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EXISTENCE AND STABILITY FOR FRACTIONAL ORDER PANTOGRAPH EQUATIONS WITH NONLOCAL CONDITIONS

ISRAR AHMAD, JUAN JOSE NIETO, GHAUS UR RAHMAN, KAMAL SHAH

Abstract. In this article we study the a coupled system of fractional pantograph differential equations (FPDEs). Using degree theory, we state necessary conditions for the existence of solutions to a coupled system of fractional partial differential equations with non-local boundary conditions. Also using tools from non-linear analysis, we establish some stability results. We illustrate our theoretical results with a test problem.

1. Introduction

Fractional calculus is gaining prominence in mathematical modeling. For example in the field of nanotechnology, it has received a lot of attention by researchers. This emerging field of mathematics has been used in the modeling of real word problems as well in applications in biological sciences, numerical and physical sciences, dynamical systems, economics and so forth; see for example [9, 12, 13, 15, 16] and the references therein. Looking at the advantage of this field over other fields of mathematics, experts have explored applications in areas including qualitative theory, stability theory, optimization and numerical simulations; see [1, 2, 7, 18, 20, 26, 28].

Fractional ordinary ordinary differential equations (FODEs) have been studied in detail including existence theory, stability analysis, and numerical solutions. Similarly, various classes, including impulsive, implicit type FODEs have been considered. On the other hand, an important class known as pantograph differential equations with fractional order derivatives have not been not properly investigated. Ordinary derivatives are local by nature and cannot be used to model some properties like memory and hereditary etc. Therefore, in some cases, FODEs provide an adequate way to model some natural systems.

The underlying theory of fractional differential equations has also applied aspects. The applied aspects in this area include: probability theory, physics, control theory, light absorption and division of cells phenomena [4, 5]. Among the attractive features of fractional differential equations, stability theory is an important and comprehensive aspects of fractional differential equations (FDEs).

Stability of FODEs and systems has been considered in several works in last two decades; see for example [3, 19, 24, 25, 27, 29]. Recently, some authors have

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investigated the fractional pantograph differential equation

$$\begin{gathered} {}^{C}D_{0^+}^{\gamma}x(t) - c(t)x(t) = \sum_{\sigma=1}^{m} a_\sigma(t)D^{\gamma_\sigma}x(r_\sigma t), \quad n-1<\gamma\le n,\ t\in[0,S_1], \\ x^{(j-1)}(0) = x_{j-1}, \quad j=1,\dots,n. \end{gathered} \tag{1.1}$$

Here $r_\sigma \in (0,1) = I_1$, $0 \le \gamma_\sigma < \sigma \le n$, while $c(t)$, $a_\sigma(t)$ $(\sigma = 1,2,\dots,m)$, are known functions and D represents Caputo derivative of order $\gamma > 0$. In [17] the authors developed numerical solutions based on Bernoulli wavelets. Vignesh et al [23] studied existence theory for the fractional pantograph differential equation

$$\begin{gathered} D^{\gamma}x(t) = P_1(t,x(t),x(rt)), \quad t\in[0,S],\ \gamma,r\in I_1, \\ x(0) = x_0. \end{gathered} \tag{1.2}$$

Here $C[0,S]$ is the usual Banach-Space of continuous functions on $[0,S]$ and P_1 is a non-linear continuous function from $[0,S]\times\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$. Where D represents Riemann-Liouville derivative. Further, the authors in [14] studied stability aspects for the FPDE

$$\begin{gathered} {}^{H}D_{0^+}^{\gamma,\beta;\psi}x(t) - P_2(t,x(t),x(P_3(t))) = 0, \quad t\in(0,d]=I_2, \\ I_{0^+}^{1-\alpha;\psi}x(0^+) = x_0\in\mathbb{R}, \quad x(t)=\phi(t), \quad -h\le t\le 0. \end{gathered} \tag{1.3}$$

Here ${}^{H}D_{0^+}^{\gamma,\beta;\psi}(\cdot)$ is the ψ-Hilfer of type $0<\gamma,\beta,\gamma\le 1$ and $I_{0^+}^{1-\alpha;\psi}x(\cdot)$ is the classical Riemann-Liouville integral with $\alpha = \gamma+\beta(1-\gamma)$. ψ is a non-linear continuous function and $P_2 : I_2\times\mathbb{R}^2\to\mathbb{R}$. Afterwards, the authors in [8] formulated a coupled system of fractional order differential equation with ψ Hilfer derivative and explored existence results.

To obtain new results for FPDEs with non-local boundary conditions using degree theory one needs further explorations. Existence, uniqueness and stability results with coupled system of FPDEs using degree theory has not been considered as far we know. Therefore, we study the coupled system of FPDEs with non-local boundary conditions,

$$\begin{gathered} {}^{C}D^{\gamma_1}x_1(t) = f_1(t,x_1(t),x_1(rt),x_2(t)), \\ {}^{C}D^{\gamma_2}x_2(t) = f_2(t,x_1(t),x_2(t),x_2(rt)), \\ g_1(x_1) = \lambda_1 x_1(0) - \mu_1 x_1(\eta) - \delta_1 x_1(1), \\ g_2(x_2) = \lambda_2 x_2(0) - \mu_2 x_2(\zeta) - \delta_2 x_2(1)). \end{gathered} \tag{1.4}$$

Here $t\in I=[0,1]$, $r\in I_1$, $\gamma_1,\gamma_2\in(0,1]$, $f_1,f_2\in C(I\times\mathbb{R}^3,\mathbb{R})$ are non-linear and $g_1,g_2 : C(I,\mathbb{R})\to\mathbb{R}$ are given maps. Here, $\zeta,\eta\in I_1$, $\lambda_j\ne\delta_j+\mu_j, \lambda_j,\delta_j,\mu_j$ $(j=1,2)$ are real and D is Caputo's derivative. We use degree theory to obtain existence and uniqueness results, and illustrate our main results through an example.

Let $X = C(I,\mathbb{R})$ be the Banach space of all continuous maps with norm $\|x_1\| = \max\{|x_1(t)| : t\in I\}$. The product space $X\times X = \{(x_1,x_2) : x_1,\, x_2\in X\}$ is also a Banach space with norm $\|(x_1,x_2)\| = \max\{\|x_1\|,\|x_2\|\}$.

2. Preliminaries

In this section, we pointed out few important results from degree theory, functional analysis and fractional calculus [9, 13, 15, 16].

Definition 2.1. Let $\gamma > 0$. Assume $x(t) \in L[0,1]$. The Riemann-Liuville fractional integral of order γ is defined as

$$I^{\gamma}_{0^+}x(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}x(\theta)\,d\theta.$$

Definition 2.2. Let $k = [\gamma]+1$. If $x(t) \in AC[0,1]$, the Caputo fractional derivative is defined as

$${}^{C}D^{\gamma}_{0^+}x(t) = \frac{1}{\Gamma(k-\gamma)}\int_0^t (t-\theta)^{k-\gamma-1}x^{(k)}(\theta)d\theta.$$

Lemma 2.3. *In fractional calculus for a function $x(t) \in AC[0,1]$, we have*

$$I^{\gamma}_{0^+}\,{}^{C}D^{\gamma}_{0^+}x(t) = x(t) + a_0 + a_1 t + a_2 t^2 + \dots + a_{k-1}t^{k-1},$$

for constants $a_j \in \mathbb{R}$, for $j = 0,1,2,\cdots,k-1$.

