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OSCILLATION OF MODIFIED EULER TYPE HALF-LINEAR DIFFERENTIAL EQUATIONS VIA AVERAGING TECHNIQUE

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Abstract. In this article, we analyze the oscillation behavior of half-linear differential equation

$$\big(r(t)t^{p-1}\Phi(x')\big)' + \frac{s(t)}{t\log^p t}\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1.$$

Applying the modified half-linear Prüfer angle and a general averaging technique over unbounded intervals, we prove an oscillation criterion for the studied equation. We point out that the presented oscillation criterion is new even in the linear case when $p = 2$.

1. Introduction

The research presented in this article belongs to the qualitative theory of half-linear equations. We intend to prove an oscillation criterion and to fill the gap in the knowledge about oscillation properties of Euler type half-linear and also linear differential equations. Before we describe our motivation in the form of results that we intend to generalize and complete by our research, we outline the main points of the theory of half-linear equations. For more details, we refer, e.g., to [1, 10]. A more comprehensive list of relevant papers is given at the end of this section.

The half-linear differential equations are equations of the form

$$\big(c(t)\Phi(x')\big)' + d(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{1.1}$$

where coefficients $c > 0, d$ are continuous functions. Equations of the form (1.1) are closely related to linear equations, non-linear equations, and also to partial differential equations. To obtain linear equations, it suffices to put $p = 2$ which leads to the equation $(c(t)x')' + d(t)x = 0$.

The main difference between linear and half-linear equations is the fact that the solution space of (1.1) is homogeneous but it is not additive. It is the origin of the designation "half-linear" and one of the reasons that many tools from the theory of linear equations are not available and that other tools have to be non-trivially modified for half-linear equations. Hence, it is more complicated to obtain results for half-linear equations. At the same time, such results are often used as a gateway to research in the field of non-linear equations (especially, those with functions that preserve sign in the place of Φ).

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Another important relation is to (elliptic) partial differential equations with p-Laplacian. The connection is caused by the fact that (1.1) can be viewed as a scalar PDE with one dimensional p-Laplacian (see the first term and the form of Φ). Then, oscillation results for the one dimensional case (half-linear equations) lead to criteria for more general PDE's (in particular, concerning the so-called weak oscillation). Of course, the motivation for the research in the field of half-linear equations is not just purely mathematical which is described above. These equations are also used in real world models (e.g., in the non-Newtonian fluid theory). In addition, its impact to non-linear equations and to PDE's is important.

In this article, we study the oscillation of half-linear equations, where we use the half-linear Sturm theory. In particular, the well-known half-linear Sturm separation theorem allows us to categorize half-linear equations as oscillatory and non-oscillatory. We call an equation oscillatory if all of its solutions are oscillatory, i.e., there does not exist the largest zero point of any solution. A basic consequence of the half-linear Sturm separation theorem is the following implication. If one non-trivial solution is oscillatory, then every non-trivial solution is oscillatory.

Next, we focus to the description of the result that we are going to prove in this paper. We treat (1.1) with

$$c(t) = \Big(\frac{t}{r(t)}\Big)^{p-1}, \quad d(t) = \frac{s(t)}{t\log^p t},$$

where $r > 0$ and s are continuous functions. Further, in the whole paper, let t be sufficiently large and let $p > 1$ be arbitrarily given. As q, we denote the real number conjugated to p, i.e., it holds $p + q = pq$, and log stands for the natural logarithm.

Now, we mention the main three motivations explicitly. The first one of our main motivations comes from [32]. In [32], it is proved the following oscillation criterion.

Theorem 1.1. *Let us consider the equation*

$$\big(r^{1-p}(t)t^{p-1}\Phi(x')\big)' + \frac{s(t)}{t\log^p t}\Phi(x) = 0, \tag{1.2}$$

where $r > 0$ and s are continuous functions such that

$$\lim_{t\to\infty} \frac{\int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau}{\sqrt{t\log t}} = 0, \quad \lim_{t\to\infty} \frac{\int_t^{t+\alpha} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t\log t}} = 0 \tag{1.3}$$

for some $\alpha > 0$. Let $R, S > 0$ be given. If

$$R^{p-1}S > q^{-p} \tag{1.4}$$

and if

$$\frac{1}{\alpha}\int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau \ge R, \quad \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau \ge S \tag{1.5}$$

for all large t, then (1.2) *is oscillatory.*

Theorem 1.1 has been improved in a certain sense in [20] as follows.

Theorem 1.2. *Let us consider*

$$\big(r(t)t^{p-1}\Phi(x')\big)' + \frac{s(t)}{t\log^p t}\Phi(x) = 0, \tag{1.6}$$

where $r > 0$ and s are continuous functions such that

$$0 < \liminf_{t\to\infty} r(t) \le \limsup_{t\to\infty} r(t) < \infty$$

and that the integral

$$\int^{\infty} \frac{s(\tau)}{\tau \log^p \tau}\, \mathrm{d}\tau$$

is convergent. Let there exist $M, \alpha > 0$ such that

$$\Big|\int_t^{t+\beta} s(\tau)\, \mathrm{d}\tau\Big| < M, \quad \Big|\int_t^{t+\beta} \frac{s(\tau)}{\tau\log\tau}\, \mathrm{d}\tau\Big| < \frac{M}{t\log t}, \quad \beta \in [0,\alpha],$$

for all large t. If there exist $R, S > 0$ satisfying (1.4) and if the inequalities

$$\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\, \mathrm{d}\tau \ge R, \quad \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\, \mathrm{d}\tau \ge S$$

hold for all large t, then (1.6) is oscillatory.

The last main motivation is given by the main result of [23] which reads as follows.

