_\alpha^{-\alpha} \quad \text{in } \Omega. \tag{4.9}$$

By Lemma 2.9, $(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha}) \in H_0^1(\Omega)$, and there exists a positive constant c' such that $(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha}) \leq c'\vartheta_\alpha$ a.e. in Ω. Thus, from (4.9), the weak maximum principle gives $z \leq q(-\Delta)^{-1}z + c(-\Delta)^{-1}(\vartheta_\alpha^{-\alpha}) \leq q(-\Delta)^{-1}z + c_0\vartheta_\alpha$ in Ω, with $c_0 = cc'$. Then (4.8) holds for $k=0$. Now suppose that (4.8) holds for $k=j$, i.e., that for some positive constant c_j,

$$z \leq q^{2^j} (-\Delta)^{-2^j} z + c_j \vartheta_\alpha \quad \text{in } \Omega. \tag{4.10}$$

Then, since $(-\Delta)^{-2^j}$ is a positive operator on $H_0^1(\Omega)$, from (4.10) we obtain

$$\begin{aligned} z &\leq q^{2^j} (-\Delta)^{-2^j} (q^{2^j} (-\Delta)^{-2^j} z + c_j \vartheta_\alpha) + c_j \vartheta_\alpha \\ &= q^{2^{j+1}} (-\Delta)^{-2^{j+1}} z + q^{2^j} c_j (-\Delta)^{-2^j} (\vartheta_\alpha) + c_j \vartheta_\alpha \text{ in } \Omega; \end{aligned} \tag{4.11}$$

note that, for some positive constant c'', $d_\Omega \leq c''\vartheta_\alpha$ in Ω. Also, since $0 \leq \vartheta_\alpha \in L^\infty(\Omega)$, there exist a positive constant c_j''' such that $(-\Delta)^{-2^j}(\vartheta_\alpha) \leq c_j''' d_\Omega$ in Ω. Thus (4.11) implies that (4.8) holds for $k = j+1$. Then (4.8) holds for any $k \in \mathbb{N} \cup \{0\}$. By a bootstrap argument we have, for k large enough, $(-\Delta)^{-2^k} z \in C^1(\overline{\Omega})$, and so, for such a k we have, for some positive constant $\widetilde{c}_k$, $(-\Delta)^{-2^k} z \leq \widetilde{c}_k d_\Omega \leq \widetilde{c}_k c'' \vartheta_\alpha$ in Ω. Then, from (4.8), we obtain, for some positive constant c, $z \leq c\vartheta_\alpha$ in Ω. Thus $z \in X_\alpha$.

Consider now the general case $\zeta = S_v$, with v non necessarily nonnegative. Write $v = v^+ - v^-$, and consider the solution z_1 (respectively z_2) of the problem $-\Delta z_1 = \mu m z_1 + v^+$ in Ω, $z_1 = 0$ on $\partial\Omega$ (resp. $-\Delta z_2 = m z_2 + v^-$ in Ω, $z_2 = 0$ on $\partial\Omega$). Thus $z := z_1 - z_2$ is the solution of $-\Delta z = \mu m z + \zeta$ in Ω, $z = 0$ on $\partial\Omega$, and the general case of (iii) follows from the previous one. □

Lemma 4.5. *Let $u \in D_\alpha$ and let $h \in L^{(2^*)'}(\Omega)$. Then:*

(i) *There exists a unique weak solution $z \in H_0^1(\Omega)$ to the problem*

$$\begin{gathered} -\Delta z + \alpha a u^{-\alpha-1} z = h \quad \text{in } \Omega, \\ z = 0 \text{ on } \partial\Omega. \end{gathered} \tag{4.12}$$

(ii) *If $h \geq 0$ in Ω, then $z \geq 0$ in Ω.*

(iii) *If $h \geq 0$ in Ω and $h \not\equiv 0$ in Ω, then, for any δ positive and small enough, there exists a positive constant c such that $z \geq c d_{\Omega_\delta}$ in Ω_δ.*

Proof. (i) and (ii) follow from Lemma 4.4. To prove (iii), suppose $h \geq 0$ in Ω and $h \not\equiv 0$ in Ω. Then, there exist $\varepsilon > 0$, and a measurable set $E \subset \Omega$ such that $|E| > 0$ and $h \geq \varepsilon$ in E. Thus $h \geq \varepsilon \chi_E$ in Ω. Let $\widetilde{z} \in H_0^1(\Omega)$ be the solution (given by the part (i) of the lemma) to the problem

$$\begin{gathered} -\Delta \widetilde{z} + \alpha a u^{-\alpha-1} \widetilde{z} = \varepsilon \chi_E \quad \text{in } \Omega, \\ \widetilde{z} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.13}$$

By (ii), $\widetilde{z} \geq 0$ a.e. in Ω. Also, $-\Delta\widetilde{z} \leq \varepsilon\chi_E$ in Ω, $\widetilde{z} = 0$ on $\partial\Omega$, and so $\widetilde{z} \leq (-\Delta)^{-1}(\varepsilon\chi_E)$ a.e. in Ω, and thus $\widetilde{z} \in L^\infty(\Omega)$. Moreover, since, for some positive constants c and c', $u \geq c'\vartheta_\alpha \geq cd_\Omega$ in Ω, we have also that $\varepsilon\chi_E - \alpha a u^{-\alpha-1}\widetilde{z} \in L^\infty_{\rm loc}(\Omega)$. Then, for δ positive and small enough (such that $\Omega_\delta \neq \emptyset$), the inner elliptic estimates (as stated in [27, Theorem 8.24]) give that $\widetilde{z} \in C^1(\overline{\Omega_\delta})$. Since $-\Delta\widetilde{z} + \alpha a u^{-\alpha-1}\widetilde{z} = \varepsilon\chi_E$ in $D'(\Omega_\delta)$ and $0 \leq (\alpha a u^{-\alpha-1})\big|_{\Omega_\delta} \in L^\infty(\Omega_\delta)$, the strong maximum principle, and the Hopf boundary lemma (as stated in [27, Theorem 9.6], and in [42, Theorem 1.1], respectively) imply that, for some positive constant c', $\widetilde{z} \geq c'd_{\Omega_\delta}$ in Ω_δ. Now, since $h \geq \varepsilon\chi_E$ in Ω, (ii) implies $z \geq \widetilde{z}$ a.e. in Ω. Then $z \geq c'd_{\Omega_\delta}$ in Ω_δ. □

Lemma 4.6. *Let $u \in D_\alpha$ and let $m \in L^\infty(\Omega)$ such that $0 \leq m \not\equiv 0$ in Ω. Let ρ be a nonnegative function in $L^\infty_{\rm loc}(\Omega)$ such that $\rho\varphi \in L^1(\Omega)$ for any $\varphi \in H^1_0(\Omega)$, and let $h \in H^1_0(\Omega)$ be a weak solution of the problem*

$$\begin{gathered} -\Delta h + \alpha a u^{-\alpha-1}h = \lambda m h + \rho \quad \text{in } \Omega, \\ h = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.14}$$

such that, for δ positive and small enough, there exists a positive constant c such that $h \geq cd_{\Omega_\delta}$ in Ω_δ. Then $\lambda \leq \mu_{m,u}$. If in addition, $\rho \not\equiv 0$ in Ω, then $\lambda < \mu_{m,u}$.