Definition 2.4 ([22])**.** The spectral radius of a matrix $\mathscr{M} \in \mathbb{C}^{n\times n}$ is defined by

$$\Upsilon(\mathscr{M}) = \max\{|\beta_1|, |\beta_2|, \dots, |\beta_n|\},$$

where $\beta_1, \beta_2, \dots, \beta_n$ are the eigenvalues. A matrix $\mathscr{M}$ is said to converge to zero if the spectral radius satisfies $\Upsilon(\mathscr{M}) < 1$.

Definition 2.5 ([22])**.** A system of fractional order pantograph equations (1.4) is Ulam-Hyers stable, if for all $C = (C_{f_1,f_2}) > 0$ such that, for some ordered pair $\epsilon = (\epsilon_1, \epsilon_2)$ of positive real numbers and for each $t \in I$ and if for any solution $(x_1, x_2) \in X \times X$ of the inequalities

$$\begin{aligned} &|{}^{C}D^{\gamma_1}x_1(t) - f_1(t, x_1(t), x_1(rt), x_2(t))| < \epsilon_1,\\ &|{}^{C}D^{\gamma_2}x_2(t) - f_2(t, x_1(t), x_2(t), x_2(rt))| < \epsilon_2, \end{aligned} \tag{2.1}$$

there exist a unique solution $(\bar{x_1}, \bar{x_2})$, belonging to $X \times X$, of system (1.4) that satisfy the relation

$$\|(\bar{x_1}, \bar{x_2}) - (x_1, x_2)\| \le C\epsilon\,.$$

Definition 2.6 ([22])**.** A system of fractional order pantograph equations (1.4) is generalized Ulam-Hyers stable, if for all $C = (C_{f_1,f_2}) > 0$, φ belonging to $C(I,\mathbb{R})$ along with $\varphi(0) = 0$, for all $t \in I$ and a solution $(x_1, x_2) \in X \times X$ of (1.4), there is a unique solution $(\bar{x_1}, \bar{x_2}) \in X \times X$ of (1.4) which satisfies

$$\|(\bar{x_1}, \bar{x_2}) - (x_1, x_2)\| \le C\,\varphi(\epsilon).$$

Theorem 2.7 ([22])**.** *Assume the operators $T_1, T_2 : X \times X \to X$ such that for the operator system*

$$\begin{aligned} T_1(x_1, x_2) &= x_1,\\ T_2(x_1, x_2) &= x_2, \end{aligned} \tag{2.2}$$

for all $x_i, \bar{x}_i \in X$ for $i = 1,2$, the following system of inequations holds

$$\begin{aligned} \|T_1(x_1, x_2) - T_1(\bar{x_1}, \bar{x_2})\| &\le \ell_1\|x_1 - \bar{x_1}\| + \ell_2\|x_2 - \bar{x_2}\|,\\ \|T_2(x_1, x_2) - T_2(\bar{x_1}, \bar{x_2})\| &\le \ell_3\|x_1 - \bar{x_1}\| + \ell_4\|x_2 - \bar{x_2}\|, \end{aligned} \tag{2.3}$$

where $(x_1, x_2), (\bar{x_1}, \bar{x_2}) \in X \times X$ are exact and approximate solutions, respectively and $\ell_1, \ell_2, \ell_3, \ell_4 > 0$. Moreover, if the matrix

$$\mathscr{M} = \begin{bmatrix} \ell_1 & \ell_2\\ \ell_3 & \ell_4 \end{bmatrix}, \tag{2.4}$$

converges to 0, then the fixed points for (2.2) are stable in the sense of Ulam-Hyers.

Fir our proofs we let $P(X)$ be the power set, and $\mathbb{J}$ be the sub collection of $P(X)$ that contain all bounded sets. Also we adopt the terminologies and results from [6].

Definition 2.8. For non-compactness the Kuratowski measure $\Im : \mathbb{J} \to R^+$ is defined as

$$\Im(J) = \inf\{p_1 > 0 : J \in \mathbb{J} \text{ has sets of finite cover of diameter less or equal to } p_1\, t\}.$$

Proposition 2.9. *The Kuratowski measure $\Im$ satisfies the following properties:*

(i) *J is relatively compact if and only if $\Im(J) = 0$.*
(ii) *J is a semi-norm, i.e. $\Im(lJ) = |l|\Im(J), l$ is real and $\Im(J_1) + \Im(J_2) \geq \Im(J_1 + J_2)$.*
(iii) *$J_1 \subset J_2$ implies $\Im(J_1) \leq \Im(J_2)$, $\Im(J_1 \cup J_2) = \max\{\Im(J_1), \Im(J_2)\}$.*
(iv) *$\Im(conv J) = \Im(J)$.*
(v) *$\Im(\bar{J}) - \Im(J) = 0$.*

Definition 2.10. Let $\mathbb{Z}$ is a subspace of X, then a function $\mathbb{H}$ from $\mathbb{Z}$ into X that is bounded and continuous. $\mathbb{H}$ is $\Im$-Lipschitz if there is an $L \geq 0$ such that

$$\Im(\mathbb{H}(J)) \leq L\Im(J), \quad \forall J \subset \mathbb{Z} \text{ bounded}.$$

$\mathbb{H}$ is $\Im$-contraction if L< 1. $\mathbb{H}$ is $\Im$-condensing if

$$\Im(\mathbb{H}(J)) < \Im(J), \forall J \subset \mathbb{Z} \text{ bounded with } \Im(J) > 0.$$

Proposition 2.11. *If the functions $\mathbb{H}, \mathbb{F} : \mathbb{Z} \to X$ are $\Im$-Lipschitz with real positive constants L_1 and L_2, respectively, then $\mathbb{H} + \mathbb{F} : \mathbb{Z} \to X$ are $\Im$-Lipschitz with $L_1 + L_2$.*

Proposition 2.12. *If $\mathbb{H} : \mathbb{Z} \to X$ is compact, then $\mathbb{H}$ is $\Im$-Lipschitz having Lipschitz constant equal to zero.*

Proposition 2.13. *If $\mathbb{H} : \mathbb{Z} \to X$ is Lipschitz along a constant $\Im$, so $\mathbb{H}$ is $\Im$-Lipschitz with same constant $\Im$.*

For our main result the following theorem due to Isaia [11] plays an important role.