Theorem 1.3. *Let $f > 0$ be a continuously differentiable function and $g \ge 1$ be a continuous function such that*

$$\lim_{t\to\infty} f'(t)g(t) = 0, \quad \lim_{t\to\infty} \frac{f(t)g^2(t)}{t} = 0. \tag{1.7}$$

Let us consider the equation

$$\big(t^{\alpha-1} r^{1-p}(t)\Phi(x')\big)' + t^{\alpha-1-p} s(t)\Phi(x) = 0, \tag{1.8}$$

where $\alpha \in \mathbb{R}\setminus\{p\}$ and $r > 0$ and s are continuous functions satisfying

$$\limsup_{t\to\infty} \frac{\int_t^{t+f(t)} r(\tau)\, \mathrm{d}\tau}{f(t)g(t)} < \infty, \quad \limsup_{t\to\infty} \frac{\int_t^{t+f(t)} |s(\tau)|\, \mathrm{d}\tau}{f(t)g(t)} < \infty.$$

Let

$$\begin{aligned} M(r,f) &:= \liminf_{t\to\infty} \frac{1}{f(t)}\int_t^{t+f(t)} r(\tau)\, \mathrm{d}\tau \in \mathbb{R},\\ M(s,f) &:= \liminf_{t\to\infty} \frac{1}{f(t)}\int_t^{t+f(t)} s(\tau)\, \mathrm{d}\tau \in \mathbb{R}. \end{aligned}$$

If $(M(r,f))^{p-1}M(s,f) > p^{-p}|p-\alpha|^p$, then (1.8) is oscillatory.

We remark that the power $1-p$ of r in (1.2) and in (1.8) is considered only from technical reasons (technical parts of proofs are more transparent in this case). Of course, since r is positive, it does not mean any impact to the used methods.

The aim of this paper to prove an oscillation criterion which generalizes Theorems 1.1 and 1.2 and which corresponds Theorem 1.3 in the omitted case $\alpha = p$. Although Theorem 1.2 is proved using the Riccati technique, Theorem 1.3 is proved via a modification of the adapted Prüfer angle. In this paper, we obtain the announced criterion (which corresponds Theorem 1.3) applying a different modification of the Prüfer angle.

We emphasize that the main result of this paper is not a generalization of any known fact about linear equations, i.e., we obtain a result which has not been known for linear equations. The corresponding result for linear equations is explicitly formulated as a corollary of the main result at the end of this paper (see Corollary 4.4). We also mention another consequence and a very simple example of an equation whose oscillation properties have not been known. These corollaries together with the example declare the significant novelty of the main result.

In this paragraph, we give a short literature overview. The basics of the oscillation theory of the treated half-linear equations are presented, e.g., in [1, 10]. Concerning linear and half-linear equations which are strongly close to the considered Euler type equations, the oscillation behavior is studied in many papers. We point out at least [9, 12, 13, 16, 17, 19, 22, 31, 37, 47] in the case of differential equations, [28, 30, 33, 38, 43] in the case of difference equations, and papers [21, 34, 35, 45, 48] in the case of dynamic equations on time scales. In addition, a similar study of oscillation properties of Euler type equations can be found in [2, 3, 18, 26, 49]. The oscillation of perturbed half-linear Euler type equations is analyzed, e.g., in [5, 6, 14, 15, 25, 29, 40, 41, 42, 44] in the case of differential equations and in [36, 50] in the case of linear and non-linear difference equations.

This article is organized as follows. The used preliminaries are collected in Section 2. Especially, in Section 2, it is derived the used modification of the adapted Prüfer angle. Auxiliary results (about an averaging function of the Prüfer angle) can be found in Section 3. The presented oscillation criterion is formulated and proved in Section 4 together with two corollaries and an illustrative example.

2. Preliminaries

Now, we describe the equation for the used modification of the half-linear Prüfer angle. To introduce this modification, we recall the concept of half-linear trigonometric functions. Let

$$\pi_p := \frac{2\pi}{p \sin \frac{\pi}{p}}.$$

The half-linear sine function $\sin_p$ is defined as the odd $2\pi_p$-periodic extension of the solution of the problem

$$(\Phi(x'))' + (p-1)\Phi(x) = 0, \quad x(0) = 0, \quad x'(0) = 1.$$

The half-linear sine function is continuously differentiable. Its derivative is called the half-linear cosine function and it is denoted by $\cos_p$. We collect only properties of the half-linear functions which we will use later. At first, we mention the half-linear Pythagorean identity

$$|\sin_p x|^p + |\cos_p x|^p = 1, \quad x \in \mathbb{R}, \tag{2.1}$$

which implies

$$|\cos_p x|^p \le 1, \quad |\sin_p x|^p \le 1, \quad x \in \mathbb{R}. \tag{2.2}$$

In addition, we have

$$|\Phi(\cos_p x) \sin_p x| \le 1, \quad x \in \mathbb{R}. \tag{2.3}$$

We will also use the fact that the functions $y = |\sin_p x|^p$, $y = |\cos_p x|^p$, and $y = \Phi(\cos_p x)\sin_p x$ are periodic and continuously differentiable on $\mathbb{R}$ (see, e.g., [8]). Hence, there exists a constant $C > 0$ for which

$$\big| |\sin_p x_1|^p - |\sin_p x_2|^p \big| \le C |x_1 - x_2|, \quad x_1, x_2 \in \mathbb{R}, \tag{2.4}$$

$$\big| |\cos_p x_1|^p - |\cos_p x_2|^p \big| \le C |x_1 - x_2|, \quad x_1, x_2 \in \mathbb{R}, \tag{2.5}$$

$$\left|\Phi\left(\cos_p x_1\right)\sin_p x_1-\Phi\left(\cos_p x_2\right)\sin_p x_2\right|\le C\left|x_1-x_2\right|,\quad x_1,x_2\in\mathbb{R}. \tag{2.6}$$

For other properties, we refer to [10, Section 1.1.2]).

We consider the half-linear equation

$$\Big(\big(\frac{t}{r(t)}\big)^{p-1}\Phi(x')\Big)'+\frac{s(t)}{t\log^p t}\,\Phi(x)=0, \tag{2.7}$$

where $r>0$ and s are continuous functions. Applying the Riccati type transformation

$$w(t)=\Big(\frac{t}{r(t)}\Big)^{p-1}\Phi\Big(\frac{x'(t)}{x(t)}\Big)$$

to (2.7), we obtain the so-called Riccati equation

$$w'(t)+\frac{s(t)}{t\log^p t}+(p-1)\left(r^{1-p}(t)\,t^{p-1}\right)^{\frac{1}{1-p}}|w(t)|^{\frac{p}{p-1}}=0,$$

i.e.,

$$w'(t)+\frac{s(t)}{t\log^p t}+(p-1)\,\frac{r(t)}{t}\,|w(t)|^q=0.$$

For details, see [10, Section 1.1.4].

