Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 50, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MATHEMATICAL METHODS FOR THE RANDOMIZED NON-AUTONOMOUS BERTALANFFY MODEL

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Abstract. In this article we analyze the randomized non-autonomous Bertalanffy model

$$x'(t,\omega)=a(t,\omega)x(t,\omega)+b(t,\omega)x(t,\omega)^{2/3},\quad x(t_0,\omega)=x_0(\omega),$$

where $a(t,\omega)$ and $b(t,\omega)$ are stochastic processes and $x_0(\omega)$ is a random variable, all of them defined in an underlying complete probability space. Under certain assumptions on a, b and x_0, we obtain a solution stochastic process, $x(t,\omega)$, both in the sample path and in the mean square senses. By using the random variable transformation technique and Karhunen-Loève expansions, we construct a sequence of probability density functions that under certain conditions converge pointwise or uniformly to the density function of $x(t,\omega)$, $f_{x(t)}(x)$. This permits approximating the expectation and the variance of $x(t,\omega)$. At the end, numerical experiments are carried out to put in practice our theoretical findings.

1. Introduction and motivation

Bertalanffy model [2] is a biological ordinary differential equation model that describes the relationship between the metabolism and the growth of an organism. The metabolism is divided into anabolism (synthesis) and catabolism (destruction). The model assumes that the body weight $W(t)$, at the time instant t, of an animal is the result of the counteraction of the processes of anabolism and catabolism,

$$W'(t)=\eta W^m(t)-\kappa W^n(t),$$

where η and κ are the constants of anabolism and catabolism, respectively; both constants are proportional to some power of the body weight at the time t. This model follows the law of allometry: the rate of change of weight depends on the constants of anabolism and catabolism via a power of the weight.

The surface rule states that the dependence of anabolism on body weight takes the power $m=2/3$ [1, 26] (however, other exponents have been suggested, see [18]). Bertalanffy justified, by using physiological facts and mathematical considerations, that the rate of catabolism should have the power $n=1$. Bertalanffy model thus

2010 *Mathematics Subject Classification.* 34F05, 60H35, 60H10, 65C30.
Key words and phrases. Random non-autonomous Bertalanffy model; random differential equation; random variable transformation technique; Karhunen-Loève expansion; probability density function.

Submitted July 20, 2019. Published May 26, 2020.

becomes

$$W'(t) = \eta W^{2/3}(t) - \kappa W(t).$$

In this article, we want to perform a mathematical study of the randomized non-autonomous Bertalanffy model,

$$\begin{gathered} x'(t,\omega) = a(t,\omega)x(t,\omega) + b(t,\omega)x(t,\omega)^{2/3}, \quad t \in [t_0,T],\ \omega \in \Omega, \\ x(t_0,\omega) = x_0(\omega), \quad \omega \in \Omega. \end{gathered} \tag{1.1}$$

In this setting, we consider an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The set Ω contains the outcomes, that will be generically denoted by ω, the set $\mathcal{F}$ is the σ-algebra of events and $\mathbb{P}$ is the probability measure. In (1.1), we are also considering the stochastic processes

$$a = \{a(t,\omega) : t \in [t_0,T],\, \omega \in \Omega\}, \quad b = \{b(t,\omega) : t \in [t_0,T],\, \omega \in \Omega\}$$

and the random variable $x_0(\omega)$, all of them defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The solution to the stochastic system, $x(t,\omega)$, becomes a stochastic process.

The mathematical term that encompasses equations such as (1.1) is *random differential equation* (RDE) [20, Ch. 8], [22, 23]. RDEs are differential equations in which randomness enters via the input data (coefficients and initial conditions). Any probability distribution for the input data is allowed. RDEs can be interpreted in a sample path sense or in a random L^p sense. In both senses, one of the main goals is to compute or approximate the solution stochastic process $x(t,\omega)$ and to calculate its main statistical information, say the expectation, $\mathbb{E}[x(t,\omega)]$, and the variance, $\mathbb{V}[x(t,\omega)]$ (uncertainty quantification). Both statistics can be computed or approximated if one is able to compute or approximate the probability density function of the solution process, $f_{x(t)}(x)$, since all one-dimensional moments and probabilities can be determined:

$$\begin{gathered} \mathbb{E}[x(t,\omega)^k] = \int_{\mathbb{R}} x^k f_{x(t)}(x)\,\mathrm{d}x, \\ \mathbb{P}(a_1 \le x(t,\omega) \le a_2) = \int_{a_1}^{a_2} f_{x(t)}(x)\,\mathrm{d}x. \end{gathered}$$

This target has been achieved in [5, 6, 7, 8, 10, 12, 13], by using the Random Variable Transformation technique [15, Lemma 4.12], although the inputs are random variables (time invariant) rather than stochastic processes. The random autonomous Bertalanffy model has been studied, with applications to fish weight growth modelling, in [5]. The recent paper [4], that has become the main reference for our article, studied the random non-autonomous inhomogeneous linear differential equation, by focusing on approximating the probability density function of the solution stochastic process. Its main contribution has been using Karhunen-Loève expansions [15, Ch. 5] to construct the approximating sequence of density functions. Other mathematical studies of the Bertalanffy model, in the sense of stochastic differential equations (SDEs) of Itô type [16], have been carried out in [17, 21]. In SDEs of Itô type, the uncertainty is introduced in the deterministic differential equation by means of stochastic perturbations driven by the white noise process.

The goal of this paper is to perform a mathematical study of the random non-autonomous Bertalanffy model (1.1). It is proved that, under certain conditions, there is a solution stochastic process in the sample path and the mean square senses. This is done by relating the random non-autonomous Bertalanffy model to the random non-autonomous linear differential equation from [4], via the usual change of variables performed in Bernoulli differential equations. The Random Variable Transformation technique permits relating the probability density functions of the solution processes to the random non-autonomous Bertalanffy model and the random non-autonomous linear differential equation. The Karhunen-Loève expansions of the coefficient stochastic processes in the RDE (1.1) allow us to obtain an approximating sequence of probability density functions for the density $f_{x(t)}(x)$ of the solution process $x(t,\omega)$. A numerical and computational treatment of the theoretical results obtained will be performed at the end.

2. Solving the random non-autonomous Bertalanffy model

There are different ways of interpreting the random differential equation (1.1). One way, which strongly uses the deterministic theory on differential equations, is the sample path interpretation [20, p. 440], [23, p. 2, SP problem]: fixed $\omega \in \Omega$, the random problem (1.1) becomes a deterministic problem, so one looks for stochastic processes $x(t,\omega)$ with absolutely continuous sample paths that solve (1.1). Another way consists in using $\mathrm{L}^p(\Omega)$ random calculus (see [22, Ch. 4] and [15, Section 5.5] for an introduction). We will study solutions to (1.1) in the mean square sense, according to the definition in [22, p. 118] or [23, p. 3, L^p problem].

Consider the stochastic process

$$x(t,\omega) = \Big(x_0(\omega)^{1/3}\mathrm{e}^{\frac{1}{3}\int_{t_0}^t a(s,\omega)\,\mathrm{d}s} + \frac{1}{3}\int_{t_0}^t b(s,\omega)\mathrm{e}^{\frac{1}{3}\int_s^t a(r,\omega)\,\mathrm{d}r}\,\mathrm{d}s\Big)^3. \tag{2.1}$$

This stochastic process comes from randomizing the deterministic solution to the deterministic non-autonomous Bertalanffy model. The integrals in (2.1) are considered in the Lebesgue sense, for each $\omega \in \Omega$ fixed (sample path Lebesgue integral), or in the mean square sense.

2.1. Solution stochastic process with absolutely continuous sample paths.

By the classical theory on deterministic differential equations [11, p. 28–30], the following theorem holds.

Theorem 2.1 (Sample path solution to the random Bertalanffy model)**.** *Suppose that the processes $a(\cdot,\omega), b(\cdot,\omega) \in \mathrm{L}^1([t_0,T])$, for a.e. $\omega \in \Omega$. Then the stochastic process $x(t,\omega)$ given by (2.1) satisfies that, for a.e. $\omega \in \Omega$, $x(\cdot,\omega)$ is absolutely continuous on $[t_0,T]$ and satisfies (1.1) for a.e. $t \in [t_0,T]$.*

If $a(\cdot,\omega)$ and $b(\cdot,\omega)$ are continuous on $[t_0,T]$, then $x(\cdot,\omega)$ is in $C^1([t_0,T])$ and satisfies (1.1) for all $t \in [t_0,T]$.

2.2. Solution stochastic process in the mean square sense.

Consider the random linear differential equation

$$\begin{gathered} y'(t,\omega) = \frac{1}{3}a(t,\omega)y(t,\omega) + \frac{1}{3}b(t,\omega), \quad t \in [t_0,T],\ \omega \in \Omega, \\ y(t_0,\omega) = x_0(\omega)^{1/3}, \quad \omega \in \Omega. \end{gathered} \tag{2.2}$$

Notice that, if a stochastic process $y(t,\omega)$ is a solution to (2.2) in the $\mathrm{L}^6(\Omega)$ sense, then the stochastic process defined as $x(t,\omega) = y(t,\omega)^3$ is a solution to (1.1) in the $\mathrm{L}^2(\Omega)$ sense. Indeed, if $y(t,\omega)$ is differentiable in the $\mathrm{L}^6(\Omega)$ sense, then $x(t,\omega)$

is differentiable in the $\mathrm{L}^2(\Omega)$ sense, by [25, Lemma 3.14], and by the product rule, $x(t,\omega)$ solves (1.1) in the $\mathrm{L}^2(\Omega)$ sense.

