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A NONLINEAR MATHEMATICAL MODEL FOR TWO-PHASE FLOW IN NANOPOROUS MEDIA

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Abstract. We propose a mathematical model for the two-phase flow nanoporous media. Unlike classical models, our model suppose that the rock permeability depends on the gradient of pressure. Using usual laws of flows in porous media, we obtain a system of two nonlinear partial differential equations: the first is elliptic and the second is parabolic degenerate. We study a regularized version of our model, obtained by adding a "vanishing" term to the coefficient causing the degeneracy. We prove the existence of a weak solution of the regularized model. Our approach consists essentially to use the Rothe's method coupled with Galerkin's method.

1. Introduction

Modeling flow (of shale gas for instance) in nanoporous rocks is becoming an interesting and challenging point for many researchers. A nanoporous media is characterized by an extremely low permeability on the order of a nanodarcy ($\approx 10^{-21}$ m^2) or less. During the exploitation of those kind of porous medium (rocks), there appears very large pressure gradient at the boundaries of pores causing their extension or completely their destruction, this phenomena generates a big increase of the rock permeability. In 2012, Barenblatt et al. [13] proposed a one dimensional mathematical model describing fluid and gas flow in nanoporous media using a new formulation of permeability of the rock supposing that it depends on the pressure gradient (see also [4]). Inspired by the previous work, we propose a three dimensional mathematical model for two-phase flow in nanoporous media. Supposing the rock permeability depending on the gradient of pressure, using mass conservation, Darcy's law, capillary pressure, introducing the concept of global pressure, some functional coefficients (mobilities, fractional fluxes) and using total velocity $\mathbf{u}$ of the phases; we obtain the following system describing the flow of two incompressible, immiscible fluids in nanoporous media:

$$
\begin{gathered}
-\operatorname{div}\big(\lambda(s)K(\nabla p)\nabla p\big) = q, \\
\phi\frac{\partial s}{\partial t} - \operatorname{div}\big(\lambda_w(s)K(\nabla p)\nabla p + \Lambda(s)p_c'(s)K(\nabla p)\nabla s\big) = q_w,
\end{gathered} \tag{1.1}
$$

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where s, the saturation of the wetting phase, and p, the global pressure, are the unknowns. This system is degenerate because the coefficient $\Lambda(s)$ vanishes for $s = 0$ and $s = 1$.

In this article, we prove the existence of weak solution for a regularized version of the above system to which we associate boundary and initial conditions.

This work is organized as follows: In this section, we complete this Introduction by presenting the derivation of our model. In Sections 2 and 3, we precise the hypotheses on the data, regularize the system by adding a term guarantying the coerciveness of the parabolic equation, extend outside $[0, 1]$ the functional coefficient depending on s and give the definition of a weak solution of the regularized system. In Section 4, we discretize in time our system and give the Definition of its weak discrete time solution. In Section 5 we give Galerkin's approximations of this weak solution and prove its existence using a monotonicity method for the pressure and Brouwder Fixed Point Theorem for the saturation. Section 6 is devoted to uniform estimates that allow us to pass to the limit on Galerkin's approximations in Section 7. We give in Section 8, different uniform estimates on discrete time solutions which permit us to prove their compactness in Section 9 and to pass to the limit in Section 10, making the step time goes to zero to obtain our main result, Theorem 3.2. This work finishes by Section 11, proving a maximum principle showing that the solution s obtained is a "true" saturation.

1.1. **Flow equations.** The mass balance equation for each of the fluid phases is

$$\phi(x)\frac{\partial(\rho_\alpha s_\alpha)}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{u}_\alpha) = \rho_\alpha q_\alpha, \quad \alpha = w, n, \tag{1.2}$$

where $\alpha = w$ denotes the wetting phase (e.g. water), $\alpha = n$ indicates the non wetting phase (e.g. oil or air), ϕ is the porosity of the medium Ω which depends only on x; $\rho_\alpha, s_\alpha, \mathbf{u}_\alpha$ and q_α are respectively the density, (reduced) saturation, volumetric velocity and external volumetric flow of the α phase.

The Darcy-Muskat's law is

$$\mathbf{u}_\alpha = -K\frac{K_{r\alpha}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha \mathbf{g}), \quad \alpha = w, n, \tag{1.3}$$

where K is the absolute permeability (of the nanoporous medium), p_α, μ_α and $K_{r\alpha}$ are the pressure, the viscosity and relative permeability of the α phase, respectively.

Several discussions with petroleum engineers and porous media specialists show us that assuming the absolute permeability a function of pressure gradient seems to be a good choice.

In this work, we suppose that the absolute permeability is a function of the pressure gradient (of the wetting phase p_w), more precisely, in order to control that dependency (to control the deformation at the edge of pores), we adopt the following new formulation of the rock permeability

$$K(\nabla p_w) = \kappa_1 \frac{|\nabla p_w|}{1 + \eta|\nabla p_w|} + \kappa_2 \tag{1.4}$$

with $\kappa_1 > 0$, $\kappa_2 > 0$, $\eta > 0$ three constants. Here η is a positive control constant. The constant κ_2 ensures the coerciveness of our model. Concerning the choice of K, see the Remark 11.2 at the end of this paper.

In addition to the above equations, we suppose the customary property of saturations

$$s_w + s_n = 1, \tag{1.5}$$

and introduce the capillary pressure function

$$p_n - p_w = p_c. \tag{1.6}$$

To separate the pressure and saturation equations, we introduce the phase mobility functions

$$\lambda_\alpha(x, s_\alpha) = \frac{K_{r\alpha}(x, s_\alpha)}{\mu_\alpha}, \quad \alpha = w, n,$$

and the total mobility

$$\lambda(x, s_w) = \lambda_w(x, s_w) + \lambda_n(x, s_w).$$

The fractional flow functions are defined by

$$f_\alpha(x, s_w) = \frac{\lambda_\alpha(x, s_w)}{\lambda(x, s_w)}, \quad \alpha = w, n;$$

finally, we define the total velocity

$$\mathbf{u} = \mathbf{u}_w + \mathbf{u}_n. \tag{1.7}$$

In what follows, we re-write the equations in term of primary variables, the total velocity $\mathbf{u}$, the pressure of wetting phase p_w and the saturation of the wetting phase s_w. Under the assumptions that fluids are incompressible (ρ_α is constant), summing up equations (1.2) for $\alpha = w, n$, we obtain

$$\phi \frac{\partial}{\partial t}(s_w + s_n) + \operatorname{div}(\mathbf{u}_w + \mathbf{u}_n) = q_w + q_n,$$

using (1.5) and (1.7), we obtain

$$\operatorname{div} \mathbf{u} = q = q_w + q_n. \tag{1.8}$$

Also, concerning the total velocity, we have

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}_w + \mathbf{u}_n \\
&= -K(\nabla p_w)\lambda_w(s_w)(\nabla p_w - \rho_w \mathbf{g}) - K_w(\nabla p)\lambda_n(s_w)(\nabla p_n - \rho_n \mathbf{g}) \\
&= -K(\nabla p_w)\lambda(s_w)\Big[\frac{\lambda_w(s_w)}{\lambda(s_w)}(\nabla p_w - \rho_w \mathbf{g}) + \frac{\lambda_n(s_w)}{\lambda(s_w)}(\nabla p_n - \rho_n \mathbf{g})\Big],
\end{aligned}$$

since $p_c = p_n - p_w$, we have $\nabla p_n = \nabla p_w + \nabla p_c$ and

$$\begin{aligned}
\mathbf{u} &= -K(\nabla p_w)\lambda(s_w)\Big[\frac{\lambda_w(s_w)}{\lambda(s_w)}(\nabla p_w - \rho_w \mathbf{g}) + \frac{\lambda_n(s_w)}{\lambda(s_w)}(\nabla p_n - \rho_n \mathbf{g})\Big] \\
&= -K(\nabla p_w)\lambda(s_w)\Big[\frac{\lambda_w(s_w)}{\lambda(s_w)}(\nabla p_w - \rho_w \mathbf{g}) + \frac{\lambda_n(s_w)}{\lambda(s_w)}(\nabla p_w + \nabla p_c - \rho_n \mathbf{g})\Big] \\
&= -K(\nabla p_w)\lambda(s_w)\Big[\nabla p_w + f_n(s_w)\nabla p_c - \mathbf{g}\{f_w(s_w)\rho_w + f_n(s_w)\rho_n\}\Big],
\end{aligned}$$

as a result, we have the equation

$$\mathbf{u} = -K(\nabla p_w)\lambda(s_w)\Big[\nabla p_w + f_n(s_w)\nabla p_c - \mathbf{g}\{f_w(s_w)\rho_w + f_n(s_w)\rho_n\}\Big]. \tag{1.9}$$

Remark 1.1. We can obtain expressions

$$\mathbf{u}_w = -K(\nabla p_w)\lambda_w(s_w)(\nabla p_w - \rho_w \mathbf{g}), \tag{1.10}$$

$$\mathbf{u}_n = -K(\nabla p_w)\lambda_n(s_w)(\nabla p_w + \nabla p_c - \rho_n \mathbf{g}). \tag{1.11}$$

So finally, we obtain the system of equations

$$\begin{gathered}
\operatorname{div} \mathbf{u} = q_w + q_n = q, \\
\mathbf{u} = -K(\nabla p_w)\lambda(s_w)\big[\nabla p_w + f_n(s_w)\nabla p_c - \mathbf{g}\{f_w(s_w)\rho_w + f_n(s_w)\rho_n\}\big], \\
q_w = \phi \frac{\partial s_w}{\partial t} - \operatorname{div}\Big(K(\nabla p_w)\lambda_w(s_w)(\nabla p_w - \rho_w \mathbf{g})\Big),
\end{gathered} \tag{1.12}$$

where the primary unknowns are p_w, s_w and $\mathbf{u}$. Taking $\mathbf{g} = \mathbf{0}$, the system is written as

$$\begin{gathered}
-\operatorname{div}\Big(\lambda(s_w)K(\nabla p_w)\nabla p_w\Big) - \operatorname{div}\Big(f_n(s_w)\lambda(s_w)K(\nabla p_w)\nabla p_c\Big) = q, \\
\phi \frac{\partial s_w}{\partial t} - \operatorname{div}\Big(\lambda_w(s_w)K(\nabla p_w)\nabla p_w\Big) = q_w.
\end{gathered} \tag{1.13}$$

We introduce as in [8], the global pressure

$$p = p_n - \int_0^s \Big(f_w \frac{\partial p_c}{\partial s}\Big)(x,\xi)\, d\xi, \tag{1.14}$$

with $s = s_w$. Making use of the definition of global and capillary pressure, and the concept of the Differentiation of Integrals (see for example [18, page 213]) we can write

$$\nabla p = \nabla p_n - \nabla \int_0^s \Big(f_w \frac{\partial p_c}{\partial s}\Big)(x,\xi)\, d\xi = \nabla p_n - \nabla s(x)\Big(f_w \frac{\partial p_c}{\partial s}\Big)(x, s(x)) - \gamma_1(x, s(x)),$$

with

$$\gamma_1(x,s) = \int_0^s \frac{\partial}{\partial x}\Big(f_w \frac{\partial p_c}{\partial s}\Big)(x,\xi)\, d\xi.$$

Now, we have

$$\begin{aligned}
\nabla p_w &= \nabla p_n - \nabla p_c = \nabla p + \nabla \int_0^s f_w(\xi)\frac{\partial p_c}{\partial s}(x,\xi)\, d\xi - \nabla p_c \\
&= \nabla p + f_w(s)\nabla p_c(s) + \gamma_1(x,s) - \nabla p_c \\
&= \nabla p + (f_w(s) - 1)\nabla p_c(s) + \gamma_1(x,s) \\
&= \nabla p - f_n(s)\nabla p_c(s) + \gamma_1(x,s),
\end{aligned}$$

so

$$K(\nabla p_w) = K\Big(\nabla p - f_n(s)\nabla p_c(s) + \gamma_1(x,s)\Big) \doteq \overline{K}(\nabla p, s, \nabla s).$$

This leads to the system

$$\begin{gathered}
-\operatorname{div}\Big(\lambda(s)\overline{K}(\nabla p, s, \nabla s)(\nabla p + \gamma_1(x,s))\Big) = q, \\
\phi \frac{\partial s}{\partial t} - \operatorname{div}\Big(\lambda_w(s)\overline{K}(\nabla p, s, \nabla s)\nabla p - \overline{K}(\nabla p, s, \nabla s)\Lambda(s)p_c'(s)\nabla s \\
+ \lambda_w(s)\overline{K}(\nabla p, s, \nabla s)\gamma_1(s)\Big) = q_w,
\end{gathered}$$

where $\Lambda(s) = \lambda_w(s)\lambda_n(s).\lambda(s)$. From a theoretical point of view, and in order to simplify the model, we are going to neglect the term γ_1 and assume that

$$\overline{K}(\nabla p, s, \nabla s) \approx K(\nabla p).$$

Consequently, we obtain the system

$$-\operatorname{div}\Big(\lambda(s)K(\nabla p)\nabla p\Big) = q,$$
$$\phi\frac{\partial s}{\partial t} - \operatorname{div}\Big(\lambda_w(s)K(\nabla p)\nabla p + \Lambda(s)p_c'(s)K(\nabla p)\nabla s\Big) = q_w,$$

to which we must add initial and boundary conditions. Therefore, we consider the system:

Find (p,s) solving the equations

$$-\operatorname{div}\Big(\lambda(s)K(\nabla p)\nabla p\Big) = q \quad \text{in } \Omega_T \doteq \Omega\times]I[, \tag{1.15}$$
$$\phi\frac{\partial s}{\partial t} - \operatorname{div}\Big(\lambda_w(s)K(\nabla p)\nabla p + \Lambda(s)p_c'(s)K(\nabla p)\nabla s\Big) = q_w \quad \text{in } \Omega_T, \tag{1.16}$$
$$p(x,t) = 0 \quad \text{et} \quad s(x,t) = 0 \qquad \text{on } \partial\Omega\times[I], \tag{1.17}$$
$$s(x,0) = s_0(x) \quad \text{in } \Omega, \tag{1.18}$$

where $\Omega\subset\mathbb{R}^3$ represents the nanoporous medium, supposed to be bounded, connected and Lipschitz domain, $I\doteq]0,T[$ is the time interval, and with the following expression of the absolute permeability given in page 2:

$$K(\nabla p) = \kappa_1\frac{|\nabla p|}{1+\eta|\nabla p|} + \kappa_2, \quad \text{with } \kappa_1>0,\ \kappa_2>0,\ \eta>0 \text{ three constants,}$$

and $q=q(x,t)$, $q_w=q_w(x,t)$, $s_0=s_0(x)$ three given functions.

In all that follows, we will denote by (S) the system of equations (1.15) and (1.16), with boundary conditions (1.17) and the initial condition (1.18).

1.2. **Functional setting.** We denote by V the Sobolev space $H_0^1(\Omega)$, equipped with the inner product $(u,v)_V = \int_\Omega \nabla u\cdot\nabla v\,dx$ and the gradient norm $\|u\|_V = \big[\int_\Omega |\nabla u|^2\,dx\big]^{1/2}$, its dual is indicated by $V^\star$. For $1\le p<\infty$ and B a Banach space, we denote $L^p(I;B)$ the Bochner space (of classes with respect to equivalence a.e.) of Bochner integrable functions $u: I\longrightarrow B$ satisfying $\int_0^T \|u(t)\|_B^p\,dt < +\infty$. This space is a Banach space if endowed with the norm $\|u\|_{L^p(0,T;B)} = \big(\int_0^T \|u(t)\|_B^p\,dt\big)^{1/p}$. For $p=\infty$, this norm is $\|u\|_{L^\infty(I;B)} = \operatorname{ess\,sup}_{t\in I}\|u(t)\|_B$. Following [16], we denote $W(0,T)$, the Sobolev-Bochner space

$$W(0,T) = W^{2,2}(0,T;V,V^\star) = \Big\{u\in L^2(0,T,V)\,\big|\,u' = \frac{du}{dt}\in L^2(0,T;V^\star)\Big\}.$$

Equipped by the norm $\|u\|_W = \big(\|u\|^2_{L^2(I;V)} + \|u'\|^2_{L^2(I;V^\star)}\big)^{1/2}$, $W(0,T)$ is a Hilbert space which is continuously embedded in $C([0,T];L^2(\Omega))$, equipped with the norm of uniform convergence. Proofs of the above facts can be found in [10, 14, 16].

2. Hypotheses

- (H1) The porosity ϕ belongs to $W^{1,+\infty}(\Omega)$, and for two constants, ϕ_* and ϕ^* we have
$$0<\phi_*\le\phi(x)\le\phi^*<+\infty, \quad \text{a.e.} \quad x\in\Omega. \tag{2.1}$$
- (H2) $\lambda_\alpha(x,s)$, $\alpha=n,w$ are measurable in $x\in\Omega$ and continuous in $s\in[0,1]$, and satisfies $\lambda_n(x,1)=0$, $\lambda_n(x,s)>0$ for $s<1$, $\lambda_w(x,0)=0$, $\lambda_w(x,s)>0$ for $s>0$; and there exist two constants λ_*, λ^* such that
$$0<\lambda_*\le\lambda(x,s)\le\lambda^*<+\infty, \quad x\in\Omega, \quad s\in[0,1], \tag{2.2}$$

where $\lambda(s) = \lambda_w(s) + \lambda_n(s)$.

(H3) $p_c \in W^{1,\infty}([0,1])$ and p'_c is a continuous function, also there exits two constants p'_{c*}, and p'^{*}_{c} such that
$$0 < p'_{c*} \le p'_c(s) \le p'^{*}_{c} < +\infty.$$

(H4) The initial saturation s_0 is in $L^2(\Omega)$, the functions q and q_w are positive functions in $L^2(\Omega_T)$.