Applying the substitution

$$v(t)=w(t)\cdot\log^{p-1}t$$

and the modified Prüfer transformation

$$x(t)=\rho(t)\sin_p\varphi(t),\quad \left(r^{1-p}(t)\,t^{p-1}\right)^{q-1}x'(t)=\frac{\rho(t)}{\log t}\cos_p\varphi(t),$$

i.e.,

$$x(t)=\rho(t)\sin_p\varphi(t),\quad x'(t)=\frac{r(t)}{t\log t}\,\rho(t)\cos_p\varphi(t),$$

we obtain the equation for the adapted Prüfer angle φ in the form

$$\begin{aligned}&\varphi'(t)\\&=\frac{1}{t\log t}\Big[r(t)|\cos_p\varphi(t)|^p-\Phi\left(\cos_p\varphi(t)\right)\sin_p\varphi(t)+\frac{1}{p-1}\,s(t)\,|\sin_p\varphi(t)|^p\Big].\end{aligned} \tag{2.8}$$

Concerning details about the derivation of (2.8), we refer to [32] (where (2.7) is treated without the power $1-p$ in the first term).

3. Auxiliary results

Let f be a positive and continuously differentiable function and let g be a positive and continuous function which are defined for all large t and for which

$$\lim_{t\to\infty}f'(t)g(t)=0, \tag{3.1}$$

$$\lim_{t\to\infty}\frac{f(t)g^2(t)}{t\log t}=0, \tag{3.2}$$

$$\lim_{t\to\infty}\frac{f(t)g(t)}{t}=0, \tag{3.3}$$

$$g_{\inf}:=\liminf_{t\to\infty}g(t)>0. \tag{3.4}$$

Especially, (3.2) and (3.4) give

$$\lim_{t\to\infty}\frac{f(t)g(t)}{t\log t}=0, \tag{3.5}$$

i.e.,
$$\lim_{t\to\infty} \frac{t\log t}{f(t)g(t)} = \infty. \tag{3.6}$$
We consider (2.7), where $r > 0$ and s are continuous functions satisfying
$$r[f,g] := \limsup_{t\to\infty} \frac{\int_t^{t+f(t)} r(\tau)\,\mathrm{d}\tau}{f(t)g(t)} < \infty, \tag{3.7}$$
$$s[f,g] := \limsup_{t\to\infty} \frac{\int_t^{t+f(t)} |s(\tau)|\ \mathrm{d}\tau}{f(t)g(t)} < \infty. \tag{3.8}$$

Let F be a continuous function defined on an interval in the form $[T,\infty)$. For such a function F and the given function f, we define the averaging function by the formula
$$\operatorname{ave}[F,f](t) := \frac{1}{f(t)} \int_t^{t+f(t)} F(\tau)\,\mathrm{d}\tau, \quad t\in[T,\infty). \tag{3.9}$$
For a solution φ of (2.8), we consider properties of $\operatorname{ave}[\varphi,f]$ in the following three lemmas.

Lemma 3.1. *Let φ be a solution of (2.8) on an interval $[t_0,\infty)$. Then, the inequality*
$$\Delta\left(\varphi,\operatorname{ave}[\varphi,f]\right) := \limsup_{t\to\infty} \frac{t\log t}{f(t)g(t)} \max_{\tau\in[t,t+f(t)]} |\varphi(\tau) - \operatorname{ave}[\varphi,f](t)| < \infty \tag{3.10}$$
holds.

Proof. Based on the continuity of φ, we have (see (2.2), (2.3), (3.4), (3.7), and (3.8) and consider (2.8))
$$\begin{aligned}
&\limsup_{t\to\infty} \frac{t\log t}{f(t)g(t)} \max_{\tau\in[t,t+f(t)]} |\varphi(\tau) - \operatorname{ave}[\varphi,f](t)| \\
&\le \limsup_{t\to\infty} \frac{t\log t}{f(t)g(t)} \int_t^{t+f(t)} |\varphi'(\sigma)|\,\mathrm{d}\sigma \\
&\le \limsup_{t\to\infty} \frac{t\log t}{f(t)g(t)} \int_t^{t+f(t)} \frac{1}{\sigma\log\sigma} \Big[r(\sigma)|\cos_p\varphi(\sigma)|^p \\
&\quad + |\Phi\left(\cos_p\varphi(\sigma)\right)\sin_p\varphi(\sigma)| + |s(\sigma)|\ \frac{|\sin_p\varphi(\sigma)|^p}{p-1}\Big]\,\mathrm{d}\sigma \\
&\le \limsup_{t\to\infty} \frac{t\log t}{f(t)g(t)} \int_t^{t+f(t)} \frac{1}{t\log t}\Big[r(\sigma) + 1 + \frac{|s(\sigma)|}{p-1}\Big]\,\mathrm{d}\sigma \\
&= \limsup_{t\to\infty} \frac{1}{f(t)g(t)} \int_t^{t+f(t)} r(\sigma)\,\mathrm{d}\sigma + \limsup_{t\to\infty} \frac{1}{g(t)} \\
&\quad + \limsup_{t\to\infty} \frac{1}{f(t)g(t)} \int_t^{t+f(t)} \frac{|s(\sigma)|}{p-1}\,\mathrm{d}\sigma \\
&= r[f,g] + \frac{1}{g_{\inf}} + \frac{s[f,g]}{p-1} < \infty.
\end{aligned}$$
The proof is complete. □

In particular, from (3.10) (i.e., from the statement of Lemma 3.1), we obtain the next auxiliary result.