Let $y(t,\omega)$ be a stochastic process solution to (2.2) in the $\mathrm{L}^6(\Omega)$ sense. By [20, Thm. 8–20] and [23, Thm. 3(a)], there exists a stochastic process $\varphi(t,\omega)$, measurable on $[t_0,T]\times\Omega$, equivalent to $y(t,\omega)$ (meaning that $\varphi(t,\cdot)=y(t,\cdot)$ a.s., for all $t\in[t_0,T]$), such that $\varphi(\cdot,\omega)$ is absolutely continuous on $[t_0,T]$ and solves (2.2) in the sample path sense, for each $\omega\in\Omega$. By [4, Thm. 1.3], up to equivalence,

$$y(t,\omega)=x_0(\omega)^{1/3}\mathrm{e}^{\frac{1}{3}\int_{t_0}^t a(s,\omega)\,\mathrm{d}s}+\frac{1}{3}\int_{t_0}^t b(s,\omega)\mathrm{e}^{\frac{1}{3}\int_s^t a(r,\omega)\,\mathrm{d}r}\,\mathrm{d}s \tag{2.3}$$

Thus, (2.3) is the unique candidate, up to equivalence, that solves (2.2) in the $\mathrm{L}^6(\Omega)$ sense. We are going to show that, under some conditions on the stochastic processes $a(t,\omega)$ and $b(t,\omega)$, the stochastic process $y(t,\omega)$ given by (2.3) is indeed a solution to (2.2) in the $\mathrm{L}^6(\Omega)$ sense. From this fact, (2.1) will solve the random Bertalanffy model (1.1) in the mean square sense.

Theorem 2.2 (Mean square solution to the random Bertalanffy model)**.** *Suppose*

(i) *x_0, a and b are independent.*
(ii) *$x_0\in\mathrm{L}^4(\Omega)$, and a and b are $\mathrm{L}^{12}(\Omega)$ continuous.*
(iii) *There exist $r>12$ and $\delta>0$ such that*

$$\sup_{s,s^*\in[-\delta,\delta]}\mathbb{E}\big[\mathrm{e}^{r\int_{x+s}^{t+s^*}a(u)\,\mathrm{d}u}\big]<\infty,$$

for each $t_0\le x\le t\le T$.

Then the stochastic process $y(t,\omega)$ defined by (2.3) *(with mean square Riemann integrals) is differentiable in the $\mathrm{L}^6(\Omega)$ sense and satisfies the random problem* (2.2)*. As a consequence, the stochastic process $x(t,\omega)$ defined by* (2.1) *is differentiable in the mean square sense and satisfies the random Bertalanffy model* (1.1)*.*

The proof of the above theorem is straightforward from [9]. Denote $y_0(\omega)=x_0(\omega)^{1/3}$, which belongs to $\mathrm{L}^{12}(\Omega)$. We apply the theory from [9, Section 3] adapted to $\mathrm{L}^6(\Omega)$ calculus.

Example 2.3 (Applications of Theorem 2.2)**.** Let us see some examples of processes $a(t,\omega)$ for which the hypotheses of Theorem 2.2 fulfill.

• Let $[t_0,T]=[0,1]$ and $a(t,\omega)$ be a standard Brownian motion on $[0,1]$ [15, p. 185–186]. We have $a(t+h,\omega)-a(t,\omega)\sim\mathrm{Normal}(0,h)$, so

$$\|a(t+h,\omega)-a(t,\omega)\|_{\mathrm{L}^{12}(\Omega)}=\|\sqrt{h}\,Z\|_{\mathrm{L}^{12}(\Omega)}=\sqrt{h}\,\|Z\|_{\mathrm{L}^{12}(\Omega)}\to 0\quad\text{as } h\to 0,$$

being $Z\sim\mathrm{Normal}(0,1)$. This shows that $a(t,\omega)$ is continuous in the $\mathrm{L}^{12}(\Omega)$ sense. Another way of checking the $\mathrm{L}^{12}(\Omega)$-continuity of the Brownian motion uses [25, Lemma 3.11]: $a(t,\omega)$ is continuous in the $\mathrm{L}^{12}(\Omega)$ sense if and only if the function $\mathbb{E}[a(t_1,\omega)\cdots a(t_{12},\omega)]$ defined on $\mathbb{R}^{12}$ is continuous on the diagonal $(t,\dots,t)\in[t_0,T]^{12}$. This is clear by [22, p. 28].

On the other hand, since $a(t,\omega)$ is a Gaussian process, its mean square integral is normally distributed [22, Thm. 4.6.4]. Bearing in mind the moment generating function of a normal distribution,

$$\mathbb{E}\big[\mathrm{e}^{r\int_{x+s}^{t+s^*}a(u,\omega)\,\mathrm{d}u}\big]<\infty.$$

• If $a(t,\omega)$ is a standard Brownian bridge on $[0,1]$, [15, p. 193–195], the same holds, as $a(t,\omega)$ is a Gaussian process with stationary increments.

• An example of a non-Gaussian process is given by $a(t,\omega)=t\,U(\omega)$, $t\in[t_0,T]$, being U any bounded random variable. Indeed, $a(t,\omega)$ is $\mathrm{L}^{12}(\Omega)$-continuous, since

$$\|a(t+h,\omega)-a(t,\omega)\|_{\mathrm{L}^{12}(\Omega)}=|h|\|U\|_{\mathrm{L}^{12}(\Omega)}\xrightarrow{h\to 0}0.$$

On the other hand,

$$\mathbb{E}\Big[\mathrm{e}^{r\int_{x+s}^{t+s^*}a(u,\omega)\,\mathrm{d}u}\Big]\le\mathbb{E}\Big[\mathrm{e}^{r\|U\|_{\mathrm{L}^\infty(\Omega)}\int_{x+s}^{t+s^*}u\,\mathrm{d}u}\Big]=\mathrm{e}^{r\|U\|_{\mathrm{L}^\infty(\Omega)}\frac{(t+s^*)^2-(x+s)^2}{2}}<\infty,$$

so the hypotheses of Theorem 2.2 hold for $a(t,\omega)$.

3. Obtaining the probability density function of the solution stochastic process

The aim of this section is at approximating the probability density function of the solution stochastic process to the random Bertalanffy model, $x(t,\omega)$ given by (2.1). To achieve this goal, we will use existing results on the random linear differential equation from [4], together with a version of the Random Variable Transformation technique and the Karhunen-Loève expansions of both processes $a(t,\omega)$ and $b(t,\omega)$ from (1.1).

Lemma 3.1 (Random Variable Transformation technique [14, p. 115])**.** *Let X be an absolutely continuous random variable with density f_X and with support D_X contained in an open set $D\subseteq\mathbb{R}$. Let $g:D\to\mathbb{R}$ be such that $D=\cup_{i=1}^n D_i$ and $g_i=g|_{D_i}$ is injective and $C^1(D_i)$ with non-vanishing derivative. Then the random variable $Y=g(X)$ is absolutely continuous, with density function given by*

$$f_Y(y)=\begin{cases}\sum_{i:y\in g(D_i)}f_X(g_i^{-1}(y))|\frac{\mathrm{d}g_i^{-1}(y)}{\mathrm{d}y}|, & y\in g(D),\\ 0, & y\notin g(D).\end{cases}$$

Lemma 3.2 (Karhunen-Loève Theorem [15, Thm. 5.28])**.** *Consider a stochastic process $\{X(t,\omega):t\in\mathcal{T},\,\omega\in\Omega\}$ in $\mathrm{L}^2(\mathcal{T}\times\Omega)$. Then*

$$X(t,\omega)=\mu(t)+\sum_{j=1}^\infty\sqrt{\nu_j}\,\phi_j(t)\xi_j(\omega),\tag{3.1}$$

where the sum converges in $\mathrm{L}^2(\mathcal{T}\times\Omega)$, $\mu(t)=\mathbb{E}[X(t)]$, $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis of $\mathrm{L}^2(\mathcal{T})$, $\{(\nu_j,\phi_j)\}_{j=1}^\infty$ is the set of pairs of (nonnegative) eigenvalues and eigenvectors of the operator

$$\mathcal{C}:\mathrm{L}^2(\mathcal{T})\to\mathrm{L}^2(\mathcal{T}),\quad \mathcal{C}f(t)=\int_{\mathcal{T}}\mathrm{Cov}[X(t),X(s)]f(s)\,\mathrm{d}s,\tag{3.2}$$

and $\{\xi_j\}_{j=1}^\infty$ is a sequence of random variables with zero expectation, unit variance and pairwise uncorrelated. In (3.2), $\mathrm{Cov}[\cdot,\cdot]$ stands for the covariance operator. Moreover, if $\{X(t):t\in\mathcal{T}\}$ is a Gaussian process, then $\{\xi_j\}_{j=1}^\infty$ are independent and Gaussian.