Proof. Let $v := -\log(h)$, and let $\psi \in C^\infty_c(\Omega)$. Since, for δ positive and small enough, $h \geq cd_{\Omega_\delta}$ in Ω_δ, and since ψ has compact support, we have $h^{-1}\psi^2 \in H^1_0(\Omega)$ and $\rho h^{-1}\psi^2 \in H^1_0(\Omega)$. Note that, by Lemma 4.1, $\alpha a u^{-\alpha-1}h\psi \in L^1(\Omega)$. Now we proceed as in [31, Remark 2.2 iv]. We take $h^{-1}\psi^2$ as a test function in (4.14) and, after a computation, we obtain

$$\lambda\int_\Omega m\psi^2 = \int_\Omega (|\nabla\psi|^2 + \alpha a u^{-\alpha-1}\psi^2) - \int_\Omega \rho h^{-1}\psi^2 - \int_\Omega |\nabla\psi + \psi\nabla v|^2$$

and so $\lambda\int_\Omega m\psi^2 \leq \int_\Omega(|\nabla\psi|^2 + \alpha a u^{-\alpha-1}\psi^2)$. Now, for $\varphi \in W_m$, i.e., for $\varphi \in H^1_0(\Omega)$ such that $\int_\Omega m\varphi^2 = 1$, consider a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset C^\infty_c(\Omega)$ such that $\{\varphi_j\}_{j\in\mathbb{N}}$ converges to φ in $H^1_0(\Omega)$. Then $\{\varphi_j\}_{j\in\mathbb{N}}$ converges to φ strongly in $L^2(\Omega)$ and, taking a subsequence if necessary, we can assume that $\{\varphi_j\}_{j\in\mathbb{N}}$ converges to φ a.e. in Ω. Thus, for all j,

$$\lambda\int_\Omega m\varphi_j^2 = \int_\Omega (|\nabla\varphi_j|^2 + \alpha a u^{-\alpha-1}\varphi_j^2) - \int_\Omega \rho h^{-1}\varphi_j^2 - \int_\Omega |\nabla\varphi_j + \varphi_j\nabla v|^2.$$

In particular, $\lambda\int_\Omega m\varphi_j^2 \leq \int_\Omega(|\nabla\varphi_j|^2 + \alpha a u^{-\alpha-1}\varphi_j^2)$, and so

$$\lambda\int_\Omega m\varphi^2 \leq \int_\Omega \big(|\nabla\varphi|^2 + \alpha a u^{-\alpha-1}\varphi^2\big).$$

Therefore

$$\lambda \leq \int_\Omega \big(|\nabla\varphi|^2 + \alpha a u^{-\alpha-1}\varphi^2\big),$$

and thus, since this holds for any $\varphi \in W_m$, we conclude that $\lambda \leq \mu_{m,u}$.

If $\lambda = \mu_{m,u}$ then, for $\varphi \in H_0^1(\Omega)$ and for $\{\varphi_j\}_{j\in\mathbb{N}}$ as above, we have, for all j,

$$\begin{aligned}
&\mu_{m,u}\int_\Omega m\varphi_j^2 + \int_\Omega \rho h^{-1}\varphi_j^2 \\
&= \lambda\int_\Omega m\varphi_j^2 + \int_\Omega \rho h^{-1}\varphi_j^2 \\
&\le \lambda\int_\Omega m\varphi_j^2 + \int_\Omega \rho h^{-1}\varphi_j^2 + \int_\Omega |\nabla\varphi_j + \varphi_j\nabla v|^2 \\
&= \int_\Omega (|\nabla\varphi_j|^2 + \alpha a u^{-\alpha-1}\varphi_j^2),
\end{aligned} \tag{4.15}$$

and so

$$\mu_{\lambda,u}\int_\Omega m\varphi_j^2 + \int_\Omega \rho h^{-1}\varphi_j^2 \le \int_\Omega (|\nabla\varphi_j|^2 + \alpha a u^{-\alpha-1}\varphi_j^2). \tag{4.16}$$

Now, by Lemma 4.1, for all j, we have

$$\begin{aligned}
|\int_\Omega \alpha a u^{-\alpha-1}\varphi_j^2 - \int_\Omega \alpha a u^{-\alpha-1}\varphi^2| &\le \int_\Omega \alpha a u^{-\alpha-1}|\varphi_j - \varphi||\varphi_j + \varphi| \\
&\le c\|\varphi_j - \varphi\|_{H_0^1(\Omega)}\|\varphi_j + \varphi\|_{H_0^1(\Omega)}
\end{aligned}$$

where c is a positive constant independent of j. Thus

$$\lim_{j\to\infty}\int_\Omega \alpha a u^{-\alpha-1}\varphi_j^2 = \int_\Omega \alpha a u^{-\alpha-1}\varphi^2. \tag{4.17}$$

Now we take $\liminf_{j\to\infty}$ in (4.16), to obtain, from Fatou's lemma,

$$\mu_{\lambda,u}\int_\Omega m\varphi^2 + \int_\Omega \rho h^{-1}\varphi^2 \le \int_\Omega |\nabla\varphi|^2 + \int_\Omega \alpha a u^{-\alpha-1}\varphi^2 \tag{4.18}$$

for any $\varphi \in H_0^1(\Omega)$. Let w be as in Lemma 4.3. By taking $\varphi = w$ in (4.18), we obtain $\int_\Omega \rho h^{-1}w^2 \le 0$ and so $\rho w^2 = 0$ a.e. in Ω. Since, by Lemma 4.3, $w > 0$ a.e. in Ω, it follows that $\rho = 0$ a.e. in Ω. □

5. An application of the implicit function theorem

Let us recall some results from [31] and [33].

Lemma 5.1 ([31, Lemma 3.5]). *For any $\lambda_0 > 0$ there exists a constant $c_{\lambda_0} > 0$ such that $\|u\|_\infty < c_{\lambda_0}$ whenever $\lambda \ge \lambda_0$ and $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ is a weak solution of problem* (1.1).

Lemma 5.2 ([31, Lemma 4.8]). *Let $\lambda_0 > 0$, let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence in $[\lambda_0,\infty)$ and, for $j \in \mathbb{N}$, let $w_j \in H_0^1(\Omega)\cap L^\infty(\Omega)$ be a weak solution of problem* (1.1) *with $\lambda = \lambda_j$. Then:*

- (i) *$\{w_j\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$.*
- (ii) *If, additionally, $\{w_j\}_{j\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$ to some $w \in H_0^1(\Omega)\cap L^\infty(\Omega)$, and $\lim_{j\to\infty}\lambda_j = \lambda$ for some $\lambda \in [\lambda_0,\infty)$; then w is a weak solution of* (1.1)*, and there exists a positive constant c such that $w \ge cd_\Omega$ in Ω.*

Remark 5.3. The assertion (ii) of Lemma 5.2 holds also for $\lambda_0 = 0$, provided that $\{w_j\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$. Indeed, in the proof of [31, Lemma 4.8], the only use of the condition $\lambda_0 > 0$ is to guarantee that $\{w_j\}_{j\in\mathbb{N}}$ be bounded in $L^\infty(\Omega)$.

Definition 5.4. Let $h : \Omega \to \mathbb{R}$ be a measurable function such that $h\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$. We say that $u : \Omega \to \mathbb{R}$ is a weak subsolution (respectively a weak supersolution) of (1.4) if $u \leq 0$ on $\partial\Omega$ and $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \leq \int_\Omega h\varphi$ (resp. $u \geq 0$ on $\partial\Omega$ and $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \geq \int_\Omega h\varphi$) for any nonnegative $\varphi \in H_0^1(\Omega)$.