In what follows, we put
$$\Lambda_\varepsilon(x,s) = \Lambda(x,s) + \varepsilon \quad \text{with } \Lambda(x,s) = \frac{\lambda_w(x,s)\lambda_n(x,s)}{\lambda(x,s)} \text{ and } \varepsilon > 0.$$

Remark 2.1. Hypothesis (H1) permits us, among other things, to put $\langle \phi\partial_t s, v\rangle := \langle \partial_t s, \phi v\rangle$ for $v \in V = H^1_0(\Omega)$ to give sense to $\phi\partial_t s$ knowing that $\partial_t s \in V^\star$. This because $\phi \in W^{1,\infty}(\Omega)$ implies that $\phi v \in V$, for all $v \in V$. Also, we should note that, during the entire work, inequality (2.1) is used to obtain different estimations on the equation of saturation.

3. Regularization: system (S_ε)

We extend the coefficients of identities (1.15), (1.16) outside $[0,1]$ as continuous functions in s by putting
$$\lambda_{w\star}(x,s) = \begin{cases} \lambda_w(x,s), & x \in \Omega, s \in [0,1], \\ \lambda_w(x,1), & x \in \Omega, s \ge 1, \\ \lambda_w(x,0), & x \in \Omega, s \le 0, \end{cases} \quad \lambda_{n\star}(x,s) = \begin{cases} \lambda_n(x,s), & x \in \Omega, s \in [0,1], \\ \lambda_n(x,1), & x \in \Omega, s \ge 1, \\ \lambda_n(x,0), & x \in \Omega, s \le 0. \end{cases}$$
The capillary pressure p_c is extended outside $[0,1]$ in the same way. Also, we put
$$\lambda_\star(x,s) = \lambda_{w\star}(x,s) + \lambda_{n\star}(x,s), \quad \Lambda_\star(x,s) = \frac{\lambda_{w\star}(x,s)\lambda_{n\star}(x,s)}{\lambda_{w\star}(x,s) + \lambda_{n\star}(x,s)},$$
$$\Lambda_\varepsilon(x,s) = \Lambda_\star(x,s) + \varepsilon, \quad \varepsilon > 0.$$
Substituting these functions in the system (S) (equations (1.15)–(1.18)), we obtain the system (S_ε).

3.1. Weak solution of system (S_ε).

Definition 3.1. A weak solution of system (S_ε) is a couple (p,s) such that
$$(p,s) \in L^2(I;V) \times L^2(I;V), \partial_t s \in L^2(I;V^\star), \tag{3.1}$$
$$\int_\Omega \lambda_\star(s) K(\nabla p)\nabla p \cdot \nabla\varphi \, dx = (q,\varphi), \quad \forall \varphi \in V, \text{ a.e. } t \in I, \tag{3.2}$$
$$\begin{aligned} &\int_0^T \Big\langle \frac{\phi\partial s}{\partial t}, \psi \Big\rangle \, dt + \int_{\Omega_T} \lambda_{w\star}(s) K(\nabla p)\nabla p \cdot \nabla\psi \, dx\, dt \\ &+ \int_{\Omega_T} \Lambda_\varepsilon(s) p'_c(s) K(\nabla p)\nabla s \cdot \nabla\psi \, dx\, dt \\ &= \int_0^T (q_w,\psi)\, dt, \quad \forall \psi \in L^2(I;V), \end{aligned} \tag{3.3}$$
$$s(x,0) = s_0(x). \tag{3.4}$$

Theorem 3.2. *Under the hypothesis* (H1)–(H4)*, Problem* (S_ε) *has at least one weak solution in the sense of Definition* 3.1.

For simplicity, we omit subindex $\star$, standing for extension of λ, λ_n, and λ_w outside $[0,1]$.

4. Time discretization of system (S_ε)

To show the existence of a weak solution of the system (S_ε) in the sense of definition 3.1, we use the method of Rothe (semi-discretization in time) coupled with Galerkin's method. To do this, for each positive integer n, we divide the interval $I=]0,T[$ into $N=2^n$ subintervals and we set $\alpha=\frac{T}{N}=2^{-n}T$ and put $t_j=j\alpha$ and $I_j=(t_{j-1},t_j]$ for any integer j, $j=1,\dots,N$. We approach the time derivative $\frac{\partial s}{\partial t}$ by the time difference operator

$$\partial_t^\alpha s(x,t)=\frac{s(x,t+\alpha)-s(x,t)}{\alpha}.$$

If $w=w(x,t)$ is a function, the average in time over I_j is

$$w_\alpha(x,t)=\frac{1}{\alpha}\int_{I_j} w(x,\tau)\,d\tau,\quad t\in I_j. \tag{4.1}$$

The value of $w_\alpha(\cdot)$ on the interval I_j is denoted by $w_{\alpha j}(\cdot)$. Also, for any linear space H, we define

$$\ell^\alpha(I;H)=\{v\in L^\infty(I;H): v \text{ is constant in time on each subinterval } I_j\subset I\}.$$

The value of a function $v^\alpha(\cdot)$ from the space $\ell^\alpha(I;H)$ on the interval I_j is constant and it is equal to $v^\alpha(t_j)(\cdot)$ which will be denoted by $v^{j\alpha}(\cdot)$, i.e.

$$v^\alpha(x,t)=\sum_{j=1}^N v^\alpha(x,t_j)\chi_{]t_{j-1},t_j]}(t)=\sum_{j=1}^N v^{j\alpha}(x)\chi_{]t_{j-1},t_j]}(t).$$

We define also the function $\widetilde{v}^\alpha$ by

$$\begin{aligned}\widetilde{v}^\alpha(x,t)&=\sum_{j=1}^N\Big[\frac{v^\alpha(x,t_j)-v^\alpha(x,t_{j-1})}{\alpha}(t-t_{j-1})+v^\alpha(x,t_{j-1})\Big]\chi_{[t_{j-1},t_j[}(t)\\ &=\sum_{j=1}^N\Big[\frac{v^{j\alpha}(x)-v^{(j-1)\alpha}(x)}{\alpha}(t-t_{j-1})+v^{(j-1)\alpha}(x)\Big]\chi_{[t_{j-1},t_j[}(t),\end{aligned}$$

where we put $\widetilde{v}^{0\alpha}(x)=\widetilde{v}(x,0)=v_0(x)$, a given function supposed hereafter to play the role of the initial condition.

Remark 4.1. Performing simple calculations, one can easily see that

$$\|w_\alpha\|^2_{L^2(\Omega_T)}=\alpha\sum_{j=1}^N\|w_{\alpha j}\|^2_{L^2(\Omega)},\quad \|w_\alpha\|_{L^2(\Omega_T)}\le\|w\|_{L^2(\Omega_T)},$$

$$\|v^\alpha\|^2_{L^2(I;V)}=\alpha\sum_{j=1}^N\|\nabla v^{j\alpha}\|^2_{L^2(\Omega)},\quad \|\widetilde{v}^\alpha\|^2_{L^2(I;V)}\le 5\alpha\sum_{j=1}^N\|\nabla v^{j\alpha}\|^2_{L^2(\Omega)},$$

$$\|v^\alpha-\widetilde{v}^\alpha\|^2_{L^2(I;X)}=\frac{\alpha}{3}\sum_{j=1}^N\|v^{j\alpha}-v^{(j-1)\alpha}\|^2_X,$$

$$\frac{\partial\widetilde{v}^\alpha}{\partial t}=\sum_{j=1}^N\frac{v^{j\alpha}-v^{(j-1)\alpha}}{\alpha}\chi_{[t_{j-1},t_j[},\text{ a.e.},$$

$$\|\frac{\partial \widetilde{v}^{\alpha}}{\partial t}\|^2_{L^2(I;V)} = \frac{1}{\alpha}\sum_{j=1}^{N}\|v^{j\alpha} - v^{(j-1)\alpha}\|^2_V.$$

Definition 4.2. A discrete time solution is a couple of functions

$$(p^{\alpha}, s^{\alpha}) \in \ell^{\alpha}(I;V)\times \ell^{\alpha}(I;V)$$

which satisfies

$$\begin{gathered}\int_{\Omega}\lambda(s^{(j-1)\alpha})K(\nabla p^{j\alpha})\nabla p^{j\alpha}\cdot\nabla\varphi\,dx = (q_{\alpha j},\varphi),\\ \forall\varphi\in V,\ t\in I_j,\ j=1,\dots,N,\end{gathered} \tag{4.2}$$

$$\begin{aligned}&\int_0^T(\phi\partial_t^{-\alpha}s^{\alpha},\psi)\,dt + \int_{\Omega_T}\lambda_w(s^{\alpha})K(\nabla p^{\alpha})\nabla p^{\alpha}\cdot\nabla\psi\,dx\,dt\\ &+\int_{\Omega_T}\Lambda_{\varepsilon}(s^{\alpha})p_c'(s^{\alpha})K(\nabla p^{\alpha})\nabla s^{\alpha}\cdot\nabla\psi\,dx\,dt\\ &=\int_0^T(q_{w\alpha},\psi)\,dt,\quad \forall\psi\in\ell^{\alpha}(I;V).\end{aligned} \tag{4.3}$$

Regarding the first term in (4.3), we have $\int_0^T(\phi\partial_t^{-\alpha}s^{\alpha},\psi)\,dt = \int_0^T\langle\frac{\partial\tilde{s}^{\alpha}}{\partial t},\psi\rangle\,dt$ this because $\partial_t^{-\alpha}s^{\alpha} = \frac{\partial\tilde{s}^{\alpha}}{\partial t}$. In fact,

$$\begin{aligned}\partial_t^{-\alpha}s^{\alpha}(x,t) &= \frac{s^{\alpha}(x,t)-s^{\alpha}(x,t-\alpha)}{\alpha}\\ &= \frac{\sum_{j=1}^{N}(s^{\alpha}(x,t_j)-s^{\alpha}(x,t_j-\alpha)\chi_{[t_{j-1},t_j[}(t)}{\alpha}\\ &= \frac{\sum_{j=1}^{N}(s^{j\alpha}(x)-s^{(j-1)\alpha}(x))\chi_{[t_{j-1},t_j[}(t)}{\alpha}\\ &= \frac{\partial\widetilde{s}^{\alpha}}{\partial t}(x,t).\end{aligned}$$

Let us re-write the integral identity (4.3) in an equivalent form. By taking the test function in the form $\chi_{I_j}(t)\varphi(x)$, where χ_{I_j} is the characteristic function of the interval $I_j = [t_{j-1},t_j[= [(j-1)\alpha, j\alpha[= [j'\alpha, j\alpha[$, and φ is a function in the space V, we then obtain

$$\begin{aligned}&\int_{I_j}\Big(\phi\frac{s^{\alpha}(t)-s^{\alpha}(t-\alpha)}{\alpha},\varphi\Big)\,dt + \int_{I_j}\big(\lambda_w(s^{\alpha})K(\nabla p^{\alpha})\nabla p^{\alpha},\nabla\varphi\big)\,dt\\ &+\int_{I_j}\big(\Lambda_{\epsilon}(s^{\alpha})p_c'(s^{\alpha})K(\nabla p_{\alpha})\nabla s_{\alpha},\nabla\varphi\big)\,dt\\ &=\int_{I_j}(q_{w\alpha},\varphi)\,dt.\end{aligned}$$

Since $s^{\alpha}(\cdot,t)$ is constant with respect to t on the interval I_j and it is equal to $s^{\alpha}(\cdot,t_j)$, the same thing is true for $p^{\alpha}(\cdot,t)$, so, we obtain the following integral identity

$$\begin{aligned}&\big(\phi s^{j\alpha},\varphi\big) + \alpha\Big(\lambda_w(s^{j\alpha})K(\nabla p^{j\alpha})\nabla p^{j\alpha},\nabla\varphi\Big)\\ &+\alpha\Big(\Lambda_{\epsilon}(s^{j\alpha})p_c'(s^{j\alpha})K(\nabla p^{j\alpha})\nabla s^{j\alpha},\nabla\varphi\Big)\\ &=\big(\phi s^{j'\alpha},\varphi\big) + \alpha(q_{w\alpha j},\varphi),\quad\forall\varphi\in V.\end{aligned} \tag{4.4}$$

5. Galerkin's approximations of discrete time solutions

To use the Galerkin procedure in determining the solution at the level $t_1 = \alpha$, we choose an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ in V. Therefore the subspaces $H_d = \langle e_1, \cdots, e_d\rangle$, $d \in \mathbb{N}$, spanned by these functions are denses in V, and then we look for functions, written as

$$p^{1\alpha}(\cdot) \in V \quad \text{and} \quad s_d^{1\alpha}(x) = \sum_{i=1}^{d} \sigma_i^1 e_i(x),$$

where $\{\sigma_i^1\}_{i=1}^d$ are unknowns real coefficients, and satisfying, for all $\varphi \in V$,

$$\int_\Omega \lambda(s_0) K(\nabla p^{1\alpha}) \nabla p^{1\alpha} \cdot \nabla\varphi \, dx = (q_{\alpha 1}, \varphi), \tag{5.1}$$

and, for all $\psi \in H_d$,

$$\begin{aligned} &\big(\phi s_d^{1\alpha}, \psi\big) + \alpha\Big(\lambda_w(s_d^{1\alpha}) K(\nabla p^{1\alpha}) \nabla p^{1\alpha}, \nabla\psi\Big) \\ &+ \alpha\kappa_1 \Big(\Lambda_\epsilon(s_d^{1\alpha}) p_c'(s_d^{1\alpha}) K(\nabla p^{1\alpha}) \nabla s_d^{1\alpha}, \nabla\psi\Big) \\ &= \big(\phi s_0, \psi\big) + \alpha(q_{w\alpha 1}, \psi). \end{aligned} \tag{5.2}$$

To be brief, instead of $(s_d^{1\alpha})$, we denote (s_d).

5.1. Existence of Galerkin's approximations.

Before giving the proof of existence, we give the following statement.

Remark 5.1. The finite dimensional space H_d is equipped with the three equivalent norms defined, for $v = \sum_{k=1}^d \alpha_k e_k \in H_d$, by

$$|v|_{\mathbb{R}^d} = \Big[\sum_{k=1}^{d} \alpha_k^2\Big]^{1/2}, \quad |v|_2 = \Big[\int_\Omega |v|^2 \, dx\Big]^{1/2}, \quad \|v\|_V = \Big[\int_\Omega |\nabla v|^2 \, dx\Big]^{1/2}.$$

Let us explain the first step of the existence of Galerkin's approximation. In the beginning, we have to find $p^{1\alpha}$ solution of

$$\kappa_1 \int_\Omega \lambda(s_0) \frac{|\nabla p^{1\alpha}|}{1+\eta|\nabla p^{1\alpha}|} \nabla p^{1\alpha} \cdot \nabla\varphi \, dx + \kappa_2 \int_\Omega \lambda(s_0) \nabla p^{1\alpha} \cdot \nabla\varphi \, dx = (q_{\alpha 1}, \varphi),$$

for all $\varphi \in V$, which is equivalent to

$$\int_\Omega \lambda(s_0) K_{12}(\nabla p^{1\alpha}) \cdot \nabla\varphi \, dx = (q_{\alpha 1}, \varphi), \quad \forall \varphi \in V,$$

with $K_{12}(x) \doteq K_1(x) + K_2(x)$ where ($| \cdot |$ stands for the Euclidean norm)

$$\mathbb{R}^3 \ni x \longmapsto K_1(x) = \kappa_1 \frac{|x| x}{1+\eta|x|} \in \mathbb{R}^3, \quad \mathbb{R}^3 \ni x \longmapsto K_2(x) = \kappa_2 x \in \mathbb{R}^3. \tag{5.3}$$

Consider now the operator A from $V = H_0^1(\Omega)$ into its dual $V^\star = H^{-1}(\Omega)$, given by

$$\langle A(p), v\rangle = \int_\Omega \lambda(s_0) K_{12}(\nabla p) \cdot \nabla v \, dx, \quad \forall v \in V.$$

Notice that in fact $A(p) \in L^2(\Omega)$, for all $p \in V$.