3.1. **Main results.**

Let $a(t,\omega)$ and $b(t,\omega)$ be stochastic processes in $\mathrm{L}^2([t_0,T]\times\Omega)$. According to Lemma 3.2, we can expand both $a(t,\omega)$ and $b(t,\omega)$ via a Karhunen-Loève expansion:

$$a(t,\omega)=\mu_a(t)+\sum_{j=1}^\infty\sqrt{\nu_j}\,\phi_j(t)\xi_j(\omega),\quad b(t,\omega)=\mu_b(t)+\sum_{j=1}^\infty\sqrt{\gamma_j}\,\psi_j(t)\eta_j(\omega),\tag{3.3}$$

respectively. The summation symbol in both expansions will be always written up to ∞, although it could be possible that their corresponding covariance integral operators $\mathcal{C}$ have only a finite number of nonzero eigenvalues, so that the summation symbol finishes at an index $J < \infty$. As the more complex case arises when the sum arrives at infinity, we will always write the Karhunen-Loève expansions of $a(t,\omega)$ and $b(t,\omega)$ up to infinity.

Notice that, from $a, b \in \mathrm{L}^2([t_0,T] \times \Omega)$, we have $a(\cdot,\omega), b(\cdot,\omega) \in \mathrm{L}^1([t_0,T])$, therefore the process $x(t,\omega)$ has absolutely continuous sample paths and solves the random Bertalanffy model (1.1), by Theorem 2.1. Under the stricter assumptions of Theorem 2.2, the process $x(t,\omega)$ will be a mean square solution too.

For convenience, let us consider the truncation of the Karhunen-Loève expansions (3.3) of the stochastic processes a and b of common order N

$$a_N(t,\omega) = \mu_a(t) + \sum_{j=1}^{N} \sqrt{\nu_j}\, \phi_j(t)\xi_j(\omega), \quad b_N(t,\omega) = \mu_b(t) + \sum_{j=1}^{N} \sqrt{\gamma_j}\, \psi_j(t)\eta_j(\omega).$$

This gives a truncation of the solution stochastic process $x(t,\omega)$ given by (2.1),

$$x_N(t,\omega) = \Big(x_0(\omega)^{1/3} \mathrm{e}^{\frac{1}{3}\int_{t_0}^{t} a_N(s,\omega)\,\mathrm{d}s} + \frac{1}{3}\int_{t_0}^{t} b_N(s,\omega) \mathrm{e}^{\frac{1}{3}\int_{s}^{t} a_N(r,\omega)\,\mathrm{d}r}\,\mathrm{d}s \Big)^3, \quad (3.4)$$

We also obtain a truncation of the solution stochastic process $y(t,\omega)$ to the random linear differential equation, given by (2.3):

$$y_N(t,\omega) = x_0(\omega)^{1/3} \mathrm{e}^{\frac{1}{3}\int_{t_0}^{t} a_N(s,\omega)\,\mathrm{d}s} + \frac{1}{3}\int_{t_0}^{t} b_N(s,\omega) \mathrm{e}^{\frac{1}{3}\int_{s}^{t} a_N(r,\omega)\,\mathrm{d}r}\,\mathrm{d}s. \quad (3.5)$$

Naturally, the relation between both truncations (3.4) and (3.5) is that $x_N(t,\omega) = y_N(t,\omega)^3$.

We denote, as in [4], the following vectors in bold letters, $\boldsymbol{\xi}_N = (\xi_1,\dots,\xi_N)$ and $\boldsymbol{\eta}_M = (\eta_1,\dots,\eta_M)$, understanding this as a random vector or as a deterministic real vector, depending on the context. Denote

$$K_a(t,\boldsymbol{\xi}_N) = \int_{t_0}^{t} \Big(\mu_a(s) + \sum_{j=1}^{N} \sqrt{\nu_j}\, \phi_j(s)\xi_j \Big)\,\mathrm{d}s,$$
$$S_b(s,\boldsymbol{\eta}_N) = \mu_b(s) + \sum_{i=1}^{N} \sqrt{\gamma_i}\, \psi_i(s)\eta_i.$$

Suppose that x_0 and $(\xi_1,\dots,\xi_N,\eta_1,\dots,\eta_N)$ are absolutely continuous and independent, for each $N \geq 1$. Let $y_0(\omega) = x_0(\omega)^{1/3}$ be the initial condition of the random linear differential equation (2.2). By Lemma 3.1 applied with the transformation mapping $g(x) = x^{1/3}$ on $D = \mathbb{R}\backslash\{0\}$, with domain partition $D = D_1 \cup D_2$, being $D_1 = (0,\infty)$ and $D_2 = (-\infty,0)$, we have $y_0(\omega)$ is absolutely continuous, with density function

$$f_{y_0}(y) = f_{x_0}(y^3)3y^2, \quad (3.6)$$

for $y \in \mathbb{R}$.

By using the version of the Random Variable Transformation technique of [4, Lemma 2.1], in [4, Expression (10)] it was obtained the probability density function of $y_N(t,\omega)$,

$$f_{y_N(t)}(y)$$

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} f_{y_0}\Big(y\,\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}-\frac{1}{3}\int_{t_0}^{t} S_b(s,\boldsymbol{\eta}_N)\mathrm{e}^{-\frac{1}{3}K_a(s,\boldsymbol{\xi}_N)}\,\mathrm{d}s\Big)\\
&\quad\times \mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)} f_{\boldsymbol{\xi}_N,\boldsymbol{\eta}_N}(\boldsymbol{\xi}_N,\boldsymbol{\eta}_N)\,\mathrm{d}\boldsymbol{\xi}_N\,\mathrm{d}\boldsymbol{\eta}_N\\
&=\mathbb{E}\Big[f_{y_0}\Big(y\,\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}-\frac{1}{3}\int_{t_0}^{t} S_b(s,\boldsymbol{\eta}_N)\mathrm{e}^{-\frac{1}{3}K_a(s,\boldsymbol{\xi}_N)}\,\mathrm{d}s\Big)\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}\Big]\\
&=\mathbb{E}\Big[f_{x_0}\Big(\Big\{y\,\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}-\frac{1}{3}\int_{t_0}^{t} S_b(s,\boldsymbol{\eta}_N)\mathrm{e}^{-\frac{1}{3}K_a(s,\boldsymbol{\xi}_N)}\,\mathrm{d}s\Big\}^3\Big)\\
&\quad\times 3\Big\{y\,\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}-\frac{1}{3}\int_{t_0}^{t} S_b(s,\boldsymbol{\eta}_N)\mathrm{e}^{-\frac{1}{3}K_a(s,\boldsymbol{\xi}_N)}\,\mathrm{d}s\Big\}^2\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}\Big],
\end{aligned}
$$

for $y \in \mathbb{R}$.

Since $x_N(t,\omega) = y_N(t,\omega)^3$, by Lemma 3.1 with the transformation mapping $g(x) = x^3$ on $D = \mathbb{R}\backslash\{0\}$, with domain partition $D = D_1 \cup D_2$, being $D_1 = (0,\infty)$ and $D_2 = (-\infty, 0)$, we have that $x_N(t,\omega)$ is an absolutely continuous random variable for each $t \in [t_0, T]$, with density function

$$
\begin{aligned}
&f_{x_N(t)}(x)\\
&= f_{y_N(t)}\big(x^{1/3}\big)\frac{1}{3x^{2/3}}\\
&= \frac{1}{x^{2/3}}\mathbb{E}\Big[f_{x_0}\Big(\Big\{x^{1/3}\,\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}-\frac{1}{3}\int_{t_0}^{t} S_b(s,\boldsymbol{\eta}_N)\mathrm{e}^{-\frac{1}{3}K_a(s,\boldsymbol{\xi}_N)}\,\mathrm{d}s\Big\}^3\Big)\\
&\quad\Big\{x^{1/3}\,\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}-\frac{1}{3}\int_{t_0}^{t} S_b(s,\boldsymbol{\eta}_N)\mathrm{e}^{-\frac{1}{3}K_a(s,\boldsymbol{\xi}_N)}\,\mathrm{d}s\Big\}^2\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}\Big],
\end{aligned}
\tag{3.7}
$$

for $0 \neq x \in \mathbb{R}$. Density functions are defined up to sets of Lebesgue measure 0, so the fact that $f_{x_N(t)}(x)$ is not defined at $x = 0$ is not a problem.

Under some assumptions, for instance, by taking into account [4, Thm. 2.9, Thm. 2.12], the random variable $y(t,\omega)$ is absolutely continuous for each $t \in [t_0, T]$, with density function

$$
f_{y(t)}(y) = \lim_{N\to\infty} f_{y_N(t)}(y), \tag{3.8}
$$

for all $y \in \mathbb{R}$. Bearing in mind that $x(t,\omega) = y(t,\omega)^3$, by Lemma 3.1 with the transformation mapping $g(x) = x^3$ on $D = \mathbb{R}\backslash\{0\}$, we get that $x(t,\omega)$ is an absolutely continuous random variable, for each $t \in [t_0, T]$, with density function

$$
f_{x(t)}(x) = f_{y(t)}\big(x^{1/3}\big)\frac{1}{3x^{2/3}}, \tag{3.9}
$$

for $0 \neq x \in \mathbb{R}$. By combining (3.7), (3.8) and (3.9),

$$
f_{x(t)}(x) = f_{y(t)}\big(x^{1/3}\big)\frac{1}{3x^{2/3}} = \lim_{N\to\infty} f_{y_N(t)}\big(x^{1/3}\big)\frac{1}{3x^{2/3}} = \lim_{N\to\infty} f_{x_N(t)}(x),
$$

for all $0 \neq x \in \mathbb{R}$.

