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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO COUPLED SEMILINEAR PARABOLIC SYSTEMS WITH BOUNDARY DEGENERACY

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Abstract. This article concerns the asymptotic behavior of solutions to coupled semilinear parabolic systems with boundary degeneracy. For the problem in a bounded domain, it is proved that there exist both nontrivial global and blowing-up solutions if the degeneracy is not strong, while any nontrivial solution must blow up in a finite time if the degeneracy is enough strong. For the problem in an unbounded domain, blowing-up theorems of Fujita type are established. It is shown that the critical Fujita curve is determined by the strength of degeneracy. In particular, it is infinite if the degeneracy is enough strong.

1. Introduction

As a typical parabolic equation with boundary degeneracy,

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial u}{\partial x}\Big)+f(x,t,u),\quad 0<x<1,\, t>0,\quad (\lambda>0) \tag{1.1}$$

is degenerate at $x=0$, a portion of the lateral boundary. It is well known that (1.1) can be used to describe some models, such as the Budyko-Sellers climate model ([18]), the Black-Scholes model coming from the option pricing problem [3], and a simplified Crocco-type equation coming from the study on the velocity field of a laminar flow on a flat plate ([4]). In recent years, the null controllability of the control system governed by (1.1) was studied in [1, 4, 5, 6, 8, 9, 16, 21, 24, 25], and it was shown that the null controllability depends on the degenerate exponent. In particular, for the control system governed by

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial u}{\partial x}\Big)+h(x,t)\chi_\omega,\quad (x,t)\in(0,1)\times(0,T),$$

it was proved that the control system is null controllable if $0<\lambda<2$, while not if $\lambda\ge 2$, where h is the control function, ω is a subinterval of $(0,1)$, and χ_ω is the characteristic function of ω. Although the system is not null controllable for $\lambda\ge 2$, it was shown in [19] that the system is approximately controllable in $L^2(0,1)$ for any $\lambda>0$. In [20], the author proved that the asymptotic behavior of solutions

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to (1.1) depends on the degenerate exponent. Precisely, the following problem was studied

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u}{\partial x}\Big) + u^p, \quad 0 < x < 1,\, t > 0, \tag{1.2}$$

$$\Big(x^\lambda \frac{\partial u}{\partial x}\Big)(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{1.4}$$

where $\lambda > 0$, $p > 1$, and u_0 is a nonnegative function. For problem (1.2)–(1.4), it was proved that there exist both nontrivial global and blowing-up solutions if the degeneracy is not strong that $\lambda < 2$, while the nontrivial solution must blow up in a finite time if the degeneracy is so strong that $\lambda \geq 2$. Furthermore, the blowing-up theorems of Fujita type were also established in [20] for the following problem in an unbounded domain

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u}{\partial x}\Big) + u^p, \quad x > 0,\, t > 0, \tag{1.5}$$

$$\Big(x^\lambda \frac{\partial u}{\partial x}\Big)(0,t) = 0, \quad t > 0, \tag{1.6}$$

$$u(x,0) = u_0(x), \quad x > 0. \tag{1.7}$$

It was shown that the critical Fujita exponent is

$$p_c = \begin{cases} 3 - \lambda, & 0 < \lambda < 2, \\ +\infty, & \lambda \geq 2. \end{cases}$$

That is to say, in the case $0 < \lambda < 2$, any nontrivial solution to problem (1.5)–(1.7) must blow up in a finite time if $1 < p \leq 3 - \lambda$, while there are both nontrivial global and blowing-up solutions to problem (1.5)–(1.7) if $p > 3 - \lambda$. As to the case $\lambda \geq 2$, any nontrivial solution to (1.5)–(1.7) must blow up in a finite time for $p > 1$. In 1966, Fujita [11] proved that for the Cauchy problem of the semilinear equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad x \in \mathbb{R}^n,\ t > 0,$$

any nontrivial solution must blow up in a finite time if $1 < p < 1 + n/2$, whereas there exist both nontrivial global and blowing-up solutions when $p > 1 + n/2$. For this problem, $p_c = 1 + n/2$ is called the critical Fujita exponent, and the critical case $p = p_c$ was proved to belong to the blowing-up case in [12, 13]. Fujita revealed an important topic of nonlinear partial differential equations. And there have been a great number of extensions of Fujita's results in several directions since then, including similar results for numerous of quasilinear parabolic equations and systems in various of geometries with nonlinear sources or nonhomogeneous boundary conditions, see the survey papers [7, 14] and also the recent papers [2, 15, 17, 22, 23, 26, 27, 28, 29]. In particular, Escobedo and Herrero in [10] investigated the Cauchy problem of the coupled semilinear parabolic system

$$\frac{\partial u}{\partial t} = \Delta u + v^p, \quad \frac{\partial v}{\partial t} = \Delta v + u^q, \quad x \in \mathbb{R}^n,\ t > 0,\ (p, q > 1),$$

and proved that the critical Fujita curve is

$$(pq)_c = 1 + \frac{2}{n}\max\{p+1, q+1\}.$$

That is to say, any nontrivial solution must blow up in a finite time if $pq \le 1 + 2/n \max\{p+1, q+1\}$, whereas there exist both nontrivial global and blowing-up solutions when $pq > 1 + 2/n \max\{p+1, q+1\}$.

A natural question of [20] is that how about the parabolic systems with the boundary degeneracy, which is solved in this paper. More precisely, in this paper we study the asymptotic behavior of solutions to the following two coupled parabolic systems with the boundary degeneracy

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u}{\partial x}\Big) + v^p, \quad 0 < x < 1, \quad t > 0, \tag{1.8}$$
$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial v}{\partial x}\Big) + u^q, \quad 0 < x < 1, \quad t > 0, \tag{1.9}$$
$$\Big(x^\lambda \frac{\partial u}{\partial x}\Big)(0,t) = \Big(x^\lambda \frac{\partial v}{\partial x}\Big)(0,t) = 0, \quad t > 0, \tag{1.10}$$
$$u(1,t) = v(1,t) = 0, \quad t > 0, \tag{1.11}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad 0 < x < 1, \tag{1.12}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u}{\partial x}\Big) + v^p, \quad x > 0, \quad t > 0, \tag{1.13}$$
$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial v}{\partial x}\Big) + u^q, \quad x > 0, \quad t > 0, \tag{1.14}$$
$$\Big(x^\lambda \frac{\partial u}{\partial x}\Big)(0,t) = \Big(x^\lambda \frac{\partial v}{\partial x}\Big)(0,t) = 0, \quad t > 0, \tag{1.15}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x > 0, \tag{1.16}$$

where $p, q > 1$ and $\lambda > 0$. For problem (1.8)–(1.12) in a bounded domain, it is shown that $\lambda = 2$ is a threshold in the sense that there exist both nontrivial global and blowing-up solutions if $\lambda < 2$, while the nontrivial solution must blow up in a finite time if $\lambda \ge 2$. As for problem (1.13)–(1.16) in an unbounded domain, it is shown that $\lambda = 2$ is also a threshold such that the critical Fujita curve is

$$(pq)_c = \begin{cases} 1 + (2-\lambda)\max\{p+1, q+1\}, & 0 < \lambda < 2, \\ +\infty, & \lambda \ge 2. \end{cases}$$

Furthermore, the critical curve $pq = (pq)_c$ belongs to the blowing-up case if $0 < \lambda < 2$. That is to say, if the degeneracy is not strong that $0 < \lambda < 2$, any nontrivial solution to (1.13)–(1.16) blows up in a finite time when $pq \le (pq)_c$, whereas there exist both nontrivial global solution and blowing-up solution when $pq > (pq)_c$. If the degeneracy is so strong that $\lambda \ge 2$, any nontrivial solution to (1.13)–(1.16) must blow up. The methods for the system in this paper are similar to the ones in [20] for the single equation. For the blowing-up of solutions to (1.8)–(1.12) in a bounded domain and problem (1.13)–(1.16) in an unbounded domain, we apply the methods of weighted energy estimates to determine the interaction of the degenerate diffusions and the reactions, and the key is to choose appropriate weights. To prove the global existence of nontrivial solutions, we construct suitable self-similar supersolutions. Since the problems in this paper are on coupled parabolic systems with boundary degeneracy, some complicated estimates are needed. In particular, for the critical case $pq = (pq)_c$ when $0 < \lambda < 2$, it is shown that it belongs to the blowing-up case by a series of elaborate energy estimates.