Lemma 5.5 ([33, Lemma 4])**.** *Let $\lambda > 0$, and suppose that u and v are two nonnegative weak supersolutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem* (1.1)*. Then there exists a weak solution $z \in H_0^1(\Omega) \cap C(\overline{\Omega})$ of problem* (1.1) *such that $z \leq \min\{u, v\}$ in Ω.*

Lemma 5.6 ([31, Lemma 2.9])**.** *For any nonnegative ζ in $L^\infty(\Omega)$, and for any positive weak solution u of the problem*

$$\begin{gathered} -\Delta u = au^{-\alpha} + \zeta \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

the following statements hold:

(i) *If $1 < \alpha < 3$ then there exists a positive constant c such that $u \leq c d_\Omega^{\frac{2}{1+\alpha}}$ in Ω.*
(ii) *If $0 < \alpha \leq 1$ and $\gamma \in (0, 1)$ then there exists a positive constant c such that $u \leq c d_\Omega^\gamma$ in Ω.*

The next lemma deals with the existence of maximal solutions to problem (1.1).

Lemma 5.7. *Let $\lambda \in [0, \Lambda]$. Then there exists a weak solution $v_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem* (1.1) *such that, if $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of problem* (1.1) *satisfying $w \geq v_\lambda$ in Ω, then $w = v_\lambda$ in Ω.*

Proof. When $\lambda = 0$, the lemma holds because, by Remark 1.1, problem (1.1) has a unique weak solution $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Let us consider the case when $\lambda \in (0, \Lambda]$. Let $\mathcal{S}_\lambda$ be the set of weak solutions $\zeta \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to problem (1.1). Thus $\mathcal{S}_\lambda$ is nonempty and, by Lemma 5.1, there exists a constant $C > 0$ such that $u \leq C$ for any $u \in \mathcal{S}_\lambda$. Then $\mathcal{I}_\lambda := \{\int_\Omega u : u \in \mathcal{S}_\lambda\}$ is bounded. Let $\beta := \sup \mathcal{I}_\lambda$. Thus $0 < \beta < \infty$. Let $\{u_j\}_{j \in \mathbb{N}} \subset \mathcal{S}_\lambda$ be a maximizing sequence for $\mathcal{I}_\lambda$, i.e., such that $\lim_{j \to \infty} \int_\Omega u_j = \beta$. Now, by Lemma 5.2, $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$, and so, taking a subsequence if necessary, we can assume that there exists $v_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $\{u_j\}_{j \in \mathbb{N}}$ converges strongly in $L^2(\Omega)$ to v_λ and $\{\nabla u_j\}_{j \in \mathbb{N}}$ converges weakly in $L^2(\Omega, \mathbb{R}^n)$ to ∇v_λ. Since $u_j \leq C$ for all $j \in \mathbb{N}$, $v_\lambda \in L^\infty(\Omega)$; and, since $\{u_j\}_{j \in \mathbb{N}}$ converges to v_λ in $L^1(\Omega)$, $\int_\Omega v_\lambda = \beta$. By Lemma 5.2, v_λ is a weak solution of (1.1). Suppose now that $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.1) such that $w \geq v_\lambda$. Then $\beta = \int_\Omega v_\lambda \leq \int_\Omega w$ and, from the definition of β, $\int_\Omega w \leq \beta$. Thus $\int_\Omega v_\lambda = \int_\Omega w$; which (since $v_\lambda \leq w$) implies $v_\lambda = w$. □

Remark 5.8. By Remark 1.1, $v_\lambda \in C(\overline{\Omega})$, and there exists a constant $c_1 > 0$ such that $v_\lambda \geq c_1 d_\Omega$ in Ω.

Remark 5.9. Note that, by Lemma 3.5, u_λ and v_λ belong to D_α for any $\lambda \in [0, \Lambda]$.

Lemma 5.10. *For $\lambda \in (0, \Lambda]$, let u_λ and v_λ be as given by Remark 1.1 v), and Lemma 5.7 respectively. Then, for any $r, t \in [0, 1]$ such that $r + t = 1$, $ru_\lambda + tv_\lambda$ is a weak supersolution of* (1.1).

Proof. As u_λ and v_λ are weak solutions of (1.1), then, for any $\varphi \in H_0^1(\Omega)$, both $(au_\lambda^{-\alpha} + f(\lambda,x,u_\lambda))\varphi$ and $(av_\lambda^{-\alpha} + f(\lambda,x,v_\lambda))\varphi$, belong to $L^1(\Omega)$, and so

$$r(au_\lambda^{-\alpha} + f(\lambda,x,u_\lambda))\varphi + t(av_\lambda^{-\alpha} + f(\lambda,x,v_\lambda))\varphi \in L^1(\Omega).$$

Note that, by (H1), H2) and (H9), $s \to as^{-\alpha} + f(\lambda,x,s)$ is strictly convex for any $\lambda \in (0,\infty)$ and for a.e. $x \in \Omega$. Then

$$\begin{gathered} -\Delta(ru_\lambda + tv_\lambda) = r(au_\lambda^{-\alpha} + f(\lambda,x,u_\lambda)) + t(av_\lambda^{-\alpha} + f(\lambda,x,v_\lambda)) \\ \geq a(ru_\lambda + tv_\lambda)^{-\alpha} + f(\lambda,x,ru_\lambda + tv_\lambda) \quad \text{in } \Omega, \\ ru_\lambda + tv_\lambda = 0 \quad \text{on } \partial\Omega, \\ ru_\lambda + tv_\lambda > 0 \quad \text{in } \Omega. \end{gathered}$$

□

From now on, for any $\lambda \in [0,\Lambda]$, u_λ will denote the minimal solution of problem (1.1) given by Remark 1.1 (v); and v_λ will denote a maximal solution, for the same problem, given by Lemma 5.7. We also set

$$\psi_\lambda := v_\lambda - u_\lambda. \tag{5.1}$$

Note that, by the multiplicity result given in Remark 1.1 (iii), $v_\lambda \neq u_\lambda$ for any $\lambda \in (0,\Lambda)$. Also, note that, if for $\lambda = \Lambda$ there exist at least two weak solutions of problem (1.1), then $v_\Lambda \neq u_\Lambda$.

Lemma 5.11. *Let $\lambda \in [0,\Lambda]$, and let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a weak solution of problem* (1.1)*. Then there exist $\gamma \in (0,1)$, and a positive constant c, such that $u \leq cd_\Omega^\gamma$ in Ω.*

Proof. The lemma follows from Lemma 5.6, taking $\varepsilon = 0$ and $\zeta = f(\lambda,\cdot,u)$. □

Lemma 5.12. *(i) For any $\lambda \in (0,\Lambda)$, we have $\psi_\lambda \geq 0$ in Ω, $|\{\psi_\lambda > 0\}| > 0$, and*

$$-\Delta(u_\lambda + t\psi_\lambda) \geq a(u_\lambda + t\psi_\lambda)^{-\alpha} + f(\lambda,\cdot,u_\lambda + t\psi_\lambda) \quad \text{in } \Omega \tag{5.2}$$

for any $t \in [0,1]$.