The goal is to find out under which conditions on the stochastic processes $a(t,\omega)$ and $b(t,\omega)$ and on the random variable $x_0(\omega)$ from (1.1), the solution stochastic process $x(t,\omega)$ given by (2.1) is an absolutely continuous random variable, for each $t \in [t_0, T]$, with density function satisfying

$$
f_{x(t)}(x) = \lim_{N\to\infty} f_{x_N(t)}(x), \tag{3.10}
$$

for each $0 \neq x \in \mathbb{R}$. For this purpose, we will use results on the random linear differential equation (2.2) [4, Thm. 2.9, Thm. 2.12], that establish under which conditions the limit (3.8) is justified.

In the next theorem we use the following assumptions:

Theorem 3.3. *Assume the following hypotheses:*

(1) $a, b \in \mathrm{L}^2([t_0, T] \times \Omega)$;
(2) x_0 *and* $(\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N)$ *are absolutely continuous and independent,* $N \geq 1$;
(3) *the density function of* x_0, f_{x_0}, *is continuous on* $\mathbb{R}$ *and* $f_{x_0}(x) \leq \frac{C}{|x|^{2/3}}$, *for* $x \neq 0$;
(4) $\|\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}\|_{\mathrm{L}^2(\Omega)} \leq C$ *for all* $N \geq 1$ *and all* $t \in [t_0, T]$.

Then, for all $0 \neq x \in \mathbb{R}$ *and* $t \in [t_0, T]$, *the sequence* $\{f_{x_N(t)}(x)\}_{N=1}^{\infty}$ *given by* (3.7) *converges to the density* $f_{x(t)}(x)$ *of the solution process* $x(t, \omega)$ *given by* (2.1).

Proof. Since $y_0(\omega) = x_0(\omega)^{1/3}$, by hypothesis (2), y_0 and $(\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N)$ are absolutely continuous and independent, for $N \geq 1$. By (3.6) and hypothesis (3), $f_{y_0}(y)$ is continuous on $\mathbb{R}$ and

$$f_{y_0}(y) \leq \frac{C}{|y^3|^{2/3}} 3y^2 = 3C,$$

for $y \neq 0$, therefore bounded. The hypotheses of [4, Thm. 2.9] are fulfilled for (2.2), therefore $y(t, \omega)$ is an absolutely continuous random variable for each $t \in [t_0, T]$, with density function satisfying (3.8). Then, $x(t, \omega)$ is absolutely continuous and verifies (3.10). □

Theorem 3.4. *Assume the following hypotheses:*

(1) $a, b \in \mathrm{L}^2([t_0, T] \times \Omega)$;
(2) $x_0, \eta_1, (\xi_1, \dots, \xi_N, \eta_2, \dots, \eta_N)$ *are absolutely continuous and independent,* $N \geq 1$;
(3) *the density function of* η_1, f_{η_1}, *is continuous and bounded on* $\mathbb{R}$;
(4) $\xi_1, \xi_2, \dots$ *have compact support in* $[-A, A]$ $(A > 0)$ *and* $\psi_1 > 0$ *on* (t_0, T).

Then, for each $0 \neq x \in \mathbb{R}$ *and* $t \in (t_0, T]$, *the sequence* $\{f_{x_N(t)}(x)\}_{N=1}^{\infty}$ *given by* (3.7) *converges to the density* $f_{x(t)}(x)$ *of the solution process* $x(t, \omega)$ *given by* (2.1).

Proof. By [4, Thm. 2.12], $y(t, \omega)$ is an absolutely continuous random variable for each $t \in [t_0, T]$, with density function satisfying (3.8). Then, $x(t, \omega)$ is absolutely continuous and verifies the desired limit (3.10). □

By using the following lemma, we will establish a theorem similar to Theorem 3.3, but which substitutes the continuity hypothesis in (3) by a.e. continuity. This is important, as in (3) we will allow densities with some discontinuities in $\mathbb{R}$, such as the uniform distribution, exponential distribution, etc.

Lemma 3.5. *Let* U *and* V *be two independent random variables. If* U *is absolutely continuous, then* $U + V$ *is absolutely continuous.*

Proof. For any Borel set A, by the convolution formula [3, p. 266] we have $\mathbb{P}(U+V \in A) = \int_{\mathbb{R}} \mathbb{P}(U \in A - v)\mathbb{P}_V(\mathrm{d}v)$, where $\mathbb{P}_V = \mathbb{P} \circ V^{-1}$ is the law of V. If A is null, then $A - v$ is null, so $\mathbb{P}(U \in A - v) = 0$. Thus, if A is null, then $\mathbb{P}(U + V \in A) = 0$. By the Radon-Nikodym Theorem [27, Ch. 14], $U + V$ has a density. □

Theorem 3.6. *Assume the following five hypotheses:*

1. $a, b \in \mathrm{L}^2([t_0, T] \times \Omega)$;
2. $x_0, \xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N$ *are absolutely continuous and independent,* $N \geq 1$;
3. *the density function of* x_0, f_{x_0}, *is a.e. continuous on* $\mathbb{R}$ *and* $f_{x_0}(x) \leq C/|x|^{2/3}$, *for a.e.* $x \neq 0$;
4. $\|\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}\|_{\mathrm{L}^2(\Omega)} \leq C$ *for all* $N \geq 1$ *and all* $t \in [t_0, T]$;
5. $\psi_1(t) \neq 0$ *for all* $t \in (t_0, T)$.

Then, for all $0 \neq x \in \mathbb{R}$ *and* $t \in [t_0, T]$, *the sequence* $\{f_{x_N(t)}(x)\}_{N=1}^{\infty}$ *given by* (3.7) *converges to the density* $f_{x(t)}(x)$ *of the solution process* $x(t, \omega)$ *given by* (2.1).

Proof. The proof is analogous to Theorem 3.3, but with a slight modification. We analyze the proof of [4, Thm. 2.9]. In the notation of [4, Thm. 2.9], $xY - Z$ is absolutely continuous, by Lemma 3.5, (2) and (5). Then the probability that $xY-Z$ belongs to the discontinuity set of f_{x_0} is 0. Recalling that in [4, Thm. 2.9] one has $xY_N(\omega) - Z_N(\omega) \to xY(\omega) - Z(\omega)$ a.s. as $N \to \infty$, by the Continuous Mapping Theorem [24, p. 7, Thm. 2.3] it follows $|f_0(xY_N(\omega)-Z_N(\omega))-f_0(xY(\omega)-Z(\omega))|^2 \to 0$ a.s. as $N \to \infty$. With this fact, the proof of [4, Thm. 2.9] is applicable, as we did in Theorem 3.3. □

Finally, we can establish results on uniform convergence of $\{f_{x_N(t)}(x)\}_{N=1}^{\infty}$, as a consequence of [4, Thms. 2.4, 2.7].

Theorem 3.7. *Assume the following four hypotheses:*

1. $a, b \in \mathrm{L}^2([t_0, T] \times \Omega)$;
2. x_0 *and* $(\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N)$ *are absolutely continuous and independent,* $N \geq 1$;
3. *the function* $f_{x_0}(x^3)x^2$ *is Lipschitz on* $\mathbb{R}$;
4. *there exist* p, q *with* $2 \leq p \leq \infty$ *and* $4 \leq q \leq \infty$ *such that* $1/p + 2/q = 1/2$,

$$\|\mu_b\|_{\mathrm{L}^p(t_0,T)} + \sum_{j=1}^{\infty} \sqrt{\gamma_j}\, \|\psi_j\|_{\mathrm{L}^p(t_0,T)} \|\eta_j\|_{\mathrm{L}^p(\Omega)} < \infty,$$

$$\|\mathrm{e}^{-\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}\|_{\mathrm{L}^q(\Omega)} \leq C, \quad \text{for all } N \geq 1,\ t \in [t_0, T].$$

Then the sequence $\{f_{x_N(t)}(x)\}_{N=1}^{\infty}$ *given by* (3.7) *converges in* $\mathrm{L}^\infty(J \times [t_0, T])$ *for every bounded set* $J \subseteq \mathbb{R}\backslash[-\delta, \delta]$, *for every* $\delta > 0$, *to the density* $f_{x(t)}(x)$ *of the solution process* $x(t, \omega)$ *given by* (2.1).