Lemma 3.2. *If φ is a solution of* (2.8) *on an interval* $[t_0, \infty)$*, then*

$$\lim_{t\to\infty} |\varphi(t) - \operatorname{ave}[\varphi, f](t)| = 0. \tag{3.11}$$

Proof. From (3.10) in Lemma 3.1, it follows that

$$\limsup_{t\to\infty} \frac{t\log t}{f(t)g(t)} |\varphi(t) - \operatorname{ave}[\varphi, f](t)| < \infty. \tag{3.12}$$

To prove (3.11), it suffices to consider (3.12) together with (3.6). □

For a solution φ of (2.8) on an interval $[t_0, \infty)$, where t_0 is sufficiently large, we show the fundamental property of the derivative of $\operatorname{ave}[\varphi, f]$ in the following lemma.

Lemma 3.3. *Let φ be a solution of* (2.8) *on an interval* $[t_0, \infty)$*. Then,*

$$\begin{aligned}
&\lim_{t\to\infty} \Big| t\log t \frac{\partial \operatorname{ave}[\varphi, f](t)}{\partial t} - |\cos_p(\operatorname{ave}[\varphi, f](t))|^p \operatorname{ave}[r, f](t) \\
&+ \Phi(\cos_p(\operatorname{ave}[\varphi, f](t))) \sin_p(\operatorname{ave}[\varphi, f](t)) \\
&- \frac{1}{p-1} |\sin_p(\operatorname{ave}[\varphi, f](t))|^p \operatorname{ave}[s, f](t) \Big| = 0.
\end{aligned} \tag{3.13}$$

Proof. Without loss of generality, we can assume that $t_0 > 1$ and that the functions f, g, r, s are defined for all $t > t_0$. For $t \in (t_0, \infty)$, it holds (see (3.9))

$$\begin{aligned}
&\frac{\partial \operatorname{ave}[\varphi, f](t)}{\partial t} \\
&= \Big(\frac{1}{f(t)} \int_t^{t+f(t)} \varphi(\tau)\, \mathrm{d}\tau \Big)' \\
&= -\frac{f'(t)}{f^2(t)} \int_t^{t+f(t)} \varphi(\tau)\, \mathrm{d}\tau + \frac{1}{f(t)} \left[(1 + f'(t))\, \varphi(t + f(t)) - \varphi(t) \right] \\
&= \frac{1}{f(t)} \int_t^{t+f(t)} \varphi'(\tau)\, \mathrm{d}\tau + \frac{f'(t)}{f(t)} \Big(\varphi(t + f(t)) - \frac{1}{f(t)} \int_t^{t+f(t)} \varphi(\tau)\, \mathrm{d}\tau \Big) \\
&= \frac{1}{f(t)} \int_t^{t+f(t)} \varphi'(\tau)\, \mathrm{d}\tau + \frac{f'(t)}{f(t)} \Big(\varphi(t + f(t)) - \operatorname{ave}[\varphi, f](t) \Big).
\end{aligned} \tag{3.14}$$

At the same time, we have (see (3.1) and (3.10) in Lemma 3.1)

$$\begin{aligned}
&\limsup_{t\to\infty} t\log t \Big| \frac{f'(t)}{f(t)} \Big(\varphi(t + f(t)) - \operatorname{ave}[\varphi, f](t) \Big) \Big| \\
&\le \lim_{t\to\infty} |f'(t)g(t)| \cdot \limsup_{t\to\infty} \frac{t\log t}{f(t)g(t)} \max_{\tau\in[t, t+f(t)]} |\varphi(\tau) - \operatorname{ave}[\varphi, f](t)| = 0.
\end{aligned} \tag{3.15}$$

From (3.14) and (3.15), it follows that

$$\lim_{t\to\infty} \Big| t\log t \frac{\partial \operatorname{ave}[\varphi, f](t)}{\partial t} - \frac{t\log t}{f(t)} \int_t^{t+f(t)} \varphi'(\tau)\, \mathrm{d}\tau \Big| = 0. \tag{3.16}$$

By direct calculations, one can verify that

$$\lim_{t\to\infty} g(t) \frac{(t + f(t)) \log(t + f(t)) - t\log t}{t\log t} = 0. \tag{3.17}$$

Indeed, from (3.3) (consider also (3.4)), it is seen that

$$
\begin{aligned}
0 &\le \liminf_{t\to\infty} g(t)\, \frac{(t+f(t))\log(t+f(t)) - t\log t}{t\log t} \\
&\le \limsup_{t\to\infty} g(t)\, \frac{(t+f(t))\log(t+f(t)) - t\log t}{t\log t} \\
&\le \limsup_{t\to\infty} g(t)\, \frac{\left(1+\frac{\varepsilon}{g(t)}\right) t\log\left(\left(1+\frac{\varepsilon}{g(t)}\right)t\right) - t\log t}{t\log t} \\
&= \limsup_{t\to\infty} g(t)\, \frac{\left(1+\frac{\varepsilon}{g(t)}\right)\left(1+\frac{1}{\log t}\cdot\log\left(1+\frac{\varepsilon}{g(t)}\right)\right) - 1}{1} = \varepsilon,
\end{aligned}
$$

where $\varepsilon > 0$ is arbitrary and where it suffices to use the well-known limit

$$\lim_{y\to\infty} y\log\left(1+\frac{\varepsilon}{y}\right) = \varepsilon$$

if $\limsup_{t\to\infty} g(t) = \infty$.

We have (see also (2.2), (2.3), (3.4), (3.7), and (3.8) with (2.8))