Theorem 2.14. *Let $\mathbb{H} : X \to X$ be $\Im$-condensing, and*

$$\mathbb{S} = \{y \in X : \exists \mu \in I \text{ such that } y = \mu \mathbb{H} y\}.$$

For a bounded set $\mathbb{S} \in X$ there exists a positive real number ϱ such that $\mathbb{S} \subset B_\varrho(0)$. If the following equation holds then the degree of $\mathbb{H}$ is one and has one fixed point,

$$\deg(I - \mu\mathbb{H}, B_\varrho(0), 0) = 1, \quad \forall \mu \in I.$$

The collection of all fixed points of $\mathbb{H}$ is denoted by $B_\varrho(0)$. Furthermore, to establish our main results, we need the following assumptions:

(A1) There exist constants $K_1, K_2 \in [0, 1)$ such that for $x_1, \bar{x_1}, x_2$ and $\bar{x_2} \in X$,
$$|g_1(\bar{x_1}) - g_1(x_1)| \leq K_1\|\bar{x_1} - x_1\|, \quad |g_2(\bar{x_2}) - g_2(x_2)| \leq K_2\|\bar{x_2} - x_2\|.$$
(A2) There exist constants $C_{g_1}, C_{g_2}, M_{g_1}, M_{g_2} > 0$ such that for $x_1, x_2 \in X$,
$$|g_1(x_1)| \leq C_{g_1}\|x_1\|^{q_1} + M_{g_1}, \quad |g_2(x_2)| \leq C_{g_2}\|x_2\|^{q_2} + M_{g_2}.$$

(A3) There exist constants $C^{(j)}_{(f_1)}, C^{(j)}_{(f_2)}$ $(j=1,2)$, , M_{f_1}, M_{f_2} such that for $t \in I$, $x_1, x_2 \in X$,

$$|f_1(t, x_1(\theta), x_1(r\theta), x_2(\theta))| \le 2C^1_{f_1}\|x_1\|^{q_1} + C^2_{f_1}\|x_2\|^{q_2} + M_{f_1},$$
$$|f_2(t, x_1(\theta), x_2(\theta), x_2(r\theta))| \le C^1_{f_2}\|x_1\|^{q_1} + 2C^2_{f_2}\|x_2\|^{q_2} + M_{f_2}.$$

(A4) For arbitrary $t \in I, x_1, \bar{x_1}, x_2$ and $\bar{x_2} \in X$ there exist positive constants L_{f_1}, L_{f_2} such that

$$|f_1(t, \bar{x_1}(t), \bar{x_1}(rt), \bar{x_2}(t)) - f_1(t, x_1(t), x_1(rt), x_2(t))| \le L_{f_1}(2|\bar{x_1} - x_1| + |\bar{x_2} - x_2|),$$
$$|f_2(t, \bar{x_1}(t), \bar{x_2}(t), \bar{x_2}(rt)) - f_2(t, x_1(t), x_2(t), x_2(rt))| \le L_{f_2}(|\bar{x_1} - x_1| + 2|\bar{x_2} - x_2|).$$

3. Main results

Theorem 3.1. *Let $x_1(t) \in C(I, \mathbb{R})$ and $w_1(t) \in L(I, \mathbb{R})$ be the solution for the problem*

$$D^{\gamma_1}_{0^+} x_1(t) = w_1(t), \quad t \in I,\ \gamma_1 \in (0,1], g_1(x_1) = \lambda_1 x_1(0) - \mu_1 x_1(\eta) - \delta_1 x_1(1), \tag{3.1}$$

is as follow:

$$\begin{aligned} x_1(t) &= \frac{g_1(x_1)}{\lambda_1 - (\delta_1 + \mu_1)} + \frac{\delta_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^1 (1-\theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta \\ &\quad + \frac{\mu_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^\eta (\eta - \theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta \\ &\quad + \frac{1}{\Gamma(\gamma_1)} \int_0^t (t-\theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta. \end{aligned} \tag{3.2}$$

Proof. Integrating on both sides of (3.1) and with the help of Lemma 2.3, we have

$$x_1(t) = I^{\gamma_1} w_1(t) + a_0 = \frac{1}{\Gamma(\gamma_1)} \int_0^t (t-\theta) w_1(\theta) d\theta + a_0, \tag{3.3}$$

using boundary conditions in (3.1), we obtain

$$\begin{gathered} x_1(0) = a_0, \quad x_1(\eta) = \frac{1}{\Gamma(\gamma_1)} \int_0^\eta (\eta - \theta)^{\gamma_1 - 1} w_1(\theta) d\theta + a_0, \\ x_1(1) = \frac{1}{\Gamma(\gamma_1)} \int_0^1 (1-\theta)^{\gamma_1 - 1} w_1(\theta) d\theta + a_0. \end{gathered} \tag{3.4}$$

Putting all the values of (3.4), in the relation given in (3.1), yields

$$\begin{aligned} g_1(t) &= a_0(\lambda_1 - (\delta_1 + \mu_1)) - \frac{\mu_1}{\Gamma(\gamma_1)} \int_0^\eta (\eta - \theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta \\ &\quad - \frac{\delta_1}{\Gamma(\gamma_1)} \int_0^1 (1-\theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta \\ &\quad + \frac{1}{\Gamma(\gamma_1)} \int_0^t (t-\theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta. \end{aligned} \tag{3.5}$$

The value of a_0 from (3.5), is

$$\begin{aligned} a_0(t) &= \frac{g_1(x_1)}{\lambda_1 - (\delta_1 + \mu_1)} + \frac{\delta_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^1 (1-\theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta \\ &\quad + \frac{\mu_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^\eta (\eta - \theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta. \end{aligned} \tag{3.6}$$

Putting the value of a_0, in (3.3), we obtain the solution

$$
\begin{aligned}
x_1(t) = {} & \frac{g_1(x_1)}{\lambda_1 - (\delta_1 + \mu_1)} + \frac{\delta_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^1 (1-\theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta \\
& + \frac{\mu_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^\eta (\eta-\theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta \\
& + \frac{1}{\Gamma(\gamma_1)} \int_0^t (t-\theta)^{\gamma_1 - 1} w_1(\theta)\, d\theta.
\end{aligned}
\tag{3.7}
$$

Using similar approach for $x_2(t) \in C(I, \mathbb{R})$ and $w_2(t) \in L(I, \mathbb{R})$ the solution for the problem

$$
\begin{gathered}
D_{0^+}^{\gamma_2} x_2(t) = w_2(t), \ t \in I, \ \gamma_2 \in (0,1], \\
g_2(x_2) = \lambda_2 x_2(0) - \mu_2 x_2(\zeta) - \delta_2 x_2(1),
\end{gathered}
\tag{3.8}
$$

is

$$
\begin{aligned}
x_2(t) = {} & \frac{g_2(x_2)}{\lambda_2 - (\delta_2 + \mu_2)} + \frac{\delta_2}{(\lambda_2 - (\delta_2 + \mu_2))\Gamma(\gamma_2)} \int_0^1 (1-\theta)^{\gamma_2 - 1} w_2(\theta)\, d\theta \\
& + \frac{\mu_2}{(\lambda_2 - (\delta_2 + \mu_2))\Gamma(\gamma_2)} \int_0^\zeta (\zeta-\theta)^{\gamma_2 - 1} w_1(\theta)\, d\theta \\
& + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-\theta)^{\gamma_2 - 1} w_2(\theta)\, d\theta.
\end{aligned}
\tag{3.9}
$$