Proof. By (3.6), f_{y_0} is Lipschitz on $\mathbb{R}$. As a consequence of [4, Thm. 2.4], the limit (3.8) holds in $\mathrm{L}^\infty(J \times [t_0, T])$. Then

$$\begin{aligned} &\|f_{x_N(t)}(x) - f_{x(t)}(x)\|_{\mathrm{L}^\infty(J\times[t_0,T])} \\ &= \|\big(f_{y_N(t)}(x^{1/3}) - f_{y(t)}(x^{1/3})\big)\frac{1}{3x^{2/3}}\|_{\mathrm{L}^\infty(J\times[t_0,T])} \\ &\leq \frac{1}{3\delta^{2/3}}\|f_{y_N(t)}(x^{1/3}) - f_{y(t)}(x^{1/3})\|_{\mathrm{L}^\infty(J\times[t_0,T])} \to 0 \quad \text{as } N \to \infty \end{aligned} \tag{3.11}$$

This concludes the proof. □

Theorem 3.8. *Assume that*

1. $a, b \in \mathrm{L}^2([t_0, T] \times \Omega)$, $x_0 \in \mathrm{L}^{2/3}(\Omega)$;

(2) *$x_0, \eta_1, (\xi_1, \dots, \xi_N, \eta_2, \dots, \eta_N)$ are absolutely continuous and independent, $N \geq 1$;*
(3) *the density function of η_1, f_{η_1}, is Lipschitz on $\mathbb{R}$;*
(4) *$\xi_1, \xi_2, \dots$ have compact support in $[-A, A]$ $(A > 0)$ and $\psi_1 > 0$ on (t_0, T).*

Then, for each fixed $t \in (t_0, T]$, the sequence $\{f_{x_N(t)}(x)\}_{N=1}^{\infty}$ given by (3.7) converges in $\mathrm{L}^\infty(J)$ for every bounded set $J \subseteq \mathbb{R}\backslash[-\delta, \delta]$, for every $\delta > 0$, to the density $f_{x(t)}(x)$ of the solution process $x(t, \omega)$ given by (2.1).

Proof. Since $y_0(\omega) = x_0(\omega)^{1/3}$, it follows that $\mathbb{E}[y_0^2] = \mathbb{E}[x_0^{2/3}] < \infty$ by hypothesis (1). Then, $y_0 \in \mathrm{L}^2(\Omega)$ and therefore hypothesis (1) of [4, Thm. 2.7] holds. In fact, the four hypotheses of [4, Thm. 2.7] are fulfilled. Hence, fixed $t \in (t_0, T]$, the limit in (3.8) holds in $\mathrm{L}^\infty(J)$, then (3.11) follows and we are done. $\square$

3.2. **Comments on the hypotheses of the theorems.** We comment on some examples where the hypotheses of the previously established theorems hold. We point out that these comments will be very useful later in the examples exhibited in Section 4.

(1) In [4, pp. 29, 30], it was proved that, if $a(t, \omega)$ is Gaussian or if $\xi_1, \xi_2, \dots$ have a common compact support, then given any $c \in \mathbb{R}$, there is a constant $C > 0$ such that the inequality $\mathbb{E}[\mathrm{e}^{c\, K_a(t, \boldsymbol{\xi}_N)}] \leq C$ holds for all $N \geq 1$ and $t \in [t_0, T]$.
(2) The continuity on $\mathbb{R}$ is satisfied, for instance, by the density function of the distributions Normal(μ, σ^2), $\mu \in \mathbb{R}$ and $\sigma^2 > 0$; Beta(α, β), $\alpha > 1$ and $\beta > 1$; Gamma(α, β), $\alpha > 1$ and $\beta > 0$. The a.e. continuity from Theorem 3.6 is satisfied in more cases: Beta(α, β), $\alpha \geq 1$ and $\beta \geq 1$; Uniform(α, β), $\alpha < \beta$; Gamma(α, β), $\alpha \geq 1$ and $\beta > 0$ (in particular, Exponential(β)); truncated normal distribution; etc.
(3) In hypothesis (3) of Theorem 3.3, the hypotheses f_{x_0} continuous on $\mathbb{R}$ and $f_{x_0}(x) \leq C/|x|^{2/3}$ are independent, merely because there are unbounded continuous density functions.
(4) Hypothesis (4) of Theorem 3.7 is satisfied, for example, when $b(t, \omega)$ is a standard Brownian motion or a Brownian bridge, as it was seen in [4, pp. 30–31] with $p = 3$.

3.3. **Approximation of the expectation and variance of the solution stochastic process.** We have seen that, under some assumptions, (3.10) holds. We would like to derive conditions under which the expectation and variance of $x(t, \omega)$ can be approximated:

$$\mathbb{E}[x_N(t, \omega)] = \int_{\mathbb{R}} x f_{x_N(t)}(x)\, \mathrm{d}x \to \int_{\mathbb{R}} x f_{x(t)}(x)\, \mathrm{d}x = \mathbb{E}[x(t, \omega)] \tag{3.12}$$

and

$$\begin{aligned} \mathbb{V}[x_N(t, \omega)] &= \int_{\mathbb{R}} x^2 f_{x_N(t)}(x)\, \mathrm{d}x - \mathbb{E}[x_N(t, \omega)]^2 \\ &\to \int_{\mathbb{R}} x^2 f_{x(t)}(x)\, \mathrm{d}x - \mathbb{E}[x(t, \omega)]^2 = \mathbb{V}[x(t, \omega)]. \end{aligned} \tag{3.13}$$

as $N \to \infty$. From [4, p. 13], $y_N(t, \omega) \to y(t, \omega)$ a.s., therefore $x_N(t, \omega) \to x(t, \omega)$ a.s. This implies that $x_N(t, \omega) \to x(t, \omega)$ in probability. By [3, p. 338 Corollary] or

[25, Thm. 2.4], if we check that

$$\sup_{N\geq 1}\mathbb{E}[|x_N(t,\omega)|^{2+\epsilon}]<\infty \tag{3.14}$$

for some $\epsilon>0$, then (3.12), (3.13) and $x(t,\omega)\in \mathrm{L}^2(\Omega)$ will follow.

In what follows, we will use a consequence of Jensen's inequality [3, p. 80]: if $a_1,\dots,a_m\geq 0$ and $p\geq 1$, then $(a_1+\dots+a_m)^p\leq m^{p-1}(a_1^p+\dots+a_m^p)$. For ease of notation, we will denote by C_ϵ any positive constant only depending on ϵ.

By the triangular, Jensen's, and Hölder's inequalities, we have

$$\begin{aligned}
&\mathbb{E}[|x_N(t)|^{2+\epsilon}]\\
&=\mathbb{E}\Big[\Big|x_0^{1/3}\,\mathrm{e}^{\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}+\frac{1}{3}\int_{t_0}^t S_b(s,\boldsymbol{\eta}_N)\mathrm{e}^{\frac{1}{3}(K_a(t,\boldsymbol{\xi}_N)-K_a(s,\boldsymbol{\xi}_N))}\,\mathrm{d}s\Big|^{3(2+\epsilon)}\Big]\\
&\leq C_\epsilon\Big(\mathbb{E}\big[\big|x_0^{1/3}\,\mathrm{e}^{\frac{1}{3}K_a(t,\boldsymbol{\xi}_N)}\big|^{3(2+\epsilon)}\big]\\
&\quad+\mathbb{E}\Big[\Big|\frac{1}{3}\int_{t_0}^t S_b(s,\boldsymbol{\eta}_N)\mathrm{e}^{\frac{1}{3}(K_a(t,\boldsymbol{\xi}_N)-K_a(s,\boldsymbol{\xi}_N))}\,\mathrm{d}s\Big|^{3(2+\epsilon)}\Big]\Big)\\
&\leq C_\epsilon\Big(\mathbb{E}\big[|x_0|^{2+\epsilon}\,\mathrm{e}^{(2+\epsilon)K_a(t,\boldsymbol{\xi}_N)}\big]\\
&\quad+\mathbb{E}\Big[\int_{t_0}^t |S_b(s,\boldsymbol{\eta}_N)|^{3(2+\epsilon)}\mathrm{e}^{(2+\epsilon)(K_a(t,\boldsymbol{\xi}_N)-K_a(s,\boldsymbol{\xi}_N))}\,\mathrm{d}s\Big]\Big).
\end{aligned}\tag{3.15}$$

If $\xi_1,\xi_2,\dots$ have compact support in a common interval, then there is a constant $C_\epsilon>0$ such that $\mathrm{e}^{(2+\epsilon)K_a(t,\boldsymbol{\xi}_N)}\leq C_\epsilon$, for all $N\geq 1$ and $t\in[t_0,T]$. This is a consequence of [4, p. 30]. In this case, from (3.15),

$$\mathbb{E}[|x_N(t)|^{2+\epsilon}]\leq C_\epsilon\Big(\mathbb{E}[|x_0|^{2+\epsilon}]+\mathbb{E}\Big[\int_{t_0}^t |S_b(s,\boldsymbol{\eta}_N)|^{3(2+\epsilon)}\,\mathrm{d}s\Big]\Big).$$

Thus, it suffices to have $\|x_0\|_{\mathrm{L}^{2+\epsilon}(\Omega)}<\infty$ and

$$\|\mu_b\|_{\mathrm{L}^{3(2+\epsilon)}([t_0,T])}+\sum_{i=1}^\infty\sqrt{\gamma_i}\|\psi_i\|_{\mathrm{L}^{3(2+\epsilon)}([t_0,T])}\|\eta_i\|_{\mathrm{L}^{3(2+\epsilon)}(\Omega)}<\infty,$$

to ensure that (3.14) holds.