(ii) If for $\lambda = \Lambda$ there exist at least two positive solutions of (1.1)*, then the assertions made in (i) hold also for $\lambda = \Lambda$.*

Proof. To prove (i), note that for any $\varphi \in H_0^1(\Omega)$, $|(u_\lambda + t\psi_\lambda)^{-\alpha}\varphi| \leq au_\lambda^{-\alpha}|\varphi| \in L^1(\Omega)$, and also note that, by (H3),

$$|f(\lambda,\cdot,u_\lambda + t\psi_\lambda)\varphi| \leq \|f(\lambda,\cdot,.)\|_{L^\infty(\Omega\times[0,M])}|\varphi| \in L^1(\Omega),$$

with $M := \|u_\lambda\|_\infty + \|\psi_\lambda\|_\infty$. Thus $\big(a(u_\lambda + t\psi_\lambda)^{-\alpha} + f(\lambda,\cdot,u_\lambda + t\psi_\lambda)\big)\varphi \in L^1(\Omega)$. By Remark 1.1 (v), $u_\lambda \leq v_\lambda$; and then $\psi_\lambda \geq 0$ in Ω. If ψ_λ is identically zero in Ω, then $u_\lambda = v_\lambda$ in Ω. If $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.1), then $u_\lambda \leq w$ in Ω, i.e., $v_\lambda \leq w$ in Ω, and thus $w = v_\lambda$ in Ω, which, for $\lambda \in (0,\Lambda)$, contradicts Remark 1.1 (iii). Also, for $t \in [0,1]$, Lemma 5.10 gives (5.2). Thus (i) holds. The same argument gives (ii). □

Remark 5.13. Let $M > 0$, $\lambda \geq 0$. From (H9) we have, for a.a., $x \in \Omega$, $0 \leq \frac{\partial f}{\partial s}(\lambda,x,\cdot) \leq \frac{\partial f}{\partial s}(\lambda,x,M)$ on $[0,M]$; and from (H8), $\|\frac{\partial f}{\partial s}(\lambda,\cdot,M)\|_\infty < \infty$. Therefore we have $\|\frac{\partial f}{\partial s}(\lambda,\cdot,\cdot)\|_{L^\infty(\Omega\times(0,M))} < \infty$.

Lemma 5.14. *For $\lambda \in [0, \Lambda]$, the following statements hold: (i)*

$$(-\alpha a u_\lambda^{-\alpha-1} + \frac{\partial f}{\partial s}(\lambda, \cdot, u_\lambda))\psi_\lambda \varphi \in L^1(\Omega)$$

for any $\varphi \in H_0^1(\Omega)$.

(ii) $-\Delta\psi_\lambda \geq \big(-\alpha a u_\lambda^{-\alpha-1} + \frac{\partial f}{\partial s}(\lambda, x, u_\lambda)\big)\psi_\lambda$ in Ω.

Proof. (i) follows from Lemma 4.1. It is clear that (ii) holds when $\lambda = 0$. If $\lambda \in (0, \Lambda]$, from Lemma 5.12, we have, for any $t \in (0,1)$ and $\varepsilon > 0$,

$$\begin{aligned} -\Delta(u_\lambda + t\psi_\lambda) &\geq a(u_\lambda + t\psi_\lambda)^{-\alpha} + f(\lambda, \cdot, u_\lambda + t\psi_\lambda) \\ &\geq a(u_\lambda + \varepsilon + t\psi_\lambda)^{-\alpha} + f(\lambda, \cdot, u_\lambda + t\psi_\lambda) \text{ in } \Omega. \end{aligned}$$

Also,

$$\begin{gathered} -\Delta u_\lambda = a u_\lambda^{-\alpha} + f(\lambda, \cdot, u_\lambda) \quad \text{in } \Omega, \\ u_\lambda = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

and so

$$-\Delta(t\psi_\lambda) \geq a\big((u_\lambda + \varepsilon + t\psi_\lambda)^{-\alpha} - u_\lambda^{-\alpha}\big) + f(\lambda, \cdot, u_\lambda + t\psi_\lambda) - f(\lambda, \cdot, u_\lambda) \quad \text{in } \Omega$$

Taking $\varepsilon = t\eta$ with $\eta > 0$, and dividing by t, we obtain

$$-\Delta\psi_\lambda \geq a t^{-1}\big((u_\lambda + t(\eta + \psi_\lambda))^{-\alpha} - u_\lambda^{-\alpha}\big) + t^{-1}(f(\lambda, \cdot, u_\lambda + t\psi_\lambda) - f(\lambda, \cdot, u_\lambda)),$$

i.e., for any nonnegative $\varphi \in H_0^1(\Omega)$, it holds that

$$\begin{aligned} \int_\Omega \langle \nabla\psi_\lambda, \nabla\varphi \rangle \geq & \int_\Omega a t^{-1}((u_\lambda + t(\eta + \psi_\lambda))^{-\alpha} - u_\lambda^{-\alpha})\varphi \\ & + \int_\Omega t^{-1}(f(\lambda, \cdot, u_\lambda + t\psi_\lambda) - f(\lambda, \cdot, u_\lambda))\varphi. \end{aligned} \tag{5.3}$$

Now, $\lim_{t\to 0^+} a t^{-1}((u_\lambda + t(\eta + \psi_\lambda))^{-\alpha} - u_\lambda^{-\alpha})\varphi = -\alpha a u_\lambda^{-\alpha-1}\varphi(\eta + \psi_\lambda)$ a.e. in Ω. Also, from the mean value theorem,

$$|a t^{-1}\big((u_\lambda + t(\eta + \psi_\lambda))^{-\alpha} - u_\lambda^{-\alpha}\big)\varphi| = |\alpha a (u_\lambda + \theta(\eta + \psi_\lambda))^{-\alpha-1}\varphi(\eta + \psi_\lambda)|$$

for some θ such that $0 \leq \theta \leq t$; and so, taking into account Lemma 4.1,

$$|a t^{-1}((u_\lambda + t(\eta + \psi_\lambda))^{-\alpha} - u_\lambda^{-\alpha})\varphi| \leq \alpha a u_\lambda^{-\alpha-1}\varphi(\eta + \psi_\lambda) \in L^1(\Omega).$$

Then the Lebesgue dominated convergence theorem gives

$$\lim_{t\to 0^+} \int_\Omega a t^{-1}\big((u_\lambda + t(\eta + \psi_\lambda))^{-\alpha} - u_\lambda^{-\alpha}\big)\varphi = -\int_\Omega \alpha a u_\lambda^{-\alpha-1}\varphi(\eta + \psi_\lambda).$$

On the other hand, $\lim_{t\to 0^+} t^{-1}(f(\lambda, \cdot, u_\lambda + t\psi_\lambda) - f(\lambda, \cdot, u_\lambda))\varphi = \frac{\partial f}{\partial s}(\lambda, \cdot, u_\lambda)\varphi\psi_\lambda$ a.e. in Ω. Let $M := \|u_\lambda + \psi_\lambda\|_\infty$. Then, by Remark 5.13, $\|\frac{\partial f}{\partial s}(\lambda, \cdot, .)\|_{L^\infty(\Omega\times(0,M))} < \infty$. Thus, the mean value theorem gives, for some $\widetilde{\theta}$ such that $0 \leq \widetilde{\theta} \leq t$,

$$\begin{aligned} |t^{-1}(f(\lambda, \cdot, u_\lambda + t\psi_\lambda) - f(\lambda, \cdot, u_\lambda))\varphi| &= \big|\frac{\partial f}{\partial s}(\lambda, \cdot, u_\lambda + \widetilde{\theta}\psi_\lambda)\varphi\psi_\lambda\big| \\ &\leq \|\frac{\partial f}{\partial s}(\lambda, \cdot, \cdot)\|_{L^\infty(\Omega\times(0,M))}|\varphi\psi_\lambda| \in L^1(\Omega), \end{aligned}$$

where in the last inequality we have used Remark 5.13. The Lebesgue dominated convergence theorem now gives

$$\lim_{t\to 0^+} \int_\Omega t^{-1}(f(\lambda, \cdot, u_\lambda + t\psi_\lambda) - f(\lambda, \cdot, u_\lambda))\varphi = \int_\Omega \frac{\partial f}{\partial s}(\lambda, \cdot, u_\lambda)\psi_\lambda\varphi.$$

Now we take the limit as $t \to 0^+$ in (5.3), to obtain

$$-\Delta\psi_\lambda \geq -\alpha a u_\lambda^{-\alpha-1}(\psi_\lambda+\eta) + \frac{\partial f}{\partial s}(\lambda,\cdot,u_\lambda)\psi_\lambda \text{ in } \Omega. \tag{5.4}$$

Since (5.4) holds for any $\eta > 0$, the lemma follows. $\square$

For $\lambda \in [0,\Lambda]$, let $m_\lambda := \frac{\partial f}{\partial s}(\lambda,\cdot,u_\lambda)$. Since $u_\lambda \in C(\overline{\Omega})$, Remark 5.13 gives $m_\lambda \in L^\infty(\Omega)$; and, by (H9), $m_\lambda > 0$ in Ω. Observe also that $u_\lambda \in D_\alpha$. Thus we can define $\mu_\lambda := \mu_{m_\lambda,u_\lambda}$.