We have the following results:

(1) Operator A is coercive in the sense that $\frac{\langle A(p),p\rangle}{\|p\|_V} \to +\infty$ when $\|p\|_V \to +\infty$. This because, $K_{12}x\cdot x \geq \kappa_2|x|^2$, for all $x\in\mathbb{R}^3$, and therefore

$$\frac{\langle A(p),p\rangle}{\|p\|} = \frac{\int_\Omega \lambda(s_0)K_{12}(\nabla p)\cdot\nabla p\,dx}{\|\nabla p\|_{L^2(\Omega)}} \geq \frac{\kappa_2\lambda_*\|\nabla p\|_{L^2}^2}{\|\nabla p\|_{L^2(\Omega)}} = \kappa_2\lambda_*\|p\|_V.$$

(2) Operator A is monotone, i.e., for all $p,q\in V$: $\langle A(p)-A(q),p-q\rangle\geq 0$. In fact, for p, $q\in V$, we have

$$\begin{aligned}
&\langle A(p)-A(q),p-q\rangle\\
&=\langle A(p),p-q\rangle-\langle A(q),p-q\rangle\\
&=\int_\Omega \lambda(s_0)[K_{12}(\nabla p)-K_{12}(\nabla q)]\cdot\nabla(p-q)\,dx\\
&=\int_\Omega \lambda(s_0)(K_1(\nabla p)-K_1(\nabla q))\cdot\nabla(p-q)\,dx+\kappa_2\int_\Omega\lambda(s_0)|\nabla(p-q)|^2\,dx.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&(K_1(\nabla p)-K_1(\nabla q))\cdot\nabla(p-q)\quad\text{(Cauchy-Schwarz)}\\
&=\kappa_1\Big\{\frac{|\nabla p|\nabla p}{1+\eta|\nabla p|}-\frac{|\nabla q|\nabla q}{1+\eta|\nabla q|}\Big\}\cdot\nabla(p-q)\\
&\geq\kappa_1\Big\{\frac{|\nabla p|^3}{1+\eta|\nabla p|}-\frac{|\nabla p|^2|\nabla q|}{1+\eta|\nabla p|}+\frac{|\nabla q|^3}{1+\eta|\nabla q|}-\frac{|\nabla p||\nabla q|^2}{1+\eta|\nabla q|}\Big\}\\
&=\kappa_1(|\nabla p|-|\nabla q|)\Big\{\frac{|\nabla p|^2}{1+\eta|\nabla p|}-\frac{|\nabla q|^2}{1+\eta|\nabla q|}\Big\}.
\end{aligned}$$

Let us now consider the real function $\mathbb{R}_+\ni\xi\longmapsto f(\xi)=\frac{\xi^2}{1+\eta\xi}\in\mathbb{R}_+$. We have

$$f'(\xi)=\frac{2\xi+\eta\xi^2}{(1+\eta\xi)^2}>0,\quad\forall\xi>0.$$

Putting $\xi=|\nabla p|$ and $\sigma=|\nabla q|$, we see, by using the Mean Value Theorem, that

$$\frac{|\nabla p|^2}{1+\eta|\nabla p|}-\frac{|\nabla q|^2}{1+\eta|\nabla q|}=f(\xi)-f(\sigma)=(\xi-\sigma)f'(c_{\xi\eta})$$

where $c_{\xi\eta}$ is a point between ξ and η. We conclude that

$$(K_1(\nabla p)-K_1(\nabla q))\cdot\nabla(p-q)\geq\kappa_1(|\nabla p|-|\nabla q|)^2f'(c_{\xi\eta})\geq 0,\quad\forall p,q\in V.$$

This implies that

$$\langle A(p)-A(q),p-q\rangle\geq\kappa_2\lambda_*\|p-q\|_V^2,\quad\forall p,q\in V,$$

showing that A is in fact strongly monotone, see, for instance [12] or [14].
(3) A is bounded. Let $p\in V$ with $\|p\|_V\leq M$, we have $\|A(p)\|_{V^\star}\leq M'$. In fact, if $p\in V$ with $\|p\|_V\leq M$, we have

$$\begin{aligned}
\langle A(p),p\rangle&=\int_\Omega\lambda(s_0)K_{12}(\nabla p)\cdot\nabla p\,dx\\
&\leq\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\lambda^*\|\nabla p\|_{L^2(\Omega)}^2\\
&=\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\lambda^*\|p\|_V^2\leq\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\lambda^*M^2=C,
\end{aligned}$$

and

$$\begin{aligned}
\|A(p)\|_{V^\star} &= \sup_{\substack{q\in V\\ \|q\|_V\le 1}} |\langle A(p), q\rangle| \\
&\le \sup_{\substack{q\in V\\ \|q\|\le 1}} \Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\int_\Omega \lambda(s^0)|\nabla p|\,|\nabla q|\,dx \\
&\le \sup_{\substack{q\in V\\ \|q\|\le 1}} \lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\|\nabla p\|_{L^2(\Omega)}\,\|\nabla q\|_{L^2(\Omega)} \\
&\le \lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\|\nabla p\|_{L^2(\Omega)} \le \lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)M = M'.
\end{aligned}$$

Operator A is hemicontinuous. Let p, q, $r\in V$ be three functions. Let us prove that the application defined from $\mathbb{R}$ into $\mathbb{R}$ by $\theta\mapsto\langle A(p+\theta q), r\rangle$ is continuous. To see that, we consider a real sequence $(\theta_n)_n$ converging to θ. First, because of the continuity of the function $x\longmapsto K_{12}(x)$, we have

$$K_{12}(\nabla p+\theta_n\nabla q)\cdot\nabla r \xrightarrow[\text{a.e. } x\in\Omega]{} K_{12}(\nabla p+\theta\nabla q)\cdot\nabla r.$$

Second, since the sequence $(\theta_n)_n$ is convergent, there exists a constant $M>0$ such that $|\theta_n|\le M$, $\forall n\in\mathbb{N}$. Then, we obtain

$$\begin{aligned}
\Big|K_{12}(\nabla p+\theta_n\nabla q)\cdot\nabla r\Big| &\le \Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)|\nabla p+\theta_n\nabla q|\,|\nabla r| \\
&\le \Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\Big(|\nabla p|+M|\nabla q|\Big)\,|\nabla r|\in L^1(\Omega).
\end{aligned}$$

Using the Lebesgue's Dominated Convergence Theorem, we see that

$$\int_\Omega \lambda(s_0)K_{12}(\nabla p+\theta_n\nabla q)\cdot\nabla r\,dx \xrightarrow[n\to+\infty]{} \int_\Omega \lambda(s_0)K_{12}(\nabla p+\theta\nabla q)\cdot\nabla r\,dx.$$

This means $\langle A(p+\theta_n q), r\rangle \xrightarrow[n\to+\infty]{} \langle A(p+\theta q), r\rangle$, which is the hemicontinuity of A.

As a result, the operator A is bounded, hemicontinuous, monotone and coercive. Consequently, for $q_{\alpha 1}\in L^2(\Omega)$, there exists $p^{1\alpha}$ solution of (5.1), see [12] or [14].

Now, to prove the existence of $s_d(=s_d^{1\alpha})$ solution to (5.2), we use a variant of Brouwer's Fixed Point Theorem which asserts that a continuous mapping P from $\mathbb{R}^d$ into itself satisfying, for some $\rho>0$, $P(\sigma)\cdot\sigma\ge 0$, for all σ, $|\sigma|=\rho$, has at least a zero $\sigma_0\in\mathbb{R}^d$ with $|\sigma_0|\le\rho$, see, for instance, [12, page 53].

Let us therefore consider the operator $\mathbb{R}^d\ni\sigma\mapsto P(\sigma)=\beta\in\mathbb{R}^d$ where $\beta\doteq(\beta_1,\dots,\beta_d)$, defined, for $k=1,\dots,d$, by

$$\begin{aligned}
\beta_k = &\int_\Omega \phi\frac{s_d^\sigma-s_0}{\alpha}e_k\,dx + \int_\Omega \lambda_w(s_d^\sigma)K(\nabla p^{1\alpha})\nabla p^{1\alpha}\cdot\nabla e_k\,dx \\
&+\int_\Omega \Lambda_\varepsilon(s_d^\sigma)p_c'(s_d^\sigma)K(\nabla p^{1\alpha})\nabla s_d^\sigma\cdot\nabla e_k\,dx - (q_{w\alpha 1}, e_k).
\end{aligned}$$

Here $s_d^\sigma=\sum_{l=1}^d\sigma_l e_l$, for $\sigma=(\sigma_1,\dots,\sigma_d)$.

The operator P has the following properties:

(1) P is continuous. Let $\{\sigma^m\}_{m=1}^\infty \doteq \{(\sigma_1^m,\dots,\sigma_d^m)\}_{m=1}^\infty$ a sequence in $\mathbb{R}^d$ converging in this space to $\sigma\doteq(\sigma_1,\dots,\sigma_d)$. We have to prove that the sequence $\{P(\sigma^m)\}_{m=1}^\infty$ is converging to $P(\sigma)$. We do have the following convergences

$$s_d^{\sigma^m}(x)=\sum_{l=1}^d\sigma_l^m e_l(x) \xrightarrow[m\to\infty]{\text{a.e. } x} \sum_{l=1}^d\sigma_l e_l(x)=s_d^\sigma(x),$$

$$\nabla s_d^{\sigma^m}(x) = \sum_{l=1}^d \sigma_l^m \nabla e_l(x) \xrightarrow[m\to\infty]{\text{a.e. } x} \sum_{l=1}^d \sigma_l \nabla e_l(x) = \nabla s_d^\sigma(x).$$

Using the continuity of the concerned functions

$$\lambda_w(s_d^{\sigma^m}(x)) \xrightarrow[m\to\infty]{\text{a.e. } x} \lambda_w(s_d^\sigma(x)), \quad \Lambda_\varepsilon(s_d^{\sigma^m}(x)) \xrightarrow[m\to\infty]{\text{a.e. } x} \Lambda_\varepsilon(s_d^\sigma(x)),$$
$$p_c'(s_d^{\sigma^m}(x)) \xrightarrow[m\to\infty]{\text{a.e. } x} p_c'(s_d^\sigma(x)).$$

Which implies that (we omit the variable x)

$$\lambda_w(s_d^{\sigma^m})K(\nabla p^{1\alpha})\nabla p^{1\alpha}\cdot\nabla e_k \xrightarrow[m\to\infty]{\text{a.e. } x} \lambda_w(s_d^\sigma)K(\nabla p^{1\alpha})\nabla p^{1\alpha}\cdot\nabla e_k,$$
$$\Lambda_\varepsilon(s_d^{\sigma^m})p_c'(s_d^{\sigma^m})K(\nabla p^{1\alpha})\nabla s_d^{\sigma^m}\cdot\nabla e_k \xrightarrow[m\to\infty]{\text{a.e. } x} \Lambda_\varepsilon(s_d^\sigma)p_c'(s_d^\sigma)K(\nabla p^{1\alpha})\nabla s_d^\sigma\cdot\nabla e_k,$$

We have also the estimate

$$|\lambda_w(s)| + |\Lambda_\varepsilon(s)| + |p_c'(s)| + |\sigma^m|_{\mathbb{R}^d} \le M, \quad \forall s\in\mathbb{R},\ \forall m\in\mathbb{N}.$$

The constant M depends, among other things, on λ_*, λ^*, ${p'}_c^*$, and ε. Also

$$\begin{aligned}|\lambda_w(s_d^{\sigma^m})K(\nabla p^{1\alpha})\nabla p^{1\alpha}\cdot\nabla e_k| &\le \Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)M|\nabla p^{1\alpha}\cdot\nabla e_k|\\ &\le \Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)M|\nabla p^{1\alpha}|\,|\nabla e_k| \in L^1(\Omega),\end{aligned}$$

and

$$\begin{aligned}|\Lambda_\varepsilon(s_d^{\sigma^m})p_c'(s_d^{\sigma^m})K(\nabla p^{1\alpha})\nabla s_d^{\sigma^m}\cdot\nabla e_k| &\le \Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)M^2|\nabla s_d^{\sigma^m}|\,|\nabla e_k|\\ &\le \Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)M^3\sum_{l=1}^d|\nabla\varphi_l|\,|\nabla e_k| \in L^1(\Omega),\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem, $\lim_{m\to\infty} P(\sigma^m) = P(\sigma)$, which proves the continuity of P.

(2) There exists $\rho > 0$ such that $P(\sigma)\cdot\sigma \ge 0$ for all $\sigma\in\mathbb{R}^d$ with $|\sigma| = \rho$. Here the central dot stands for the classical dot (scalar) product in $\mathbb{R}^d$. We have

$$\begin{aligned}P(\sigma)\cdot\sigma = \sum_{k=1}^d \sigma_k\Big(&\int_\Omega \phi\frac{s_d^\sigma - s_0}{\alpha}e_k\,dx + \int_\Omega \lambda_w(s_d^\sigma)K(\nabla p^{1\alpha})\nabla p^{1\alpha}\cdot\nabla e_k\,dx\\ &+ \int_\Omega \Lambda_\varepsilon(s_d^\sigma)p_c'(s_d^\sigma)K(\nabla p^{1\alpha})\nabla s_d^\sigma\cdot\nabla e_k\,dx - (q_{w\alpha1}, e_k)\Big)\end{aligned}$$

and

$$\begin{aligned}&P(\sigma)\cdot\sigma\\ &= \int_\Omega \phi\frac{s_d^\sigma - s_0}{\alpha}\Big(\sum_{k=1}^d\sigma_k e_k\Big)dx + \int_\Omega \lambda_w(s_d^\sigma)K(\nabla p^{1\alpha})\nabla p^{1\alpha}\cdot\Big(\sum_{k=1}^d\sigma_k\nabla e_k\Big)dx\\ &\quad+ \int_\Omega \Lambda_\varepsilon(s_d^\sigma)p_c'(s_d^\sigma)K(\nabla p^{1\alpha})\nabla s_d^\sigma\cdot\Big(\sum_{k=1}^d\sigma_k\nabla e_k\Big)dx - \Big(q_{w\alpha1}, \Big(\sum_{k=1}^d\sigma_k e_k\Big)\Big),\end{aligned}$$

which is equivalent to

$$\begin{aligned}P(\sigma)\cdot\sigma = &\int_\Omega \phi\frac{s_d^\sigma - s_0}{\alpha}s_d^\sigma\,dx + \int_\Omega \lambda_w(s_d^\sigma)K(\nabla p^{1\alpha})\nabla p^{1\alpha}\cdot\nabla s_d^\sigma\,dx\\ &+ \int_\Omega \Lambda_\varepsilon(s_d^\sigma)p_c'(s_d^\sigma)K(\nabla p^{1\alpha})\nabla s_d^\sigma\cdot\nabla s_d^\sigma\,dx - (q_{w\alpha1}, s_d^\sigma).\end{aligned}$$

Let us estimate each one of the four terms of the above equality. Concerning the first and second terms, we have the estimates

$$\begin{aligned}\int_\Omega \phi\frac{s_d^\sigma - s_0}{\alpha} s_d^\sigma\,dx &= \int_\Omega \frac{\phi}{\alpha}(s_d^\sigma)^2\,dx - \int_\Omega \frac{\phi}{\alpha} s_0 s_d^\sigma\,dx\\ &\ge \frac{\phi_*}{\alpha}\|s_d^\sigma\|_{L^2(\Omega)}^2 - \frac{\phi^*}{\alpha}\|s_0\|_{L^2(\Omega)}\,\|s_d^\sigma\|_{L^2(\Omega)}\\ &\ge \frac{\phi_*}{\alpha}\|s_d^\sigma\|_{L^2(\Omega)}^2 - C_p\frac{\phi^*}{\alpha}\|s_0\|_{L^2(\Omega)}\,\|\nabla s_d^\sigma\|_{L^2(\Omega)},\end{aligned}$$

where C_p is Poincaré's constant, and

$$\Big|\int_\Omega \lambda_w(s_d^\sigma)K(\nabla p^{1\alpha})\nabla p^{1\alpha}\cdot\nabla s_d^\sigma\,dx\Big| \le \lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\|\nabla p^{1\alpha}\|_{L^2(\Omega)}\,\|\nabla s_d^\sigma\|_{L^2(\Omega)}.$$

For the third term, we have the estimate

$$\int_\Omega \Lambda_\varepsilon(s_d^\sigma)p_c'(s_d^\sigma)K(\nabla p^{1\alpha})|\nabla s_d^\sigma|^2 \ge \varepsilon p_{c*}'\kappa_2\|\nabla s_d^\sigma\|_{L^2(\Omega)}^2.$$

Concerning the fourth term,

$$\Big|\int_\Omega q_{w\alpha 1}s_d^\sigma\Big| \le \|q_{w\alpha 1}\|_{L^2(\Omega)}\,\|s_d^\sigma\|_{L^2(\Omega)} \le C_p\|q_{w\alpha 1}\|_{L^2(\Omega)}\,\|\nabla s_d^\sigma\|_{L^2(\Omega)}.$$

Collecting the previous estimates, we see that

$$\begin{aligned}P(\sigma)\cdot\sigma &\ge \frac{\phi_*}{\alpha}\|s_d^\sigma\|_{L^2(\Omega)}^2\,dx - C_p\frac{\phi^*}{\alpha}\|s_0\|_{L^2(\Omega)}\,\|\nabla s_d^\sigma\|_{L^2(\Omega)}\\ &\quad - \lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\|\nabla p^{1\alpha}\|_{L^2(\Omega)}\,\|\nabla s_d^\sigma\|_{L^2(\Omega)} + \varepsilon p_{c*}'\kappa_2\|\nabla s_d^\sigma\|_{L^2(\Omega)}^2\\ &\quad - C_p\|q_{w\alpha 1}\|_{L^2(\Omega)}\,\|\nabla s_d^\sigma\|_{L^2(\Omega)}\\ &= \|\nabla s_d^\sigma\|_{L^2(\Omega)}\Big(\varepsilon p_{c*}'\kappa_2\|\nabla s_d^\sigma\|_{L^2(\Omega)} - C_p\frac{\phi^*}{\alpha}\|s_0\|_{L^2(\Omega)}\\ &\quad - \lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\|\nabla p^{1\alpha}\|_{L^2(\Omega)} - C_p\|q_{w\alpha 1}\|_{L^2(\Omega)}\Big) + \frac{\phi_*}{\alpha}\|s_d^\sigma\|_{L^2(\Omega)}^2.\end{aligned}$$

Thus $P(\sigma)\cdot\sigma\ge 0$ if

$$\begin{aligned}\|\nabla s_d^\sigma\| \ge \Big[&C_p\frac{\phi^*}{\alpha}\|s_0\|_{L^2(\Omega)} + \lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\|\nabla p^{1\alpha}\|_{L^2(\Omega)}\\ &+ C_p\|q_{w\alpha}^1\|_{L^2(\Omega)}\Big]\{\varepsilon p_{c*}'\kappa_2\}^{-1} \doteq \rho_0.\end{aligned}$$

Let us recall here that $\{e_i\}_{i=1}^\infty$ being an orthonormal basis in V, implies that, if $s_d^\sigma = \sum_{l=1}^d \sigma_l e_l$, one has $|\sigma|_{\mathbb{R}^d} = \|s_d^\sigma\|_V$. Therefore, $P(\sigma)\cdot\sigma\ge 0$ for all $\sigma\in\mathbb{R}^d$ with $|\sigma|=\rho$ for all $\rho\ge\rho_0$.

We are now in a position to apply the variant of Brouwer's Fixed Point Theorem mentioned above: there exists $\sigma_0=(\sigma_{01},\dots,\sigma_{0d})\in\mathbb{R}^d$, with $|\sigma_0|\le\rho$, such that $P(\sigma_0)=0$. It is now easy to see that $s_d^{1\alpha} = s_d^{\sigma_0} = \sum_{l=1}^d \sigma_{0l}e_l$ is a solution of (5.2).

6. Uniform estimates on Galerkin's approximations

Proposition 6.1. *Let $(p^{1\alpha}, s_d^{1\alpha})$ a solution to the system (5.1)–(5.2) at the time level $t_1=\alpha$. For the functions $(s_d^{1\alpha})_{d\ge 1}$ the following estimate holds,*

$$\|s_d^{1\alpha}\|_V \le C,\quad \forall d\ge 1, \tag{6.1}$$

where C is a positive constant independent of d.