This article is organized as follows. Some preliminaries and main results are stated in Section 2. Problem (1.8)–(1.12) in a bounded domain and problem (1.13)–(1.16) in an unbounded domain are studied in Section 3 and Section 4, respectively.

2. Preliminaries and main results

The subsolutions, supersolutions, as well as solutions to problems (1.8)–(1.12) and (1.13)–(1.16) are defined as follows.

Definition 2.1. Let $0 < T \le +\infty$. A pair of nonnegative functions (u, v) is called a subsolution (supersolution, solution) to problem (1.8)–(1.12) in $(0, T)$, if

(i) For any $0 < \tilde{T} < T$, $u, v \in L^\infty((0,1) \times (0, \tilde{T}))$, and $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, x^{\lambda/2}\frac{\partial u}{\partial x}, x^{\lambda/2}\frac{\partial v}{\partial x} \in L^2((0,1) \times (0, \tilde{T}))$.

(ii) For any $0 < \tilde{T} < T$ and any nonnegative functions $\varphi, \phi \in C^1([0,1] \times [0, \tilde{T}])$ vanishing at $x = 1$,
$$
\begin{aligned}
&\int_0^{\tilde{T}} \int_0^1 \Big(\frac{\partial u}{\partial t}(x,t)\varphi(x,t) + x^\lambda \frac{\partial u}{\partial x}(x,t)\frac{\partial \varphi}{\partial x}(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le (\ge, =) \int_0^{\tilde{T}} \int_0^1 v^p(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$
and
$$
\begin{aligned}
&\int_0^{\tilde{T}} \int_0^1 \Big(\frac{\partial v}{\partial t}(x,t)\phi(x,t) + x^\lambda \frac{\partial v}{\partial x}(x,t)\frac{\partial \phi}{\partial x}(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le (\ge, =) \int_0^{\tilde{T}} \int_0^1 u^q(x,t)\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

(iii) $u(1,\cdot), v(1,\cdot) \le (\ge, =) 0$ in $(0, T)$ and $u(\cdot, 0) \le (\ge, =) u_0(\cdot)$, $v(\cdot, 0) \le (\ge, =) v_0(\cdot)$ in $(0,1)$ in the sense of trace.

Definition 2.2. Let $0 < T \le +\infty$. A pair of nonnegative functions (u, v) is called a subsolution (supersolution, solution) to problem (1.13)–(1.16) in $(0, T)$, if

(i) For any $0 < \tilde{T} < T$, $u, v \in L^\infty((0, +\infty) \times (0, \tilde{T}))$, and $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$, $x^{\lambda/2}\frac{\partial u}{\partial x}$, $x^{\lambda/2}\frac{\partial v}{\partial x}$ in $L^2((0,R) \times (0, \tilde{T}))$ for each $R > 0$.

(ii) For any $0 < \tilde{T} < T$ and any nonnegative $\varphi, \phi \in C^1([0, +\infty) \times [0, \tilde{T}])$ vanishing when x is large,
$$
\begin{aligned}
&\int_0^{\tilde{T}} \int_0^{+\infty} \Big(\frac{\partial u}{\partial t}(x,t)\varphi(x,t) + x^\lambda \frac{\partial u}{\partial x}(x,t)\frac{\partial \varphi}{\partial x}(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le (\ge, =) \int_0^{\tilde{T}} \int_0^{+\infty} v^p(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$
and
$$
\begin{aligned}
&\int_0^{\tilde{T}} \int_0^{+\infty} \Big(\frac{\partial v}{\partial t}(x,t)\phi(x,t) + x^\lambda \frac{\partial v}{\partial x}(x,t)\frac{\partial \phi}{\partial x}(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le (\ge, =) \int_0^{\tilde{T}} \int_0^{+\infty} u^q(x,t)\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

(iii) $u(\cdot, 0) \le (\ge, =) u_0(\cdot)$, $v(\cdot, 0) \le (\ge, =) v_0(\cdot)$ in $(0, +\infty)$ in the sense of trace.

If (u, v) is a solution to (1.8)–(1.12) (or to (1.13)–(1.16)) in $(0, +\infty)$, it is said that (u, v) is a global solution in time. Otherwise, there exists $T > 0$ such that (u, v) is a solution in $(0, T)$ and satisfies

$$\|u(\cdot,t)\|_{L^\infty(0,1)} + \|v(\cdot,t)\|_{L^\infty(0,1)} \to +\infty, \quad \text{as } t \to T^-,$$
$$(\text{or } \|u(\cdot,t)\|_{L^\infty(0,+\infty)} + \|v(\cdot,t)\|_{L^\infty(0,+\infty)} \to +\infty, \quad \text{as } t \to T^-),$$

and it is said that (u, v) blows up in a finite time.

Similarly to [20], one can establish the well-posedness and the comparison principles for problems (1.8)–(1.12) and (1.13)–(1.16).

Proposition 2.3. (i) *For any $0 \le u_0, v_0 \in L^\infty(0,1)$ with $x^{\lambda/2}u_0', x^{\lambda/2}v_0' \in L^2(0,1)$, problem* (1.8)–(1.12) *admits at least one solution locally in time.*

(ii) *Assume that $(\hat u, \hat v)$ and $(\check u, \check v)$ are a supersolution and a subsolution to problem* (1.8)–(1.12) *in $(0, T)$, respectively. Then $(\check u, \check v) \le (\hat u, \hat v)$ in $(0, 1) \times (0, T)$.*

Proposition 2.4. (i) *For any $0 \le u_0, v_0 \in L^\infty(0,+\infty)$ with $x^{\lambda/2}u_0', x^{\lambda/2}v_0' \in L^2((0,R))$ for each $R > 0$, problem* (1.13)–(1.16) *admits at least one solution locally in time.*

(ii) *Assume that $(\hat u, \hat v)$ and $(\check u, \check v)$ are a supersolution and a subsolution to problem* (1.13)–(1.16) *in $(0, T)$, respectively. Then $(\check u, \check v) \le (\hat u, \hat v)$ in $(0, +\infty) \times (0, T)$.*

The main results of the paper are as follows.