Definition 5.15. For $\lambda \in [0,\Lambda]$, let $m_\lambda := \frac{\partial f}{\partial s}(\lambda,\cdot,u_\lambda)$, and let $\mu_\lambda := \mu_{m_\lambda,u_\lambda}$.

Remark 5.16. Since $u_\lambda \in C(\overline{\Omega})$, Remark 5.13 gives $m_\lambda \in L^\infty(\Omega)$; and, by (H9), $m_\lambda > 0$ a.e. in Ω. Observe also that $u_\lambda \in D_\alpha$. Thus μ_λ is well defined.

Lemma 5.17. *(i) $\mu_\lambda > 1$ for any$\lambda \in (0,\Lambda)$.*

(ii) If, for $\lambda = \Lambda$, there exist at least two weak solutions of problem (1.1)*, then $\mu_\Lambda > 1$.*

Proof. Suppose that either $\lambda = \Lambda$ and there exist at least two weak solutions of problem (1.1), or $\lambda \in (0,\Lambda)$. Let $\psi_\lambda \in H_0^1(\Omega)$ be as defined in (5.1). By Lemma 5.12, $\psi_\lambda \geq 0$ in Ω and $|\{x \in \Omega : \psi_\lambda(x) > 0\}| > 0$. Also, by Lemma 5.14, $\alpha a u_\lambda^{-\alpha-1}\psi_\lambda\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, and

$$\begin{gathered} -\Delta\psi_\lambda + \alpha a u_\lambda^{-\alpha-1}\psi_\lambda \geq m_\lambda\psi_\lambda \quad \text{in } \Omega, \\ \psi_\lambda = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{5.5}$$

Let $z \in H_0^1(\Omega)$ be the weak solution, given by Lemma 4.5, to the problem

$$\begin{gathered} -\Delta z + \alpha a u_\lambda^{-\alpha-1} z = m_\lambda\psi_\lambda \quad \text{in } \Omega, \\ z = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{5.6}$$

By (H9), $m_\lambda\psi_\lambda \geq 0$ in Ω, and then Lemma 4.5 gives $z \geq 0$ in Ω. Now,

$$\begin{gathered} -\Delta(\psi_\lambda - z) + \alpha a u_\lambda^{-\alpha-1}(\psi_\lambda - z) = h_\lambda \quad \text{in } \Omega, \\ \psi_\lambda - z = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{5.7}$$

where $h_\lambda := \alpha a u_\lambda^{-\alpha-1}\psi_\lambda + a v_\lambda^{-\alpha} + f(\lambda,\cdot,v_\lambda) - (a u_\lambda^{-\alpha} + f(\lambda,\cdot,u_\lambda)) - m_\lambda z$. By Remark 5.9, u_λ and v_λ belong to D_α and then by Lemma 3.3, $a v_\lambda^{-\alpha} + f(\lambda,\cdot,v_\lambda)$ and $a u_\lambda^{-\alpha} + f(\lambda,\cdot,u_\lambda)$ belong to $(H_0^1(\Omega))'$. Since $\psi_\lambda \in L^\infty(\Omega)$, Lemma 4.1 gives $\alpha a u_\lambda^{-\alpha-1}\psi_\lambda \in (H_0^1(\Omega))'$. Also, $z \in H_0^1(\Omega)$ and $m_\lambda \in L^\infty(\Omega)$. Then $h_\lambda \in (H_0^1(\Omega))'$. From (5.5) and (5.6) we conclude that $h_\lambda \geq 0$ a.e. in Ω. Then, taking into account (5.7), Lemma 4.5 gives $\psi_\lambda \geq z$ in Ω. Also,

$$\begin{gathered} -\Delta z + \alpha a u_\lambda^{-\alpha-1} z = m_\lambda z + \rho \quad \text{in } \Omega, \\ z = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{5.8}$$

with $\rho := m_\lambda(\psi_\lambda - z)$. Since $\psi_\lambda \geq z$ in Ω, and, by (H9), $m_\lambda > 0$ a.e. in Ω, we have $\rho \geq 0$ a.e. in Ω. We claim that $\rho \not\equiv 0$ in Ω. To see this, by way of contradiction, let us suppose $\rho = 0$. From (5.8) we have

$$\begin{gathered} -\Delta z + \alpha a u_\lambda^{-\alpha-1} z = m_\lambda z \quad \text{in } \Omega, \\ z = 0 \quad \text{on } \partial\Omega; \end{gathered} \tag{5.9}$$

which, jointly with (5.6), gives $z = \psi_\lambda$ in Ω. Then (5.9) reads

$$-\Delta\psi_\lambda = -\alpha a u_\lambda^{-\alpha-1}\psi_\lambda + m_\lambda\psi_\lambda \quad \text{in } \Omega,$$

$$\psi_\lambda = 0 \quad \text{on } \partial\Omega;$$

that is,

$$\begin{gathered}-\Delta(v_\lambda - u_\lambda) = -\alpha a u_\lambda^{-\alpha-1}(v_\lambda - u_\lambda) + m_\lambda(v_\lambda - u_\lambda) \quad \text{in } \Omega,\\ v_\lambda - u_\lambda = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Now, taking into account the equations satisfied by u_λ and v_λ, we obtain

$$\begin{aligned}&a v_\lambda^{-\alpha} + f(\lambda,\cdot,v_\lambda) - [a u_\lambda^{-\alpha} + f(\lambda,\cdot,u_\lambda)]\\ &= -\alpha a u_\lambda^{-\alpha-1}(v_\lambda - u_\lambda) + m_\lambda(v_\lambda - u_\lambda)\\ &= -\alpha a u_\lambda^{-\alpha-1}(v_\lambda - u_\lambda) + \frac{\partial f}{\partial s}(\lambda,\cdot,u_\lambda)(v_\lambda - u_\lambda) \quad \text{a.e. in } \Omega,\end{aligned}$$

which contradicts the fact that, by (H9), $s \to a s^{-\alpha} + f(\lambda,x,s)$ is strictly convex on $(0,\infty)$ for a.e. $x \in \Omega$; which ends the proof that $\rho \not\equiv 0$ in Ω. Thus $0 \le m_\lambda z + \rho \not\equiv 0$ in Ω, then, since $m_\lambda z + \rho \in (H_0^1(\Omega))'$ from (5.8) and Lemma 4.5 (iv), we obtain that, for δ positive and small enough, there exists a positive constant c such that $z \ge c d_{\Omega_\delta}$ in Ω_δ, and so, from (5.8) and Lemma 4.6, $\mu_\lambda > 1$. □

Lemma 5.18. *Assume that either of the following conditions holds: (i)* $0 < \lambda < \Lambda$. *(ii)* $\lambda = \Lambda$ *and, for* $\lambda = \Lambda$*, there exist at least two weak solutions of problem* (1.1). *Then* $(D_u T)(\lambda, u_\lambda) : X_\alpha \to X_\alpha$ *is bijective.*