Proof. By writing (5.2) with e_i as the test function, then multiplying by σ_i^1 and summing i from 1 to d, we have

$$\begin{aligned} &\int_\Omega \phi \frac{s_d - s_0}{\alpha} s_d\,dx + \int_\Omega \lambda_w(s_d) K(\nabla p^{1\alpha}) \nabla p^{1\alpha} \cdot \nabla s_d\,dx \\ &+ \int_\Omega \Lambda_\varepsilon(s_d) p_c'(s_d) K(\nabla p^{1\alpha}) |\nabla s_d|^2\,dx = (q_{w\alpha 1}, s_d), \end{aligned} \tag{6.2}$$

concerning the first term, we have

$$\int_\Omega \phi(s_d - s_0) s_d\,dx = \int_\Omega \phi(s_d)^2\,dx - \int_\Omega \phi s_0 \times s_d\,dx$$

with

$$\int_\Omega \phi(s_d)^2\,dx \ge \phi_* \int_\Omega (s_d)^2\,dx = \phi_* \|s_d\|^2_{L^2(\Omega)}$$

and by Young's Inequality, for $\beta > 0$, we obtain

$$\Big| \int_\Omega \phi s_0\, s_d\,dx \Big| \le \phi^* C_p \frac{1}{2\beta} \|s_0\|^2_{L^2(\Omega)} + \phi^* C_p \frac{\beta}{2} \|\nabla s_d\|^2_{L^2(\Omega)}.$$

For the second term of (6.2), using Young's Inequality, with $\beta_1 > 0$, we obtain

$$\begin{aligned} &\Big| \int_\Omega \lambda_w(s_d) K(\nabla p^{1\alpha}) \nabla p^{1\alpha} \cdot \nabla s_d\,dx \Big| \\ &\le \lambda^* \Big(\frac{\kappa_1}{\eta} + \kappa_2 \Big) \|\nabla p^{1\alpha}\|_{L^2(\Omega)} \|\nabla s_d\|_{L^2(\Omega)}, \\ &\le \frac{1}{2\beta_1} \Big(\lambda^* \Big[\frac{\kappa_1}{\eta} + \kappa_2 \Big] \Big)^2 \|\nabla p^{1\alpha}\|^2_{L^2(\Omega)} + \frac{\beta_1}{2} \|\nabla s_d\|^2_{L^2(\Omega)}. \end{aligned}$$

Concerning the third term of (6.2), we have

$$\int_\Omega \Lambda_\varepsilon(s_d) p_c'(s_d) K(\nabla p^{1\alpha}) |\nabla s_d|^2\,dx \ge \varepsilon p_{c*}' \kappa_2 \|\nabla s_d\|^2_{L^2(\Omega)},$$

and finally for the last term and using again Young's Inequality, for $\beta_2 > 0$, we have

$$\begin{aligned} \Big| \int_\Omega q_{w\alpha 1} s_d\,dx \Big| &\le \|q_{w\alpha 1}\|_{L^2(\Omega)} \|s_d\|_{L^2(\Omega)} \\ &\le C_p \|q_{w\alpha 1}\|_{L^2(\Omega)} \|\nabla s_d\|_{L^2(\Omega)} \\ &\le \frac{1}{2\beta_2} (C_p \|q^1_{w\alpha}\|_{L^2(\Omega)})^2 + \frac{\beta_2}{2} \|\nabla s_d\|^2_{L^2(\Omega)}. \end{aligned}$$

By taking into account all the previous estimates, we have

$$\begin{aligned} &\alpha \varepsilon p_{c*}' \kappa_2 \|\nabla s_d\|^2_{L^2(\Omega)} + \phi_* \|s_d\|^2_{L^2(\Omega)} \\ &\le \phi^* C_p \frac{1}{2\beta} \|s_0\|^2_{L^2(\Omega)} + \phi^* C_p \frac{\beta}{2} \|\nabla s_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2\beta_1} \Big(\lambda^* \Big[\frac{\kappa_1}{\eta} + \kappa_2 \Big] \Big)^2 \|\nabla p^{1\alpha}\|^2_{L^2(\Omega)} \\ &\quad + \frac{\alpha\beta_1}{2} \|\nabla s_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2\beta_2} (C_p \|q_{w\alpha 1}\|_{L^2(\Omega)})^2 + \frac{\alpha\beta_2}{2} \|\nabla s_d\|^2_{L^2(\Omega)}, \end{aligned}$$

which implies that

$$\begin{aligned} &\Big(\alpha \varepsilon p_{c*}' \kappa_2 - \phi^* C_p \frac{\beta}{2} - \frac{\alpha\beta_1}{2} - \frac{\alpha\beta_2}{2} \Big) \|\nabla s_d\|^2_{L^2(\Omega)} \\ &\le \phi^* C_p \frac{1}{2\beta} \|s_0\|^2_{L^2(\Omega)} + \frac{\alpha}{2\beta_1} \Big(\lambda^* \Big[\frac{\kappa_1}{\eta} + \kappa_2 \Big] \Big)^2 \|\nabla p^{1\alpha}\|^2_{L^2(\Omega)} + \frac{\alpha}{2\beta_2} (C_p \|q_{w\alpha 1}\|_{L^2(\Omega)})^2 . \end{aligned}$$

Taking

$$\beta = \frac{\alpha \varepsilon p_{c*}' \kappa_2}{2 C_p \phi^*}, \quad \beta_1 = \beta_2 = \frac{\varepsilon p_{c*}' \kappa_2}{2},$$

we obtain

$$\frac{1}{4} \alpha \varepsilon p_{c*}' \kappa_2 \|\nabla s_d\|^2_{L^2(\Omega)}$$

$$\le \phi^* C_p \frac{1}{2\beta}\|s_0\|^2_{L^2} + \frac{\alpha}{2\beta_1}\Big(\lambda^*\Big[\frac{\kappa_1}{\eta}+\kappa_2\Big]\Big)^2\|\nabla p^{1\alpha}\|^2_{L^2(\Omega)} + \frac{\alpha}{2\beta_2}C_p^2\|q_{w\alpha 1}\|^2_{L^2(\Omega)},$$

then estimate (6.1) holds. □

7. Passing to the limit (with respect of d) in Galerkin's approximations

We proved previously that

$$\|\nabla s_d^{1\alpha}\|_{L^2(\Omega)} \le C, \quad \forall d \ge 1\,.$$

Since the sequence $\{s_d^{1\alpha}\}_{d=1}^{\infty}$ is bounded in V (associated with the norm of gradient) we can extract a subsequence (denoted in the same symbol) such that

$$s_d^{1\alpha} \rightharpoonup s^{1\alpha} \text{ weakly in } V \quad \text{and} \quad s_d^{1\alpha} \to s^{1\alpha} \text{ a.e. in } \Omega, \tag{7.1}$$

more precisely, by the Rellich-Kondrachov theorem $s_d^{1\alpha} \to s^{1\alpha}$ strongly in $L^2(\Omega)$ and by the inverse of the Dominated Convergence theorem of Lebesgue, we can extract a subsequence which converge almost everywhere. Let d_0 a positive integer, since the sequence of linear spaces H_d are nested, we have

$$\begin{aligned}
&\big(\phi s_d^{1\alpha}, \psi\big) + \alpha\big(\lambda_w(s_d^{1\alpha})K(\nabla p^{1\alpha})\nabla p^{1\alpha}, \nabla\psi\big)\\
&+ \alpha\big(\Lambda_\epsilon(s_d^{1\alpha})p_c'(s_d^{1\alpha})K(\nabla p^{1\alpha})\nabla s_d^{1\alpha}, \nabla\psi\big)\\
&= \Big(\phi s_0, \psi\Big) + \alpha(q_{w\alpha 1}, \psi)_{\Gamma_N}, \quad \forall \psi \in H_{d_0},\ \forall d \ge d_0.
\end{aligned} \tag{7.2}$$

Let us fix ψ in H_{d_0}. Using the convergences (7.1) and taking into account that the functions λ_w, Λ_ε and p_c' are bounded and continuous, by making d goes to infinity in the equation of saturation (7.2) and using the Dominated Convergence Theorem of Lebesgue, one obtains

$$\begin{aligned}
&\big(\phi s^{1\alpha}, \psi\big) + \alpha\big(\lambda_w(s^{1\alpha})K(\nabla p^{1\alpha})\nabla p^{1\alpha}, \nabla\psi\big)\\
&+ \alpha\big(\Lambda_\epsilon(s^{1\alpha})p_c'(s^{1\alpha})K(\nabla p^{1\alpha})\nabla s^{1\alpha}, \nabla\psi\big)\\
&= \big(\phi s_0, \psi\big) + \alpha(q_{w\alpha 1}, \psi)_{\Gamma_N}, \quad \forall \psi \in H_{d_0}.
\end{aligned} \tag{7.3}$$

Now, using the density of $\cup_{d=1}^{\infty} H_d$ in V, we see that the previous integral identity is satisfied for all $\psi \in V$. This makes an end to the proof of existence of the couple $(p^{1\alpha}, s^{1\alpha})$ solution of the system $(S)_\varepsilon$ at the time level $t_1 = \alpha$.

Note that the same reasoning permits us to prove inductively the existence of the discrete time solution $(p^{j\alpha}, s^{j\alpha})$ at each time level $t_j = j\alpha$ for $j = 2, \dots, N$.

Knowing the functions $p^{j\alpha}$, $s^{j\alpha}$ at levels $j = 1, \dots, N$, we construct the Rothe's functions p^α and s^α which are in $\ell^\alpha(I, V)$, see the beginning of Section 4. We construct also $\widetilde{s}^\alpha$ as explained there, with $\widetilde{s}^{0\alpha}(0) = s_0$, the initial condition.

8. Uniform estimates for discrete time solutions

Lemma 8.1. *Let (p^α, s^α) be a time discrete solution of (S_ε) in the sense of Definition 4.2. Then, there exists a positive constant C (independent of α) such that*

$$\|p^\alpha\|_{L^2(I;V)} \le C, \quad \forall \alpha > 0, \tag{8.1}$$

$$\|s^\alpha\|_{L^2(I;V)} \le C, \quad \forall \alpha > 0, \tag{8.2}$$

$$\|\widetilde{s}^\alpha\|_{L^2(I;V)} \le C, \quad \forall \alpha > 0, \tag{8.3}$$

$$\sum_{j=1}^{N} \|s^{j\alpha}(\cdot) - s^{j'\alpha}(\cdot)\|^2_{L^2(\Omega)} \le C. \tag{8.4}$$

Proof. Let us begin by the equation of pressure. Testing Equation (4.2) with $\varphi = p^{j\alpha}$, for $j = 1, \dots, N$, we obtain

$$\int_\Omega \lambda(s^{j'\alpha})K(\nabla p^{j\alpha})\nabla p^{j\alpha}\cdot\nabla p^{j\alpha}\,dx = (q_{\alpha j}, p^{j\alpha}), t \in I_j,$$

which implies that
$$\lambda_*\kappa_2\|\nabla p^{j\alpha}\|^2_{L^2(\Omega)} \le \|q_{\alpha j}\|_{L^2(\Omega)}\,\|p^{j\alpha}\|_{L^2(\Omega)} \le C_p\|q_{\alpha j}\|_{L^2(\Omega)}\,\|\nabla p^{j\alpha}\|_{L^2(\Omega)}.$$
Using Young inequality, we obtain
$$\lambda_*\kappa_2\|\nabla p^{j\alpha}\|^2_{L^2(\Omega)} \le \frac{1}{2\beta}C_p^2\|q_{\alpha j}\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|\nabla p^{j\alpha}\|^2_{L^2(\Omega)}, \quad \beta>0,$$
and choosing $\beta = \lambda_*\kappa_2$, we obtain
$$\frac{\lambda_*\kappa_2}{2}\|\nabla p^{j\alpha}\|^2_{L^2(\Omega)} \le \frac{1}{2\lambda_*\kappa_2}C_p^2\|q_{\alpha j}\|^2_{L^2(\Omega)},$$
this shows that
$$\frac{\lambda_*\kappa_2}{2}\alpha\sum_{j=1}^N\|\nabla p^{j\alpha}\|^2_{L^2(\Omega)} \le \frac{C_p^2}{2\lambda_*\kappa_2}\alpha\sum_{j=1}^N\|q_{\alpha j}\|^2_{L^2(\Omega)},$$
it results that
$$\|p^\alpha\|^2_{L^2(I;V)} \le \Big(\frac{C_p}{\lambda_*\kappa_2}\Big)^2\|q\|^2_{L^2(I;L^2(\Omega))}.$$

Remark 8.2. If $q\in L^\infty(I;L^2(\Omega))$, then $p^\alpha\in L^\infty(I;V)$ with
$$\|p^\alpha\|^2_{L^\infty(I;V)} \le \Big(\frac{C_p}{\lambda_*\kappa_2}\Big)^2\|q\|^2_{L^\infty(I;L^2(\Omega))}.$$

Concerning the equation of saturation, we test Equation (4.4) with $\varphi = s^{j\alpha}$, for $j = 1,\dots,N$, and obtain
$$\begin{aligned}
&\big(\phi(s^{j\alpha}-s^{j'\alpha}), s^{j\alpha}\big) + \alpha\Big(\lambda_w(s^{j\alpha})K(\nabla p^{j\alpha})\nabla p^{j\alpha}, \nabla s^{j\alpha}\Big)\\
&+\alpha\Big(\Lambda_\epsilon(s^{j\alpha})p_c'(s^{j\alpha})K(\nabla p^{j\alpha})\nabla s^{j\alpha}, \nabla s^{j\alpha}\Big)\\
&=\alpha(q_{w\alpha j}, s^{j\alpha}).
\end{aligned}$$
For the first term, using the identity $a(a-b)=\frac12[a^2-b^2+(a-b)^2]$, we obtain
$$\int_\Omega \phi(s^{j\alpha}(x)-s^{j'\alpha}(x))s^{j\alpha}(x)\,dx = \int_\Omega\phi\frac12\big[(s^{j\alpha}(x))^2-(s^{j'\alpha}(x))^2+(s^{j\alpha}(x)-s^{j'\alpha}(x))^2\big]\,dx.$$
Consequently
$$\begin{aligned}
&\sum_{j=1}^N\int_\Omega\phi(s^{j\alpha}(x)-s^{j'\alpha}(x))s^{j\alpha}(x)\,dx\\
&=\sum_{j=1}^m\int_\Omega\phi\frac12\Big[(s^{j\alpha}(x))^2-(s^{j'\alpha}(x))^2+(s^{j\alpha}(x)-s^{j'\alpha}(x))^2\Big]\,dx\\
&=\frac12\int_\Omega\phi(s^{N\alpha}(x))^2\,dx - \frac12\int_\Omega\phi(s^0(x))^2\,dx + \frac12\sum_{j=1}^N\int_\Omega\phi|(s^{j\alpha}(x)-s^{j'\alpha}(x))|^2\,dx.
\end{aligned}$$
Concerning the second term, summing j from 1 to N, we have
$$\Big|\sum_{j=1}^N\alpha\Big(\lambda_w(s^{j\alpha})K(\nabla p^{j\alpha})\nabla p^{j\alpha},\nabla s^{j\alpha}\Big)\Big| \le \lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\sum_{j=1}^N\alpha\|\nabla p^{j\alpha}\|_{L^2(\Omega)}\,\|\nabla s^{j\alpha}\|_{L^2(\Omega)}.$$
Using Hölder's Inequality, we obtain
$$\begin{aligned}
&\lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\sum_{j=1}^N\alpha\|\nabla p^{j\alpha}\|_{L^2(\Omega)}\,\|\nabla s^{j\alpha}\|_{L^2(\Omega)}\\
&\le\lambda^*\Big(\frac{\kappa_1}{\eta}+\kappa_2\Big)\Big(\sum_{j=1}^N\alpha\|\nabla p^{j\alpha}\|^2_{L^2(\Omega)}\Big)^{1/2}\Big(\sum_{j=1}^N\alpha\|\nabla s^{j\alpha}\|^2_{L^2(\Omega)}\Big)^{1/2}
\end{aligned}$$

$$= \lambda^* \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big) \|p^\alpha\|_{L^2(I;V)} \, \|s^\alpha\|_{L^2(I;V)},$$

then applying Young's Inequality, for $\beta > 0$, we obtain

$$\Big| \sum_{j=1}^{N} \alpha\big(\lambda_w(s^{j\alpha}) K(\nabla p^{j\alpha}) \nabla p^{j\alpha}, \nabla s^{j\alpha}\big)\Big| \leq \Big[\lambda^* \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big)\Big]^2 \frac{1}{2\beta} \|p^\alpha\|^2_{L^2(V)} + \frac{\beta}{2} \|s^\alpha\|^2_{L^2(V)}.$$

Summing the third term from $j = 1$ to N, we obtain

$$\begin{aligned} \sum_{j=1}^{N} \alpha\Big(\Lambda_\epsilon(s^{j\alpha}) p_c'(s^{j\alpha}) K(\nabla p^{j\alpha}) \nabla s^{j\alpha}, \nabla s^{j\alpha}\Big) &\geq \varepsilon \kappa_2 p'_{c*} \sum_{j=1}^{N} \alpha \int_\Omega |\nabla s^{j\alpha}|^2 \, dx \\ &= \varepsilon \kappa_2 p'_{c*} \sum_{j=1}^{N} \alpha \|\nabla s^{j\alpha}\|^2_{L^2(\Omega)} \\ &= \varepsilon \kappa_2 p'_{c*} \|s^\alpha\|^2_{L^2(I;V)}. \end{aligned}$$

Finally, for the second member and using Hölder's Inequality, we obtain

$$\begin{aligned} \Big| \sum_{j=1}^{N} \alpha(q_{w\alpha j}, s^{j\alpha})\Big| &\leq \sum_{j=1}^{N} \alpha \|q_{w\alpha j}\|_{L^2(\Omega)} \, \|s^{j\alpha}\|_{L^2(\Omega)} \\ &\leq C_p \sum_{j=1}^{N} \alpha \|q_{w\alpha j}\|_{L^2(\Omega)} \, \|\nabla s^{j\alpha}\|_{L^2(\Omega)} \\ &\leq C_p \Big(\sum_{j=1}^{N} \alpha \|q_{w\alpha}(t_j)\|^2_{L^2(\Omega)}\Big)^{1/2} \Big(\sum_{j=1}^{N} \alpha \|\nabla s^{j\alpha}\|^2_{L^2(\Omega)}\Big)^{1/2} \\ &= C_p \|q_{w\alpha}\|_{L^2(I;L^2(\Omega))} \, \|s^\alpha\|_{L^2(I;V)}. \end{aligned}$$