Theorem 2.5. *Assume that $0 < \lambda < 2$, $p, q > 1$ and $0 \le u_0, v_0 \in L^\infty(0,1)$ with $x^{\lambda/2}u_0', x^{\lambda/2}v_0' \in L^2(0,1)$. Then the solution to* (1.8)–(1.12) *exists globally in time if (u_0, v_0) is suitably small, while blows up in a finite time if (u_0, v_0) is suitably large.*

Theorem 2.6. *Assume that $\lambda \ge 2$ and $p, q > 1$. Then for any nontrivial $0 \le u_0, v_0 \in L^\infty(0,1)$ with $x^\lambda u_0', x^\lambda v_0' \in L^2(0,1)$, the solution to* (1.8)–(1.12) *must blow up in a finite time.*

Theorem 2.7. *Assume that $0 < \lambda < 2$, $0 \le u_0, v_0 \in L^\infty(0,+\infty)$ with $x^{\lambda/2}u_0'$, $x^{\lambda/2}v_0' \in L^2((0,R))$ for each $R > 0$, and (u_0, v_0) is nontrivial.*

(i) *If $p, q > 1$ and $pq < 1 + (2-\lambda)\max\{p+1, q+1\}$, then the solution to* (1.13)–(1.16) *must blow up in a finite time.*

(ii) *If $p, q > 1$ and $pq = 1 + (2-\lambda)\max\{p+1, q+1\}$, then the solution to t* (1.13)–(1.16) *must blow up in a finite time.*

(iii) *If $p, q > 1$ and $pq > 1 + (2-\lambda)\max\{p+1, q+1\}$, then the solution to* (1.13)–(1.16) *exists globally in time if (u_0, v_0) is suitably small, while blows up in a finite time if (u_0, v_0) is suitably large.*

Theorem 2.8. *Assume that $\lambda \ge 2$. Then for a nontrivial $0 \le u_0, v_0 \in L^\infty(0,+\infty)$ with $x^{\lambda/2}u_0', x^{\lambda/2}v_0' \in L^2((0,R))$ for each $R > 0$, the solution to* (1.13)–(1.16) *must blow up in a finite time.*

3. Problem in a bounded domain

In this section, we prove Theorems 2.5 and 2.6 for problem (1.8)–(1.12) in a bounded domain.

Proof of Theorem 2.5. First consider the global case. To show the existence of a global solution to (1.8)–(1.12), we study self-similar supersolutions to (1.8) and (1.9) of the form

$$\hat{u}(x,t) = (t+\tau)^{-(p+1)/(pq-1)}U((t+\tau)^{-1/(2-\lambda)}x), \quad x \in [0,1],\ t \geq 0, \tag{3.1}$$

$$\hat{v}(x,t) = (t+\tau)^{-(q+1)/(pq-1)}V((t+\tau)^{-1/(2-\lambda)}x), \quad x \in [0,1],\ t \geq 0, \tag{3.2}$$

where $\tau > 0$ will be determined later. If $U, V \in C^2((0, \tau^{-1/(2-\lambda)}))$ solve

$$(r^\lambda U'(r))' + \frac{1}{2-\lambda} rU'(r) + \frac{p+1}{pq-1}U(r) + V^p(r) \leq 0, \quad 0 < r < \tau^{-1/(2-\lambda)}, \tag{3.3}$$

$$(r^\lambda V'(r))' + \frac{1}{2-\lambda} rV'(r) + \frac{q+1}{pq-1}V(r) + U^q(r) \leq 0, \quad 0 < r < \tau^{-1/(2-\lambda)}, \tag{3.4}$$

then $(\hat{u}, \hat{v})$ given by (3.1) and (3.2) is a supersolution to (1.8) and (1.9). Set

$$U(r) = V(r) = \frac{1}{2-\lambda}\Big(\frac{1}{\tau} - r^{2-\lambda}\Big), \quad 0 \leq r \leq \tau^{-1/(2-\lambda)}.$$

For $0 < r < \tau^{1/(2-\lambda)}$, from direct calculations we have

$$\begin{aligned}
&(r^\lambda U'(r))' + \frac{1}{2-\lambda} rU'(r) + \frac{p+1}{pq-1}U(r) + V^p(r) \\
&= -1 - \frac{1}{2-\lambda}r^{2-\lambda} + \frac{p+1}{(pq-1)(2-\lambda)}(\frac{1}{\tau} - r^{2-\lambda}) + \frac{1}{(2-\lambda)^p}\Big(\frac{1}{\tau} - r^{2-\lambda}\Big)^p, \\
&(r^\lambda V'(r))' + \frac{1}{2-\lambda} rV'(r) + \frac{q+1}{pq-1}V(r) + U^q(r) \\
&= -1 - \frac{1}{2-\lambda}r^{2-\lambda} + \frac{q+1}{(pq-1)(2-\lambda)}(\frac{1}{\tau} - r^{2-\lambda}) + \frac{1}{(2-\lambda)^q}\Big(\frac{1}{\tau} - r^{2-\lambda}\Big)^q.
\end{aligned}$$

Hence $(\hat{u}, \hat{v})$ is a supersolution to (1.8) and (1.9) for each $\tau \geq \tau_0$, where

$$\tau_0 = \max\Big\{\frac{p+1}{(pq-1)(2-\lambda)} + \frac{1}{(2-\lambda)^p} + 1, \frac{q+1}{(pq-1)(2-\lambda)} + \frac{1}{(2-\lambda)^q} + 1\Big\}.$$

It is noted that

$$\begin{gathered}
\Big(x^\lambda \frac{\partial \hat{u}}{\partial x}\Big)(0,t) = 0, \quad \hat{u}(1,t) \geq 0, \quad t > 0, \\
\Big(x^\lambda \frac{\partial \hat{v}}{\partial x}\Big)(0,t) = 0, \quad \hat{v}(1,t) \geq 0, \quad t > 0.
\end{gathered}$$

Therefore, $(\hat{u}, \hat{v})$ is a supersolution to (1.8)–(1.12) if

$$u_0(x) \leq \hat{u}(x,0), \quad v_0(x) \leq \hat{v}(x,0), \quad 0 < x < 1 \quad (\tau \geq \tau_0). \tag{3.5}$$

Thanks to Proposition 2.3 (ii), one gets that the solution to (1.8)–(1.12) exists globally in time if (u_0, v_0) satisfies (3.5).

Now we turn to the blowing-up case. Without loss of generality, it is assumed that $p \geq q > 1$. Set

$$\zeta(x) = \begin{cases} 2, & 0 < x \leq \frac{1}{2}, \\ 1 + \cos(2x-1)\pi, & \frac{1}{2} < x \leq 1. \end{cases}$$

It is clear that $\zeta \in C^1\big([0,1]\big)$ is piecewise smooth and satisfies $\zeta'(0) = 0$, $\zeta(1) = 0$, and for $1/2 < x \leq 1$,

$$\big(x^\lambda \zeta'(x)\big)' = -2\lambda\pi x^{\lambda-1}\sin(2x-1)\pi - 4\pi^2 x^\lambda \cos(2x-1)\pi \geq -4\pi^2\zeta(x).$$

Assume that (u,v) is a solution to (1.8)–(1.12) in $(0,+\infty)$. It follows from Definition 2.1 that (u,v) satisfies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \big(u(x,t)+v(x,t)\big)\zeta(x)\,\mathrm{d}x\\
&=\int_0^1 \big(u(x,t)+v(x,t)\big)(x^\lambda\zeta'(x))'\,\mathrm{d}x+\int_0^1 v^p(x,t)\zeta(x)\,\mathrm{d}x+\int_0^1 u^q(x,t)\zeta(x)\,\mathrm{d}x\\
&\ge -4\pi^2\int_0^1 \big(u(x,t)+v(x,t)\big)\zeta(x)\,\mathrm{d}x+\int_0^1 v^p(x,t)\zeta(x)\,\mathrm{d}x\\
&\quad+\int_0^1 u^q(x,t)\zeta(x)\,\mathrm{d}x,\quad t>0.
\end{aligned}\tag{3.6}$$