□

Corollary 3.2. *In light of Theorem 3.1, the solution of the problem* (1.4) *is*

$$
\begin{aligned}
& x_1(t) \\
& = \frac{g_1(x_1)}{\lambda_1 - (\delta_1 + \mu_1)} \\
& \quad + \frac{\delta_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^1 (1-\theta)^{\gamma_1 - 1} f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta))\, d\theta \\
& \quad + \frac{\mu_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^\eta (\eta-\theta)^{\gamma_1 - 1} f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta))\, d\theta \\
& \quad + \frac{1}{\Gamma(\gamma_1)} \int_0^t (t-\theta)^{\gamma_1 - 1} f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta))\, d\theta, \\
& x_2(t) \\
& = \frac{g_2(x_2)}{\lambda_2 - (\delta_2 + \mu_2)} \\
& \quad + \frac{\delta_2}{\lambda_2 - (\delta_2 + \mu_2)} \frac{1}{\Gamma(\gamma_2)} \int_0^1 (1-\theta)^{\gamma_2 - 1} f_2(\theta, x_1(\theta), x_2(\theta), x_2(r\theta))\, d\theta \\
& \quad + \frac{\mu_2}{(\lambda_2 - (\delta_2 + \mu_2))\Gamma(\gamma_2)} \int_0^\zeta (\zeta-\theta)^{\gamma_2 - 1} f_2(\theta, x_1(\theta), x_2(\theta), x_2(r\theta))\, d\theta \\
& \quad + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-\theta)^{\gamma_2 - 1} f_2(\theta, x_1(\theta), x_2(\theta), x_2(r\theta)) d\theta.
\end{aligned}
\tag{3.10}
$$

Lemma 3.3. *Let f_1, f_2, f_3 be nonlinear continuous the functions. Then $(x_1, x_2) \in X \times X$ is the required solution of the integral equations* (3.10)*, if and only if (x_1, x_2) is a solution of* (1.4).

Proof. If (x_1, x_2) is the solution of (3.10), then differentiating both sides of (3.10), we obtain (1.4). However, If (x_1, x_2) is a solution of (1.4), thus (x_1, x_2) is the solution of (3.10). $\square$

It is sufficient for the existence of FPDEs (1.4), to prove that integral system (3.10) has at least one solution $(x_1, x_2) \in X \times X$. The operators $H, F, T : X \times X \to X \times X$ are defined by

$$\begin{gathered} H(x_1, x_2)(t) = (H_1(x_1(t)), H_2(x_2(t))), \\ F(x_1, x_2)(t) = (F_1((x_1(t)), (x_2(t))), F_2(x_1(t), x_2(t)) \\ T(x_1, x_2) = H(x_1, x_2) + F(x_1, x_2), \end{gathered}$$

where

$$\begin{gathered} H_1 x_1(t) = \frac{g_1(x_1)}{\lambda_1 - \delta_1 - \mu_1}, \quad H_2 x_2(t) = \frac{g_2(x_2)}{\lambda_2 - \delta_2 - \mu_2}, \end{gathered}$$

$$\begin{aligned} &F_1(x_1, x_2)(t) \\ &= \frac{\delta_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^1 (1 - \theta)^{\gamma_1 - 1} f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta)) d\theta \\ &\quad + \frac{\mu_1}{(\lambda_1 - (\delta_1 + \mu_1))\Gamma(\gamma_1)} \int_0^\eta (-v + \eta)^{\gamma_1 - 1} f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta)) d\theta \\ &\quad + \frac{1}{\Gamma(\gamma_1)} \int_0^t (t - \theta)^{\gamma_1 - 1} f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta)) d\theta, \\ &F_2(x_1, x_2)(t) \\ &= \frac{\delta_2}{(\lambda_2 - (\delta_2 + \gamma_2))\Gamma(\gamma_2)} \int_0^1 (1 - \theta)^{\gamma_2 - 1} f_2(\theta, x_1(\theta), x_2(\theta)), x_2(r\theta)) d\theta \\ &\quad + \frac{\mu_2}{(\lambda_2 - (\delta_2 + \mu_2))\Gamma(\gamma_2)} \int_0^\zeta (\zeta - \theta)^{\gamma_2 - 1} f_2(\theta, x_1(\theta), x_2(\theta)), x_2(r\theta)) d\theta \\ &\quad + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t - \theta)^{\gamma_2 - 1} f_2(\theta, x_1(\theta), x_2(\theta)), x_2(r\theta)) d\theta. \end{aligned}$$

The continuity of the functions $f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta))$, $f_2(\theta, x_1(\theta), x_2(\theta)), x_2(r\theta))$ imply that T is well-defined. We can also write (3.10) in operator form as

$$(x_1, x_2) = T(x_1, x_2) = H(x_1, x_2) + F(x_1, x_2), \tag{3.11}$$

the fixed points of (3.11) are solutions of (3.10).

Lemma 3.4. *In view of assumptions* (A1) *and* (A2)*, the operator* $H : X \times X \to X \times X$*, is Lipschitz and satisfies*

$$\|H(x_1, x_2)\| \leq C\|(x_1, x_2)\|^q + M. \tag{3.12}$$

Proof. For $(x_1, x_2), (\bar{x_1}, \bar{x_2}) \in X \times X$, using (A1) and (3.11), we obtain

$$\begin{gathered} \|H_1 x_1 - H_1 \bar{x_1}\| \leq \frac{1}{|\lambda_1 - \delta_1 - \mu_1|} |g_1(x_1) - g_1(\bar{x_1})| \leq \frac{K_1}{|\lambda_1 - \delta_1 - \mu_1|} \|x_1 - \bar{x_1}\|, \\ \|H_2 x_2 - H_2 \bar{x_2}\| \leq \frac{K_2}{|\lambda_2 - (\delta_2 + \mu_2)|} \|x_2 - \bar{x_2}\|. \end{gathered}$$

This implies

$$\|H(x_1, x_2) - H(\bar{x_1}, \bar{x_2})\|$$

$$\begin{aligned}
&\le \max\Big(\frac{K_1}{|\lambda_1-(\delta_1+\mu_1)|},\frac{K_2}{|\lambda_2-(\delta_2+\mu_2)|}\Big)\|(x_1,x_2)-(\bar{x_1},\bar{x_2})\| \\
&\le h\|(x_1,x_2)-(\bar{x_1},\bar{x_2})\|,
\end{aligned}$$

where

$$h=\max\Big(\frac{K_1}{|\lambda_1-\delta_1-\mu_1)|},\frac{K_2}{|\lambda_2-\delta_2-\mu_2)|}\Big).$$

So with proposition (2.11), H is $\Im$-Lipschitz of positive value h. Using assumption (A2) for growth condition, we have

$$\begin{gathered}
|H_1x_1(t)|=\|\frac{g_1(x_1)}{\lambda_1-(\delta_1+\mu_1)}\|\le\frac{C_{g_1}\|x_1\|^{q_1}}{|\lambda_1-(\delta_1+\mu_1)|}+M_{g_1}, \\
|H_2x_2(t)|=|\frac{g_1(x_2)}{\lambda_2-(\delta_2+\mu_2)}|\le\frac{C_{g_2}\|x_2\|^{q_2}}{|\lambda_2-(\delta_2+\mu_2)|}+M_{g_2}.
\end{gathered}$$

Consequently

$$\|H(x_1,x_2)\|\le\mathbf{C}\|(x_1,x_2)\|^q+\mathbf{M},$$

where

$$\mathbf{C}=\max\Big(\frac{C_{g_1}}{|\lambda_1-\delta_1-\mu_1)|},\frac{C_{g_2}}{|\lambda_2-\delta_2-\mu_2)|}\Big),$$