Otherwise, if $\xi_1,\xi_2,\dots$ do not have compact support in a common interval, one continues from (3.15) by applying Hölder's inequality with exponents $r_1=1+\delta>1$ and $r_2=(1+\delta)/\delta$:

$$\begin{aligned}
&\mathbb{E}[|x_N(t)|^{2+\epsilon}]\\
&\leq C_\epsilon\Big(\mathbb{E}\big[|x_0|^{(2+\epsilon)(1+\delta)}\big]^{\frac{1}{1+\delta}}\mathbb{E}\big[\mathrm{e}^{(2+\epsilon)r_2K_a(t,\boldsymbol{\xi}_N)}\big]^{1/r_2}\\
&\quad+\int_{t_0}^t\mathbb{E}\big[|S_b(s,\boldsymbol{\eta}_N)|^{3(2+\epsilon)(1+\delta)}\big]^{\frac{1}{1+\delta}}\mathbb{E}\big[\mathrm{e}^{(2+\epsilon)r_2(K_a(t,\boldsymbol{\xi}_N)-K_a(s,\boldsymbol{\xi}_N))}\big]^{1/r_2}\,\mathrm{d}s\Big).
\end{aligned}$$

If $a(t,\omega)$ is a Gaussian process, then, by the first paragraph of Subsection 3.2 and Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}[|x_N(t)|^{2+\epsilon}]\\
&\leq C_{\epsilon,\delta}\Big(\mathbb{E}\big[|x_0|^{(2+\epsilon)(1+\delta)}\big]^{\frac{1}{1+\delta}}+\int_{t_0}^t\mathbb{E}\big[|S_b(s,\boldsymbol{\eta}_N)|^{3(2+\epsilon)(1+\delta)}\big]^{\frac{1}{1+\delta}}\,\mathrm{d}s\Big)
\end{aligned}$$

$$\leq C_{\epsilon,\delta}\Big(\mathbb{E}\big[|x_0|^{(2+\epsilon)(1+\delta)}\big]^{\frac{1}{1+\delta}}+\Big(\int_{t_0}^t \mathbb{E}\big[|S_b(s,\boldsymbol{\eta}_N)|^{3(2+\epsilon)(1+\delta)}\big]\,\mathrm{d}s\Big)^{\frac{1}{1+\delta}}\Big).$$

Thereby, if $\|x_0\|_{\mathrm{L}^{2+s}(\Omega)}<\infty$ and

$$\|\mu_b\|_{\mathrm{L}^{6+s}([t_0,T])}+\sum_{i=1}^\infty \sqrt{\gamma_i}\|\psi_i\|_{\mathrm{L}^{6+s}([t_0,T])}\|\eta_i\|_{\mathrm{L}^{6+s}(\Omega)}<\infty,$$

Inequality (3.14) holds. Summarizing, the following theorem has been established.

Theorem 3.9. *If $a(t,\omega)$ is a Gaussian process or $\xi_1,\xi_2,\dots$ have a common compact support, if $\|x_0\|_{\mathrm{L}^{2+s}(\Omega)}<\infty$ and if*

$$\|\mu_b\|_{\mathrm{L}^{6+s}([t_0,T])}+\sum_{i=1}^\infty \sqrt{\gamma_i}\|\psi_i\|_{\mathrm{L}^{6+s}([t_0,T])}\|\eta_i\|_{\mathrm{L}^{6+s}(\Omega)}<\infty \tag{3.16}$$

for some $s>0$, then $x(t,\omega)\in\mathrm{L}^2(\Omega)$ and $x_N(t,\omega)$ tends in $\mathrm{L}^2(\Omega)$ to $x(t,\omega)$, for each $t\in[t_0,T]$. As a consequence, $\mathbb{E}[x_N(t,\omega)]\to\mathbb{E}[x(t,\omega)]$ and $\mathbb{V}[x_N(t,\omega)]\to\mathbb{V}[x(t,\omega)]$ as $N\to\infty$, for each $t\in[t_0,T]$.

4. Numerical examples

In this section we show examples where the theoretical findings of this paper are illustrated. We choose specific stochastic processes $a(t,\omega)$ and $b(t,\omega)$ (via their Karhunen-Loève expansions) and an initial condition $x_0(\omega)$ in the random Bertalanffy model (1.1), and then we compute the approximating density function $f_{x_N(t)}(x)$ given by (3.7) for different values of $N\geq 1$. We also check that it converges to a function, which will be $f_{x(t)}(x)$, as an application of Theorems 3.3, 3.4, 3.6, 3.7 or 3.8. We inform the reader that (1)–(4) listed in Subsection 3.2 will be extensively used throughout this section to check that the hypotheses of the involved theorems hold within the context of each example.

To compute $f_{x_N(t)}(x)$, we have used the software Mathematica. We have programmed a Monte Carlo procedure to compute the expectation in (3.7): fixed $N\geq 1$, we obtain M realizations of each random variable $\xi_1,\dots,\xi_N,\eta_1,\dots,\eta_N$:

$$\text{realizations of } \xi_1\text{: } \xi_1^{(1)},\dots,\xi_1^{(M)},$$
$$\dots$$
$$\text{realizations of } \eta_N\text{: } \eta_N^{(1)},\dots,\eta_N^{(M)}.$$

We denote $\boldsymbol{\xi}_N^{(i)}=(\xi_1^{(i)},\dots,\xi_N^{(i)})$ and $\boldsymbol{\eta}_N^{(i)}=(\eta_1^{(i)},\dots,\eta_N^{(i)})$, $i=1,\dots,M$. Then

$$\begin{aligned} &f_{x_N(t)}(x)\\ &\approx \frac{1}{x^{2/3}}\frac{1}{M}\sum_{i=1}^M\Big[f_{x_0}\Big(\Big\{x^{1/3}\mathrm{e}^{-\frac13 K_a(t,\boldsymbol{\xi}_N^{(i)})}-\frac13\int_{t_0}^t S_b(s,\boldsymbol{\eta}_N^{(i)})\mathrm{e}^{-\frac13 K_a(s,\boldsymbol{\xi}_N^{(i)})}\,\mathrm{d}s\Big\}^3\Big)\\ &\quad\times\Big\{x^{1/3}\mathrm{e}^{-\frac13 K_a(t,\boldsymbol{\xi}_N^{(i)})}-\frac13\int_{t_0}^t S_b(s,\boldsymbol{\eta}_N^{(i)})\mathrm{e}^{-\frac13 K_a(s,\boldsymbol{\xi}_N^{(i)})}\,\mathrm{d}s\Big\}^2\mathrm{e}^{-\frac13 K_a(t,\boldsymbol{\xi}_N^{(i)})}\Big], \end{aligned} \tag{4.1}$$

with convergence as $M\to\infty$, by the Law of Large Numbers. We take M large enough in such a way that expression (4.1) coincides for orders of truncation M and $M'>M$ (convergence in (4.1)).

We perform three examples in what follows. In the first example, both processes $a(t,\omega)$ and $b(t,\omega)$ are Gaussian. In the second and third examples, the processes involved may not be Gaussian.

Example 4.1. Let

$$a(t,\omega)=\sum_{j=1}^{\infty}\frac{\sqrt{2}}{(j-\frac{1}{2})\pi}\sin\big(t(j-\frac{1}{2})\pi\big)\xi_j(\omega),$$

where $\xi_1,\xi_2,\dots$ are independent and Normal$(0,1)$ random variables, be a standard Brownian motion on $[t_0,T]=[0,1]$ [15, Exercise 5.12]. Let

$$b(t,\omega)=\sum_{i=1}^{\infty}\frac{\sqrt{2}}{i\pi}\sin(ti\pi)\eta_i(\omega),$$

where $\xi_1,\xi_2,\dots,\eta_1,\eta_2,\dots$ are independent and Normal$(0,1)$ distributed random variables, be a standard Brownian bridge on $[0,1]$ [15, Example 5.30]. The sums defining $a(t,\omega)$ and $b(t,\omega)$ converge in $\mathrm{L}^2([0,1]\times\Omega)$. Let $x_0\sim$ Beta$(4,6)$. It is assumed x_0, $\xi_1,\xi_2,\dots$ and $\eta_1,\eta_2,\dots$ to be independent.

Since $a(\cdot,\omega)$ and $b(\cdot,\omega)$ are continuous on $[0,1]$, by Theorem 2.1, the stochastic process $x(t,\omega)$ given by (2.1) has $C^1([t_0,T])$ sample paths that satisfy the random Bertalanffy model (1.1). Moreover, by Theorem 2.2 and the subsequent Example 2.3, $x(t,\omega)$ is a solution to the random Bertalanffy model (1.1) in the mean square sense.

The hypotheses of Theorems 3.3 and 3.7 are satisfied. Indeed, f_{x_0} is continuous on $\mathbb{R}$ by (2) in subsection 3.2 and $f_{x_0}(x)\leq C/|x|^{2/3}$ holds because f_{x_0} is continuous with compact support, therefore (3) of Theorem 3.3 fulfills. Hypothesis (4) of Theorem 3.3 is satisfied because $a(t,\omega)$ is Gaussian and (1). Regarding Theorem 3.7, hypothesis (3) holds, since $g(x):=f_{x_0}(x^3)x^2=504x^{11}(1-x^3)^5\mathbf{1}_{(0,1)}(x)$ is differentiable on $\mathbb{R}$, with $g'(x)=504x^{10}(1-x^3)^4(11(1-x^3)-15x^3)\mathbf{1}_{(0,1)}(x)$ bounded on $\mathbb{R}$. Moreover, (4) is fulfilled by (1) in subsection 3.2 applied to $a(t,\omega)$ and (4) in subsection 3.2 applied to $b(t,\omega)$.