$$
\begin{aligned}
&\Big|\frac{t\log t}{f(t)} \int_t^{t+f(t)} \varphi'(\tau)\,\mathrm{d}\tau - \frac{1}{f(t)}\int_t^{t+f(t)} \tau\log\tau\cdot\varphi'(\tau)\,\mathrm{d}\tau\Big| \\
&\le \frac{t\log t}{f(t)} \int_t^{t+f(t)} \Big(\frac{\tau\log\tau}{t\log t} - 1\Big)\,|\varphi'(\tau)|\,\mathrm{d}\tau \\
&= \frac{t\log t}{f(t)} \int_t^{t+f(t)} \frac{\tau\log\tau - t\log t}{t\log t}\,|\varphi'(\tau)|\,\mathrm{d}\tau \\
&= \frac{1}{f(t)} \int_t^{t+f(t)} \frac{\tau\log\tau - t\log t}{\tau\log\tau}\Big|r(\tau)|\cos_p\varphi(\tau)|^p \\
&\quad - \Phi\left(\cos_p\varphi(\tau)\right)\sin_p\varphi(\tau) + \frac{1}{p-1}\,s(\tau)\,|\sin_p\varphi(\tau)|^p\,\Big|\,\mathrm{d}\tau \\
&\le \frac{1}{f(t)} \int_t^{t+f(t)} \frac{(t+f(t))\log(t+f(t)) - t\log t}{t\log t}\left[r(\tau) + 1 + \frac{|s(\tau)|}{p-1}\right]\mathrm{d}\tau \\
&= g(t)\frac{(t+f(t))\log(t+f(t)) - t\log t}{t\log t}\Big[\frac{1}{f(t)g(t)}\int_t^{t+f(t)} r(\tau)\,\mathrm{d}\tau \\
&\quad + \frac{1}{g(t)} + \frac{1}{f(t)g(t)}\int_t^{t+f(t)} \frac{|s(\tau)|}{p-1}\,\mathrm{d}\tau\Big] \\
&\le g(t)\frac{(t+f(t))\log(t+f(t)) - t\log t}{t\log t}\Big(r[f,g] + \frac{1}{g_{\inf}} + \frac{s[f,g]}{p-1} + 1\Big)
\end{aligned}
$$

for all large t, which yields (see (3.17))

$$\lim_{t\to\infty}\Big|\frac{t\log t}{f(t)}\int_t^{t+f(t)} \varphi'(\tau)\,\mathrm{d}\tau - \frac{1}{f(t)}\int_t^{t+f(t)} \tau\log\tau\cdot\varphi'(\tau)\,\mathrm{d}\tau\Big| = 0. \tag{3.18}$$

From (3.16) and (3.18), we have

$$\lim_{t\to\infty}\Big|t\log t\,\frac{\partial\,\mathrm{ave}[\varphi,f](t)}{\partial t} - \frac{1}{f(t)}\int_t^{t+f(t)} \tau\log\tau\cdot\varphi'(\tau)\,\mathrm{d}\tau\Big| = 0. \tag{3.19}$$

Considering (2.8), we know that (3.19) implies

$$
\begin{aligned}
&\lim_{t\to\infty}\Big|t\log t\,\frac{\partial\,\mathrm{ave}[\varphi,f](t)}{\partial t}-\frac{1}{f(t)}\int_t^{t+f(t)} r(\tau)|\cos_p\varphi(\tau)|^p\,\mathrm{d}\tau\\
&+\frac{1}{f(t)}\int_t^{t+f(t)}\Phi\left(\cos_p\varphi(\tau)\right)\sin_p\varphi(\tau)\,\mathrm{d}\tau\\
&-\frac{1}{f(t)}\int_t^{t+f(t)} s(\tau)\,\frac{|\sin_p\varphi(\tau)|^p}{p-1}\,\mathrm{d}\tau\Big|=0.
\end{aligned}
\tag{3.20}
$$

We have (see (2.5), (3.2), (3.7), (3.9), and (3.10) in Lemma 3.1)

$$
\begin{aligned}
&\limsup_{t\to\infty}\Big|\,|\cos_p\left(\mathrm{ave}[\varphi,f](t)\right)|^p\,\mathrm{ave}[r,f](t)-\frac{1}{f(t)}\int_t^{t+f(t)} r(\tau)|\cos_p\varphi(\tau)|^p\,\mathrm{d}\tau\Big|\\
&=\limsup_{t\to\infty}\Big|\,|\cos_p\left(\mathrm{ave}[\varphi,f](t)\right)|^p\Big(\frac{1}{f(t)}\int_t^{t+f(t)} r(\tau)\,\mathrm{d}\tau\Big)\\
&\quad-\frac{1}{f(t)}\int_t^{t+f(t)} r(\tau)|\cos_p\varphi(\tau)|^p\,\mathrm{d}\tau\Big|\\
&\le\limsup_{t\to\infty}\frac{1}{f(t)}\int_t^{t+f(t)} r(\tau)\,\Big|\,|\cos_p\left(\mathrm{ave}[\varphi,f](t)\right)|^p-|\cos_p\varphi(\tau)|^p\,\Big|\,\mathrm{d}\tau\\
&\le\limsup_{t\to\infty}\frac{1}{f(t)}\int_t^{t+f(t)} r(\tau)C\left|\mathrm{ave}[\varphi,f](t)-\varphi(\tau)\right|\mathrm{d}\tau\\
&\le\limsup_{t\to\infty}C\Big(\frac{t\log t}{f(t)g(t)}\max_{\tau\in[t,t+f(t)]}\left|\varphi(\tau)-\mathrm{ave}[\varphi,f](t)\right|\Big)\\
&\quad\times\frac{f(t)g^2(t)}{t\log t}\cdot\frac{1}{f(t)g(t)}\int_t^{t+f(t)} r(\tau)\,\mathrm{d}\tau\\
&=\limsup_{t\to\infty}C\cdot\Delta\left(\varphi,\mathrm{ave}[\varphi,f]\right)\cdot\frac{f(t)g^2(t)}{t\log t}\cdot r[f,g],
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\lim_{t\to\infty}\Big|\,|\cos_p\left(\mathrm{ave}[\varphi,f](t)\right)|^p\,\mathrm{ave}[r,f](t)\\
&-\frac{1}{f(t)}\int_t^{t+f(t)} r(\tau)|\cos_p\varphi(\tau)|^p\,\mathrm{d}\tau\Big|=0.
\end{aligned}
\tag{3.21}
$$