Denote

$$\omega(t)=\int_0^1 (u(x,t)+v(x,t))\zeta(x)\,\mathrm{d}x,\quad t\ge 0.$$

It follows from (3.6) and Hölder's inequality that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\omega(t)\\
&\ge -4\pi^2\omega(t)+\int_0^1 u^q(x,t)\zeta(x)\,\mathrm{d}x+\int_0^1 v^p(x,t)\zeta(x)\,\mathrm{d}x\\
&\ge -4\pi^2\omega(t)+\Big(\int_0^1 \zeta(x)\,\mathrm{d}x\Big)^{1-q}\Big(\int_0^1 u(x,t)\zeta(x)\,\mathrm{d}x\Big)^q\\
&\quad+\Big(\int_0^1 \zeta(x)\,\mathrm{d}x\Big)^{1-p}\Big(\int_0^1 v(x,t)\zeta(x)\,\mathrm{d}x\Big)^p\\
&\ge -4\pi^2\omega(t)+2^{1-p}\Big\{\Big(\int_0^1 u(x,t)\zeta(x)\,\mathrm{d}x\Big)^q+\Big(\int_0^1 v(x,t)\zeta(x)\,\mathrm{d}x\Big)^p\Big\}.
\end{aligned}\tag{3.7}$$

If $\omega(t)\ge 2$, then

$$\int_0^1 u(x,t)\zeta(x)\,\mathrm{d}x\ge 1\quad\text{or}\quad\int_0^1 v(x,t)\zeta(x)\,\mathrm{d}x\ge 1.$$

It follows from [17, Lemma 3.6] that

$$\Big(\int_0^1 u(x,t)\zeta(x)\,\mathrm{d}x\Big)^q+\Big(\int_0^1 v(x,t)\zeta(x)\,\mathrm{d}x\Big)^p\ge 2^{-p}\omega^q(t),\quad\text{if } \omega(t)\ge 2.\tag{3.8}$$

If (u_0,v_0) is sufficiently large such that

$$\omega(0)\ge 2,\quad \omega^{q-1}(0)\ge 2^{2+2p}\pi^2,\tag{3.9}$$

one gets from (3.7) and (3.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega(t)\ge 2^{-2p}\omega^{q-1}(t),\quad t>0.$$

Since $p\ge q>1$, there exists $T>0$ such that

$$\omega(t)=\int_0^1 (u(x,t)+v(x,t))\zeta(x)\,\mathrm{d}x\to+\infty,\quad\text{as } t\to T^-,$$

which leads to

$$\|u(\cdot,t)\|_{L^\infty(0,1)}+\|v(\cdot,t)\|_{L^\infty(0,1)}\to+\infty,\quad\text{as } t\to T^-.$$

That is to see, if (u_0, v_0) satisfies (3.9), then (u, v) blows up in a finite time. The proof of Theorem 2.5 is complete. $\square$

Proof of Theorem 2.6. Without loss of generality, it is assumed that $p \geq q > 1$. For $0 < \delta < 1$, set

$$\zeta_\delta(x) = \begin{cases} \frac{\lambda-1}{\delta} 2^{\lambda-1-\delta} x^{-\delta} - \frac{\lambda-1-\delta}{\delta} 2^{\lambda-1} - 1, & 0 < x < 1/2, \\ x^{1-\lambda} - 1, & 1/2 \leq x \leq 1. \end{cases}$$

It is clear that $\zeta_\delta(x) \in C^1\big((0,1]\big)$ is piecewise smooth, and

$$\int_0^1 \zeta_\delta(x)\,\mathrm{d}x \leq M_1 \tag{3.10}$$

$$\big(x^\lambda \zeta_\delta'(x)\big)' = \begin{cases} -(\lambda-1)(\lambda-1-\delta)2^{\lambda-1-\delta} x^{\lambda-2-\delta}, & 0 < x < 1/2, \\ 0, & 1/2 \leq x \leq 1, \end{cases}$$

where $M_1 > 0$ is a constant depending only on λ but independent of δ. Therefore, there exists a constant $M_2 > 0$ depending only on λ but independent of δ such that

$$\big(x^\lambda \zeta_\delta'(x)\big)' \geq -M_2 \delta \zeta_\delta(x), \quad 0 < x < 1. \tag{3.11}$$

Assume that (u, v) is a solution to (1.8)–(1.12) in $(0, +\infty)$. By the similar argument in [20, Theorem 2.2], one can get that (u, v) satisfies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \big(u(x,t) + v(x,t)\big)\zeta_\delta(x)\,\mathrm{d}x \\ &= \int_0^1 \big(u(x,t) + v(x,t)\big)(x^\lambda \zeta_\delta'(x))'\,\mathrm{d}x + \int_0^1 v^p(x,t)\zeta_\delta(x)\,\mathrm{d}x \\ &\quad + \int_0^1 u^q(x,t)\zeta_\delta(x)\,\mathrm{d}x \\ &\geq -M_2\delta \int_0^1 \big(u(x,t) + v(x,t)\big)\zeta_\delta(x)\,\mathrm{d}x + \int_0^1 v^p(x,t)\zeta_\delta(x)\,\mathrm{d}x \\ &\quad + \int_0^1 u^q(x,t)\zeta_\delta(x)\,\mathrm{d}x, \quad t > 0. \end{aligned} \tag{3.12}$$

We denote

$$\omega(t) = \int_0^1 (u(x,t) + v(x,t))\zeta_\delta(x)\,\mathrm{d}x, \quad t > 0.$$

It follows from (3.12) and Hölder's inequality that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\omega(t) \\ &\geq -M_2\delta\omega(t) + \int_0^1 u^q(x,t)\zeta_\delta(x)\,\mathrm{d}x + \int_0^1 v^p(x,t)\zeta_\delta(x)\,\mathrm{d}x \\ &\geq -M_2\delta\omega(t) + \Big(\int_0^1 \zeta_\delta(x)\,\mathrm{d}x\Big)^{1-q}\Big(\int_0^1 u(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^q \\ &\quad + \Big(\int_0^1 \zeta_\delta(x)\,\mathrm{d}x\Big)^{1-p}\Big(\int_0^1 v(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^p \\ &\geq -M_2\delta\omega(t) + M_1^{1-q}\Big(\int_0^1 u(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^q + M_1^{1-p}\Big(\int_0^1 v(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^p \end{aligned}$$

$$\geq -M_2\delta\omega(t) + M_1^{1-p}\Big\{\Big(\int_0^1 u(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^q + \Big(\int_0^1 v(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^p\Big\}. \quad (3.13)$$

If $\omega(t) < 2$, then

$$\int_0^1 u(x,t)\zeta_\delta(x)\,\mathrm{d}x < 2, \quad \int_0^1 v(x,t)\zeta_\delta(x)\,\mathrm{d}x < 2.$$

It follows from [17, Lemma 3.6] that

$$\Big(\int_0^1 u(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^q + \Big(\int_0^1 v(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^p \geq 2^{-2p}\omega^p(t), \quad \text{if } \omega(t) < 2. \quad (3.14)$$

Similarly to the proof of (3.8), one can get

$$\Big(\int_0^1 u(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^q + \Big(\int_0^1 v(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^p \geq 2^{-p}\omega^q(t), \quad \text{if } \omega(t) \geq 2. \quad (3.15)$$

Owing to $\inf_{0<\delta<1}\{\omega(0)\} > 0$, there exists sufficiently small $0 < \delta < 1$ such that

$$M_2\delta \leq 2^{-2p-1}M_1^{1-p}\omega^{p-1}(0), \quad M_2\delta \leq 2^{-p-1}M_1^{1-p}\omega^{q-1}(0). \quad (3.16)$$

If $\omega(0) < 2$, we claim that there exists $\tilde{T} > 0$ such that $\omega(\tilde{T}) \geq 2$. Otherwise,

$$\omega(t) < 2, \quad t \in [0, +\infty). \quad (3.17)$$

It follows from (3.13), (3.14), (3.16) and (3.17) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega(t) \geq 2^{-2p-1}M_1^{1-p}\omega^p(t), \quad t > 0.$$

Since $p \geq q > 1$, there exists $\hat{T} > 0$ such that

$$\omega(t) = \int_0^1 (u(x,t) + v(x,t))\zeta_\delta(x)\,\mathrm{d}x \to +\infty, \quad \text{as } t \to \hat{T}^-,$$

which contradicts (3.17). Therefore, one can assume that $\omega(0) \geq 2$. Then one gets from (3.13), (3.15) and (3.16) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega(t) \geq 2^{-p-1}M_1^{1-p}\omega^q(t), \quad t > 0.$$

Since $p \geq q > 1$, there exists $T > 0$ such that

$$\omega(t) = \int_0^1 (u(x,t) + v(x,t))\zeta_\delta(x)\,\mathrm{d}x \to +\infty, \quad \text{as } t \to T^-,$$

which leads to

$$\|u(\cdot,t)\|_{L^\infty(0,1)} + \|v(\cdot,t)\|_{L^\infty(0,1)} \to +\infty, \quad \text{as } t \to T^-.$$

That is to say, (u, v) must blow up in a finite time. The proof of Theorem 2.6 is complete. $\square$

4. Problem in an unbounded domain

In this section, we prove Theorems 2.7 and 2.8 for problem (1.13)–(1.16) in an unbounded domain. It is clear that Theorem 2.8 follows from Theorem 2.6, Propositions 2.4 and 2.3. We only prove Theorem 2.7.