Hence, the sequence $\{f_{x_N(t)}(x)\}$ converges in $\mathrm{L}^\infty(J\times[t_0,T])$ for every bounded set $J\subseteq\mathbb{R}\backslash[-\delta,\delta]$, for every $\delta>0$, to the density $f_{x(t)}(x)$ of the solution process $x(t,\omega)$.

The assumptions of Theorem 3.9 are fulfilled. Indeed, by taking $s=1$ in (3.16),

$$\begin{aligned}
&\|\mu_b\|_{\mathrm{L}^7([0,1])}+\sum_{i=1}^{\infty}\sqrt{\gamma_i}\|\psi_i\|_{\mathrm{L}^7([0,1])}\|\eta_i\|_{\mathrm{L}^7(\Omega)}\\
&=M_7\sum_{i=1}^{\infty}\frac{\sqrt{2}}{i\pi}\Big(\int_0^1|\sin(ti\pi)|^7\,\mathrm{d}t\Big)^{1/7}\\
&=M_7\sum_{i=1}^{\infty}\frac{1}{i\pi}\frac{2\sqrt[14]{2}\sqrt[7]{1/i}}{\sqrt[7]{35\pi}}<\infty,
\end{aligned}\tag{4.2}$$

where $M_7=\|Z\|_{\mathrm{L}^7(\Omega)}$, being $Z\sim$ Normal$(0,1)$. Hence, $\mathbb{E}[x_N(t,\omega)]\to\mathbb{E}[x(t,\omega)]$ and $\mathbb{V}[x_N(t,\omega)]\to\mathbb{V}[x(t,\omega)]$ as $N\to\infty$.

In Figure 1, we observe the graph of $f_{x_N(0.5)}(x)$, $x \in \mathbb{R}$, for $N = 1, 2, 3, 4, 5, 6$. The theoretical convergence of the sequence $\{f_{x_N(0.5)}(x)\}_{N=1}^{\infty}$ agrees with the numerical results of Figure 1. In Table 1, we have presented the expectation and variance for $N = 1, 2, 3, 4, 5, 6$, by using the formulas $\mathbb{E}[x_N(0.5, \omega)] = \int_{\mathbb{R}} x f_{x_N(0.5)}(x)\,\mathrm{d}x$ and $\mathbb{V}[x_N(0.5, \omega)] = \int_{\mathbb{R}} x^2 f_{x_N(0.5)}(x)\,\mathrm{d}x - \mathbb{E}[x_N(0.5, \omega)]^2$.

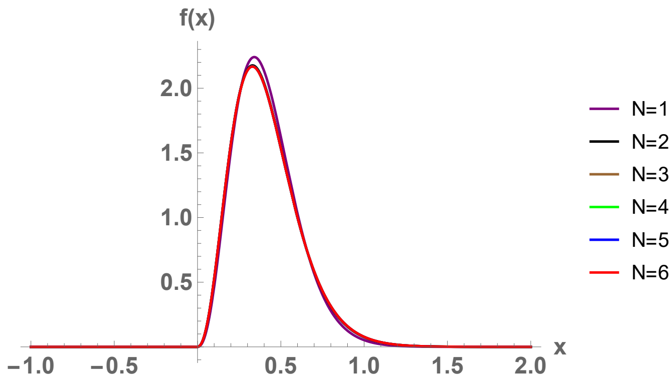

FIGURE 1. Density $f_{x_N(0.5)}(x)$ for $N = 1, 2, 3, 4, 5, 6$. Example 4.1.

TABLE 1. $\mathbb{E}[x_N(0.5, \omega)]$ and $\mathbb{V}[x_N(0.5, \omega)]$ for $N = 1, 2, 3, 4, 5, 6$ in Example 4.1

N	1	2	3	4	5	6
$\mathbb{E}[x_N(0.5, \omega)]$	0.4109	0.4139	0.4145	0.4145	0.4146	0.4146
$\mathbb{V}[x_N(0.5, \omega)]$	0.0345	0.0391	0.0396	0.0396	0.0397	0.0397

Example 4.2. We work on $[t_0, T] = [0, 1]$. Let

$$a(t, \omega) = \sum_{j=1}^{\infty} \frac{\sqrt{2}}{j^3} \sin(tj\pi)\xi_j(\omega), \tag{4.3}$$

where $\xi_1, \xi_2, \dots$ are independent with distribution Uniform$(-\sqrt{3}, \sqrt{3})$. The sum in (4.3) converges in $\mathrm{L}^2([0, 1] \times \Omega)$: given $M < N$, by using Pythagoras's Theorem, we have

$$\| \sum_{j=M+1}^{N} \frac{\sqrt{2}}{j^3} \sin(tj\pi)\xi_j(\omega) \|^2_{\mathrm{L}^2([0,1]\times\Omega)} = \sum_{j=M+1}^{N} \frac{2}{j^6},$$

and since $\sum_{j=1}^{\infty} 2/j^6 < \infty$, the partial sums $\{\sum_{j=1}^{N} (\sqrt{2}/j^3) \sin(tj\pi)\xi_j(\omega)\}_{N=1}^{\infty}$ form a Cauchy sequence in $\mathrm{L}^2([0, 1] \times \Omega)$, therefore convergent.

Let

$$b(t, \omega) = \sum_{i=1}^{\infty} \frac{\sqrt{2}}{i^4 + 6} \sin(ti\pi)\eta_i(\omega), \tag{4.4}$$

where $\eta_1, \eta_2, \dots \sim$ Normal$(0, 1)$ are independent. The sum defining $b(t, \omega)$ in (4.4) exists in $\mathrm{L}^2([0, 1] \times \Omega)$, reasoning as in (4.3).

Let x_0 follow an Exponential(2) distribution. It is assumed x_0, $\xi_1, \xi_2, \dots$ and $\eta_1, \eta_2, \dots$ to be independent.

By Theorem 2.1, the process $x(t,\omega)$ given by (2.1) has absolutely continuous sample paths that solve the random Bertalanffy model (1.1). In fact, $a(t,\omega)$ and $b(t,\omega)$ have continuous sample paths. To prove the continuity for $a(t,\omega)$, we bound

$$\frac{\sqrt{2}}{j^3}|\sin(tj\pi)\xi_j(\omega)| \le \frac{\sqrt{6}}{j^3},$$

with $\sum_{j=1}^\infty 1/j^3 < \infty$, and by using Weierstrass M-test for uniform convergence of series [19, Thm. 7.10], we deduce that the series in (4.3) converges uniformly on $[0,1]$, so $a(t,\omega)$ has continuous sample paths. To prove the continuity for $b(t,\omega)$, one has to work a bit more, since η_i is not bounded. Notice that

$$\mathbb{P}(|\eta_i| \ge i) = 2\int_i^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x \le \frac{2}{\sqrt{2\pi}}\int_i^\infty \frac{x}{i}\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x = \frac{2}{\sqrt{2\pi}\,i}\mathrm{e}^{-i^2/2} \le \mathrm{e}^{-i^2/2};$$

therefore

$$\sum_{i=1}^\infty \mathbb{P}(|\eta_i| \ge i) \le \sum_{i=1}^\infty \mathrm{e}^{-i^2/2} < \infty.$$

By Borel-Cantelli lemma [3, Thm. 4.3], for a.e. $\omega \in \Omega$, there exists an $i_0(\omega) \ge 1$ such that, for all $i \ge i_0(\omega)$, $|\eta_i(\omega)| \le i$. Thus, for all $i \ge i_0(\omega)$,

$$\frac{\sqrt{2}}{i^4+6}|\sin(ti\pi)\eta_i(\omega)| \le \frac{\sqrt{2}\,i}{i^4+6}.$$

Since $\sum_{i=1}^\infty i/(i^4+6) < \infty$, by Weierstrass M-test for uniform convergence of series, we derive that the series defining $b(t,\omega)$ in (4.4) converges uniformly on $[0,1]$, therefore $b(t,\omega)$ has continuous sample paths. As a consequence, by Theorem 2.1, $x(t,\omega)$ has $C^1([0,1])$ sample paths that solve the random Bertalanffy model (1.1).

The process $x(t,\omega)$ is a mean square solution to the random Bertalanffy model (1.1), as the hypotheses of Theorem 2.2 are accomplished. Indeed, $a(t,\omega)$ is continuous in the $\mathrm{L}^{12}(\Omega)$ sense, since

$$\|a(t+h,\omega) - a(t,\omega)\|_{\mathrm{L}^{12}(\Omega)} \le \sqrt{3}\sum_{j=1}^\infty \frac{\sqrt{2}}{j^3}|\sin((t+h)j\pi) - \sin(tj\pi)| \to 0$$

as $h \to 0$, where we have used the fact that $|\xi_j(\omega)| \le \sqrt{3}$ and, for the last limit, the Dominated Convergence Theorem [19, result 11.32, p. 321] applied to the last series. On the other hand,

$$\Big|\int_0^t a(s,\omega)\,\mathrm{d}s\Big| \le \int_0^t |a(s,\omega)|\,\mathrm{d}s \le \sum_{j=1}^\infty \frac{\sqrt{2}}{j^3}\sqrt{3} < \infty,$$

so the third hypothesis of Theorem 2.2 holds. The other process, $b(t,\omega)$, is $\mathrm{L}^{12}(\Omega)$-continuous:

$$\|b(t+h,\omega) - b(t,\omega)\|_{\mathrm{L}^{12}(\Omega)} \le M_{12}\sum_{i=1}^\infty \frac{\sqrt{2}}{i^4+6}|\sin((t+h)j\pi) - \sin(tj\pi)| \to 0$$

as $h \to 0$, where $M_{12} = \|Z\|_{\mathrm{L}^{12}(\Omega)}$, being $Z \sim$ Normal$(0,1)$. For the limit, the Dominated Convergence Theorem has been applied again. Thus, the hypotheses of Theorem 2.2 hold, as stated, and the process $x(t,\omega)$ is a mean square solution.