Proof. Note that, if $h \in X_\alpha$, then $(-\alpha a u_\lambda^{-\alpha-1} + m_\lambda)h \in (H_0^1(\Omega))'$. Indeed, for some positive constants c and c', we have, for for any $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned}\int_\Omega |(-\alpha a u_\lambda^{-\alpha-1} + m_\lambda)h\varphi| &\le c\int_\Omega \vartheta_\alpha^{-(\alpha+1)}|h\varphi| + c\int_\Omega |h\varphi|\\ &= c\int_\Omega \vartheta_\alpha^{-\alpha} d_\Omega \Big|\frac{h}{\vartheta_\alpha}\Big|\Big|\frac{\varphi}{d_\Omega}\Big| + c\int_\Omega \vartheta_\alpha d_\Omega \Big|\frac{h}{\vartheta_\alpha}\Big|\Big|\frac{\varphi}{d_\Omega}\Big|\\ &\le c'\|h\|_{X_\alpha}\|\varphi\|_{H_0^1(\Omega)},\end{aligned}$$

where we have used the Holder and the Hardy inequalities, that $u_\lambda \in D_\alpha$, $\vartheta_\alpha^{-\alpha} d_\Omega \in L^2(\Omega)$, and $\vartheta_\alpha d_\Omega \in L^\infty(\Omega)$. Similarly, we have

$$\begin{aligned}|(-\alpha a u_\lambda^{-\alpha-1} + m_\lambda)h| &\le c\vartheta_\alpha^{-(\alpha+1)}|h| + c|h|\\ &= c\vartheta_\alpha^{-\alpha}\Big|\frac{h}{\vartheta_\alpha}\Big| + c\vartheta_\alpha\Big|\frac{h}{\vartheta_\alpha}\Big|\\ &\le c'\vartheta_\alpha^{-\alpha}\|h\|_{X_\alpha}.\end{aligned}$$

Then, by Lemma 4.4, the problem

$$-\Delta z = (-\alpha a u_\lambda^{-\alpha-1} + m_\lambda)z + (-\alpha a u_\lambda^{-\alpha-1} + m_\lambda)h$$

has a solution z in X_α. Then $\psi := z + h$ belongs to X_α and satisfies

$$\psi - (-\Delta)^{-1}((-\alpha a u_\lambda^{-\alpha-1} + m_\lambda)\psi) = h.$$

Thus $(D_u T)(\lambda, u_\lambda)$ is surjective.

To see that it is injective, suppose that

$$\varphi - (-\Delta)^1([-\alpha a u_\lambda^{-\alpha-1} + m_\lambda]\varphi) = \eta - (-\Delta)^1([-\alpha a u_\lambda^{-\alpha-1} + m_\lambda]\eta)$$

for some φ and η in X_α. Thus $z := \eta - \varphi$ is a weak solution of the problem

$$\begin{gathered}-\Delta z = (-\alpha a u_\lambda^{-\alpha-1} + m_\lambda)z \quad \text{in } \Omega,\\ z = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{5.10}$$

By way of contradiction, suppose that z^+ is not identically zero on Ω. Taking z^+ as a test function in (5.10) we obtain

$$\int_\Omega (|\nabla z^+|^2 + \alpha a u_\lambda^{-\alpha-1}(z^+)^2) = \int_\Omega m_\lambda (z^+)^2,$$

and so $\mu_\lambda \leq 1$, which contradicts Lemma 5.17. Thus $z \leq 0$ in Ω; i.e., $\eta \leq \varphi$ in Ω. Interchanging the roles of η and φ, we obtain the reverse inequality. Thus $\eta = \varphi$ in Ω. □

Lemma 5.19. *For any $\lambda \in (0, \Lambda)$, there exist $\varepsilon > 0$, and a neighborhood V of u_λ in X_α, such that, for any $\sigma \in (\lambda - \varepsilon, \lambda + \varepsilon)$, there is a unique $U_\lambda(\sigma) \in V$ such that $T(\sigma, U_\lambda(\sigma)) = 0$, with $U_\lambda(\lambda) = u_\lambda$. Moreover, U_λ is a C^1 mapping from $(\lambda - \varepsilon, \lambda + \varepsilon)$ into X_α.*

Proof. The lemma follows from lemmas 3.7 and 5.18, and from the implicit function theorem, as stated in [21, Appendix B, Theorem B.1]. □

6. Proofs of main results

Lemma 6.1. *(i) $\lim_{\sigma\to\lambda} u_\sigma(x) = u_\lambda(x)$ for any $\lambda \in (0, \Lambda)$ and $x \in \Omega$.*
(ii) $\lim_{\sigma\to\Lambda^-} u_\sigma(x) = u_\Lambda(x)$ for any $x \in \Omega$.
(iii) $\lim_{\sigma\to 0^+} u_\sigma(x) = u_0(x)$ for any $x \in \Omega$.

Proof. Let $\lambda \in (0, \Lambda)$. As stated in Remark 1.1 (v), the map $\sigma \to u_\sigma$ is strictly increasing on $[0, \Lambda]$, and $u_\sigma \in H_0^1(\Omega) \cap C(\overline{\Omega})$ for any $\sigma \in [0, \Lambda]$. Then, for any $\lambda \in (0, \Lambda)$ and $x \in \Omega$, there exist the lateral limits $\underline{w}_\lambda(x) := \lim_{\sigma\to\lambda^-} u_\sigma(x)$, and $\overline{w}_\lambda(x) := \lim_{\sigma\to\lambda^+} u_\sigma(x)$. Moreover, $\underline{w}_\lambda$ and $\overline{w}_\lambda$ belong to $L^\infty(\Omega)$ and satisfy $\underline{w}_\lambda \leq u_\lambda \leq \overline{w}_\lambda$ in Ω.

Step 1: To see that $\underline{w}_\lambda = u_\lambda$ in Ω, consider an increasing sequence $\{\underline{\sigma}_j\}_{j\in\mathbb{N}} \subset [\frac{\lambda}{2}, \lambda)$ such that $\lim_{j\to\infty} \underline{\sigma}_j = \lambda$. By lemma 5.2, there exists a subsequence $\{\underline{\sigma}_{j_k}\}_{k\in\mathbb{N}}$, and a weak solution $\underline{w}_\lambda^* \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (1.1), such that $\{\nabla u_{\underline{\sigma}_{j_k}}\}_{k\in\mathbb{N}}$ converges weakly in $L^2(\Omega, \mathbb{R}^n)$ to $\nabla \underline{w}_\lambda^*$ and $\{u_{\underline{\sigma}_{j_k}}\}_{k\in\mathbb{N}}$ converges to $\underline{w}_\lambda^*$ in $L^2(\Omega)$. Taking a subsequence if necessary, we can assume that $\{u_{\underline{\sigma}_{j_k}}\}_{k\in\mathbb{N}}$ converges to $\underline{w}_\lambda^*$ a.e. in Ω. Thus $\underline{w}_\lambda^* = \underline{w}_\lambda$ in Ω, and so $\underline{w}_\lambda$ is a weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ to problem (1.1). Then, by the minimality property of u_λ stated in Remark 1.1 (v), we have $u_\lambda \leq \underline{w}_\lambda$ in Ω and so, since also $\underline{w}_\lambda \leq u_\lambda$ in Ω, we conclude that $u_\lambda = \underline{w}_\lambda$ in Ω