Proof of Theorem 2.7 (i) and (iii). Without loss of generality, it is assumed that $p \geq q > 1$. For $R > 0$, set

$$\zeta_R(x) = \begin{cases} 1, & 0 < x \leq R, \\ \frac{1}{2}\Big(1 + \cos\frac{(x-R)\pi}{R}\Big), & R < x < 2R, \\ 0, & x \geq 2R. \end{cases} \tag{4.1}$$

It is clear that $\zeta_R \in C^1\big([0,+\infty)\big)$ is piecewise smooth, and

$$\begin{aligned} (x^\lambda \zeta_R'(x))' &= -\frac{\lambda\pi}{2R}x^{\lambda-1}\sin\frac{(x-R)\pi}{R} - \frac{\pi^2}{2R^2}x^\lambda\cos\frac{(x-R)\pi}{R} \\ &\geq -2^\lambda\pi^2R^{\lambda-2}\zeta_R(x), \quad R < x < 2R. \end{aligned}$$

Assume that (u,v) is a solution to (1.13)–(1.16) in $(0,+\infty)$. It follows from Definition 2.2 that (u,v) satisfies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}\big(u(x,t) + R^\sigma v(x,t)\big)\zeta_R(x)\,\mathrm{d}x \\ &= \int_0^{+\infty}\big(u(x,t) + R^\sigma v(x,t)\big)(x^\lambda\zeta_R'(x))'\,\mathrm{d}x + R^\sigma\int_0^{+\infty}u^q(x,t)\zeta_R(x)\,\mathrm{d}x \\ &\quad + \int_0^{+\infty}v^p(x,t)\zeta_R(x)\,\mathrm{d}x \\ &\geq -2^\lambda\pi^2R^{\lambda-2}\int_0^{+\infty}\big(u(x,t) + R^\sigma v(x,t)\big)\zeta_R(x)\,\mathrm{d}x + R^\sigma\int_0^{+\infty}u^q(x,t)\zeta_R(x)\,\mathrm{d}x \\ &\quad + \int_0^{+\infty}v^p(x,t)\zeta_R(x)\,\mathrm{d}x, \quad t > 0, \end{aligned} \tag{4.2}$$

where

$$\sigma = \frac{q-p}{p+1}.$$

We denote

$$\omega_R(t) = \int_0^{+\infty}(u(x,t) + R^\sigma v(x,t))\zeta_R(x)\,\mathrm{d}x, \quad t > 0.$$

It is noted that

$$1 - q + \sigma = 1 - p - \sigma p = -\frac{pq-1}{p+1}.$$

It follows from (4.2) and Hölder's inequality that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t) \geq& -2^\lambda\pi^2R^{\lambda-2}\omega_R(t)+R^\sigma\int_0^{+\infty}u^q(x,t)\zeta_R(x)\,\mathrm{d}x+\int_0^{+\infty}v^p(x,t)\zeta_R(x)\,\mathrm{d}x\\
\geq& -2^\lambda\pi^2R^{\lambda-2}\omega_R(t)+R^\sigma\Big(\int_0^{+\infty}\zeta_R(x)\,\mathrm{d}x\Big)^{1-q}\Big(\int_0^{+\infty}u(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^q\\
&+\Big(\int_0^{+\infty}\zeta_R(x)\,\mathrm{d}x\Big)^{1-p}\Big(\int_0^{+\infty}v(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^p\\
\geq& -2^\lambda\pi^2R^{\lambda-2}\omega_R(t)+2^{1-q}R^{1-q+\sigma}\Big(\int_0^{+\infty}u(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^q\\
&+2^{1-p}R^{1-p-\sigma p}\Big(\int_0^{+\infty}R^\sigma v(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^p\\
\geq& -2^\lambda\pi^2R^{\lambda-2}\omega_R(t)+2^{1-p}R^{-(pq-1)/(p+1)}\Big(\Big(\int_0^{+\infty}u(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^q\\
&+\Big(\int_0^{+\infty}R^\sigma v(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^p\Big),\quad t>0.
\end{aligned}\tag{4.3}$$

Similarly to the proof of (3.14) and (3.15), one obtains

$$\begin{gathered}
\Big(\int_0^{+\infty}u(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^q+\Big(\int_0^{+\infty}R^\sigma v(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^p\geq 2^{-2p}\omega_R^p(t),\\
\text{if } \omega_R(t)<2,
\end{gathered}\tag{4.4}$$

$$\begin{gathered}
\Big(\int_0^{+\infty}u(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^q+\Big(\int_0^{+\infty}R^\sigma v(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^p\geq 2^{-p}\omega_R^q(t),\\
\text{if } \omega_R(t)\geq 2.
\end{gathered}\tag{4.5}$$

Since $pq<(pq)_c$, it holds that

$$-\frac{pq-1}{p+1}>\lambda-2.$$

Thanks to $\inf_{R>0}\omega_R(0)>0$, there exists $R>0$ sufficiently large such that

$$2^\lambda\pi^2R^{\lambda-2}\leq 2^{-3p}R^{-(pq-1)/(p+1)}\omega_R^{p-1}(0),\tag{4.6}$$

$$2^\lambda\pi^2R^{\lambda-2}\leq 2^{-2p}R^{-(pq-1)/(p+1)}\omega_R^{q-1}(0).\tag{4.7}$$

If $\omega_R(0)<2$, we claim that there exists $\tilde{T}>0$ such that $\omega_R(\tilde{T})\geq 2$. Otherwise,

$$\omega_R(t)<2,\quad t\in[0,+\infty).\tag{4.8}$$

It follows from (4.3), (4.4), (4.6) and (4.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t)\geq 2^{-3p}R^{-(pq-1)/(p+1)}\omega_R^p(t),\quad t>0.$$

Since $p\geq q>1$, there exists $\hat{T}>0$ such that

$$\omega_R(t)=\int_0^{+\infty}(u(x,t)+R^\sigma v(x,t))\zeta_R(x)\,\mathrm{d}x\to+\infty,\quad \text{as } t\to\hat{T}^-,$$

which contradicts (4.8). Therefore, one can assume that $\omega_R(0) \geq 2$ without loss of generality. Then one gets from (4.3), (4.5) and (4.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t) \geq 2^{-2p} R^{-(pq-1)/(p+1)} \omega_R^q(t), \quad t > 0.$$

Since $p \geq q > 1$, there exists $T > 0$ such that

$$\omega_R(t) = \int_0^{+\infty} (u(x,t) + R^\sigma v(x,t))\zeta_R(x)\,\mathrm{d}x \to +\infty, \quad \text{as } t \to T^-,$$

which leads to

$$\|u(\cdot,t)\|_{L^\infty(0,+\infty)} + \|v(\cdot,t)\|_{L^\infty(0,+\infty)} \to +\infty, \quad \text{as } t \to T^-.$$

That is to say, (u,v) must blow up in a finite time.