Then, using that $\|q_{w\alpha}\|_{L^2(I;L^2(\Omega))} \leq \|q_w\|_{L^2(I;L^2(\Omega))}$ and Young's Inequality, for $\beta_1 > 0$,

$$\Big| \sum_{j=1}^{N} \alpha(q_{wj\alpha}, s^{j\alpha})\Big| \leq \frac{C_p^2}{2\beta_1} \|q_w\|^2_{L^2(I;L^2(\Omega))} + \frac{\beta_1}{2} \|s^\alpha\|^2_{L^2(I;V)}.$$

Taking into account all the previous estimates, after reorganizing terms, we obtain

$$\begin{aligned} &\frac{1}{2} \int_\Omega \phi(s^{N\alpha}(x))^2 \, dx + \frac{1}{2} \sum_{j=1}^{N} \int_\Omega \phi |(s^{j\alpha}(x) - s^{j'\alpha}(x))|^2 \, dx + \varepsilon \kappa_2 p'_{c*} \|s^\alpha\|^2_{L^2(I;V)} \\ &\leq \Big[\lambda^* \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big)\Big]^2 \frac{1}{2\beta} \|p^\alpha\|^2_{L^2(I;V)} + \frac{\beta}{2} \|s^\alpha\|^2_{L^2(I;V)} \\ &\quad + \frac{C_p^2}{2\beta_1} \|q_w\|^2_{L^2(I;L^2(\Omega))} + \frac{\beta_1}{2} \|s^\alpha\|^2_{L^2(I;V)} + \frac{1}{2} \int_\Omega \phi(s^0(x))^2 \, dx \,. \end{aligned}$$

As a result

$$\begin{aligned} &\Big(\varepsilon \kappa_2 p'_{c*} - \frac{\beta}{2} - \frac{\beta_1}{2}\Big) \|s^\alpha\|^2_{L^2(I;V)} + \frac{1}{2} \sum_{j=1}^{N} \int_\Omega \phi |(s^{j\alpha}(x) - s^{j'\alpha}(x))|^2 \, dx \\ &\leq \frac{C_p^2}{2\beta_1} \|q_w\|^2_{L^2(I;L^2(\Omega))} + \Big[\lambda^* \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big)\Big]^2 \frac{1}{2\beta} \|p^\alpha\|^2_{L^2(I;V)} + \frac{1}{2} \phi^* \|s^0\|^2_{L^2(\Omega)} \,. \end{aligned}$$

Taking $\beta = \beta_1 = \varepsilon \kappa_2 p'_{c*}/2$ and using (8.1), we see that the estimates (8.2), (8.3) and (8.4) are valid. □

Lemma 8.3. *There exists a constant $C > 0$ (independent of α) such that*

$$\big\|\phi \frac{\partial \tilde{s}^\alpha}{\partial t}\big\|_{L^2(I;V^\star)} \leq C, \quad \forall \alpha > 0. \tag{8.5}$$

Proof. For each $j = 1, \dots, N$, let L_j^α be the linear form (and continuous) on V defined by

$$\begin{aligned} L_j^\alpha(\psi) &= \int_\Omega \phi \frac{s^{j\alpha} - s^{j'\alpha}}{\alpha} \psi \, dx \\ &= -\int_\Omega \lambda_w(s^{j\alpha}) K(\nabla p^{j\alpha}) \nabla p^{j\alpha} \cdot \nabla\psi \, dx \\ &\quad - \int_\Omega \Lambda_\varepsilon(s^{j\alpha}) p_c'(s^{j\alpha}) K(\nabla p^{j\alpha}) \nabla s^{j\alpha} \cdot \nabla\psi \, dx + \int_\Omega q_{w\alpha j} \psi \, dx, \quad \forall \psi \in V. \end{aligned}$$

It follows that for all $\psi \in V$,

$$\begin{aligned} \big|L_j^\alpha(\psi)\big| &= \Big| -\int_\Omega \lambda_w(s^{j\alpha}) K(\nabla p^{j\alpha}) \nabla p^{j\alpha} \cdot \nabla\psi \, dx \\ &\quad - \int_\Omega \Lambda_\varepsilon(s^{j\alpha}) p_c'(s^{j\alpha}) K(\nabla p^{j\alpha}) \nabla s^{j\alpha} \cdot \nabla\psi \, dx + \int_\Omega q_{w\alpha j} \psi \, dx \Big| \\ &\le \lambda^* \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big) \|\nabla p^{j\alpha}\|_{L^2} \|\nabla\psi\|_{L^2} \\ &\quad + (\varepsilon + c) p_c^{'*} \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big) \|\nabla s^{j\alpha}\|_{L^2} \|\nabla\psi\|_{L^2} + \|q_{w\alpha j}\|_{L^2} \|\psi\|_{L^2}, \end{aligned}$$

where $c = (\lambda^*)^2/\lambda_*$ and $j = 1, \dots, N$. Using Poincaré's Inequality, we obtain

$$\begin{aligned} \big|L_j^\alpha(\psi)\big| &\le \lambda^* \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big) \|\nabla p^{j\alpha}\|_{L^2} \|\nabla\psi\|_{L^2} \\ &\quad + \Big(\varepsilon + c) p_c^{'*} \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big) \|\nabla s^{j\alpha}\|_{L^2} \|\nabla\psi\|_{L^2} + C_p \|q_{w\alpha j}\|_{L^2} \|\nabla\psi\|_{L^2} \\ &= \Big[\lambda^* \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big) \|\nabla p^{j\alpha}\|_{L^2} + (\varepsilon + c) p_c^{'*} \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big) \|\nabla s^{j\alpha}\|_{L^2} \\ &\quad + C_p \|q_{w\alpha j}\|_{L^2}\Big] \|\nabla\psi\|_{L^2}, \quad \text{for } j = 1, \dots, N \end{aligned}$$

In what follows, C denotes a constant which can change from one line to another. The previous inequality can be rewritten as

$$\begin{aligned} \frac{\big|L_j^\alpha(\psi)\big|}{\|\nabla\psi\|_{L^2}} &\le \Big(\frac{\kappa_1}{\eta} + \kappa_2\Big) \big\{\lambda^* \|\nabla p^{j\alpha}\|_{L^2} + (\varepsilon + c) p_c^{'*} \|\nabla s^{j\alpha}\|_{L^2}\big\} + C_p \|q_{w\alpha j}\|_{L^2} \\ &\le C\big\{\|\nabla p^{j\alpha}\|_{L^2(\Omega)} + \|\nabla s^{j\alpha}\|_{L^2(\Omega)} + \|q_{w\alpha j}\|_{L^2(\Omega)}\big\}, \quad j = 1, \dots, N, \end{aligned}$$

where C is the maximum of constants involved in the preceding inequality. Consequently

$$\|L_j^\alpha\|_{V^\star} \le C\big\{\|\nabla p^{j\alpha}\|_{L^2(\Omega)} + \|\nabla s^{j\alpha}\|_{L^2(\Omega)} + \|q_{w\alpha j}\|_{L^2(\Omega)}\big\}, \quad j = 1, \dots, N$$

which leads to

$$\|L_j^\alpha\|_{V^\star}^2 \le 3C^2\big\{\|\nabla p^{j\alpha}\|_{L^2(\Omega)}^2 + \|\nabla s^{j\alpha}\|_{L^2(\Omega)}^2 + \|q_{w\alpha j}\|_{L^2(\Omega)}^2\big\}, \quad j = 1, \dots, N. \tag{8.6}$$

Before going further, it seems good to notice that for each $j = 1, \dots, N$, the function $\phi\, \frac{s^{j\alpha} - s^{j'\alpha}}{\alpha}$ is in $L^2(\Omega)$ and then in $V^\star$, since $L^2(\Omega) \hookrightarrow V^\star$. We can therefore consider that $[0, T] \ni t \longmapsto \phi \widetilde{s}^\alpha(\cdot, t)$ is a path in $V^\star$ and we have -in the sense of classical derivatives,

$$V^\star \ni \phi(\cdot)\, \frac{\partial \widetilde{s}^\alpha(\cdot, t)}{\partial t} = \phi(\cdot) \frac{s^{j\alpha}(\cdot) - s^{j'\alpha}(\cdot)}{\alpha}, \quad \forall t \in]t_{j'}, t_j[, \quad j = 1, \dots, N.$$

Now let us calculate

$$\begin{aligned} \int_0^T \|\phi \frac{\partial \widetilde{s}^\alpha(\cdot, t)}{\partial t}\|_{V^\star}^2 dt &= \sum_{j=0}^N \int_{t_{j'}}^{t_j} \|\phi \frac{\partial \widetilde{s}^\alpha(\cdot, t)}{\partial t}\|_{V^\star}^2 dt \\ &= \sum_{j=0}^N \int_{t_{j'}}^{t_j} \|\phi \frac{s^{j\alpha} - s^{j'\alpha}}{\alpha}\|_{V^\star}^2 dt \end{aligned}$$

$$= \sum_{j=0}^{N} \int_{t_{j'}}^{t_j} \|L_j^\alpha\|_{V^\star}^2 \, dt.$$

Using estimate (8.6), we obtain

$$\begin{aligned}
&\int_0^T \|\phi \frac{\partial \widetilde{s}^\alpha(\cdot,t)}{\partial t}\|_{V^\star}^2 \, dt \\
&\le C \sum_{j=0}^{N} \int_{t_{j'}}^{t_j} \big\{ \|\nabla p^{j\alpha}\|_{L^2(\Omega)}^2 + \|\nabla s^{j\alpha}\|_{L^2(\Omega)}^2 + \|q_{w\alpha j}\|_{L^2(\Omega)}^2 \big\} \, dt \\
&= C\big\{ \|\nabla p^\alpha\|_{L^2(\Omega_T)}^2 + \|\nabla s^\alpha\|_{L^2(\Omega_T)}^2 + \|q_{w\alpha}\|_{L^2(\Omega_T)}^2 \big\} \le C.
\end{aligned} \tag{8.7}$$

Here C does not depend on α, because we proved earlier that the sequences $\{p^\alpha\}_{\alpha>0}$, $\{s^\alpha\}_{\alpha>0}$ are bounded in $L^2(0,T;V)$ and the sequence $\{q_{w\alpha}\}_{\alpha>0}$ is bounded in $L^2(\Omega_T)$. $\square$

9. Compactness of discrete time solutions

First, we give the following remark, see for instance [3].

Remark 9.1. Let w be a function belonging to $L^2(\Omega)$ and w_α the average function in time defined by relation (4.1). Then

$$\lim_{\alpha\to 0} w_\alpha = w \quad \text{in } L^2(\Omega_T) \text{ strongly.} \tag{9.1}$$

Lemma 9.2. *Let s^α satisfy the saturation equation* (4.3)*. Then, there exists a constant C such that*

$$\frac{1}{\sqrt{\xi}} \int_\xi^T \int_\Omega \phi\{s^\alpha(x,t) - s^\alpha(x,t-\xi)\}^2 \, dx \, dt \le C, \quad \forall\, \xi > 0. \tag{9.2}$$

Proof. We follow [1] (see also [2] and [8]). Let k be fixed ($1 \le k \le N$) and let $\tau \in]k\alpha, T]$, so there exists $j \ge k+1$ such that $\tau \in I_j =]t_{j-1}, t_j]$. Let $R(\tau) =](j-k)\alpha, j\alpha]$ and take $\omega(x,t) = k\alpha\chi_{R(\tau)}(t)\partial_t^{-k\alpha} s^\alpha(x,\tau)$ as a test function in the equation of saturation (4.3). For the parabolic term, we obtain

$$\begin{aligned}
\int_I (\phi\partial_t^{-\alpha} s^\alpha, \omega)_\Omega \, dt &= \int_I \Big(\phi\partial_t^{-\alpha} s^\alpha, k\alpha\chi_{R(\tau)}(t)\partial_t^{-k\alpha} s^\alpha(x,\tau)\Big)_\Omega \, dt \\
&= \int_I \int_\Omega \phi\partial_t^{-\alpha} s^\alpha k\alpha\chi_{R(\tau)}(t)\partial_t^{-k\alpha} s^\alpha(x,\tau) \, dx \, dt \\
&= \int_\Omega \Big[\phi(x)k\alpha\partial_t^{-k\alpha} s^\alpha(x,\tau) \int_I \partial_t^{-\alpha} s^\alpha(x,t)\chi_{R(\tau)}(t) \, dt\Big] \, dx,
\end{aligned}$$

with

$$\begin{aligned}
\int_I \partial_t^{-\alpha} s^\alpha(x,t)\chi_{R(\tau)}(t) \, dt &= \int_{(j-k)\alpha}^{j\alpha} \partial_t^{-\alpha} s^\alpha(x,t) \, dt \\
&= \int_{(j-k)\alpha}^{j\alpha} \frac{s^\alpha(x,t) - s^\alpha(x,t-\alpha)}{\alpha} \, dt \\
&= \sum_{r=1}^{k} \int_{(j-k+r-1)\alpha}^{(j-k+r)\alpha} \frac{s^\alpha(x,t) - s^\alpha(x,t-\alpha)}{\alpha} \, dt \\
&= \sum_{r=1}^{k} s^\alpha(x,(j-k+r)\alpha) - s^\alpha(x,(j-k+r-1)\alpha) \\
&= s^\alpha(x,j\alpha) - s^\alpha(x,(j-k)\alpha) \\
&= s^\alpha(x,\tau) - s^\alpha(x,\tau-k\alpha) \\
&= k\alpha\partial_t^{-k\alpha} s^\alpha(x,\tau) = k\alpha\partial_t^{-k\alpha} s^\alpha(x,j\alpha),
\end{aligned}$$

which means that the parabolic term is equal to

$$\int_\Omega \phi(x)k\alpha\partial_t^{-k\alpha}s^\alpha(x,\tau)k\alpha\partial_t^{-k\alpha}s^\alpha(x,j\alpha)\,dx = \int_\Omega \phi(x)(k\alpha)^2\Big(\partial_t^{-k\alpha}s^\alpha(x,\tau)\Big)^2\,dx$$

By integrating this equality with respect to τ from $k\alpha$ to T, we obtain

$$\int_{k\alpha}^T\int_I(\phi\partial_t^{-\alpha}s^\alpha,\omega)_\Omega\,dt\,d\tau = \int_{k\alpha}^T\int_\Omega \phi(x)(k\alpha)^2\big(\partial_t^{-k\alpha}s^\alpha(x,\tau)\big)^2\,dx\,d\tau. \tag{9.3}$$

We have also

$$\begin{aligned}
&\int_{\Omega_T}\Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\nabla\omega(x,t)\,dx\,dt\\
&= \int_{\Omega_T}\Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\\
&\quad\times k\alpha\chi_{R(\tau)}(t)\nabla\partial_t^{-k\alpha}s^\alpha(x,\tau)\,dx\,dt\\
&= \int_\Omega k\alpha\nabla\partial_t^{-k\alpha}s^\alpha(x,\tau)\int_0^T\big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)\\
&\quad\times K(\nabla p^\alpha)\nabla s^\alpha\big)\chi_{R(\tau)}(t)\,dt\,dx\\
&= \int_\Omega k\alpha\nabla\partial_t^{-k\alpha}s^\alpha(x,\tau)\int_{(j-k)\alpha}^{j\alpha}\big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\big)\,dt\,dx.
\end{aligned}$$

Let

$$F(x) = \int_{(j-k)\alpha}^{j\alpha}\Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\,dt,$$
$$G(x) = k\alpha\nabla\partial_t^{-k\alpha}s^\alpha(x,\tau).$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_T}\Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\nabla\omega(x,t)\,dx\,dt\\
&\le \Big(\int_\Omega F^2(x)\,dx\Big)^{1/2}\Big(\int_\Omega G^2(x)\,dx\Big)^{1/2}.
\end{aligned} \tag{9.4}$$

Then we have

$$\begin{aligned}
\int_\Omega G^2(x)\,dx &= \int_\Omega\Big(k\alpha\frac{s^\alpha(x,\tau)-s^\alpha(x,\tau-k\alpha)}{k\alpha}\Big)^2\,dx\\
&= \int_\Omega\Big(\nabla s^\alpha(x,\tau) - \nabla s^\alpha(x,\tau-k\alpha)\Big)^2\,dx.
\end{aligned}$$

According to the inequality: $(a+b)^2 \le 2a^2+2b^2$, for all a,b real, we have

$$\begin{aligned}
\int_\Omega G^2(x)\,dx &\le \int_\Omega\Big(|\nabla s^\alpha(x,\tau)| + |\nabla s^\alpha(x,\tau-k\alpha)|\Big)^2\,dx\\
&\le \int_\Omega 2|\nabla s^\alpha(x,\tau)|^2 + 2|\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx.
\end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
&\int_\Omega F^2(x)\,dx\\
&= \int_\Omega\Big[\int_{(j-k)\alpha}^{j\alpha}1\times\Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\,dt\Big]^2\,dx\\
&\le \int_\Omega\int_{(j-k)\alpha}^{j\alpha}1^2\,dt\times\int_{(j-k)\alpha}^{j\alpha}\big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\big)^2\,dt\,dx
\end{aligned}$$

$$= k\alpha \int_\Omega \int_{(j-k)\alpha}^{j\alpha} \big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\big)^2\,dt\,dx.$$

Taking into account all the previous estimates, (9.4) implies

$$\begin{aligned}
&\int_{\Omega_T} \Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\nabla\omega(x,t)\,dx\,dt\\
&\le \sqrt{2k\alpha}\Big(\int_\Omega \int_{(j-k)\alpha}^{j\alpha}\Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)^2\,dt\,dx\Big)^{1/2}\\
&\quad\times\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2 + |\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}\\
&\le \sqrt{2k\alpha}\Big(\int_\Omega \int_{(j-k)\alpha}^{j\alpha}\Big(\lambda^*(\frac{\kappa_1}{\eta}+\kappa_2)\nabla p^\alpha + C_\varepsilon {p_c'}^*(\frac{\kappa_1}{\eta}+\kappa_2)\nabla s^\alpha\Big)^2\,dt\,dx\Big)^{1/2}\\
&\quad\times\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2 + |\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}.
\end{aligned}$$