$\mathbf{M}=\max(M_{g_1},M_{g_2})$, and $q=\max(q_1,q_2)$. □

Lemma 3.5. *If hypothesis* (A3) *holds, then the operator F from $X\times X$ to $X\times X$ is continuous and satisfies*

$$\|F(x_1,x_2)\|\le\varpi\big(\|(x_1,x_2)\|^q+M^*\big),\quad (x_1,x_2)\in X\times X, \tag{3.13}$$

where

$$\begin{gathered}
\varpi=\max\Big(\frac{C_{f_1}[|\lambda_1|+|\mu_1|+|\delta_1|]}{|\lambda_1-(\delta_1+\mu_1)|\Gamma(1+\gamma_1)},\frac{C_{f_2}[|\lambda_2|+|\gamma_2|+|\delta_2|]}{|\lambda_2-(\delta_2+\mu_2)|\Gamma(1+\gamma_2)}\Big), \\
M^*=\max(M_{f_1},M_{f_2}), \\
C_{f_1}=\max\{C^1_{f_1},C^2_{f_1}\},\quad C_{f_2}=\max\{C^1_{f_2},C^2_{f_2}\}.
\end{gathered}$$

Proof. Let

$$D_\rho=\big\{(x_1,x_2)\in X\times X:\|(x_1,x_2)\|\le\rho\big\}\subset X\times X$$

be a bounded set and let $\{z_n=(x_{1n},x_{2n})\}\in D_\rho$ be a sequence such that $z_n\to z=(x_1,x_2)$ as n tends to infinity in D_ρ. We have to show that $\|Fz_n-Fz\|\to0$, as n tends to infinity. From the continuity of $f_1(t,x_1(t),x_1(rt),x_2(t))$, it follows that $f_1(\theta,(x_1(t))_n,(x_1(rt))_n,(x_2(t))_n)\to f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta))$ as $n\to\infty$. Under assumption (A3), we have

$$\begin{aligned}
&(1-\theta)^{\gamma_1-1}|(f_1(\theta,(x_1(t))_n,(x_1(rt))_n,(x_2(t))_n)-f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta)))| \\
&\le(1-\theta)^{\gamma_1-1}[2C^1_{f_1}\rho^{q_1}+C^2_{f_1}\rho^{q_2}+M_{f_1}], \\
&(\eta-\theta)^{\gamma_1-1}|(f_1(\theta,(x_1(t))_n,(x_1(rt))_n,(x_2(t))_n)-f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta)))| \\
&\le(\eta-\theta)^{\gamma_1-1}[2C^1_{f_1}\rho^{q_1}+C^2_{f_1}\rho^{q_2}+M_{f_1}], \\
&(t-\theta)^{\gamma_1-1}|(f_1(\theta,(x_1(t))_n,(x_1(rt))_n,(x_2(t))_n)-f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta)))| \\
&\le(t-\theta)^{\gamma_1-1}[2C^1_{f_1}\rho^{q_1}+C^2_{f_1}\rho^{q_2}+M_{f_1}],
\end{aligned}$$

applying the Lebesgue dominated convergence theorem, for the preceding inequalities, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^1 (1-\theta)^{\gamma_1-1}|(f_1(\theta,(x_1(t))_n,(x_1(rt))_n,(x_2(t))_n) \\
&- f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta)))|\,d\theta = 0, \\
&\lim_{n\to\infty}\int_0^\eta (\eta-\theta)^{\gamma_1-1}|f_1(\theta,(x_1(t))_n,(x_1(rt))_n,(x_2(t))_n) \\
&- f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta))|d\theta = 0, \\
&\lim_{n\to\infty}\int_0^t (t-\theta)^{\gamma_1-1}|f_1(\theta,(x_1(t))_n,(x_1(rt))_n,(x_2(t))_n) \\
&- f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta))|\,d\theta = 0.
\end{aligned}
$$

Hence, $|F_1(x_{1n},x_{2n}) - F_1(x_1,x_2)| \to 0$ as n approaches ∞. Similarly, we have $|F_2(x_{1n},x_{2n}) - F_2(x_1,x_2)| \to 0$ as n tends to ∞. This implies $|F(x_{1n},x_{2n}) - F(x_1,x_2)| \to 0$ as n approaches ∞, due to the continuity of F. For growth conditions we proceed as follows,

$$
\begin{aligned}
&\|F_1(x_1,x_2)\| \\
&\le \frac{1}{|\lambda_1-(\delta_1+\mu_1)|\Gamma(\gamma_1)}\Big(|\delta_1|\int_0^1 (1-\theta)^{\gamma_1-1}|f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta))|d\theta \\
&\quad + |\gamma_1|\int_0^\eta (\eta-\theta)^{\gamma_1-1}|f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta))|\Big)d\theta \\
&\quad + \frac{|\mu_1|}{|\lambda_1-(\delta_1+\mu_1)|\Gamma(\gamma_1)}\int_0^t (t-\theta)^{\gamma_1-1}|f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta))|d\theta \\
&\le \Big[\frac{|\lambda_1|+|\delta_1|+|\mu_1|}{|\lambda_1-(\delta_1+\mu_1)|\Gamma(\gamma_1+1)}\Big]\Big(2C_{f_1}^1\|x_1\|^{q_2} + C_{f_1}^2\|x_2\|^{q_2} + M_{f_1}\Big).
\end{aligned} \tag{3.14}
$$

Similarly, we obtain

$$
\begin{aligned}
&\|F_2(x_1,x_2)\| \\
&\le \Big[\frac{|\lambda_2|+|\delta_2|+|\mu_2|}{|\lambda_2-(\delta_2+\mu_2)|\Gamma(\gamma_2+1)}\Big]\Big(C_{f_2}^1\|x_1\|^{q_2} + 2C_{f_2}^2\|x_2\|^{q_2} + M_{f_2}\Big).
\end{aligned} \tag{3.15}
$$

Then growth condition (3.13) follows from (3.14) and (3.15). □

Lemma 3.6. *The operator $F : X\times X \to X\times X$ is compact.*

Proof. Let the set $\mathbb{Z} \subset D_\rho \subset X\times X$ be bounded. It is necessary to prove that $F(\mathbb{Z})$ is relatively compact in $X\times X$. For any $z_n = ((x_1)_n,(x_2)_n) \in \mathbb{Z} \subset D_\rho$, growth condition (3.13) yields

$$
\|F((x_1)_n,(x_2)_n)\| \le \varpi(\rho^q + M^*),
$$

i.e. $F(\cdot)$ is uniformly bounded. To show F is equicontinuous, pick $0 \le t \le \xi \le 1$, then