Step 2: To see that $u_\lambda = \overline{w}_\lambda$, consider a decreasing sequence $\{\overline{\sigma}_j\}_{j\in\mathbb{N}} \subset (\lambda, \Lambda]$ such that $\lim_{j\to\infty} \overline{\sigma}_j = \lambda$. By lemma 5.2 there exists a subsequence $\{\overline{\sigma}_{j_k}\}_{k\in\mathbb{N}}$ and a weak solution $\overline{w}_\lambda^* \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.1) such that $\{\nabla u_{\overline{\sigma}_{j_k}}\}_{k\in\mathbb{N}}$ converges to $\nabla \overline{w}_\lambda^*$ weakly in $L^2(\Omega, \mathbb{R}^n)$ and $\{u_{\overline{\sigma}_{j_k}}\}_{k\in\mathbb{N}}$ converges to $\overline{w}_\lambda^*$ in $L^2(\Omega)$. Taking a subsequence if necessary, we can assume also that $\{u_{\overline{\sigma}_{j_k}}\}_{k\in\mathbb{N}}$ converges to $\overline{w}_\lambda^*$ a.e. in Ω. Thus $\overline{w}_\lambda^* = \overline{w}_\lambda$ and so $\overline{w}_\lambda$ is a weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (1.1). Clearly $u_\lambda \leq \overline{w}_\lambda$. We claim that $u_\lambda = \overline{w}_\lambda$ in Ω. By way of contradiction, suppose $u_\lambda(x) < \overline{w}_\lambda(x)$ for some $x \in \Omega$. As both are continuous functions, there exist $\eta > 0$ and an nonempty open set $E \subset \Omega$ such that $u_\lambda + \eta \leq \overline{w}_\lambda$ in E. Let ε and U_λ be as given by Lemma 5.19. By Using Lemma 5.19 we obtain $\varepsilon' \in (0, \varepsilon)$ such that $U_\lambda(\lambda + \varepsilon') - u_\lambda \leq \frac{\eta}{2}$ in Ω. Then $u_{\lambda+\varepsilon'} \geq \overline{w}_\lambda \geq u_\lambda + \eta > u_\lambda + \frac{\eta}{2} \geq U_\lambda(\lambda + \varepsilon')$ in E, which is impossible by the minimality property of $u_{\lambda+\varepsilon'}$ given by Remark 1.1 (v). Then $\overline{w}_\lambda^* = \overline{w}_\lambda$ in Ω, which ends the proof of (i).

To prove (ii) proceed exactly as in step 1, taking there $\lambda = \Lambda$. To prove (iii), proceed as in the first part of step 2, to obtain that $\overline{w}_0 := \lim_{\sigma\to 0^+} u_\sigma$ is a weak solution in $H_0^1(\Omega)\cap L^\infty(\Omega)$ of problem (1.1) for $\lambda = 0$. The uniqueness assertion of Remark 1.1 (i) implies $\overline{w}_0 = u_0$. $\square$

Lemma 6.2. *For any $\lambda\in(0,\Lambda)$, $\lim_{\sigma\to\lambda} u_\sigma = u_\lambda$ with convergence in X_α. Also, $\lim_{\sigma\to 0^+} u_\sigma = u_0$ and $\lim_{\sigma\to\Lambda^-} u_\sigma = u_\Lambda$, in both cases with convergence in X_α.*

Proof. Fix $p > \max\{n, 2\}$. For $0 \le \lambda < \sigma < \Lambda$, by Remark 1.1, $0 \le u_\lambda \le u_\sigma \le u_\Lambda$ in Ω, and so, in the weak sense,

$$
\begin{aligned}
-\Delta(u_\sigma - u_\lambda) &= a(u_\sigma^{-\alpha} - u_\lambda^{-\alpha}) + f(\sigma,\cdot,u_\sigma) - f(\lambda,\cdot,u_\lambda) &(6.1)\\
&\le F_{\sigma,\lambda} := f(\sigma,\cdot,u_\sigma) - f(\lambda,\cdot,u_\lambda) \text{ in } \Omega. &(6.2)
\end{aligned}
$$

We claim that

$$
\lim_{\sigma\to\lambda^+} \|F_{\sigma,\lambda}\|_p = 0. \tag{6.3}
$$

Indeed, by (H3), $f(\lambda,\cdot,u_\lambda)\ge 0$ in Ω, and, so, taking into account (H3) and (H6), $0 \le F_{\sigma,\lambda} \le f(\sigma,\cdot,u_\sigma) \le f(\Lambda,\cdot,\|u_\Lambda\|_\infty) \in L^\infty(\Omega)$. By Lemma 6.1 and (H3), $\lim_{\sigma\to\lambda^+} F_{\sigma,\lambda}^p = 0$ a.e. in Ω. Then (6.3) follows from the Lebesgue dominated convergence theorem.

Since, for some positive constant c', $\vartheta_\alpha \ge c' d_\Omega$ in Ω, and $a(u_\sigma^{-\alpha} - u_\lambda^{-\alpha}) + f(\sigma,\cdot,u_\sigma) - f(\lambda,\cdot,u_\lambda) \ge 0$ in Ω; from (6.1) and the standard elliptic estimates, for some positive constant c'', independent of λ and σ, we obtain

$$
\begin{aligned}
\|\vartheta_\alpha^{-1}(u_\sigma - u_\lambda)\|_\infty &\le (c')^{-1}\|d_\Omega^{-1}(u_\sigma - u_\lambda)\|_\infty \le (c')^{-1}\|u_\sigma - u_\lambda\|_{C^1(\overline{\Omega})}\\
&\le c''\|a(u_\sigma^{-\alpha} - u_\lambda^{-\alpha}) + f(\sigma,\cdot,u_\sigma) - f(\lambda,\cdot,u_\lambda)\|_p\\
&\le c''\|F_{\sigma,\lambda}\|_p.
\end{aligned}
$$

Thus

$$
\lim_{\sigma\to\lambda^+} \|\vartheta_\alpha^{-1}(u_\sigma - u_\lambda)\|_\infty = 0. \tag{6.4}
$$

Similarly, for $0 < \sigma < \lambda \le \Lambda$, we have

$$
-\Delta(u_\lambda - u_\sigma) = a(u_\lambda^{-\alpha} - u_\sigma^{-\alpha}) + f(\lambda,\cdot,u_\sigma) - f(\sigma,\cdot,u_\sigma) \le -F_{\sigma,\lambda} \quad \text{in } \Omega,
$$

with $0 \le -F_{\sigma,\lambda} \le f(\lambda,\cdot,u_\lambda)$; and the same arguments used to prove (6.4) apply to obtain that $\lim_{\sigma\to\lambda^-}\|F_{\sigma,\lambda}\| = 0$, and that

$$
\lim_{\sigma\to\lambda^-} \|\vartheta_\alpha^{-1}(u_\lambda - u_\sigma)\|_\infty = 0. \tag{6.5}
$$

At this point, to prove the lemma, it only remains to prove the following three facts:

$$
\lim_{\sigma\to\lambda} \|u_\sigma - u_\lambda\|_{H_0^1(\Omega)} = 0, \quad \lim_{\sigma\to 0^+} \|u_\sigma - u_0\|_{H_0^1(\Omega)} = 0, \quad \lim_{\sigma\to\Lambda^-} \|u_\sigma - u_\Lambda\|_{H_0^1(\Omega)} = 0.
$$

Since, for σ and λ in $[0,\Lambda]$, $u_\sigma - u_\lambda$ satisfies, in weak sense,

$$
-\Delta(u_\sigma - u_\lambda) = a(u_\sigma^{-\alpha} - u_\lambda^{-\alpha}) + f(\sigma,\cdot,u_\sigma) - f(\lambda,\cdot,u_\lambda) \text{ in } \Omega,
$$

the Hölder and the Poincaré inequalities give

$$
\begin{aligned}
\|u_\sigma - u_\lambda\|_{H_0^1(\Omega)}^2 = \int_\Omega |\nabla(u_\sigma - u_\lambda)|^2 &\le \int_\Omega (u_\sigma - u_\lambda)F_{\sigma,\lambda}\\
&\le c\|u_\sigma - u_\lambda\|_{H_0^1(\Omega)}\|F_{\sigma,\lambda}\|_2
\end{aligned}
$$

where c is a constant independent of both, σ and λ. Since

$$\lim_{\sigma\to\lambda}\|F_{\sigma,\lambda}\|_2=0 \text{ if } \lambda\in(0,\Lambda),\ \lim_{\sigma\to 0^+}\|F_{\sigma,0}\|_2=0, \text{ and } \lim_{\sigma\to\Lambda^-}\|F_{\sigma,\Lambda}\|_2=0,$$

the lemma follows. $\square$

The next two propositions address uniformity properties of the family $\{u_\lambda\}_{\lambda\in[0,\Lambda]}$.