In what follows, C denotes a constant which can change from one line to another.

$$\begin{aligned}
&\int_{\Omega_T} \Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\nabla\omega(x,t)\,dx\,dt\\
&\le \sqrt{2k\alpha}C\Big(\int_\Omega \int_{(j-k)\alpha}^{j\alpha}\big(\nabla p^\alpha + \nabla s^\alpha\big)^2\,dt\,dx\Big)^{1/2}\\
&\quad\times\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2 + |\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2},
\end{aligned}$$

where $C = \max\Big(\lambda^*(\frac{\kappa_1}{\eta}+\kappa_2), C_\varepsilon {p_c'}^*(\frac{\kappa_1}{\eta}+\kappa_2), \frac{\lambda^{*2}}{\lambda_*}+\varepsilon\Big)$. Then

$$\begin{aligned}
&\int_{\Omega_T} \Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\nabla\omega(x,t)\,dx\,dt\\
&\le \sqrt{2k\alpha}C\Big(\int_\Omega \int_{(j-k)\alpha}^{j\alpha}|\nabla p^\alpha|^2 + |\nabla s^\alpha|^2\,dt\,dx\Big)^{1/2}\\
&\quad\times\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2 + |\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}.
\end{aligned}$$

Now, using the fact that, for $0<p<1$ and a,b two real positive, $(a+b)^p \le a^p + b^p$, we obtain

$$\begin{aligned}
&\int_{\Omega_T} \Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\nabla\omega(x,t)\,dx\,dt\\
&\le \sqrt{2k\alpha}C\Big[\Big(\int_{\Omega_T}|\nabla p^\alpha|^2\,dx\,dt\Big)^{1/2} + \Big(\int_{\Omega_T}|\nabla s^\alpha|^2\,dx\,dt\Big)^{1/2}\Big]\\
&\quad\times\Big[\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2\,dx\Big)^{1/2} + \Big(\int_\Omega|\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}\Big].
\end{aligned}$$

Using (8.1) and (8.2), we have

$$\begin{aligned}
&\int_{\Omega_T} \Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\nabla\omega(x,t)\,dx\,dt\\
&\le 2\sqrt{2k\alpha}C\Big[\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2\,dx\Big)^{1/2} + \Big(\int_\Omega|\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}\Big].
\end{aligned}$$

Integrating this inequality with respect to τ from $k\alpha$ to T, and using estimate (8.2), we obtain

$$\int_{k\alpha}^T\int_{\Omega_T} \Big(\lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha + \Lambda_\epsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha\Big)\nabla\omega(x,t)\,dx\,dt\,d\tau$$

$$
\begin{aligned}
&\le \sqrt{2k\alpha}C\Big[\int_{k\alpha}^{T}\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2\,dx\Big)^{1/2}d\tau+\int_{k\alpha}^{T}\Big(\int_\Omega |\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}d\,\tau\Big],\\
&\le \sqrt{2k\alpha}C\Big[\int_{0}^{T}\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2\,dx\Big)^{1/2}d\tau+\int_{0}^{T-k\alpha}\Big(\int_\Omega |\nabla s^\alpha(x,s)|^2\,dx\Big)^{1/2}d\,s\Big],\\
&\le \sqrt{2k\alpha}C\Big[\int_{0}^{T}\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2\,dx\Big)^{1/2}d\tau+\int_{0}^{T}\Big(\int_\Omega |\nabla s^\alpha(x,s)|^2\,dx\Big)^{1/2}d\,s\Big],\\
&\le \sqrt{2k\alpha}C.
\end{aligned}
\tag{9.5}
$$

Finally, concerning the second term in the equation of saturation,

$$
\begin{aligned}
\int_{\Omega_T} q_{w\alpha}k\alpha\chi_{R(\tau)}(t)\partial_t^{-k\alpha}s^\alpha(x,\tau)\,dx\,dt &= \int_\Omega\int_{(j-k)\alpha}^{j\alpha} q_{w\alpha}k\alpha\partial_t^{-k\alpha}s^\alpha(x,\tau)\,dx\,dt\\
&= \int_\Omega k\alpha\partial_t^{-k\alpha}s^\alpha(x,\tau)\Big(\int_{(j-k)\alpha}^{j\alpha} q_{w\alpha}\,dt\Big)\,dx.
\end{aligned}
$$

Putting

$$
E(x)=\int_{(j-k)\alpha}^{j\alpha} q_{w\alpha}\,dt,\quad G(x)=k\alpha\partial_t^{-k\alpha}s^\alpha(x,\tau)
$$

and using Hölder's inequality, we obtain

$$
\int_{\Omega_T} q_{w\alpha}k\alpha\chi_{R(\tau)}(t)\partial_t^{-k\alpha}s^\alpha(x,\tau)\,dx\,dt \le \Big(\int_\Omega E^2(x)\,dx\Big)^{1/2}\Big(\int_\Omega G^2(x)\,dx\Big)^{1/2}.
$$

We have

$$
\Big(\int_\Omega G^2(x)\,dx\Big)^{1/2} \le C_p\Big(\int_\Omega \nabla G^2(x)\,dx\Big)^{1/2}.
$$

Using the same techniques as above, we obtain

$$
\begin{aligned}
\Big(\int_\Omega G^2(x)\,dx\Big)^{1/2} &\le C_p\Big(\int_\Omega \nabla G^2(x)\,dx\Big)^{1/2}\\
&\le C_p\sqrt{2}\Big[\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2\,dx\Big)^{1/2}+\Big(\int_\Omega |\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}\Big].
\end{aligned}
$$

Also,

$$
\begin{aligned}
\int_\Omega E^2(x)\,dx &= \int_\Omega \Big(\int_{(j-k)\alpha}^{j\alpha} q_{w\alpha}(x,t)\,dt\Big)^2\,dx\\
&\le \int_\Omega \Big(\int_{(j-k)\alpha}^{j\alpha} 1^2\,dt \times \int_{(j-k)\alpha}^{j\alpha} q_{w\alpha}^2(x,t)\,dt\Big)\,dx\\
&= \int_\Omega k\alpha \int_{(j-k)\alpha}^{j\alpha} q_{w\alpha}^2(x,t)\,dt\,dx\\
&\le \int_{\Omega_T} k\alpha q_{w\alpha}^2(x,t)\,dt\,dx \le k\alpha\int_{\Omega_T} q_w^2(x,t)\,dt\,dx.
\end{aligned}
$$

Consequently, for the second member we have

$$
\begin{aligned}
&\int_{\Omega_T} q_{w\alpha}k\alpha\chi_{R(\tau)}(t)\partial_t^{-k\alpha}s^\alpha(x,\tau)\,dx\,dt\\
&\le C_p\sqrt{2k\alpha}\Big(\int_{\Omega_T} q_w^2(x,t)\,dt\,dx\Big)^{1/2}\\
&\quad\times\Big[\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2\,dx\Big)^{1/2}+\Big(\int_\Omega |\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}\Big]\\
&= \sqrt{2k\alpha}C\Big[\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2\,dx\Big)^{1/2}+\Big(\int_\Omega |\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}\Big]
\end{aligned}
$$

with $C = C_p\big(\int_{\Omega_T} q_w^2(x,t)\,dt\,dx\big)^{1/2}$. Integrating the previous inequality with respect to τ from $k\alpha$ to T, we obtain

$$
\begin{aligned}
&\int_{k\alpha}^T\int_{\Omega_T} q_{w\alpha} k\alpha \chi_{R(\tau)}(t)\partial_t^{-k\alpha} s^\alpha(x,\tau)\,dx\,dt\,d\tau\\
&= \sqrt{2k\alpha}C\Big[\int_{k\alpha}^T\Big(\int_\Omega |\nabla s^\alpha(x,\tau)|^2\,dx\Big)^{1/2}d\tau + \int_{k\alpha}^T\Big(\int_\Omega |\nabla s^\alpha(x,\tau-k\alpha)|^2\,dx\Big)^{1/2}d\tau\Big].
\end{aligned}
$$

Using the same arguments as in (9.5), we have

$$
\int_{k\alpha}^T\int_{\Omega_T} q_{w\alpha} k\alpha \chi_{R(\tau)}(t)\partial_t^{-k\alpha} s^\alpha(x,\tau)\,dx\,dt\,d\tau \le \sqrt{2k\alpha}C. \tag{9.6}
$$

Now, taking into account (9.3), (9.5) and (9.6), one obtains

$$
\int_{k\alpha}^T\int_\Omega \phi(x)(k\alpha)^2\big(\partial_t^{-k\alpha} s^\alpha(x,\tau)\big)^2\,dx\,d\tau \le \sqrt{k\alpha}C
$$

which implies that

$$
\frac{1}{\sqrt{k\alpha}}\int_{k\alpha}^T\int_\Omega \phi(x)(k\alpha)^2\big(\partial_t^{-k\alpha} s^\alpha(x,\tau)\big)^2\,dx\,d\tau \le C, \tag{9.7}
$$

consequently, according to [1] (see also [2] and [8]), this concludes the proof of (9.2).

In our opinion, the proof can be completed as follows: For a fixed $\xi > 0$, there exists $k \ge 0$ such that $\xi \in]k\alpha,(k+1)\alpha]$. If $k \ge 1$ and since we are integrating a positive function, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{\xi}}\int_\xi^T\int_\Omega \phi\{s^\alpha(x,t)-s^\alpha(x,t-\xi)\}^2\,dx\,dt\\
&\le \frac{1}{\sqrt{k\alpha}}\int_{k\alpha}^T\int_\Omega \phi\{s^\alpha(x,t)-s^\alpha(x,t-\xi)\}^2\,dx\,dt\\
&\le \frac{2}{\sqrt{k\alpha}}\int_{k\alpha}^T\int_\Omega \phi\{s^\alpha(x,t)-s^\alpha(x,t-k\alpha)\}^2\,dx\,dt\\
&\quad + \frac{2}{\sqrt{k\alpha}}\int_{k\alpha}^T\int_\Omega \phi\{s^\alpha(x,t-k\alpha)-s^\alpha(x,t-\xi)\}^2\,dx\,dt.
\end{aligned} \tag{9.8}
$$

Since, by construction, the value of s^α on each subinterval I_j, $j=1,\dots,N$, is equal to its value at the end of I_j, we have $s^{(k+1)\alpha} = s^\alpha(\cdot,(k+1)\alpha)$ on $]k\alpha,(k+1)\alpha]$, and, for $t \in]l\alpha,(l+1)\alpha]$, $s^\alpha(t-k\alpha) = s^{(l-k+1)\alpha}$ and $s^\alpha(t-\xi) = s^{(l-k+1)\alpha}$ or $s^{(l-k)\alpha}$ (because $t-\xi$ belongs to $](l-k-1)\alpha,(l-k+1)\alpha]$). Then $s^\alpha(t-k\alpha)-s^\alpha(t-\xi) = 0$ or $s^\alpha(t-k\alpha)-s^\alpha(t-\xi) = s^{(l-k+1)\alpha} - s^{(l-k)\alpha}$. Necessarily we have

$$
\begin{aligned}
&\frac{2}{\sqrt{k\alpha}}\int_{k\alpha}^T\int_\Omega \phi\{s^\alpha(x,t-k\alpha)-s^\alpha(x,t-\xi)\}^2\,dx\,dt\\
&= \frac{2}{\sqrt{k\alpha}}\sum_{m=0}^{N-k-1}\int_{(k+m)\alpha}^{(k+m+1)\alpha}\int_\Omega \phi\{s^\alpha(x,t-k\alpha)-s^\alpha(x,t-\xi)\}^2\,dx\,dt\\
&\le \frac{2\alpha\|\phi\|_\infty}{\sqrt{k\alpha}}\sum_{m=0}^{N-k-1}\|s^{(m+1)\alpha}-s^{m\alpha}\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Since $k \ge 1$, $\alpha \le 1$ and $\sum_l \|s^{(l+1)\alpha}-s^{l\alpha}\|_{L^2}^2 \le C$ (according to (8.4)), we obtain

$$
\frac{2}{\sqrt{k\alpha}}\int_{k\alpha}^T\int_\Omega \phi\{s^\alpha(x,t-k\alpha)-s^\alpha(x,t-\xi)\}^2\,dx\,dt \le 2C\|\phi\|_\infty. \tag{9.9}
$$

Consequently, using (9.7), (9.8) and (9.9), we obtain

$$
\frac{1}{\sqrt{\xi}}\int_\xi^T\int_\Omega \phi\{s^\alpha(x,t)-s^\alpha(x,t-\xi)\}^2\,dx\,dt \le C. \tag{9.10}
$$

If $k=0$ then $\xi \le \alpha$ with $\alpha \le 1$. We can write

$$\begin{aligned}
&\frac{1}{\sqrt{\xi}} \int_\xi^T \int_\Omega \phi\{s^\alpha(x,t) - s^\alpha(x,t-\xi)\}^2\,dx\,dt \\
&= \frac{1}{\sqrt{\xi}} \int_\xi^\alpha \int_\Omega \phi\{s^\alpha(x,t) - s^\alpha(x,t-\xi)\}^2\,dx\,dt \\
&\quad + \frac{1}{\sqrt{\xi}} \sum_{k=1}^{N-1} \int_{k\alpha}^{(k+1)\alpha} \int_\Omega \phi\{s^\alpha(x,t) - s^\alpha(x,t-\xi)\}^2\,dx\,dt.
\end{aligned}$$

For $t \in]\xi,\alpha]$, we have $t-\xi \in]0,\alpha]$, and then $s^\alpha(x,t) = s^\alpha(x,t-\xi) = s^{1\alpha}$. Thus, the integral on $[\xi,\alpha]$ is zero. For $t \in]k\alpha,(k+1)\alpha]$, we distinguish two cases: The first is $t \in]k\alpha,k\alpha+\xi]$, then $t-\xi \in](k-1)\alpha,k\alpha]$, so $s^\alpha(x,t) = s^{(k+1)\alpha}$ and $s^\alpha(x,t-\xi) = s^{k\alpha}$. The second case is $t \in]k\alpha+\xi,(k+1)\alpha]$, then $t-\xi \in]k\alpha,(k+1)\alpha]$ and therefore $s^\alpha(x,t) = s^\alpha(x,t-\xi) = s^{(k+1)\alpha}$. So, we obtain (remember the estimate (8.4)):

$$\begin{aligned}
&\frac{1}{\sqrt{\xi}} \int_\xi^T \int_\Omega \phi\{s^\alpha(x,t) - s^\alpha(x,t-\xi)\}^2\,dx\,dt \\
&= \frac{1}{\sqrt{\xi}} \sum_{k=1}^{N-1} \int_{k\alpha}^{(k+1)\alpha} \int_\Omega \phi\{s^\alpha(x,t) - s^\alpha(x,t-\xi)\}^2\,dx\,dt \\
&= \frac{1}{\sqrt{\xi}} \sum_{k=1}^{N-1} \int_{k\alpha}^{k\alpha+\xi} \int_\Omega \phi\{s^\alpha(x,t) - s^\alpha(x,t-\xi)\}^2\,dx\,dt \\
&\le \frac{\xi\|\phi\|_\infty}{\sqrt{\xi}} \sum_{k=1}^{N-1} \|s^{(k+1)\alpha} - s^{k\alpha}\|^2_{L^2(\Omega)} \le C.
\end{aligned}$$

This completes the proof of estimate (9.2). $\square$

9.1. **Passing to the limit in time discretization.**

Lemma 9.3. *The sequence $(p^\alpha)_{\alpha>0}$ contains a subsequence converging weakly in $L^2(I;V)$ to a function p as α goes to zero.*

Proof. According to estimate (8.1), the sequence $(p^\alpha)_{\alpha>0}$ is bounded in $L^2(I;V)$, therefore it contains a subsequence (denoted in the same way) such that

$$p^\alpha \rightharpoonup p \in L^2(I;V) \quad \text{(weakly)}. \tag{9.11}$$

$\square$

Lemma 9.4. *The sequence $(s^\alpha)_{\alpha>0}$ contains a subsequence converging strongly in $L^2(\Omega_T)$ to a function s and a.e. in Ω_T as α goes to zero.*

Proof. Consider the set $F = \{s^\alpha : \alpha > 0\}$ and let the spaces $X = V$, $B = L^2(\Omega)$, $Y = X^\star = V^\star = H^{-1}(\Omega)$ (the dual space). It is well known that $X \underset{\text{cont.}}{\hookrightarrow} B \underset{\text{comp.}}{\hookrightarrow} Y$. We have the following

(1) F is uniformly bounded in $L^2(I;X)$, i.e., $\|s^\alpha\|_{L^2(I;X)} \le C$, with C independent of α).