$$
\begin{aligned}
&\|F_1((x_1)_n,(x_2)_n))(t) - F_1((x_1)_n,(x_2)_n))(\xi)\| \\
&\le \frac{1}{\Gamma(\gamma_1)}\int_0^t [(t-\theta)^{\gamma_1-1} - (\xi-\theta)^{\gamma_1-1}]|f_1(\theta,(x_1(t))_n,(x_1(rt))_n,(x_2(t))_n)|d\theta
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^{\xi}(\xi-\theta)^{\gamma_1-1}|f_1(\theta,(x_1(t))_n,(x_1(rt))_n,(x_2(t))_n)|d\theta,\\
&\leq \frac{1}{\Gamma(\gamma_1+1)}\big[t^{\gamma_1}-\xi^{\gamma_1}+(\xi-t)^{\gamma_1}+(\xi-t)^{\gamma_1}\big]\\
&\quad\times\Big(2C_{f_1}^1\|x_1\|^{q_1}+C_{f_1}^2\|x_2\|^{q_2}+M_{f_1}\Big),\\
&\leq \Big(\frac{\rho^q(2C_{f_1}^1+C_{f_1}^2)+M_{f_1}}{\Gamma(\gamma_1+1)}\Big)\big[t^{\gamma_1}-\xi^{\gamma_1}+(-t+\xi)^{\gamma_1}+(-\xi+t)^{\gamma_1}\big].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\|F_2((x_1)_n,(x_2)_n)(t)-F_2(((x_1)_n,(x_2)_n))(\xi)\|\\
&\leq \Big(\frac{(C_{f_2}^1+2C_{f_2}^2)\rho^q+M_{f_2}}{\Gamma(\gamma_2+1)}\Big)\big[t^{\gamma_2}-\xi^{\gamma_2}+(\xi-t)^{\gamma_2}+(t-\xi)^{\gamma_2}\big].
\end{aligned}
\tag{3.16}
$$

It follows that

$$
\begin{gathered}
\|F_1((x_1)_n,(x_2)_n)(t)-F_1((x_1)_n,(x_2)_n)(\xi)\|\to 0,\\
\|F_2((x_1)_n,(x_2)_n)(t)-F_2((x_1)_n,(x_2)_n)(\xi)\|\to 0
\end{gathered}
$$

as $t\to\xi$, which implies that $F(x_1,x_2)$ is equicontinuous. $F(x_1,x_2)$ is compact by Arzela-Ascoli theorem. Furthermore, F is $\Im$-Lipschitzen of constant zero. $\square$

Theorem 3.7. *Suppose* (A1)–(A3) *are satisfied. Then* (1.4) *has at least one solution* $(x_1,x_2)\in X\times X$*. Moreover, the solution set is bounded in* $X\times X$*.*

Proof. By Lemma (3.4), H is $\Im$-Lipschitz of real constant h, and by Lemma 3.6 F is $\Im$-Lipschitz with zero. As a result, T is $\Im$-Lipschitz of real h. Let

$$
U=\big\{(x_1,x_2)\in X\times X:\text{ there exist }\rho_1\in I,\ (x_1,x_2)=\rho_1T(x_1,x_2)\big\}.
$$

We need to show U is bounded. For $(x_1,x_2)\in U$, we have

$$
(x_1,x_2)=\rho_1T(x_1,x_2)=\rho_1\Big(H(x_1,x_2)+F(x_1,x_2)\Big),
$$

which implies that

$$
\begin{aligned}
\|x_1\|&\leq\rho_1[\|H_1x_1\|+\|F_1x_1\|]\\
&\leq\rho_1\Big[\frac{C_{g_1}||x_1||^{q_1}}{|\lambda_1-(\delta_1+\mu_1)|}+M_{g_1}\\
&\quad+\frac{|\lambda_1|+|\delta_1|+|\mu_1|}{|\lambda_1-(\delta_1+\mu_1)|\Gamma(\gamma_1+1)}(2C_{f_1}^1\|x_1\|^{q_1}+C_{f_1}^2\|x_2\|^{q_2}+M_{f_1})\Big].
\end{aligned}
\tag{3.17}
$$

In the same way we can prove that

$$
\begin{aligned}
\|x_2\|&\leq\rho_1\Big[\frac{C_{g_2}||x_2||^{q_2}}{|(\delta_2+\mu_2)-\lambda_2|}+M_{g_2}\\
&\quad+\frac{|\lambda_2|+|\delta_2|+|\mu_2|}{|\lambda_2-(\delta_2+\mu_2)|\Gamma(\gamma_2+1)}\left(C_{f_2}^1\|x_1\|^{q_1}+2C_{f_2}^2\|x_2\|^{q_2}+M_{f_2}\right)\Big].
\end{aligned}
\tag{3.18}
$$

The preceding inequalities (3.17) and (3.18) along with $q_1,q_2\in[0,1)$ yield that U is bounded in $X\times X$. If we go against the assertion and divide (3.17) by $\|x_1\|=\lambda$ and $\|x_1\|\to\infty$, we obtain the following result

$$
1\leq\lim_{\lambda\to\infty}\frac{\rho_1}{\lambda}\Big[\frac{C_{f_2}\lambda^q}{|(\delta_1+\mu_1)-\lambda_1|}+M_{g_1}
$$

$$+\frac{|\delta_1|+|\lambda_1|+|\mu_1|}{|(\delta_1+\mu_1)-\lambda_1|\Gamma(\gamma_1+1)}\Big[(2C^1_{f_1}+C^2_{f_1}]\lambda^{q_2}+M_{f_1}\Big]=0,$$

which is clearly an absurd result. Similarly, for (3.18) a contradiction occurs. Consequently, T has a minimum one fixed point. It results in the existence of at least one solution of (1.4). □

Theorem 3.8. *The operator T has unique fixed point under the conditions (A1)–(A4) and if* $\max\{d_1,d_2\}<1$, *where*

$$\begin{gathered}
d_1=\frac{K_1\Gamma(\gamma_1+1)+4L_{f_1}(|\lambda_1|+|\delta_1|+|\mu_1|)}{|\lambda_1-(\delta_1+\mu_1)|\Gamma(1+\gamma_1)},\\
d_2=\frac{K_2\Gamma(\gamma_2+1)+4L_{f_2}(|\lambda_2|+|\delta_2|+|\mu_2|)}{|\lambda_2-(\delta_2+\mu_2)|\Gamma(\gamma_2+1)}.
\end{gathered}$$

Proof. For (x_1,x_2), $(\bar{x_1},\bar{x_2})\in X\times X$, we can obtain the inequality

$$\begin{aligned}
&\|(H_1(x_1)+F_1(x_1,x_2))-(H_1(\bar{x_1})+F_1(\bar{x_1},\bar{x_2}))\|\\
&\le\frac{K_1\|x_1-\bar{x_1}\|}{|\lambda_1-(\delta_1+\mu_1)|}\\
&\quad+\frac{|\delta_1|}{|\lambda_1-(\delta_1+\mu_1)|\Gamma(\gamma_1)}\int_0^1(1-\theta)^{\gamma_1-1}|f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta))\\
&\quad-f_1(\theta,\bar{x_1}(\theta),x_1(r\theta),\bar{x_2}(\theta))|d\theta\\
&\quad+\frac{|\mu_1|}{|\lambda_1-(\delta_1+\mu_1)|\Gamma(\gamma_1)}\int_0^\eta(\eta-\theta)^{\gamma_1-1}|f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta))\\
&\quad-f_1(\theta,\bar{x_1}(\theta),x_1(r\theta),\bar{x_2}(\theta))|d\theta\\
&\quad+\int_0^t\frac{(t-\theta)^{\gamma_1-1}}{\Gamma(\gamma_1)}|f_1(\theta,x_1(\theta),x_1(r\theta),x_2(\theta))-f_1(\theta,\bar{x_1}(\theta),x_1(r\theta),\bar{x_2}(\theta))|d\theta,\\
&\le\Big[\frac{K_1}{|\lambda_1-(\delta_1+\mu_1)|}+\frac{4L_{f_1}}{\Gamma(\gamma_1+1)}\Big(\frac{|\lambda_1|+|\delta_1|+|\mu_1|}{|\lambda_1-(\delta_1+\mu_1)|}\Big)\Big]\big(\|x_1-\bar{x_1}\|+\|x_2-\bar{x_2}\|\big)\\
&\le d_1\big(\|x_1-\bar{x_1}\|+\|x_2-\bar{x_2}\|\big).
\end{aligned}$$