Next, we have (see (2.6), (3.5), and (3.10) in Lemma 3.1)

$$
\begin{aligned}
&\limsup_{t\to\infty}\Big|\Phi\left(\cos_p\left(\mathrm{ave}[\varphi,f](t)\right)\right)\sin_p\left(\mathrm{ave}[\varphi,f](t)\right)\\
&-\frac{1}{f(t)}\int_t^{t+f(t)}\Phi\left(\cos_p\varphi(\tau)\right)\sin_p\varphi(\tau)\,\mathrm{d}\tau\Big|\\
&\le\limsup_{t\to\infty}\frac{1}{f(t)}\int_t^{t+f(t)}\Big|\Phi\left(\cos_p\left(\mathrm{ave}[\varphi,f](t)\right)\right)\sin_p\left(\mathrm{ave}[\varphi,f](t)\right)\\
&\quad-\Phi\left(\cos_p\varphi(\tau)\right)\sin_p\varphi(\tau)\Big|\mathrm{d}\tau\\
&\le\limsup_{t\to\infty}\frac{1}{f(t)}\int_t^{t+f(t)} C\left|\mathrm{ave}[\varphi,f](t)-\varphi(\tau)\right|\mathrm{d}\tau
\end{aligned}
$$

$$\begin{aligned}&\le C\limsup_{t\to\infty}\Big(\frac{t\log t}{f(t)g(t)}\max_{\tau\in[t,t+f(t)]}|\varphi(\tau)-\operatorname{ave}[\varphi,f](t)|\Big)\frac{f(t)g(t)}{t\log t}\\&=C\cdot\Delta\left(\varphi,\operatorname{ave}[\varphi,f]\right)\cdot\lim_{t\to\infty}\frac{f(t)g(t)}{t\log t}=0.\end{aligned}\tag{3.22}$$

Finally, analogously as in the derivation of (3.21) (consider (2.4), (3.2), (3.8), (3.9), and (3.10) in Lemma 3.1), one can show that

$$\begin{aligned}\lim_{t\to\infty}\Big|&\frac{1}{p-1}\left|\sin_p\left(\operatorname{ave}[\varphi,f](t)\right)\right|^p\operatorname{ave}[s,f](t)\\&-\frac{1}{p-1}\cdot\frac{1}{f(t)}\int_t^{t+f(t)}s(\tau)|\sin_p\varphi(\tau)|^p\,\mathrm{d}\tau\Big|=0.\end{aligned}\tag{3.23}$$

To obtain (3.13), it suffices to use (3.20) together with (3.21), (3.22), and (3.23). $\square$

In the next lemma, we explicitly describe the connection between the oscillation of (2.7) and the unboundedness of solutions of (2.8).

Lemma 3.4. *Equation* (2.7) *is oscillatory if and only if any solution* φ *of* (2.8) *(on a neighborhood of* ∞*) is unbounded from above, i.e.,* $\limsup_{t\to\infty}\varphi(t)=\infty$*.*

Proof. It is well-known that the oscillation of all solutions of (2.7) is equivalent to the infiniteness of the corresponding adapted Prüfer angle φ given by (2.8), i.e., to the unboundedness from above of any solution of (2.8) on the maximal interval of its existence. We can refer, e.g., to [4, 5, 7, 8, 11, 24, 39, 46] for similar cases (different modifications of the adapted Prüfer angle). $\square$

Remark 3.5. In fact, to prove Lemma 3.4, it suffices to consider only (2.8) when $\sin_p\varphi(t)=0$. In addition, from the form of (2.8) in the case when $\sin_p\varphi(t)=0$, one can easily see that $\limsup_{t\to\infty}\varphi(t)=\infty$ for a solution φ of (2.8) implies $\lim_{t\to\infty}\varphi(t)=\infty$, i.e., we have the equivalence

$$\limsup_{t\to\infty}\varphi(t)=\infty\iff\lim_{t\to\infty}\varphi(t)=\infty,\tag{3.24}$$

which is used for the adapted Prüfer angle φ in the literature often.

We also need the following consequence of Theorem 1.1 and Lemma 3.4.

Lemma 3.6. *Let* $A,B>0$ *be arbitrary numbers such that*

$$A^{p-1}B>q^{-p}.\tag{3.25}$$

If ϑ *is a solution of the equation*

$$\vartheta'(t)=\frac{1}{t\log t}\Big(A|\cos_p\vartheta(t)|^p-\Phi\left(\cos_p\vartheta(t)\right)\sin_p\vartheta(t)+\frac{1}{p-1}B|\sin_p\vartheta(t)|^p\Big)\tag{3.26}$$

on an interval $[T_0,\infty)$*, then*

$$\lim_{t\to\infty}\vartheta(t)=\infty.\tag{3.27}$$

Proof. It is seen that (3.26) has the form of the equation for the adapted Prüfer angle, i.e., (2.8) for $r\equiv A$ and $s\equiv B$. Therefore (consider Lemma 3.4), if (2.7) is oscillatory for $r\equiv A$ and $s\equiv B$, then (3.27) is true (see also (3.24)). The oscillation of (2.7) with $r\equiv A$ and $s\equiv B$ follows from (3.25) (cf. (1.4)) and from Theorem 1.1, where it suffices to put $R=A$, $S=B$ and, e.g., $\alpha=1$. $\square$

4. Oscillation criterion

Now, we prove the announced result which follows. For reader's convenience, we repeat all conditions in its statement.

Theorem 4.1. *Let f be a positive and continuously differentiable function and let g be a positive and continuous function which are defined for all large t and for which the conditions* (3.1), (3.2), (3.3), *and* (3.4) *are satisfied. Let us consider* (2.7), *where $r > 0$ and s are continuous functions satisfying* (3.7) *and* (3.8). *Let the values*

$$r^f_{\mathrm{liminf}} := \liminf_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} r(\tau)\,\mathrm{d}\tau, \tag{4.1}$$

$$s^f_{\mathrm{liminf}} := \liminf_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} s(\tau)\,\mathrm{d}\tau \tag{4.2}$$

be finite. If

$$\Big(r^f_{\mathrm{liminf}}\Big)^{p-1} s^f_{\mathrm{liminf}} > q^{-p}, \tag{4.3}$$

then (2.7) *is oscillatory.*

Proof. Based on Lemma 3.4, it suffices to prove that a solution of (2.8) is unbounded from above (consider also (3.24)). Let φ be a solution of (2.8) on an interval $[t_0,\infty)$. We consider the corresponding averaging function $\mathrm{ave}[\varphi,f]$ defined in (3.9). From Lemma 3.2, we know that it suffices to show that

$$\lim_{t\to\infty} \mathrm{ave}[\varphi,f](t) = \infty. \tag{4.4}$$

Let $\delta > 0$ be an arbitrary number for which (see (4.3))

$$\Big(r^f_{\mathrm{liminf}} - 2\delta\Big)^{p-1} \Big(s^f_{\mathrm{liminf}} - 2\delta(p-1)\Big) > q^{-p}. \tag{4.5}$$