(2) $\lim_{\xi\to 0} \|\tau_\xi f - f\|_{L^2(0,T-\xi;X^\star)} = 0$ uniformly for $f \in F$. Here $(\tau_\xi f)(t) = f(t+\xi)$. In fact, using (9.2), we have

$$\phi_* \int_\xi^T \int_\Omega \big(s^\alpha(x,\tau) - s^\alpha(x,\tau-\xi)\big)^2\,dx\,d\tau \le C\sqrt{\xi}.$$

Putting $\sigma = \tau - \xi$, we obtain

$$\int_0^{T-\xi} \int_\Omega \big(s^\alpha(x,\sigma+\xi) - s^\alpha(x,\sigma)\big)^2\,dx\,d\sigma \le \frac{C}{\phi_*}\sqrt{\xi},$$

meaning that

$$\int_0^{T-\xi} \|\tau_\xi s^\alpha - s^\alpha\|_{L^2(\Omega)}^2 \, d\sigma \le \frac{C}{\phi_*}\sqrt{\xi}, \quad \forall \alpha > 0.$$

Now, since $L^2(\Omega)$ is continuously embedded in $X^\star$, we have $\|\cdot\|_{X^\star} \le c\|\cdot\|_{L^2(\Omega)}$, showing that

$$\int_0^{T-\xi} \|\tau_\xi s^\alpha - s^\alpha\|_{X^\star}^2 \, d\sigma \le \frac{C}{\phi_*}\sqrt{\xi}, \quad \forall \alpha > 0,$$

which implies that

$$\lim_{\xi \to 0} \|\tau_\xi s^\alpha - s^\alpha\|_{L^2(0,T-\xi;X^\star)} = 0.$$

Consequently, using [17, Theorem 5, p. 84], we see that F is relatively compact in $L^2(I, L^2(\Omega)) = L^2(\Omega_T)$. Therefore, from $(s^\alpha)_{\alpha>0}$, we can extract a subsequence (denoted in the same way) converging strongly in $L^2(\Omega_T)$ and a.e. in Ω_T to a function s, $s \in L^2(\Omega_T)$. □

9.2. **Consequences of estimates and the initial condition.** Since the sequences $(s^\alpha)_\alpha$ and $(\widetilde{s}^\alpha)_\alpha$ are bounded in $L^2(I;V)$, we have

$$\begin{gathered} s^\alpha \rightharpoonup s \quad \text{weakly in } L^2(I;V) \text{ and strongly in } L^2(\Omega_T), \\ \widetilde{s}^\alpha \rightharpoonup s_1 \quad \text{in } L^2(I;V) \text{ and weakly in } L^2(\Omega_T). \end{gathered}$$

Using estimate (8.4) and Remark 4.1, we see that $s^\alpha - \widetilde{s}^\alpha \to 0$ in $L^2(\Omega_T)$, so that $s = s_1$. On the one hand, from the estimate (8.5) we have $\phi\frac{\partial \widetilde{s}^\alpha}{\partial t} \rightharpoonup w$ in $L^2(I;V^\star)$. On the other hand, since $\widetilde{s}^\alpha \rightharpoonup s$ in $L^2(I;V)$, one can deduce that $\widetilde{s}^\alpha \rightharpoonup s$ in $L^2(I;L^2(\Omega))$ and $\widetilde{s}^\alpha \rightharpoonup s$ in $L^2(I;V^\star)$; consequently, $\widetilde{s}^\alpha \rightharpoonup s$ in $\mathcal{D}'(I;V^\star)$ (the space of distributions on I with values in $V^\star$) and $\partial_t(\phi\widetilde{s}^\alpha) \rightharpoonup \partial_t(\phi s)$ in $\mathcal{D}'(I;V^\star)$. Then $w = \partial_t(\phi s) = \phi\partial_t s$.

Now, since $\widetilde{s}^\alpha \rightharpoonup s$ in $L^2(I;V)$ and $\partial_t\widetilde{s}^\alpha \rightharpoonup \partial_t s$ in $L^2(I;V^\star)$, we see that $\widetilde{s}^\alpha \rightharpoonup s$ in $W(0,T)$. Let us recall that $W(0,T) \hookrightarrow_{\text{cont}} C([0,T];L^2(\Omega))$ (see, for instance, [15, 9, 14]), and, for ξ a fixed element in $\mathcal{D}(\Omega)$, consider the linear functional F_ξ defined by $W(0,T) \ni u \mapsto F_\xi(u) = \int_\Omega u(0)(x)\xi(x)\,dx \in \mathbb{R}$. If we write

$$\begin{aligned} \Big|\int_\Omega u(0)(x)\xi(x)\,dx\Big| &\le \|u(0)\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)} \\ &\le \sup_{0\le t\le T} \|u(t)\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)} \\ &\le C\|\xi\|_{L^2(\Omega)}\|u\|_{W(0,T)}, \end{aligned}$$

where C is a positive constant, We see that F_ξ is continuous with $\|F_\xi\|_{(W(0,T))^\star} \le C\|\xi\|_{L^2(\Omega)}$. Therefore,

$$\lim_{\alpha\downarrow 0} F_\xi(\widetilde{s}^\alpha) = \lim_{\alpha\downarrow 0}\int_\Omega \widetilde{s}^\alpha(0)(x)\xi(x)\,dx = \int_\Omega s_0(x)\xi(x)\,dx = F_\xi(s) = \int_\Omega s(0)(x)\xi(x)\,dx.$$

As ξ is arbitrary in $\mathcal{D}(\Omega)$, we conclude that $s(0) = s_0$, see for instance [6, Corollary 4.24]. The initial condition is thus satisfied.

10. Proof of Theorem 3.2

1. Equation of pressure. First, let us remember the approximate equations of pressure

$$\int_\Omega \lambda(s^{j'\alpha})K(\nabla p^{j\alpha})\nabla p^{j\alpha}\cdot\nabla\varphi\,dx = \int_\Omega q_{\alpha j}\,\varphi\,dx, \quad \forall\varphi\in V,\ \forall j = 1,\dots,N.$$

Let ψ in $\mathcal{D}(I;V)$, then for all $t\in[t_{j-1},t_j[$, $\psi(t)\in V$, by taking it as a test function in the equation above, we obtain

$$\int_\Omega \lambda(s^{j'\alpha})K(\nabla p^{j\alpha})\nabla p^{j\alpha}\cdot\nabla\psi(t)\,dx = \int_\Omega q_{\alpha j}\,\psi(t)\,dx, \quad \text{for } j = 1,\dots,N.$$

Integrating with respect to t from t_{j-1} to t_j and then by summing j from 1 to N, we have

$$\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\int_{\Omega}\lambda(s^{j'\alpha})K(\nabla p^{j\alpha})\nabla p^{j\alpha}\cdot\nabla\psi(t)\,dx\,dt=\sum_{i=1}^{N}\int_{t_{j-1}}^{t_j}\int_{\Omega}q_{\alpha j}\,\psi(t)\,dx\,dt,$$

which is equivalent to

$$\int_{\Omega_T}\lambda(s^{\alpha}(t-\alpha))K(\nabla p^{\alpha})\nabla p^{\alpha}\cdot\nabla\psi\,dx\,dt=\int_{\Omega_T}q_{\alpha}\,\psi\,dx\,dt\quad\text{with }\psi\in\mathcal{D}(I;V).$$

Let us make α go to zero. Using hypothesis (H2) on the function λ, Lemma 9.4 and denoting ζ the weak limit of the sequence $K(\nabla p^{\alpha})\nabla p^{\alpha}$ in $L^2(\Omega_T)$, we obtain

$$\begin{aligned}&\lim_{\alpha\to 0}\int_{\Omega_T}\lambda(s^{\alpha}(t-\alpha))K(\nabla p^{\alpha})\nabla p^{\alpha}\cdot\nabla\psi\,dx\,dt\\&=\int_{\Omega_T}\lambda(s)\zeta\cdot\nabla\psi\,dx\,dt,\quad\forall\psi\in\mathcal{D}(I;V).\end{aligned}\tag{10.1}$$

By Remark 9.1, we have

$$\lim_{\alpha\to 0}\int_{\Omega_T}q_{\alpha}\,\psi\,dx\,dt=\int_{\Omega_T}q\,\psi\,dx\,dt,\ \forall\psi\in\mathcal{D}(I;V).\tag{10.2}$$

Combining (10.1)and(10.2), we see that

$$\int_{\Omega_T}\lambda(s)\zeta\cdot\nabla\psi\,dx\,dt=\int_{\Omega_T}q\,\psi\,dx\,dt,\ \forall\psi\in\mathcal{D}(I;V),\tag{10.3}$$

which implies

$$\int_{\Omega_T}\lambda(s)\zeta\cdot\nabla\psi\,dx\,dt=\int_{\Omega_T}q\,\psi\,dx\,dt,\quad\forall\psi\in L^2(I;V).\tag{10.4}$$

Now, taking $p^{j\alpha}$ as a test function in the equation of pressure, then integrating with respect to t from t_{j-1} to t_j and then by summing j from 1 to N, we have

$$\int_{\Omega_T}\lambda(s^{\alpha}(t-\alpha))K(\nabla p^{\alpha})|\nabla p^{\alpha}|^2\,dx\,dt=\int_{\Omega_T}q_{\alpha}p^{\alpha}\,dx\,dt,\tag{10.5}$$

from (8.1), remark 9.1, lemma 9.3 and passing to the limit when α goes to zero, we have

$$\lim_{\alpha\to 0}\int_{\Omega_T}\lambda(s^{\alpha}(t-\alpha))K(\nabla p^{\alpha})|\nabla p^{\alpha}|^2\,dx\,dt=\int_{\Omega_T}qp\,dx\,dt.\tag{10.6}$$

To justify that $\lim_{\alpha\to 0}\int_{\Omega_T}q_{\alpha}p^{\alpha}\,dx\,dt=\int_{\Omega_T}qp\,dx\,dt$, we write

$$\begin{aligned}\Big|\int_{\Omega_T}q_{\alpha}p^{\alpha}-qp\Big|&=\Big|\int_{\Omega_T}q_{\alpha}p^{\alpha}-qp^{\alpha}+qp^{\alpha}-qp\Big|\\&=\Big|\int_{\Omega_T}p^{\alpha}(q_{\alpha}-q)+\int_{\Omega_T}q(p^{\alpha}-p)\Big|\\&\le\|p^{\alpha}\|\,\|q_{\alpha}-q\|+\Big|\int_{\Omega_T}q(p^{\alpha}-p)\Big|\\&=C\|q_{\alpha}-q\|+\Big|\int_{\Omega_T}q(p^{\alpha}-p)\Big|.\end{aligned}$$

For all φ in $L^2(I;V)$, we have

$$
\begin{aligned}
0 &\le \int_{\Omega_T} \lambda(s^\alpha(t-\alpha))(K(\nabla p^\alpha)\nabla p^\alpha - K(\nabla\varphi)\nabla\varphi)\cdot(\nabla p^\alpha - \nabla\varphi)\,dx\,dt \\
&= \int_{\Omega_T} \lambda(s^\alpha(t-\alpha))K(\nabla p^\alpha)|\nabla p^\alpha|^2\,dx\,dt \\
&\quad - \int_{\Omega_T} \lambda(s^\alpha(t-\alpha))K(\nabla p^\alpha)\nabla p^\alpha\cdot\nabla\varphi\,dx\,dt \\
&\quad - \int_{\Omega_T} \lambda(s^\alpha(t-\alpha))K(\nabla\varphi)\nabla\varphi\cdot(\nabla p^\alpha - \nabla\varphi)\,dx\,dt,
\end{aligned}
\tag{10.7}
$$

using (10.5), the above inequality becames

$$
\begin{aligned}
0 \le \int_{\Omega_T} q_\alpha p^\alpha\,dx\,dt &- \int_{\Omega_T} \lambda(s^\alpha(t-\alpha))K(\nabla p^\alpha)\nabla p^\alpha\cdot\nabla\varphi\,dx\,dt \\
&- \int_{\Omega_T} \lambda(s^\alpha(t-\alpha))K(\nabla\varphi)\cdot\nabla\varphi(\nabla p^\alpha - \nabla\varphi)\,dx\,dt,
\end{aligned}
$$

passing to the limit as α goes to zero, we obtain

$$
0 \le \int_{\Omega_T} qp\,dx\,dt - \int_{\Omega_T} \lambda(s)\zeta\cdot\nabla\varphi\,dx\,dt - \int_{\Omega_T} \lambda(s)K(\nabla\varphi)\nabla\varphi\cdot(\nabla p - \nabla\varphi)\,dx\,dt.
$$

Using (10.4), the previous inequality is equivalent to

$$
0 \le \int_{\Omega_T} \lambda(s)\zeta\cdot\nabla p\,dx\,dt - \int_{\Omega_T} \lambda(s)\zeta\nabla\varphi\,dx\,dt - \int_{\Omega_T} \lambda(s)K(\nabla\varphi)\nabla\varphi\cdot(\nabla p - \nabla\varphi)\,dx\,dt.
$$

Consequently,

$$
0 \le \int_{\Omega_T} \lambda(s)(\zeta - K(\nabla\varphi)\nabla\varphi)(\nabla p - \nabla\varphi)\,dx\,dt.
$$

Taking $\varphi = p - \eta\tilde\varphi$ with $\eta > 0$, we obtain

$$
\int_{\Omega_T} \lambda(s)\Big(\zeta - K(\nabla\varphi)(\nabla p - \eta\nabla\tilde\varphi)\Big)\cdot\nabla\tilde\varphi \ge 0,
\tag{10.8}
$$

when η approaches zero. Using the continuity of the operator A defined in subsection 5.1, we have

$$
\int_{\Omega_T} \lambda(s)(\zeta - K(\nabla p)\nabla p)\cdot\nabla\tilde\varphi \ge 0, \quad \forall\tilde\varphi\in L^2(I;V).
$$

Then, replacing $\tilde\varphi$ by $-\tilde\varphi$ in (10.8) and making η tend to zero, we deduce the equality

$$
\int_{\Omega_T} \lambda(s)(\zeta - K(\nabla p)\nabla p)\cdot\nabla\tilde\varphi = 0,
$$

which is exactly

$$
\int_{\Omega_T} \lambda(s)\zeta\cdot\nabla\tilde\varphi\,dx\,dt = \int_{\Omega_T} \lambda(s)K(\nabla p)\nabla p\cdot\nabla\tilde\varphi\,dx\,dt, \quad \forall\tilde\varphi\in L^2(I;V).
$$

This shows that $\zeta = K(\nabla p)\nabla p$, and as a result, (10.4) becomes

$$
\int_{\Omega_T} \lambda(s)K(\nabla p)\nabla p\cdot\nabla\psi\,dx\,dt = \int_{\Omega_T} q\,\psi\,dx\,dt, \quad \forall\psi\in L^2(I;V);
\tag{10.9}
$$

hence, p satisfies the equation of pressure.

Lemma 10.1. *The following holds,*

$$
\lim_{\alpha\to 0}\int_{\Omega_T} \lambda(s^\alpha(t-\alpha))\Big(K(\nabla p^\alpha)\nabla p^\alpha - K(\nabla p)\nabla p\Big)\Big(\nabla p^\alpha - \nabla p\Big) = 0.
$$

Proof. From (10.6) and (10.9), we deduce that

$$\begin{aligned}\lim_{\alpha\to 0}\int_{\Omega_T}\lambda(s^\alpha(t-\alpha))K(\nabla p^\alpha)|\nabla p^\alpha|^2\,dx\,dt &= \int_{\Omega_T} qp\,dx\,dt\\ &= \int_{\Omega_T}\lambda(s)\zeta\nabla p\,dx\,dt \qquad (10.10)\\ &= \int_{\Omega_T}\lambda(s)K(\nabla p)|\nabla p|^2\,dx\,dt.\end{aligned}$$

Using (10.4), (10.9), (10.10), and Lemma 9.3, we obtain

$$\begin{aligned}0 &\le \lim_{\alpha\to 0}\int_{\Omega_T}\lambda(s^\alpha(t-\alpha))\Big(K(\nabla p^\alpha)\nabla p^\alpha - K(\nabla p)\nabla p\Big)\cdot\Big(\nabla p^\alpha-\nabla p\Big)\,dx\,dt\\ &= \lim_{\alpha\to 0}\int_{\Omega_T}\lambda(s^\alpha(t-\alpha))K(\nabla p^\alpha)|\nabla p^\alpha|^2\,dx\,dt\\ &\quad - \lim_{\alpha\to 0}\int_{\Omega_T}\lambda(s^\alpha(t-\alpha))K(\nabla p^\alpha)\nabla p^\alpha\cdot\nabla p\,dx\,dt\\ &\quad - \lim_{\alpha\to 0}\int_{\Omega_T}\lambda(s^\alpha(t-\alpha))K(\nabla p)\nabla p\cdot\Big(\nabla p^\alpha-\nabla p\Big)\\ &= \int_{\Omega_T}\lambda(s)K(\nabla p)|\nabla p|^2\,dx\,dt - \int_{\Omega_T}\lambda(s)K(\nabla p)\nabla p\cdot\nabla p\,dx\,dt\\ &\quad - \int_{\Omega_T}\lambda(s)K(\nabla p)\nabla p\Big(\nabla p-\nabla p\Big) = 0. \qquad \square\end{aligned}$$

Lemma 10.2. *The sequence $(\nabla p^\alpha)_{\alpha>0}$ converges in measure to ∇p in Ω_T and a.e. for a subsequence.*

Proof. Let ε_1, $\delta>0$ be two fixed numbers, and set

$$D=\{|\nabla p^\alpha-\nabla p|\ge\delta\}\doteq\{(x,t)\in\Omega_T : |\nabla p^\alpha(x,t)-\nabla p(x,t)|\ge\delta\}.$$

Then we consider the function K_{12} introduced in subsection 5.1: $K_{12}(x)=\kappa_1\frac{|x|x}{1+\eta|x|}+\kappa_2 x$, $x\in\mathbb{R}^3$. The same method used to prove the monotony of operator A in the mentioned subsection, shows that

$$(K_{12}(x)-K_{12}(y))\cdot(x-y)\ge\kappa_2|x-y|^2,\quad \forall x,y\in\mathbb{R}^3.$$

Writing

$$\begin{aligned}&\int_{\Omega_T}\lambda(s^\alpha(t-\alpha))[K_{12}(\nabla p^\alpha)-K_{12}(\nabla p)]\cdot[\nabla p^\alpha-\nabla p]\,dx\,dt\\ &\ge \lambda_*\int_D[K_{12}(\nabla p^\alpha)-K_{12}(\nabla p)]\cdot[\nabla p^\alpha-\nabla p]\,dx\,dt\\ &\ge\lambda_*\int_D\kappa_2|\nabla p^\alpha-\nabla p|^2\,dx\,dt\ge\lambda_*\kappa_2\delta^2\ \mathrm{meas}(D),\end{aligned}$$

we see that

$$\mathrm{meas}(D)\le\frac{1}{\lambda_*\kappa_2\delta^2}\int_{\Omega_T}\lambda(s^\alpha(t-\alpha))[K_{12}(\nabla p^\alpha)-K_{12}(\nabla p)]\cdot[\nabla p^\alpha-\nabla p]\,dx\,dt.$$

Since the right-hand side of the previous inequality tends to zero as α does, this by Lemma 10.1, meas(D) can be made less than ε_1 for α sufficiently small. We conclude that meas$(\{|\nabla p^\alpha-\nabla p|\ge\delta\})\le\varepsilon_1$, for all $\varepsilon_1>0$. This proves that the sequence (∇p^α) converges in measure to ∇p. Therefore this sequence contains a subsequence, denoted in the same way, converging a.e. to ∇p in Ω_T. $\square$

Remark 10.3. Using Lemma 10.2, one can have that $K(\nabla p^\alpha)$ converge a.e. in Ω_T to $K(\nabla p)$ and since $K(\nabla p^\alpha) \le \frac{\kappa_1}{\eta} + \kappa_2$, we deduce $K(\nabla p^\alpha)$ converge strongly to $K(\nabla p)$ in $L^2(\Omega_T)$.