Similarly, we obtain

$$\|(H_2(x_2)+F_2(x_1,x_2))-(H_2(\bar{x_2})+F_2(\bar{x_1},\bar{x_2}))\|\le d_2\big(\|x_1-\bar{x_1}\|+\|x_2-\bar{x_2}\|\big).$$

Hence, the above two expressions yield

$$\begin{aligned}
\|T(x_1,x_2)-T(\bar{x_1},\bar{x_2})\|&\le\max(d_1,d_2)\big(\|(x_1,x_2)\|-\|(\bar{x_1},\bar{x_2})\|\big)\\
&=d\big(\|(x_1,x_2)\|-\|(\bar{x_1},\bar{x_2})\|\big).
\end{aligned}$$

Therefore, T is contraction as $d<1$ and so T has unique fixed point. □

Remark 3.9. It can be seen that the growth conditions in (3.7) are also valid for $q_1,q_2=1$ whenever,

$$\begin{aligned}
\max\Big\{&C_{f1}+\frac{|\lambda_1|+|\delta_1|+|\mu_1|}{|\lambda_1-(\delta_1+\mu_1)|\Gamma(\gamma_1+1)}(2C^1_{f_1}+C^2_{f_1}),\\
&C_{f_2}+\frac{|\lambda_2|+|\delta_2|+|\mu_2|}{|\lambda_2-(\delta_2+\mu_2)|\Gamma(\gamma_2+1)}(C^1_{f_2}+2C^2_{f_2})\Big\}<1.
\end{aligned}$$

4. Stability analysis

Theorem 4.1. *If the matrix $\mathscr{M}$ converges to zero and assumptions* (A1)–(A4) *hold, then system* (1.4) *is stable in the sense of Ulam-Hyers.*

Proof. For the proof we consider (x_1, x_2), $(\bar{x_1}, \bar{x_2}) \in X$, we write

$$
\begin{aligned}
&\|T_1(x_1,x_2) - T_1(\bar{x_1},\bar{x_2})\| \\
&\le \frac{|g_1(x_1) - g_1(\bar{x_1})|}{|\lambda_1 - (\delta_1+\mu_1)|} + \frac{|\delta_1|}{|\lambda_1 - (\delta_1+\mu_1)|\Gamma(\gamma_1)} \\
&\quad \times \Big| \int_0^1 (1-\theta)^{\gamma_1 - 1}(f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta)) - f_1(\theta, \bar{x_1}(\theta), \bar{x_1}(r\theta), \bar{x_2}(\theta)))d\theta \Big| \\
&\quad + \frac{|\mu_1|}{|\lambda_1 - (\delta_1+\mu_1)|\Gamma(\gamma_1)} \Big| \int_0^\eta (\eta-\theta)^{\gamma_1 - 1}(f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta)) \\
&\quad - f_1(\theta, \bar{x_1}(\theta), \bar{x_1}(r\theta), \bar{x_2}(\theta)))d\theta \Big| \\
&\quad + \frac{1}{\Gamma(\gamma_1)} \Big| \int_0^t (t-\theta)^{\gamma_1 - 1}((f_1(\theta, x_1(\theta), x_1(r\theta), x_2(\theta)) \\
&\quad - f_1(\theta, \bar{x_1}(\theta), \bar{x_1}(r\theta), \bar{x_2}(\theta)))d\theta \Big| \\
&\le \frac{K_1}{|\lambda_1 - (\delta_1+\mu_1)|}\|x_1 - \bar{x_1}\| + \Big(\frac{|\delta_1| + |\mu_1|}{|\lambda_1 - (\delta_1+\mu_1)|\Gamma(\gamma_1+1)} + \frac{1}{\Gamma(\gamma_1+1)} \Big) \\
&\quad \times L_{f_1}(2||x_1 - \bar{x_1}|| + ||x_2 - \bar{x_2}||) \\
&= \Big(\frac{K_1\Gamma(\gamma_1+1) + 4L_{f_1}(|\lambda_1| + |\delta_1| + |\mu_1|)}{|\lambda_1 - (\delta_1+\mu_1)|\Gamma(1+\gamma_1)} \Big) ||x_1 - \bar{x_1}|| \\
&\quad + \Big(\frac{2L_{f_1}(|\lambda_1| + |\delta_1| + |\mu_1|)}{|\lambda_1 - (\delta_1+\mu_1)|\Gamma(1+\gamma_1)} \Big) \|x_2 - \bar{x_2}\| \\
&\le \ell_1 ||x_1 - \bar{x_1}|| + \ell_2 \|x_2 - \bar{x_2}\|
\end{aligned} \tag{4.1}
$$

where

$$
\ell_1 = \frac{K_1\Gamma(\gamma_1+1) + 4L_{f_1}(|\lambda_1| + |\delta_1| + |\mu_1|)}{|\lambda_1 - (\delta_1+\mu_1)|\Gamma(1+\gamma_1)}, \quad \ell_2 = \frac{2L_{f_1}(|\lambda_1| + |\delta_1| + |\mu_1|)}{|\lambda_1 - (\delta_1+\mu_1)|\Gamma(1+\gamma_1)}
$$

are non-negative real numbers. Also with similar approach, we can obtain

$$
\|T_2(x_1,x_2) - T_2(\bar{x_1},\bar{x_2})\| \le \ell_3 \|x_1 - \bar{x_1}\| + \ell_4 \|x_2 - \bar{x_2}\|, \tag{4.2}
$$

where

$$
\ell_3 = \frac{K_2\Gamma(\gamma_2+1) + 4L_{f_2}(|\lambda_2| + |\delta_2| + |\mu_2|)}{|\lambda_2 - (\delta_2+\mu_2)|\Gamma(1+\gamma_2)}, \quad \ell_4 = \frac{2L_{f_2}(|\lambda_2| + |\delta_2| + |\mu_2|)}{|\lambda_2 - (\delta_2+\mu_2)|\Gamma(1+\gamma_2)}
$$

are real and non-negative. Hence, from (4.1) and (4.2), we have the two inequalities

$$
\begin{aligned}
\|T_1(x_1,x_2) - T_1(\bar{x_1},\bar{x_2})\| &\le \ell_1 \|x_1 - \bar{x_1}\| + \ell_2 \|x_2 - \bar{x_2}\|, \\
\|T_2(x_1,x_2) - T_2(\bar{x_1},\bar{x_2})\| &\le \ell_3 \|x_1 - \bar{x_1}\| + \ell_4 \|x_2 - \bar{x_2}\|.
\end{aligned} \tag{4.3}
$$

From (4.3), we have

$$
\mathscr{M} = \begin{bmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{bmatrix},
$$

Since $\mathscr{M}$ converges to 0, system (1.4) is UH stable. Consequently it is generalized UH stable. □