We turn to case (iii) that $pq > (pq)_c$. Thanks to Theorem 2.5 and Propositions 2.4 and 2.3, the solution to (1.13)–(1.16) blows up in a finite time if (u_0, v_0) is suitably large. Below we prove that there exists a nontrivial global solution to (1.13)–(1.16) if (u_0, v_0) is suitably small. Set

$$\hat{u}(x,t) = (t+1)^{-(p+1)/(pq-1)} U((t+1)^{-1/(2-\lambda)}x), \quad x \geq 0,\ t \geq 0, \tag{4.9}$$

$$\hat{v}(x,t) = (t+1)^{-(q+1)/(pq-1)} V((t+1)^{-1/(2-\lambda)}x), \quad x \geq 0,\ t \geq 0. \tag{4.10}$$

If $U, V \in C^2((0,+\infty))$ satisfy

$$(r^\lambda U'(r))' + \frac{1}{2-\lambda} rU'(r) + \frac{p+1}{pq-1} U(r) + V^p(r) \leq 0, \quad r > 0, \tag{4.11}$$

$$(r^\lambda V'(r))' + \frac{1}{2-\lambda} rV'(r) + \frac{q+1}{pq-1} V(r) + U^q(r) \leq 0, \quad r > 0, \tag{4.12}$$

then, $(\hat{u}, \hat{v})$ given by (4.9) and (4.10) is a supersolution to (1.13) and (1.14). We take

$$U(r) = V(r) = \varepsilon \mathrm{e}^{-Ar^{2-\lambda}}, \quad r \geq 0,$$

where $\varepsilon > 0$ will be determined and A is a constant such that

$$\frac{p+1}{(2-\lambda)(pq-1)} < A < \frac{1}{(2-\lambda)^2}. \tag{4.13}$$

For $r > 0$, one gets from direct calculations that

$$\begin{aligned}
&(r^\lambda U'(r))' + \frac{1}{2-\lambda} rU'(r) + \frac{p+1}{pq-1} U(r) + V^p(r) \\
&= U(r)\Big(A(2-\lambda)\Big(A(2-\lambda) - \frac{1}{2-\lambda}\Big) r^{2-\lambda} + \Big(\frac{p+1}{pq-1} - A(2-\lambda)\Big) \\
&\quad + \varepsilon^{p-1} \mathrm{e}^{-A(p-1)r^{2-\lambda}}\Big) \\
&\leq U(r)\Big(A(2-\lambda)\Big(A(2-\lambda) - \frac{1}{2-\lambda}\Big) r^{2-\lambda} + \Big(\frac{p+1}{pq-1} - A(2-\lambda)\Big) + \varepsilon^{p-1}\Big), \\
&(r^\lambda V'(r))' + \frac{1}{2-\lambda} rV'(r) + \frac{q+1}{pq-1} V(r) + U^q(r) \\
&= V(r)\Big(A(2-\lambda)\Big(A(2-\lambda) - \frac{1}{2-\lambda}\Big) r^{2-\lambda} + \Big(\frac{q+1}{pq-1} - A(2-\lambda)\Big) \\
&\quad + \varepsilon^{q-1} \mathrm{e}^{-A(q-1)r^{2-\lambda}}\Big)
\end{aligned}$$

$$\le V(r)\Big(A(2-\lambda)\Big(A(2-\lambda)-\frac{1}{2-\lambda}\Big)r^{2-\lambda}+\Big(\frac{q+1}{pq-1}-A(2-\lambda)\Big)+\varepsilon^{q-1}\Big).$$

It follows from (4.13) and $p\ge q>1$ that

$$A(2-\lambda)<\frac{1}{2-\lambda},\quad \frac{p+1}{pq-1}<A(2-\lambda),\quad \frac{q+1}{pq-1}<A(2-\lambda).$$

Therefore, $(\hat u,\hat v)$ is a supersolution to (1.13) and (1.14) for each $0<\varepsilon\le\varepsilon_0$, where

$$\varepsilon_0=\min\Big\{\Big(A(2-\lambda)-\frac{p+1}{pq-1}\Big)^{1/(p-1)},\Big(A(2-\lambda)-\frac{q+1}{pq-1}\Big)^{1/(q-1)}\Big\}.$$

Note that

$$\Big(x^\lambda\frac{\partial\hat u}{\partial x}\Big)(0,t)=0,\quad \Big(x^\lambda\frac{\partial\hat v}{\partial x}\Big)(0,t)=0,\quad t>0.$$

Therefore, $(\hat u,\hat v)$ is a supersolution to (1.13)–(1.16) if $0<\varepsilon\le\varepsilon_0$ and

$$u_0(x)\le\hat u(x,0),\quad v_0(x)\le\hat v(x,0),\quad x>0. \tag{4.14}$$

Thanks to Proposition 2.4 (ii), the solution to (1.13)–(1.16) exists globally in time if (u_0,v_0) satisfies (4.14) and $0<\varepsilon\le\varepsilon_0$. The proof of Theorem 2.7 (i) and (iii) is complete. $\square$

To prove Theorem 2.7 (ii), we need the following lemma.

Lemma 4.1. *Let $0<\lambda<2$, $p\ge q>1$ and $pq=(pq)_c$. Assume that (u,v) is a solution to* (1.13)–(1.16) *in $(0,+\infty)$. Then for any $R>0$,*

$$\omega_R(t)\le N,\quad t>0, \tag{4.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t)\ge-\frac{p-1}{p}2^{(2p+\lambda p-1)/(p-1)}N^{p/(p-1)}\pi^{2p/(p-1)}R^{\lambda-2},\quad t>0, \tag{4.16}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t)\ge{}&R^{\lambda-2}\omega_R^{1/2}(t)\Big(-2^\lambda\pi^2\Big(\int_R^{2R}(u(x,t)+R^\sigma v(x,t))\zeta_R(x)\,\mathrm{d}x\Big)^{1/2}\\&+2^{1-2p}N^{-p}\omega_R^{(2p-1)/2}(t)\Big),\quad t>0,\end{aligned} \tag{4.17}$$

where

$$\omega_R(t)=\int_0^{+\infty}(u(x,t)+R^\sigma v(x,t))\zeta_R(x)\,\mathrm{d}x,\quad t>0,$$
$$N=2^{(2p+\lambda)/(q-1)}\pi^{2/(q-1)},\quad \sigma=\frac{q-p}{p+1},$$

and $\zeta_R(x)$ is the function defined by (4.1).

Proof. Since $p\ge q>1$, it holds that $N\ge2$. It follows from (4.3) and (4.5) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t)\ge-2^\lambda\pi^2R^{\lambda-2}\omega_R(t)+2^{1-2p}R^{-(pq-1)/(p+1)}\omega_R^q(t),\quad\text{if }\omega_R(t)\ge2. \tag{4.18}$$

Since $pq=(pq)_c$, one obtains

$$-\frac{pq-1}{p+1}=\lambda-2.$$

Let us prove (4.15) by a contradiction. If (4.15) is not true, there exists $t_0>0$ such that

$$\omega_R(t_0)>N\ge2,\quad 2^\lambda\pi^2\le2^{-2p}\omega_R^{q-1}(t_0). \tag{4.19}$$

Then, from (4.18) and (4.19) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t) \geq 2^{-2p}R^{\lambda-2}\omega_R^q(t), \quad t > t_0. \tag{4.20}$$

Since $p \geq q > 1$, there exists $T > t_0$ such that

$$\omega_R(t) = \int_0^{+\infty} (u(x,t) + R^\sigma v(x,t))\zeta_R(x)\,\mathrm{d}x \to +\infty, \quad \text{as } t \to T^-,$$

which leads to

$$\|u(\cdot,t)\|_{L^\infty(0,+\infty)} + \|v(\cdot,t)\|_{L^\infty(0,+\infty)} \to +\infty, \quad \text{as } t \to T^-.$$

Hence (u,v) must blow up in a finite time, which is a contradiction. Thus (4.15) is proved.