2. Equation of saturation. First we recall a well known result.

Lemma 10.4 (A discrete integration by parts formula)**.** *For $\alpha > 0$, $T > 0$ two real numbers and Φ a smooth real function defined on the interval $[0,T]$, let us put*

$$\partial_t^\alpha \Phi(t) = \frac{\Phi(t+\alpha) - \Phi(t)}{\alpha}, \quad t \in [0, T-\alpha].$$

If $\alpha < T$ and Ψ is a real smooth function defined on the same interval $[0,T]$, then

$$\begin{aligned} &\int_\alpha^T \Phi(t)\partial_t^{-\alpha}\Psi(t)\,dt \\ &= \frac{1}{\alpha}\int_{T-\alpha}^T (\Phi\Psi)(t)\,dt - \frac{1}{\alpha}\int_0^\alpha (\Phi\Psi)(t)\,dt - \int_0^{T-\alpha} \Psi(t)\partial_t^\alpha \Phi(t)\,dt. \end{aligned} \tag{10.11}$$

Proof. For $0 < t < T - \alpha$, we can write

$$(\Phi\Psi)(t+\alpha) - (\Phi\Psi)(t) = \alpha\Phi(t+\alpha)\partial_t^\alpha\Psi(t) + \alpha\Psi(t)\partial_t^\alpha\Phi(t).$$

Integrating the left-hand side on $[0, T-\alpha]$, we obtain

$$\int_0^{T-\alpha} (\Phi\Psi)(t+\alpha)\,dt - \int_0^{T-\alpha} (\Phi\Psi)(t)\,dt = \int_{T-\alpha}^T (\Phi\Psi)(s)\,ds - \int_0^\alpha (\Phi\Psi)(t)\,dt.$$

Now, integrating on the right-hand side, we have

$$\begin{aligned} &\alpha\int_0^{T-\alpha} \Phi(t+\alpha)\partial_t^\alpha\Psi(t)\,dt + \alpha\int_0^{T-\alpha} \Psi(t)\partial_t^\alpha\Phi(t) \\ &= \alpha\int_\alpha^T \Phi(t)\partial_t^{-\alpha}\Psi(t)\,dt + \alpha\int_0^{T-\alpha} \Psi(t)\partial_t^\alpha\Phi(t). \end{aligned}$$

Putting the results together, we obtain formula (10.11). □

Now, let us remember the approximate equation of saturation

$$\begin{aligned} &\int_0^T (\phi\partial_t^{-\alpha} s^\alpha, \psi)\,dt + \int_{\Omega_T} \lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha \cdot \nabla\psi\,dx\,dt \\ &+ \int_{\Omega_T} \Lambda_\varepsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha \cdot \nabla\psi\,dx\,dt \\ &= \int_0^T (q_{w\alpha}, \psi)\,dt, \quad \forall\psi \in \ell^\alpha(I;V). \end{aligned}$$

Following [3], the pressure equation in the weak sense of Definition 3.1 can be seen to hold since $\cup_{n=1}^\infty \ell^\alpha(I;V)$ (remember that $\alpha = \frac{T}{N} = \frac{T}{2^n}$) is dense in $L^2(I;V)$. Also, for the equation of saturation, and for all $\psi \in \cup_{n=1}^\infty \ell^\alpha(I;V)$, we have for the second term, using the same technique when passing to the limit in the equation of pressure, we obtain

$$\lim_{\alpha\to 0}\int_{\Omega_T} \lambda_w(s^\alpha)K(\nabla p^\alpha)\nabla p^\alpha \cdot \nabla\psi\,dx\,dt = \int_{\Omega_T} \lambda_w(s)K(\nabla p)\nabla p \cdot \nabla\psi\,dx\,dt,$$

for the third term, using Remark 10.3, we obtain

$$\lim_{\alpha\to 0}\int_{\Omega_T} \Lambda_\varepsilon(s^\alpha)p_c'(s^\alpha)K(\nabla p^\alpha)\nabla s^\alpha \cdot \nabla\psi\,dx\,dt = \int_{\Omega_T} \Lambda_\varepsilon(s)p_c'(s)K(\nabla p)\nabla s \cdot \nabla\psi\,dx\,dt.$$

For the last term, using Remark 9.1, we have

$$\lim_{\alpha\to 0}\int_0^T (q_{w\alpha}, \psi)\,dt = \int_0^T (q_w, \psi)\,dt.$$

It follows from (4.3) that

$$\begin{aligned}
&\lim_{\alpha\to 0}\int_0^T (\phi\partial_t^{-\alpha}s^\alpha,\psi)\,dt + \int_{\Omega_T}\lambda_w(s)K(\nabla p)\nabla p\cdot\nabla\psi\,dx\,dt\\
&+\int_{\Omega_T}\Lambda_\varepsilon(s)p_c'(s)K(\nabla p)\nabla s\cdot\nabla\psi\,dx\,dt\\
&=\int_0^T (q_w,\psi),\quad \forall\psi\in\cup_{n=1}^\infty \ell^\alpha(I;V).
\end{aligned}$$

For any $\psi\in L^2(I;V)$, $\psi_\alpha\in\ell^\alpha(I;V)$, and because $s^\alpha(\cdot,t)$ is constant over each interval $I_j=(t_{j-1},t_j]$, we observe that

$$\int_0^T (\phi\partial_t^{-\alpha}s^\alpha,\psi)\,dt = \int_0^T (\phi\partial_t^{-\alpha}s^\alpha,\psi_\alpha)\,dt, \tag{10.12}$$

then, identity (4.3) can be written as

$$\begin{aligned}
\int_0^T (\phi\partial_t^{-\alpha}s^\alpha,\psi_\alpha)\,dt &= -\int_{\Omega_T}\lambda_w(s)K(\nabla p)\nabla p\cdot\nabla\psi_\alpha\,dx\,dt\\
&\quad -\int_{\Omega_T}\Lambda_\varepsilon(s)p_c'(s)K(\nabla p)\nabla s\cdot\nabla\psi_\alpha\,dx\,dt+\int_0^T (q_w,\psi_\alpha).
\end{aligned}$$

This implies that

$$\big|\int_0^T (\phi\partial_t^{-\alpha}s^\alpha,\psi)\,dt\big| \le C\|\psi\|_{L^2(I;V)},\quad \forall\psi\in L^2(I;V).$$

The sequence $(\phi\partial_t^{-\alpha}s^\alpha)$ is thus bounded in $L^2(I;V^\star)$. Consequently, for a subsequence, $(\phi\partial_t^{-\alpha}s^\alpha)$ converges weakly in $L^2(I;V^\star)$. For $\psi\in\mathcal{D}(I;V)$ and $\alpha>0$ small enough, using Formula (10.11), we have

$$\begin{aligned}
\int_0^T (\phi\partial_t^{-\alpha}s^\alpha(\cdot,t),\psi(\cdot,t))\,dt &= -\int_0^{T-\alpha}(\phi s^\alpha,\partial_t^\alpha\psi)\,dt\\
&\to -\int_0^T(\phi s,\partial_t\psi)\,dt=\int_0^T\langle\phi\partial_t s,\psi\rangle\,dt,
\end{aligned}$$

as a distribution. Therefore, $\phi\partial_t^{-\alpha}s^\alpha \rightharpoonup \phi\partial_t s$ weakly in $L^2(I;V^\star)$. Combining these results, the saturation equation holds in the weak sense of Definition 3.1 since $\cup_{n=1}^\infty\ell^\alpha(I;V)$ is dense in $L^2(I;V)$. Thus the proof of Theorem 3.2 is complete.

11. Maximum principles about weak solutions

Theorem 11.1. *If (p,s) is a weak solution of system (S_ε), then $0\le s(x,t)\le 1$ a.e. x in Ω and for all t in $[0,T]$.*

Proof. To show that $s(x,t)\ge 0$, we prove that its negative part s^- is zero on Ω_T. Let us first remark that for (p,s) a weak solution of system (S_ε), the equation of saturation (3.3) implies that

$$\begin{aligned}
&\langle\phi\partial_t s,v\rangle-\int_\Omega\lambda_w(s)K(\nabla p)\nabla p\cdot\nabla v\,dx-\int_\Omega\Lambda_\varepsilon(s)p_c'(s)K(\nabla p)\nabla s\cdot\nabla v\,dx\\
&=\int_\Omega q_w v\,dx,\quad \forall v\in V,\ \text{a.e. in } (0,T).
\end{aligned}\tag{11.1}$$

Let us fix a number $t\in(0,T]$. Since the function $\mathbb{R}\ni r\longmapsto\frac12(|r|-r)\doteq r^-\in\mathbb{R}$ is Lipschitz, it is licit to take $v=-s^-(\sigma)$, with $\sigma\in(0,t)$ non exceptional, as a test function in the Equation (11.1) written at the time point σ. We obtain

$$\langle\phi\partial_t s(\sigma),-s^-(\sigma)\rangle-\int_\Omega\lambda_w(s)K(\nabla p)\nabla p\cdot\nabla s^-\,dx-\int_\Omega\Lambda_\varepsilon(s)p_c'(s)K(\nabla p)\nabla s\cdot\nabla s^-\,dx$$

$$= -\int_\Omega q_w s^- \,dx.$$

Since $\lambda_w(s) = 0$ for $s \leq 0$, we obtain $\int_\Omega \lambda_{w\star}(s)K(\nabla p)\nabla p\cdot\nabla s^-\,dx\,dt = 0$. Also, using that

$$\begin{aligned}
-\int_\Omega \Lambda_\varepsilon(s)p_c'(s)K(\nabla p)\nabla s\cdot\nabla s^-\,dx &= \int_\Omega \Lambda_\varepsilon(s)p_c'(s)K(\nabla p)|\nabla s^-|^2\,dx \\
&\geq \varepsilon p_{c*}'\kappa_2\int_\Omega |\nabla s^-|^2\,dx,
\end{aligned}$$

and the positivity of function q_w (hypothesis (H4)), we obtain

$$\langle\phi\partial_t s(\sigma), -s^-(\sigma)\rangle + \varepsilon p_{c*}'\kappa_2\int_\Omega |\nabla s^-|^2\,dx \leq -\int_\Omega q_w s^-\,dx \leq 0.$$

We deduce that

$$\langle\phi\partial_t s(\sigma), -s^-(\sigma)\rangle \leq 0 \quad \text{a.e. } \sigma\in(0,t). \tag{11.2}$$

Note that

$$\langle\phi\partial_t s(\sigma), -s^-(\sigma)\rangle = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \phi(x)|s^-(x,\sigma)|^2\,dx \quad \text{a.e. } \sigma\in(0,T). \tag{11.3}$$

Let us now suppose that $s\in\mathcal{D}([0,T];V)$, the space of restrictions to $[0,T]$ of functions indefinitely differentiable with compact support and values in V. In this case, we can write

$$\begin{aligned}
\big\langle\phi\frac{\partial s}{\partial t}(\sigma), -s^-(\sigma)\big\rangle &= -\int_\Omega \phi(x)\frac{\partial s}{\partial t}(x,\sigma)s^-(x,\sigma)\,dx \\
&= -\int_{\Omega\cap\{s(x,\sigma)<0\}} \phi(x)\frac{\partial s}{\partial t}(x,\sigma)s^-(x,\sigma)\,dx \\
&= \int_{\Omega\cap\{s(x,\sigma)<0\}} \phi(x)\frac{\partial s^-}{\partial t}(x,\sigma)s^-(x,\sigma)\,dx \\
&= \int_{\{s<0\}} \phi(x)\frac{1}{2}\frac{\partial}{\partial t}|s^-(x,\sigma)|^2\,dx \\
&= \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \phi(x)|s^-(x,\sigma)|^2\,dx
\end{aligned}$$

Adopting the techniques used by Chipot [9, Lemma 11.2, page 203], we can prove that the (11.3) remains true for $s\in W(0,T) = H^1(0,T;V,V^\star)$. Integrating inequality (11.2), we obtain

$$\begin{aligned}
\int_0^t \Big\langle\phi\frac{\partial s}{\partial t}(\sigma), -s^-(\sigma)\Big\rangle d\sigma &= \frac{1}{2}\int_\Omega \phi(x)|s^-(x,t)|^2\,dx - \frac{1}{2}\int_\Omega \phi(x)|s^-(x,0)|^2\,dx \\
&= \frac{1}{2}\int_\Omega \phi(x)|s^-(x,t)|^2\,dx \leq 0.
\end{aligned}$$

This because $s(0) = s_0(x) \geq 0$ (hypothesis (H4)), giving $\int_\Omega \phi(x)|s^-(x,0)|^2\,dx = 0$, and the Inequality (11.2). Now, using (H1), we see that $\int_\Omega |s^-(x,t)|^2\,dx = 0$, a.e. in $t\in]0,T]$. This proves that $s(x,t)\geq 0$ a.e. in Ω_T.

To show that $s(x,t)\leq 1$ a.e., we prove that $(s-1)^+$, the positive part of $s-1$, is zero on Ω_T. Using the same techniques as before, we fix a number $t\in(0,T]$, and take $v = (s-1)^+(\sigma)$, with $\sigma\in(0,t)$ non exceptional, as a test function in (11.1) written at the time point σ. We obtain

$$\begin{aligned}
&\langle\phi\partial_t s(\sigma), (s-1)^+(\sigma)\rangle + \int_\Omega \lambda_w(s)K(\nabla p)\nabla p\cdot\nabla(s-1)^+dx \\
&+ \int_\Omega \Lambda_\varepsilon(s)p_c'(s)K(\nabla p)\nabla s\cdot\nabla(s-1)^+dx \\
&= \int_\Omega q_w(s-1)^+dx.
\end{aligned} \tag{11.4}$$

Putting $\{s > 1\} = \Omega \cap \{s(x,\sigma) > 1\}$, by the hypothesis (H2) and the definition of extensions of the coefficients in equations, see Section 3, the equation of pressure leads to

$$\begin{aligned}\int_\Omega \lambda_w(s)K(\nabla p)\nabla p\cdot\nabla(s-1)^+dx &= \int_{\{s>1\}} \lambda_w(s)K(\nabla p)\nabla p\cdot\nabla(s-1)^+dx\\ &= \int_{\{s>1\}} \lambda(s)K(\nabla p)\nabla p\cdot\nabla(s-1)^+dx\\ &= \int_{\{s>1\}} q(s-1)^+dx = \int_\Omega q(s-1)^+dx.\end{aligned}$$

Now, using the inequality

$$\int_\Omega \Lambda_\varepsilon(s)p_c'(s)K(\nabla p)\nabla s\cdot\nabla(s-1)^+dx \ge \varepsilon p_{c*}'\kappa_2\int_\Omega |\nabla(s-1)^+|^2\,dx,$$

and equation (11.4), we obtain

$$\langle\phi\partial_t s(\sigma),(s-1)^+(\sigma)\rangle + \int_\Omega q(s-1)^+dx + \varepsilon p_{c*}'\kappa_2\int_\Omega |\nabla(s-1)^+|^2\,dx \le \int_\Omega q_w(s-1)^+dx$$

Therefore, since $q - q_w = q_n$, which is a positive function, we obtain

$$\langle\phi\partial_t s(\sigma),(s-1)^+(\sigma)\rangle + \varepsilon p_{c*}'\kappa_2\int_\Omega |\nabla(s-1)^+|^2\,dx \le -\int_\Omega q_n(s-1)^+dx \le 0.$$

Consequently,

$$\langle\phi\partial_t s(\sigma),(s-1)^+(\sigma)\rangle \le 0 \quad \text{a.e. } \sigma\in(0,t). \tag{11.5}$$

To go further, we note as above that

$$\begin{aligned}\langle\phi\partial_t s(\sigma),(s-1)^+(\sigma)\rangle &= \langle\phi\partial_t(s-1)(\sigma),(s-1)^+(\sigma)\rangle\\ &= \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \phi(x)|(s-1)^+(x,\sigma)|^2\,dx \quad \text{a.e. } \sigma\in(0,T).\end{aligned} \tag{11.6}$$

This can be seen using the denseness for $s\in\mathcal{D}([0,T];V)$ in $W(0,T)$.

Integrating the previous (11.5), we obtain

$$\begin{aligned}&\int_0^t \Big\langle\phi\frac{\partial s}{\partial t}(\sigma),(s-1)^+(\sigma)\Big\rangle d\sigma\\ &= \frac{1}{2}\int_\Omega \phi(x)|(s-1)^+(x,t)|^2\,dx - \frac{1}{2}\int_\Omega \phi(x)|(s-1)^+(x,0)|^2\,dx\\ &= \frac{1}{2}\int_\Omega \phi(x)|(s-1)^+(x,t)|^2\,dx \le 0.\end{aligned}$$

This because $s(0) = s_0(x) \le 1$ (hypothesis (H4)), giving $\int_\Omega \phi(x)|(s-1)^+(x,0)|^2\,dx = 0$, and the Inequality (11.5). Now, using Hypothesis (H1), we see that $\int_\Omega |(s-1)^+(x,t)|^2\,dx = 0$, a.e. in $t\in]0,T]$. This proves that $s(x,t)\le 1$ a.e. in Ω_T. $\square$

Remark 11.2. To finish, we mention that all results of this paper are in fact true for a family of absolute permeability, in the sense that our results remain true if we replace the expression of absolute rock permeability given in page 2 by any continuous function $K:\mathbb{R}^3\longrightarrow\mathbb{R}$, bounded from below and above by positive constants with $K(x)x$ monotone.

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