Example 4.2. Consider the coupled system with non-local boundary conditions,

$$
\begin{gathered}
{}^{C}D^{\frac{9}{10}}x_1(t) = \frac{1}{3} + \frac{1}{e^{\pi t}(t+80)}\sqrt{|x_1(rt)|} + \sin\sqrt{|x_2(t)|} + \sqrt{|\sin x_1(t)|}, \\
{}^{C}D^{\frac{2}{3}}x_2(t) = \frac{1}{2} + \frac{1}{(t+90)}(|x_1(t)| + \sin\sqrt{|x_2(rt)|} + \sin|x_2(t)|, \; t \in [0,1], \\
x_1(0) - x_1(1) - x_1\big(\frac{1}{2}\big) = \frac{\sin x_1(t)}{30}, \quad x_2(0) - x_2(1) - x_2\big(\frac{1}{2}\big) = \frac{\sin x_2(t)}{t+70}.
\end{gathered} \tag{4.4}
$$

Here

$$
\begin{aligned}
f_1(t,x_1(t),x_1(rt),x_2(t)) &= \frac{1}{3} + \frac{1}{e^{\pi t}(t+80)}\big(\sqrt{|x_1(rt)|} + \sin\sqrt{|x_2(t)|} \\
&\quad + \sqrt{|\sin x_1(t)|}\big), \\
f_2(t,x_1(t),x_2(t),x_2(rt)) &= \frac{1}{2} + \frac{1}{(t+90)}(|x_1(t)| + \sin\sqrt{|x_2(rt)|} + \sin|x_2(t)|,
\end{aligned}
$$

$\gamma_1 = 9/10$, $\gamma_2 = 2/3$, $\lambda_1 = \lambda_2 = 4$, $\delta_1 = \delta_2 = 1$, $\mu_1 = \mu_2 = 0$, $\eta = \zeta = 1/2$, and $r \in I$. Since $\lambda_l \neq \delta_l + \mu_l$ $(l = 1,2)$, and

$$
\begin{aligned}
&\|f_1(t,x_1(t),x_1(rt),x_2(t)) - f_1(t,\bar{x_1}(t),\bar{x_1}(rt),\bar{x_2}(t))\| \\
&\leq \frac{1}{80}[2\|x_1 - \bar{x_1}\| + \|x_2 - \bar{x_2}\|] \qquad , \\
&\|f_2(t,x_1(t),x_2(t),x_2(rt) - f_2(t,\bar{x_1}(t),\bar{x_2}(t),\bar{x_2}(rt))\| \\
&\leq \frac{1}{90}[\|x_1 - \bar{x_1}\| + 2\|x_2 - \bar{x_2}\|],
\end{aligned}
$$

it follows that $K_1 = 1/30$, $K_2 = 1/70$, $M_{g_1} = 0 = M_{g_2}$, $M_{f_1} = 1/3$, $M_{f_1} = 1/2$, $L_{f_1} = 1/80$, $L_{f_2} = 1/90$, $C^1_{f_1} = C^2_{f_1} = 1/80$, $C^1_{f_2} = C^2_{f_2} = 1/90$, $1 = q_1 = q_2$. So all assumption of Theorem (3.8) are fulfilled, $d_1 = 0.0978 < 1$, and $d_2 = 0.0868 < 1$. So, system (4.4) has unique solution. Also for stability, the matrix

$$
\mathscr{M} = \begin{bmatrix} 0.0978 & 0.0433 \\ 0.0868 & 0.0410 \end{bmatrix}
$$

has eigenvalues equal to 0.1370 and 0.0018. Clearly $\Upsilon(\mathscr{M}) < 1$, and hence, by Theorem 4.1, matrix $\mathscr{M}$ converges to zero. Therefore, the system (4.4) is stable in the sense to Ulam-Hyers. Also, it is generalized Ulam-Hyers stable.

Conclusion. In this research, we obtained some results regarding the qualitative theory for a delay differential system via degree degree theory from topology and standard functional analysis.

Authors contribution: Equal contributions have been done by all the authors.

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Addendum posted by the editor on July 26, 2020

After this article was published, a reader indicated that there were some mistakes in this article. The editor contacted the authors and sent them detailed comments, one of the authors tried unsuccessfully to address the concerns several times. The editor therefore decided to post this addendum based on the detailed comments.

The paper has some important errors, as well as numerous typos which are not given in this addendum.

Definition 2.2, needs at least $x^{(k-1)} \in AC$ (Absolutely Continuous), then the Caputo derivative is defined almost everywhere (a.e.), stronger condition is needed if it is wanted to exist for all t.

Lemma 2.3, needs (at least) $x^{(k-1)} \in AC$.

Theorem 3.1, What is given is correct, but it proves that if $x_1 \in AC$ is a solution of (3.1) then x_1 is a solution of (3.2), it is not proving what is required. It should assume w_1 continuous, otherwise $x_1(0) = a_0$ could be not valid in the proof. But it is necessary to prove, for Corollary 3.2, that a solution $x_1 \in C[0,1]$ of (3.2) is in $AC[0,1]$ so that the Caputo derivative exists a.e. and it can be a solution of (3.1) for a.e. t. If w is continuous then $I^\gamma w$ is continuous but it does not follow that $I^\gamma w \in AC$ without extra conditions, a result shown by G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals. I., *Math. Z.* 27 (1928), 565–606.

Lemma 3.3 is not valid with given definition of Caputo derivative. It is known to experts that one does not have equivalence (if and only if), which is why the following book uses a different definition.

Diethelm, Kai, The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type. Lecture Notes in Mathematics, 2004. Springer-Verlag, Berlin, 2010

The lack of equivalence has been pointed out explicitly in the recent papers:

Cichon, Mieczyslaw; Salem, Hussein A. H., On the lack of equivalence between differential and integral forms of the Caputo-type fractional problems. J. Pseudo-Differ. Oper. Appl. 11 (2020), no. 4, 1869–1895.

Webb, Jeffrey R. L., Initial value problems for Caputo fractional equations with singular nonlinearities. Electron. J. Differential Equations 2019, Paper No. 117, 32 pp (addendum)

Theorem 4.1. 'If the matrix $\mathscr{M}$ converges to zero', it is a particular matrix here, which should be specified, and it requires some hypotheses on (4.1), (4.2) to make it true.

Example 4.2, some important brackets are omitted; the example is not correct, $\sin(x_1(t))/30$ is not a functional $g(x_1)$ on X, neither is the other term; probably $\sin(\|x_1\|)/30$ is intended.

End of addendum.

Israr Ahmad
Department of Mathematics, University of Malakand, Chakdara, Dir(L), Khyber Pakhtunkhwa, Pakistan
Email address: israrahmadjc503@gmail.com

Juan Jose Nieto
Instituto de Matemeticas, Universidade de Santiago de Compostela, Santiago de Compostela, 15782, Spain
Email address: juanjose.nieto.roig@usc.es

Ghaus ur Rahman
Department of Mathematics and Statistics, University of Swat, Khyber Pakhtunkhwa, Pakistan
Email address: ghaus957@yahoo.com

Kamal Shah
Department of Mathematics, University of Malakand, Chakdara, Dir(L), Khyber Pakhtunkhwa, Pakistan
Email address: kamalshah408@gmail.com