Theorems 3.4 and 3.8 are applicable. Indeed, hypothesis (3) of Theorem 3.4 is satisfied, because f_{η_1} is the density function of a normal distribution, which is continuous and bounded. Assumption (4) of Theorem 3.4 is accomplished, because the density function of a Uniform$(-\sqrt{3}, \sqrt{3})$ distribution has compact support and $\psi_1(t) = \sqrt{2}\sin(t\pi) > 0$ on $(0,1)$. Concerning Theorem 3.8, (3) holds because the density function of a normal distribution is Lipschitz and (4) is the same as in Theorem 3.4.

Therefore, for each $t \in (t_0, T]$, the sequence $\{f_{x_N(t)}(x)\}$ converges in $\mathrm{L}^\infty(J)$ for every bounded set $J \subseteq \mathbb{R}\backslash[-\delta, \delta]$, for every $\delta > 0$, to the density $f_{x(t)}(x)$ of the solution process $x(t,\omega)$.

The assumptions of Theorem 3.9 are fulfilled, by doing similar computations to the ones in (4.2). As a consequence, $\mathbb{E}[x_N(t,\omega)] \to \mathbb{E}[x(t,\omega)]$ and $\mathbb{V}[x_N(t,\omega)] \to \mathbb{V}[x(t,\omega)]$ as $N \to \infty$.

In Figure 2 we show the graph of $f_{x_N(0.3)}(x)$ for $N = 1,2,3,4,5,6$. In Table 2, we approximate the mean and variance of $x(t,\omega)$, with orders of truncation $N = 1,2,3,4,5,6$. The numerical results agree with our theoretical findings.

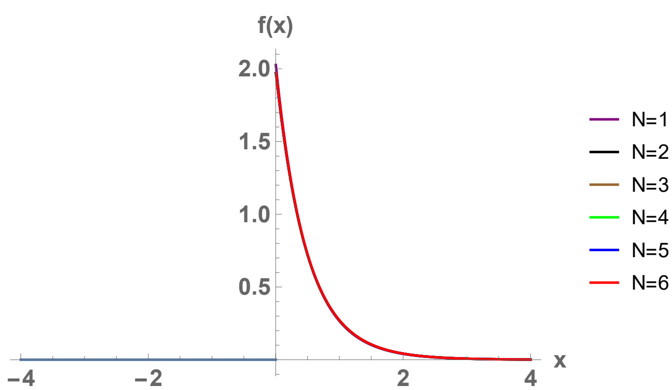

FIGURE 2. Density $f_{x_N(0.3)}(x)$ for $N = 1,2,3,4,5,6$ in Example 4.2.

TABLE 2. $\mathbb{E}[x_N(0.3,\omega)]$ and $\mathbb{V}[x_N(0.3,\omega)]$ for $N = 1,2,3,4,5,6$ in Example 4.2

N	1	2	3	4	5	6
$\mathbb{E}[x_N(0.3,\omega)]$	0.5071	0.5078	0.5076	0.5076	0.5076	0.5076
$\mathbb{V}[x_N(0.3,\omega)]$	0.2754	0.2769	0.2768	0.2768	0.2768	0.2768

Example 4.3. We work on $[t_0, T] = [0,1]$. Let

$$a(t,\omega) = \sum_{j=1}^{\infty} \frac{\sqrt{2}}{j}\sin(tj\pi)\xi_j(\omega),$$

where $\xi_1, \xi_2, \dots$ are independent with distribution Uniform$(-\sqrt{3}, \sqrt{3})$. Let

$$b(t,\omega) = \sum_{i=1}^{\infty} \frac{\sqrt{2}}{i^{\frac{3}{2}}}\sin(ti\pi)\eta_i(\omega),$$

where $\eta_1, \eta_2, \dots$ are independent having distribution Uniform$(-\sqrt{3}, \sqrt{3})$. Let the initial condition be $x_0 \sim$ Uniform$(-\sqrt{3}, \sqrt{3})$. It is assumed x_0, $\xi_1, \xi_2, \dots$ and $\eta_1, \eta_2, \dots$ to be independent. As in Example 4.2, both series converge in $\mathrm{L}^2([0,1] \times \Omega)$, as a consequence of Pythagoras's Theorem.

The process $x(t,\omega)$ has absolutely continuous sample paths that solve the random Bertalanffy model, by Theorem 1.1.

Let us see that the hypotheses of Theorem 3.6 are satisfied. The density function f_{x_0} is a.e. continuous (see (2) in subsection 3.2). The inequality $f_{x_0}(x) \leq C/|x|^{2/3}$ holds because x_0 has compact support, so (3) is satisfied. Hypotheses (4) is a consequence of (1) in subsection 3.2. Finally, (5) holds because $\psi_1(s) = \sin(s\pi) > 0$ for $s \in (0,1)$. Thus, by Theorem 3.6, the sequence $\{f_{x_N(t)}(x)\}_{N=1}^{\infty}$ converges pointwise to $f_{x(t)}(x)$.

The hypotheses of Theorem 3.9 are fulfilled, by doing similar computations to the ones in (4.2). Hence, $\mathbb{E}[x_N(t,\omega)] \to \mathbb{E}[x(t,\omega)]$ and $\mathbb{V}[x_N(t,\omega)] \to \mathbb{V}[x(t,\omega)]$ as $N \to \infty$.

In Figure 3, we observe that the sequence $\{f_{x_N(0.3)}(x)\}_{N=1}^{\infty}$ seems to converge, for $N = 1-6$. In Table 3, we approximate the expectation and variance of $x(t,\omega)$.

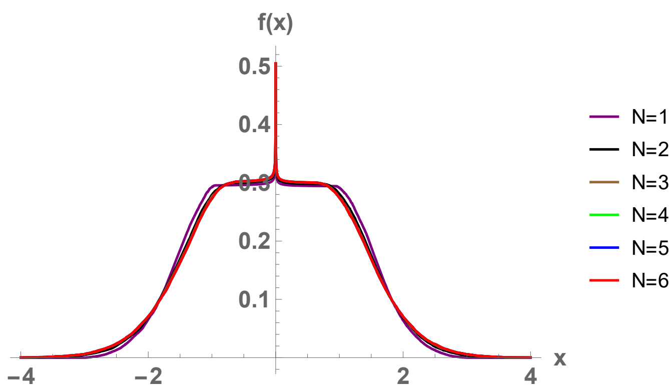

FIGURE 3. Density $f_{x_N(0.3)}(x)$ for $N = 1,2,3,4,5,6$ in Example 4.3.

TABLE 3. $\mathbb{E}[x_N(0.3,\omega)]$ and $\mathbb{V}[x_N(0.3,\omega)]$ for $N = 1,2,3,4,5,6$ in Example 4.3

N	1	2	3	4	5	6
$\mathbb{E}[x_N(0.3,\omega)]$	0.0014	0.0001	0.0001	0.0002	0.0002	0.0002
$\mathbb{V}[x_N(0.3,\omega)]$	1.1240	1.1914	1.2213	1.2298	1.2299	1.2300

Conclusions. In this paper, we have analyzed the random non-autonomous Bertalanffy model: $x'(t,\omega) = a(t,\omega)x(t,\omega) + b(t,\omega)x(t,\omega)^{2/3}$, $t \in [t_0,T]$, with initial condition $x(t_0,\omega) = x_0(\omega)$. The coefficients are stochastic processes $a(t,\omega)$ and $b(t,\omega)$ and the initial condition is a random variable $x_0(\omega)$ in an underlying complete probability space. Via the usual change of variables for solving deterministic Bernoulli differential equations, we have related the random non-autonomous

Bertalanffy model with a random non-autonomous linear differential equation, and thus we have obtained a formal solution stochastic process $x(t,\omega)$. Theorem 2.1 tells us when $x(t,\omega)$ has absolutely continuous sample paths that solve the random Bertalanffy model a.e. Theorem 2.2 gives conditions on the moments of a and b under which $x(t,\omega)$ is a mean square solution. By using existing results on the random non-autonomous linear differential equation, the Random Variable Transformation technique and Karhunen-Loève expansions, we have constructed a sequence of density functions that, under certain assumptions, converge pointwise (Theorems 3.3, 3.4 and 3.6) and uniformly (Theorems 3.7 and 3.8) to the density function of $x(t,\omega)$, $f_{x(t)}(x)$. Results on the approximation of the expectation and the variance of $x(t,\omega)$ have been achieved. Finally, these theoretical findings have been numerically assessed in the computer.

Acknowledgements. This work was supported by the Spanish Ministerio de Economía, Industria y Competitividad (MINECO), by the Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER UE) grant MTM2017–89664–P. Marc Jornet acknowledges the doctorate scholarship granted by Programa de Ayudas de Investigación y Desarrollo (PAID), Universitat Politècnica de València.

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