To prove (4.4), we use (3.13) in Lemma 3.3 which gives

$$\begin{aligned} \frac{\partial\,\mathrm{ave}[\varphi,f](t)}{\partial t} > \frac{1}{t\log t}\Big[& |\cos_p(\mathrm{ave}[\varphi,f](t))|^p\,\mathrm{ave}[r,f](t) \\ & - \Phi(\cos_p(\mathrm{ave}[\varphi,f](t)))\sin_p(\mathrm{ave}[\varphi,f](t)) \\ & + \frac{1}{p-1}|\sin_p(\mathrm{ave}[\varphi,f](t))|^p\,\mathrm{ave}[s,f](t) - \delta\Big] \end{aligned} \tag{4.6}$$

for all sufficiently large t. Applying (2.1), one can rewrite (4.6) as

$$\begin{aligned} \frac{\partial\,\mathrm{ave}[\varphi,f](t)}{\partial t} > \frac{1}{t\log t}\Big[& |\cos_p(\mathrm{ave}[\varphi,f](t))|^p\,(\mathrm{ave}[r,f](t) - \delta) \\ & - \Phi(\cos_p(\mathrm{ave}[\varphi,f](t)))\sin_p(\mathrm{ave}[\varphi,f](t)) \\ & + \frac{|\sin_p(\mathrm{ave}[\varphi,f](t))|^p\,(\mathrm{ave}[s,f](t) - \delta(p-1))}{p-1}\Big] \end{aligned} \tag{4.7}$$

for all sufficiently large t. Considering (4.1) and (4.2), (4.7) gives

$$\begin{aligned} \frac{\partial\,\mathrm{ave}[\varphi,f](t)}{\partial t} > \frac{1}{t\log t}\Big[& |\cos_p(\mathrm{ave}[\varphi,f](t))|^p\Big(r^f_{\mathrm{liminf}} - 2\delta\Big) \\ & - \Phi(\cos_p(\mathrm{ave}[\varphi,f](t)))\sin_p(\mathrm{ave}[\varphi,f](t)) \\ & + \frac{1}{p-1}|\sin_p(\mathrm{ave}[\varphi,f](t))|^p\Big(s^f_{\mathrm{liminf}} - 2\delta(p-1)\Big)\Big] \end{aligned} \tag{4.8}$$

for all sufficiently large t. Let us consider the equation

$$\begin{aligned}\theta'(t) = \frac{1}{t\log t}\Big[\,&|\cos_p\theta(t)|^p\left(r^f_{\text{liminf}} - 2\delta\right) - \Phi\left(\cos_p\theta(t)\right)\sin_p\theta(t)\\&+ \frac{1}{p-1}\,|\sin_p\theta(t)|^p\left(s^f_{\text{liminf}} - 2\delta(p-1)\right)\Big]\end{aligned} \tag{4.9}$$

which has the form of (3.26) for

$$A = r^f_{\text{liminf}} - 2\delta, \quad B = s^f_{\text{liminf}} - 2\delta(p-1). \tag{4.10}$$

Considering (4.5) and (4.10), we obtain (3.25). Thus, one can apply Lemma 3.6 which says that (see (3.27))

$$\lim_{t\to\infty}\theta(t) = \infty. \tag{4.11}$$

Comparing (4.8) with (4.9), from (4.11), we have

$$\infty \ge \limsup_{t\to\infty}\operatorname{ave}[\varphi, f](t) \ge \liminf_{t\to\infty}\operatorname{ave}[\varphi, f](t) \ge \lim_{t\to\infty}\theta(t) = \infty,$$

i.e., (4.4) is valid. The proof is complete. □

Remark 4.2. We conjecture that Theorem 4.1 cannot be substantially improved. This conjecture follows from previous results about perturbed equations in [25, 29]. In particular, if we consider (2.7) with constant coefficients r and s such that $r^{p-1}s = q^{-p}$, then (2.7) is non-oscillatory, see [10, p. 43]. See also a more general result in [27].

Remark 4.3. Now, we comment the connection between Theorem 4.1 and the basic motivations explicitly mentioned in Introduction, i.e., Theorems 1.1, 1.2, and 1.3.

Theorem 4.1 is a modification of Theorem 1.3 in the case which is not solved in Theorem 1.3. In fact, the assumptions of Theorem 4.1 are more general than the ones in the statement of Theorem 1.3 (it suffices to compare (3.1), (3.2), and (3.3) with (1.7) and (3.4) with $g \ge 1$ in Theorem 1.3).

To comment the connection between Theorem 4.1 and Theorem 1.1, we discuss the conditions on f, g as well. In Theorem 1.1, it is considered $f \equiv \alpha$ and $g(t) = \sqrt{t\log t}$. Since the method of the proof of Theorem 1.1 differs from the process used in the proof of Theorem 4.1, the condition (1.3) is more limited than (3.7) and (3.8) and, at the same time, (1.4) and (1.5) are more limited than (4.3) (see also (4.1) and (4.2) and consider unbounded f). Nevertheless, (3.2) is not true for $f \equiv \alpha$ and $g(t) = \sqrt{t\log t}$. We add that (3.1) is valid for $f \equiv \alpha$ and any g and that (3.3) and (3.4) are valid for $f \equiv \alpha$ and $g(t) = \sqrt{t\log t}$.

Theorem 1.2 does not cover the case, when r is unbounded, which is applicable in Theorem 4.1. Theorem 1.2 is based on the existence of the α-averaging values of coefficients. Nevertheless, Theorem 4.1 can be used, when any α-averaging value of the coefficient s is not possible to estimate (see also the end of Example 4.7 below).