Now we turn to (4.16). It follows from (4.3) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t) \geq -2^\lambda\pi^2R^{\lambda-2}\omega_R(t) + 2^{1-p}R^{\lambda-2}\Big(\Big(\int_0^{+\infty} u(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^q \\ + \Big(\int_0^{+\infty} R^\sigma v(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^p\Big), \quad t > 0.\end{aligned} \tag{4.21}$$

Owing to (4.15), it holds that

$$\int_0^{+\infty} u(x,t)\zeta_R(x)\,\mathrm{d}x \leq N, \quad \int_0^{+\infty} R^\sigma v(x,t)\zeta_R(x)\,\mathrm{d}x \leq N, \quad t > 0.$$

Hence from [17, Lemma 3.6] we obtain

$$\Big(\int_0^1 u(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^q + \Big(\int_0^1 R^\sigma v(x,t)\zeta_\delta(x)\,\mathrm{d}x\Big)^p \geq (2N)^{-p}\omega_R^p(t), \quad t > 0. \tag{4.22}$$

It follows from (4.21), (4.22) and Hölder's inequality that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t) &\geq -2^\lambda\pi^2R^{\lambda-2}\omega_R(t) + 2^{1-2p}N^{-p}R^{\lambda-2}\omega_R^p(t) \\ &\geq 2^{1-2p}N^{-p}R^{\lambda-2}\Big(-2^{2p+\lambda-1}N^p\pi^2\omega_R(t) + \omega_R^p(t)\Big) \\ &\geq 2^{1-2p}N^{-p}R^{\lambda-2}\Big(-\frac{p-1}{p}2^{p(2p+\lambda-1)/(p-1)}N^{p^2/(p-1)}\pi^{2p/(p-1)} \\ &\quad -\frac{1}{p}\omega_R^p(t) + \omega_R^p(t)\Big) \\ &\geq -\frac{p-1}{p}2^{(2p+\lambda p-1)/(p-1)}N^{p/(p-1)}\pi^{2p/(p-1)}R^{\lambda-2}, \quad t > 0.\end{aligned}$$

Thus (4.16) is proved.

Next we prove (4.17). It follows from (4.2) and Hölder's inequality that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\omega_R(t) &= \int_R^{2R} \big(u(x,t) + R^\sigma v(x,t)\big)(x^\lambda\zeta_R'(x))'\,\mathrm{d}x \\ &\quad + R^\sigma\int_0^{+\infty} u^q(x,t)\zeta_R(x)\,\mathrm{d}x + \int_0^{+\infty} v^p(x,t)\zeta_R(x)\,\mathrm{d}x \\ &\geq -2^\lambda\pi^2R^{\lambda-2}\int_R^{2R} \big(u(x,t) + R^\sigma v(x,t)\big)\zeta_R(x)\,\mathrm{d}x \\ &\quad + R^\sigma\int_0^{+\infty} u^q(x,t)\zeta_R(x)\,\mathrm{d}x + \int_0^{+\infty} v^p(x,t)\zeta_R(x)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\geq -2^{\lambda}\pi^2 R^{\lambda-2}\int_R^{2R}\big(u(x,t)+R^{\sigma}v(x,t)\big)\zeta_R(x)\,\mathrm{d}x\\
&\quad + R^{\sigma}\Big(\int_0^{+\infty}\zeta_R(x)\,\mathrm{d}x\Big)^{1-q}\Big(\int_0^{+\infty}u(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^{q}\\
&\quad + R^{-\sigma p}\Big(\int_0^{+\infty}\zeta_R(x)\,\mathrm{d}x\Big)^{1-p}\Big(\int_0^{+\infty}R^{\sigma}v(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^{p}\\
&\geq -2^{\lambda}\pi^2 R^{\lambda-2}\int_R^{2R}\big(u(x,t)+R^{\sigma}v(x,t)\big)\zeta_R(x)\,\mathrm{d}x\\
&\quad + 2^{1-q}R^{1-q+\sigma}\Big(\int_0^{+\infty}u(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^{q}\\
&\quad + 2^{1-p}R^{1-p-\sigma p}\Big(\int_0^{+\infty}R^{\sigma}v(x,t)\zeta_R(x)\,\mathrm{d}x\Big)^{p}\\
&\geq -2^{\lambda}\pi^2 R^{\lambda-2}\int_R^{2R}(u(x,t)+R^{\sigma}v(x,t))\zeta_R(x)\,\mathrm{d}x\\
&\quad + 2^{1-2p}N^{-p}R^{\lambda-2}\omega_R^{p}(t),\quad t>0,
\end{aligned}
$$

which leads to (4.17). □

Proof of Theorem 2.7 (ii). Assuming that (u,v) is a solution to (1.13)–(1.16) in $(0,+\infty)$, we set

$$
\Gamma=\sup_{R>0,t>0}\omega_R(t)=\sup_{t>0}\int_0^{+\infty}(u(x,t)+R^{\sigma}v(x,t))\,\mathrm{d}x. \tag{4.23}
$$

From (4.15) and the nontriviality of (u_0,v_0), it holds that $0<\Gamma<+\infty$. For $\varepsilon_0>0$, there exists $t_1\geq 0$ and $R_0>0$ such that

$$
\omega_{R_0}(t_1)\geq\Gamma-\varepsilon_0, \tag{4.24}
$$

where ε_0 will be determined below. For any $t\geq t_1$, it follows from (4.16) with $R=R_0$ and (4.24) that

$$
\begin{aligned}
\omega_{R_0}(t)\geq&\,\omega_{R_0}(t_1)-\frac{p-1}{p}2^{(2p+\lambda p-1)/(p-1)}N^{p/(p-1)}\pi^{2p/(p-1)}R_0^{\lambda-2}(t-t_1)\\
\geq&\,\Gamma-\varepsilon_0-\frac{p-1}{p}2^{(2p+\lambda p-1)/(p-1)}N^{p/(p-1)}\pi^{2p/(p-1)}R_0^{\lambda-2}(t-t_1),
\end{aligned}
$$

which, together with (4.23), leads to

$$
\begin{aligned}
&\int_{2R_0}^{4R_0}(u(x,t)+R^{\sigma}v(x,t))\zeta_{2R_0}(x)\\
&\leq\int_0^{+\infty}(u(x,t)+R^{\sigma}v(x,t))\,\mathrm{d}x-\omega_{R_0}(t)\\
&\leq\varepsilon_0+\frac{p-1}{p}2^{(2p+\lambda p-1)/(p-1)}N^{p/(p-1)}\pi^{2p/(p-1)}R_0^{\lambda-2}(t-t_1).
\end{aligned}\tag{4.25}
$$

Taking $R=2R_0$ in (4.17), we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\omega_{2R_0}(t)
$$

$$
\begin{aligned}
&\geq (2R_0)^{\lambda-2}\omega_{2R_0}^{1/2}(t)\Big(-2^{\lambda}\pi^2\Big(\int_{2R_0}^{4R_0}(u(x,t)+R^{\sigma}v(x,t))\zeta_{2R_0}(x)\,\mathrm{d}x\Big)^{1/2} \\
&\quad + 2^{1-2p}N^{-p}\omega_{2R_0}^{(2p-1)/2}(t)\Big), \quad t>t_1.
\end{aligned}
$$