Proposition 6.3. *There exists $\delta>0$ such that, for each $\lambda\in[0,\Lambda]$, $u_\lambda(x)=\|u_\lambda\|_\infty$ implies $d_\Omega(x)\geq\delta$.*

Proof. By Remark 1.1 (v), for each $\lambda\in[0,\Lambda]$, $u_0\leq u_\lambda\leq u_\Lambda$; since these functions belong to X_α, there exists $c>0$, independent of λ, such that $u_\lambda\leq c\vartheta_\alpha$ in Ω. For any $x\in\Omega$ such that $u_\lambda(x)=\|u_\lambda\|_\infty$, we have $u_\lambda(x)\geq\|u_0\|_\infty$. Taking ω_0 large enough in the definition of ϑ_α, we can assume that the function $\overline{\vartheta}_\alpha$ defined by $\vartheta_\alpha=\overline{\vartheta}_\alpha\circ d_\Omega$ is strictly increasing on $(0,\frac{1}{2}diam(\Omega)]$. Now, $\overline{\vartheta}_\alpha(d_\Omega(x))=\vartheta_\alpha(x)\geq\frac{1}{c}u_\lambda(x)\geq\frac{1}{c}\|u_0\|_\infty$; which gives $d_\Omega(x)\geq\overline{\vartheta}_\alpha^{-1}(\frac{1}{c}\|u_0\|_\infty)$. $\square$

Proposition 6.4. *The family $\{u_\lambda\}_{\lambda\in[0,\Lambda]}$ is equicontinuous on $\overline{\Omega}$.*

Proof. Let $\varepsilon>0$. Since $u_\Lambda\in C(\overline{\Omega})$ and $u_\Lambda=0$ on $\partial\Omega$, there exists $\delta_1>0$ such that $u_\Lambda\leq\frac{1}{4}\varepsilon$ in $A_{\delta_1}=:\{x\in\Omega:d_\Omega(x)\leq\delta_1\}$. For $\lambda\in[0,\Lambda]$ we have $0\leq u_0\leq u_\lambda\leq u_\Lambda$ in Ω, and so $0\leq u_\lambda\leq\frac{1}{4}\varepsilon$ in A_{δ_1}. Also, since $u_0\in D_\alpha$, $au_\lambda^{-\alpha}\leq\|a\|_\infty u_0^{-\alpha}\leq c_0\vartheta_\alpha^{-\alpha}\leq c_1$ in $\overline{\Omega_{\delta_1/2}}$, with c_0 and c_1 constants independent of λ. By (H6) and (H3), $0\leq f(\lambda,\cdot,u_\lambda)\leq f(\Lambda,\cdot,\|u_\Lambda\|_\infty)\in C(\overline{\Omega})\subset L^\infty(\Omega)$. Also, $\|u_\lambda\|_{L^\infty(\Omega)}\leq\|u_\Lambda\|_{L^\infty(\Omega)}$. Let Ω' be a $C^{1,1}$ subdomain of Ω such that $\Omega_{\delta_1}\subset\Omega'\subset\overline{\Omega'}\subset\Omega_{\frac{1}{2}\delta_1}$; by the inner elliptic estimates in [27, Theorem 8.24], for any $p\in[1,\infty)$, there exists a positive constant c_2, independent of λ, such that $\|u_\lambda\|_{W^{2,p}(\Omega'')}\leq c_2$. Take $p>n$ to get that, by [27, Theorem 7.26], $\|u_\lambda\|_{C^1(\overline{\Omega'})}\leq c_3$ with c_3 independent of λ. Let $\delta:=\min\{\delta_1,\frac{1}{2}c_3^{-1}\varepsilon\}$. If x, $y\in A_{\delta_1}$ then $|u_\lambda(x)-u_\lambda(y)|\leq\frac{1}{2}\varepsilon$. If $x,y\in\overline{\Omega_{\delta_1}}$ and $|x-y|<\delta$ then $|u_\lambda(x)-u_\lambda(y)|\leq\|u_\lambda\|_{C^1(\overline{\Omega'})}|x-y|\leq c_3\delta\leq\frac{1}{2}\varepsilon$. If $x\in A_{\delta_1}$, $y\in\Omega_{\delta_1}$ and $|x-y|<\delta$, then there exists z in the linear segment with endpoints x and y, such that $d_\Omega(z)=\delta_1$. Now, x and z belong to A_{δ_1}, z and y belong to $\overline{\Omega_{\delta_1}}$, $|x-z|\leq\delta$, and $|z-y|\leq\delta$. Thus $|u_\lambda(x)-u_\lambda(y)|\leq|u_\lambda(x)-u_\lambda(z)|+|u_\lambda(z)-u_\lambda(y)|\leq\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon$. $\square$

Proof of Theorem 1.3. Since $u_\lambda\in C(\overline{\Omega})$ for any $\lambda\in[0,\Lambda]$, and since the inclusion $i:X_\alpha\to L^\infty(\Omega)$ is continuous, the assertion (i) of the theorem follows from Lemma 6.2. To see (ii) and (iii), consider an arbitrary $\lambda\in(0,\Lambda)$, and let $\varepsilon>0$, V, and U_λ be as in Lemma 5.19. By Lemma 6.2 there exists $\varepsilon'\in(0,\varepsilon)$ such that $u_\sigma\in V$ for any $\sigma\in(\lambda-\varepsilon',\lambda+\varepsilon')$. Thus, by Lemma 5.19, $u_\sigma=U_\lambda(\sigma)$ for any $\sigma\in(\lambda-\varepsilon',\lambda+\varepsilon')$. Since $U_\lambda:(\lambda-\varepsilon,\lambda+\varepsilon)\to X_\alpha$ is a C^1 map, then $\sigma\to u_\sigma$ is a C^1 map from $(\lambda-\varepsilon,\lambda+\varepsilon)$ into X_α, and this holds for any $\lambda\in(0,\Lambda)$. Then $\sigma\to u_\sigma$ is a C^1 map from $(0,\Lambda)$ into X_α. Since the inclusions $i:X_\alpha\to L^\infty(\Omega)$ and $j:X_\alpha\to H_0^1(\Omega)$ are linear and continuous, and taking into account that $u_\sigma\in C(\overline{\Omega})$ for any σ, the theorem follows. $\square$

Proof of Theorem 1.4. By way of contradiction, suppose that for $\lambda=\Lambda$ there exist at least two weak solutions of problem (1.1). Then, taking into account Lemmas 3.7 and 5.18, Lemma 5.19 gives a nonempty interval $I:=(\Lambda-\varepsilon,\Lambda+\varepsilon)$, and a differentiable function $U_\Lambda:I\to X_\alpha$, such that $(\lambda,U_\Lambda(\lambda))\in\mathbb{R}\times D_\alpha$ for any $\lambda\in I$

and $T(\lambda, U_\Lambda(\lambda)) = 0$ for any $\lambda \in I$, in contradiction with the fact that no positive weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (1.1) exists if $\lambda > \Lambda$. $\square$

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