To illustrate the novelty of Theorem 4.1, we give two simple corollaries and a trivial example which are not covered by any previously known oscillation result. For reader's convenience, in the corollaries below, we recall all conditions as in the statement of Theorem 4.1.

Corollary 4.4. *Let f be a positive and continuously differentiable function and let g be a positive and continuous function which are defined for all large t and for* *which the conditions* (3.1)*,* (3.2)*,* (3.3)*, and* (3.4) *are satisfied. Let us consider the equation*

$$\Big(\frac{t}{r(t)}x'\Big)' + \frac{s(t)}{t\log^2 t}\,x = 0, \tag{4.12}$$

where $r > 0$ *and* s *are continuous functions satisfying* (3.7) *and* (3.8)*. Let the values* $r^f_{\rm liminf}$ *and* $s^f_{\rm liminf}$*, which are defined in* (4.1) *and* (4.2)*, respectively, be finite. If*

$$4r^f_{\rm liminf}s^f_{\rm liminf} > 1, \tag{4.13}$$

then (4.12) *is oscillatory.*

The above corollary follows from Theorem 4.1 for $p = 2$. Corollary 4.4 gives new results in many cases. In particular, we have the following new result.

Corollary 4.5. *Let* $a > 1$*. Let us consider the equation*

$$(tx')' + \frac{s(t)}{t\log^2 t}\,x = 0, \tag{4.14}$$

where s *is a non-negative, continuous, and bounded function. If*

$$4\liminf_{t\to\infty}\frac{\log_a t}{t}\int_t^{t+\frac{t}{\log_a t}} s(\tau)\,\mathrm{d}\tau > 1, \tag{4.15}$$

then (4.14) *is oscillatory.*

Proof. It suffices to consider Corollary 4.4 for $r \equiv g \equiv 1$ and $f(t) = t/\log_a t$. Especially, (4.13) reduces to (4.15). Note that (3.8), i.e.,

$$\limsup_{t\to\infty}\frac{\int_t^{t+f(t)}|s(\tau)|\,\mathrm{d}\tau}{f(t)} = \limsup_{t\to\infty}\frac{\int_t^{t+f(t)}s(\tau)\,\mathrm{d}\tau}{f(t)} < \infty, \tag{4.16}$$

follows from the boundedness of s and that

$$f'(t) = \frac{\log_a t - \frac{1}{\log a}}{\log_a^2 t}$$

for all large t gives (3.1) ((3.2), (3.3), (3.4), and (3.7) are evident). □

Remark 4.6. In fact, in the statement of Corollary 4.5, for any non-negative and continuous function s, the validity of (4.16) can be assumed based on the famous Sturm half-linear (also called the Sturm–Picone) comparison theorem, see, e.g., [10, Theorem 1.2.4].

In the example below, we mention a simple equation which is not covered by any previous result.

Example 4.7. For $\gamma_1 > 1/4$, $\gamma_2 \in [-\gamma_1, -\gamma_1 + 1/4)$ and for all large t, we put

$$s(t) := \begin{cases} \gamma_1 + \gamma_2\,(t - 2^n)\,, & t \in [2^n, 2^n + 1)\,; \\ \gamma_1 + \gamma_2, & t \in \big[2^n + 1, 2^n + \frac{2^n}{n^2} - 1\big)\,; \\ \gamma_1 + \gamma_2\,\big(2^n + \frac{2^n}{n^2} - t\big)\,, & t \in \big[2^n + \frac{2^n}{n^2} - 1, 2^n + \frac{2^n}{n^2}\big)\,; \\ \gamma_1, & t \in \big[2^n + \frac{2^n}{n^2}, 2^{n+1}\big)\,, \end{cases}$$

where $n \in \mathbb{N}$ is large. For this function, let us consider (4.14). We use Corollary 4.5 for $a = 2$. Since

$$\gamma_1 \geq \limsup_{t\to\infty}\frac{\log_2 t}{t}\int_t^{t+\frac{t}{\log_2 t}} s(\tau)\,\mathrm{d}\tau$$

$$
\begin{aligned}
&\geq \liminf_{t\to\infty} \frac{\log_2 t}{t} \int_t^{t+\frac{t}{\log_2 t}} s(\tau)\,\mathrm{d}\tau \\
&= \liminf_{n\to\infty} \frac{\log_2 2^n}{2^n} \int_{2^n}^{2^n+\frac{2^n}{\log_2 2^n}} s(\tau)\,\mathrm{d}\tau \\
&= \liminf_{n\to\infty} \frac{n}{2^n} \int_{2^n}^{2^n\left(1+\frac{1}{n}\right)} s(\tau)\,\mathrm{d}\tau \\
&\geq \liminf_{n\to\infty} \frac{n}{2^n} \Big(\int_{2^n}^{2^n\left(1+\frac{1}{n^2}\right)} (\gamma_1+\gamma_2)\,\mathrm{d}\tau + \int_{2^n\left(1+\frac{1}{n^2}\right)}^{2^n\left(1+\frac{1}{n}\right)} \gamma_1\,\mathrm{d}\tau \Big) \\
&= \lim_{n\to\infty} \frac{n}{2^n}\cdot\frac{2^n}{n^2}\,(\gamma_1+\gamma_2) + \lim_{n\to\infty} \frac{n}{2^n}\cdot 2^n\Big(\frac{1}{n}-\frac{1}{n^2}\Big)\gamma_1 = \gamma_1,
\end{aligned}
$$

we have

$$
\lim_{t\to\infty} \frac{\log_2 t}{t} \int_t^{t+\frac{t}{\log_2 t}} s(\tau)\,\mathrm{d}\tau = \gamma_1 > \frac{1}{4}.
$$

Hence, see (4.15), Corollary 4.5 implies the oscillation of the considered equation. Evidently,

$$
\liminf_{t\to\infty} \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau = \gamma_1+\gamma_2 < \frac{1}{4}
$$

for any $\alpha > 0$, which means that one cannot use Theorems 1.1 and 1.2. We remark that the oscillation of the considered equation is known for $\gamma_2 \geq -\gamma_1 + 1/4$, consider, e.g., [10, Theorem 1.2.4 and p. 43].

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