Fix $\varepsilon_0 \in (0,\Gamma)$ and $M_0>0$ such that

$$
2^{\lambda}\pi^2(\varepsilon_0+M_0)^{1/2} \leq 2^{-2p}N^{-p}(\Gamma-\varepsilon_0)^{(2p-1)/2}.
$$

Owing to (4.23)–(4.25), it holds that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\omega_{2R_0}(t) \geq 2^{\lambda-2-2p}N^{-p}R_0^{\lambda-2}(\Gamma-\varepsilon_0)^p, \quad t_1<t<t_2, \tag{4.26}
$$

where

$$
t_2 = t_1 + \frac{p}{p-1}2^{(2p+\lambda p-1)/(1-p)}N^{p/(1-p)}\pi^{2p/(1-p)}M_0R_0^{2-\lambda}.
$$

It follows from (4.24) and (4.26) that

$$
\omega_{2R_0}(t_2) \geq \omega_{2R_0}(t_1) + 2^{\lambda-2-2p}N^{-p}R_0^{\lambda-2}(\Gamma-\varepsilon_0)^p(t_2-t_1) \geq \Gamma-\varepsilon_0+\eta_0,
$$

where

$$
\eta_0 = \frac{p}{p-1}2^{(2p^2+2p+\lambda-3)/(1-p)}N^{p^2/(1-p)}\pi^{2p/(1-p)}M_0(\Gamma-\varepsilon_0)^p.
$$

Repeating the procedure, one obtains that for any positive integer i,

$$
\omega_{2^iR_0}(t_{i+1}) \geq \omega_{2^iR_0}(t_i)+\eta_0 \geq \omega_{2^{i-1}R_0}(t_i)+\eta_0 \geq \Gamma-\varepsilon_0+i\eta_0,
$$

where

$$
t_{i+1} = t_i + \frac{p}{p-1}2^{(2p+\lambda p-1)/(1-p)}N^{p/(1-p)}\pi^{2p/(1-p)}(2^{i-1}R_0)^{2-\lambda}M_0.
$$

Therefore,

$$
\sup_{t>0}\int_0^{+\infty}(u(x,t)+R^{\sigma}v(x,t))\,\mathrm{d}x = +\infty,
$$

which contradicts $0<\Gamma<+\infty$. The proof of Theorem 2.7 (ii) is complete. $\square$

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References

[1] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli; *Carleman estimates for degenerate parabolic operators with applications to null controllability*, J. Evol. Equ., 6(2) (2006), 161–204.

[2] D. Andreucci, G. R. Cirmi, S. Leonardi, A. F. Tedeev; *Large time behavior of solutions to the Neumann problem for a quasilinear second order degenerate parabolic equation in domains with noncompact boundary*, J. Differential Equations, 174(2) (2001), 253–288.

[3] F. Black, M. Scholes; *The pricing of options and corporate liabilities*, J. Polit. Econ., 81(3) (1973) 637–654.

[4] P. Cannarsa, P. Martinez, J. Vancostenoble; *Null controllability of degenerate heat equations*, Adv. Differential Equations, 10(2) (2005), 153–190.

[5] P. Cannarsa, P. Martinez, J. Vancostenoble; *Carleman estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim., 47 (1) (2008) 1–19.

[6] P. Cannarsa, L. de Teresa; *Controllability of 1-D coupled degenerate parabolic equations*, Electron. J. Differential Equations, 2009 (2009) no. 73, 21 pp.

[7] K. Deng, H. A. Levine; *The role of critical exponents in blow-up theorems: the sequel*, J. Math. Anal. Appl., 243 (1) (2000), 85–126.

[8] R. M. Du, J. Eichhorn, Q. Liu, C. P. Wang; *Carleman estimates and null controllability of a class of singular parabolic equations*, Adv. Nonlinear Anal., 8(1) (2019), 1057–1082.

[9] R. M. Du, C. P. Wang; *Null controllability of a class of systems governed by coupled degenerate equations*, Appl. Math. Lett., 26(1) (2013), 113–119.

[10] M. Escobedo, M. A. Herrero; *Boundedness and blow up for a semilinear reaction-diffusion system*, J. Differential Equations, 89(1) (1991), 176–202.

[11] H. Fujita; *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\lambda}$, J. Fac. Sci. Univ. Tokyo Sect., I 13 (1966) 109–124.

[12] K. Hayakawa; *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan Acad, 49 (1973), 503–505.

[13] K. Kobayashi, T. Siaro, H.Tanaka; *On the blowing up problem for semilinear heat equations*, J. Math. Soc. Japan, 29(1) (1977), 407–424.

[14] H. A. Levine; *The role of critical exponents in blow-up theorems*, SIAM Rev., 32(2) (1990), 262–288.

[15] H. L. Li, X. Y. Wang, Y. Y. Nie, H. He; *Asymptotic behavior of solutions to a degenerate quasilinear parabolic equation with a gradient term*, Electron. J. Differential Equations, 2015(298) (2015), 1–12.

[16] P. Martinez, J. Vancostenoble; *Carleman estimates for one-dimensional degenerate heat equations*, J. Evol. Equ., 6(2) (2006), 325–362.

[17] Y. Na, Y.Y. Nie, X. Zhou; *Asymptotic behavior of solutions to a class of coupled semilinear parabolic systems with gradient terms*, J. Nonlinear Sci. Appl., 10(11) (2017), 5813–5824.

[18] G. R. North, L. Howard, D. Pollard, B. Wielicki; *Variational formulation of Budyko-Sellers climate models*, J. Atmospheric Sci., 36 (2) (1979) 255–259.

[19] C. P. Wang; *Approximate controllability of a class of semilinear systems with boundary degeneracy*, Journal of Evolution Equations, 10(1) (2010), 163–193.

[20] C. P. Wang; *Asymptotic behavior of solutions to a class of semilinear parabolic equations with boundary degeneracy*, Proc. Amer. Math. Soc., 141(9) (2013) 3125–3140.

[21] C. P. Wang, R. M. Du; *Carleman estimates and null controllability for a class of degenerate parabolic equations with convection terms*, SIAM J. Control Optim., 52 (3) (2014), 1457–1480.

[22] C. P. Wang, S. N. Zheng; *Critical Fujita exponents of degenerate and singular parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A, 136(2) (2006), 415–430.

[23] C. P. Wang, S. N. Zheng, Z. J. Wang; *Critical Fujita exponents for a class of quasilinear equations with homogeneous Neumann boundary data*, Nonlinearity, 20(6) (2007), 1343–1359.

[24] C. P. Wang, Y. N. Zhou, R. M. Du, Q. Liu; *Carleman estimate for solutions to a degenerate convection-diffusion equation*, Discrete Contin. Dyn. Syst. Ser. B, 23(10) (2018), 4207–4222.

[25] J. N. Xu, C. P. Wang, Y. Y. Nie; *Carleman estimate and null controllability of a cascade degenerate parabolic system with general convection terms*, Electron. J. Differential Equations, 2018 (2018) no. 195, 20 pp.

[26] S. N. Zheng; *Global existence and global non-existence of solutions to a reaction-diffusion system*, Nonlinear Anal., 39 (3) (2000), 327–340.

[27] S. N. Zheng, X. F. Song, Z. X. Jiang; *Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux*, J. Math. Anal. Appl, 298(1) (2004), 308–324.

[28] S. N. Zheng, C. P. Wang; *Large time behaviour of solutions to a class of quasilinear parabolic equations with convection terms*, Nonlinearity, 21 (9) (2008), 2179–2200.

[29] Q. Zhou, Y. Y. Nie, X. Y. Han; *Large time behavior of solutions to semilinear parabolic equations with gradient*, J. Dyn. Control Syst., 22 (1) (2016), 191–205.

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