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EXISTENCE OF SOLUTIONS FOR A PROBLEM WITH MULTIPLE SINGULAR WEIGHTED p-LAPLACIANS AND VANISHING POTENTIALS

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Abstract. This work establishes the existence of positive solutions to a quasilinear singular elliptic equations involving the (p-q)-Laplacian operator with singularities and a vanishing potential. We adapt the penalization method developed by del Pino and Felmer and we consider an auxiliary problem whose corresponding functional satisfies the geometry of the mountain-pass theorem; then, we prove that the Palais-Smale sequences are bounded in a Sobolev space; after that, we show that the auxiliary problem has a solution. Finally, we use the Moser iteration scheme to obtain an appropriate estimate and we conclude that the solution to the auxiliary problem is also a solution to the original problem.

1. Introduction and main result

In this paper we consider the quasilinear elliptic equation involving the (p-q)-Laplacian operator with singularities

$$\begin{gathered}
-\operatorname{div}\Big(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\Big)-\operatorname{div}\Big(\frac{|\nabla u|^{q-2}\nabla u}{|x|^{cq}}\Big)\\
+\frac{P(x)|u|^{p-2}u}{|x|^{ap^*(a,b)}}+\frac{Q(x)|u|^{q-2}u}{|x|^{cq^*(c,d)}}=f(u), \\
u(x)>0,\quad u\in D^{1,p}_{a,b}(\mathbb{R}^N)\cap D^{1,q}_{c,d}(\mathbb{R}^N),
\end{gathered} \tag{1.1}$$

where $x\in\mathbb{R}^N$, $N\geqslant 3$, $2\leqslant q\leqslant p<N$, $a<(N-p)/p$, $a\leqslant b<a+1$, and $c<(N-q)/q$, $c\leqslant d<c+1$, and the critical exponents are defined by $p^*(a,b)\coloneqq Np/[N-p(a+1-b)]$ and $q^*(c,d)\coloneqq Nq/[N-q(c+1-d)]$. The Sobolev spaces $D^{1,p}_{a,b}(\mathbb{R}^N)$ and $D^{1,q}_{c,d}(\mathbb{R}^N)$ are defined as the completion of the function space $C_0^\infty(\mathbb{R}^N)$ with respect to the norms

$$\begin{gathered}
\|u\|_{D^{1,p}_{a,b}(\mathbb{R}^N)}\coloneqq\Big(\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{P(x)|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\Big)^{1/p},\\
\|u\|_{D^{1,q}_{c,d}(\mathbb{R}^N)}\coloneqq\Big(\int_{\mathbb{R}^N}\frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{Q(x)|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big)^{1/q}.
\end{gathered}$$

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The nonlinearity $f\colon \mathbb{R}\to\mathbb{R}$ is a continuous and nonnegative function that is not a pure power and can be subcritical at infinity and supercritical at the origin. More precisely,

- (H1) $\limsup_{s\to 0^+} \frac{|x|^{bp^*(a,b)}sf(s)}{s^{p^*(a,b)}} < +\infty$.
- (H2) There exists $\tau\in(p,p^*(a,b))$ such that $\limsup_{s\to+\infty}\frac{|x|^{bp^*(a,b)}sf(s)}{s^\tau}=0$.
- (H3) There exists $\theta>p$ such that $0\leqslant \theta F(s)\leqslant sf(s)$ for every $s\in\mathbb{R}^+$, where we use the notation $F(s):=\int_0^s f(t)\,\mathrm{d}t$.
- (H4) $f(t)=0$ for every $t\leqslant 0$.

The following properties are easily seen: under hypothesis (H1) there exists $c_1\in\mathbb{R}^+$ such that $|sf(s)|\leqslant c_1|s|^{p^*(a,b)}/|x|^{bp^*(a,b)}$ for s close to zero; and under hypothesis (H2) there exists $c_2\in\mathbb{R}^+$ such that $|sf(s)|\leqslant c_2|s|^\tau/|x|^{bp^*(a,b)}$ for s large enough. Combining these results and defining $c_0:=\max\{c_1,c_2\}$, we have the pair of inequalities

$$|sf(s)|\leqslant c_0\frac{|s|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}\quad\text{and}\quad |sf(s)|\leqslant c_0\frac{|s|^\tau}{|x|^{bp^*(a,b)}}\quad (s\in\mathbb{R}). \tag{1.2}$$

Hypothesis (H3) extends a well known Ambrosetti-Rabinowitz condition and is used to show that the energy functional satisfies the Palais-Smale condition. Recall that a functional $J\colon D^{1,p}_{a,b}(\mathbb{R}^N)\cap D^{1,q}_{c,d}(\mathbb{R}^N)\to\mathbb{R}$ is said to verify the Palais-Smale condition at the level α if any sequence $(u_n)_{n\in\mathbb{N}}\subset D^{1,p}_{a,b}(\mathbb{R}^N)\cap D^{1,q}_{c,d}(\mathbb{R}^N)$ such that $J(u_n)\to\alpha$ and $J'(u_n)\to 0$, as $n\to+\infty$, possess a convergent subsequence. Hypothesis (H3) also allows us to study the asymptotic behavior of the solution to problem (1.1).

As an example of a nonlinearity f satisfying the above set of hypotheses, for $\sigma>p^*(a,b)$ and for $\tau\in(p,p^*(a,b))$ assumed in hypothesis (H2), we define

$$f(t)=\begin{cases} t^{\sigma-1}, & \text{if } 0\leqslant t\leqslant 1;\\ t^{\tau-1}, & \text{if } 1\leqslant t.\end{cases}$$

We also assume that the functions $P,Q\colon\mathbb{R}^N\to\mathbb{R}$ are continuous and nonnegative. Moreover, the following set of hypotheses on the potential functions P and Q is used.

- (H5) $P\in L^{N/p(a+1-b)}_{ap^*(a,b)-bp}(\mathbb{R}^N)$ and $Q\in L^{N/q(c+1-d)}_{cq^*(c,d)-dq}(\mathbb{R}^N)$
- (H6) $P(x)\leqslant P_\infty$ and $Q(x)\leqslant Q_\infty$ for every $x\in B_1(0)$, where $P_\infty,Q_\infty\in\mathbb{R}^+$ are positive constants and $B_1(0)$ denotes the unitary ball centered at the origin.
- (H7) There exist constants $\Lambda\in\mathbb{R}^+$ and $R_0>1$ such that
$$\frac{1}{R_0^{p^2(a+1-b)/(p-1)}}\inf_{|x|\geqslant R_0}|x|^{p^2(a+1-b)/(p-1)}P(x)\geqslant\Lambda.$$

As an example of a potential function P satisfying this set of hypotheses, for $\Lambda\in\mathbb{R}^+$ and $R_0>1$ assumed in hypothesis (H6), we define

$$P(x)=\begin{cases} 0, & \text{if } |x|\leqslant R_0-1;\\ \frac{\Lambda(|x|-R_0+1)}{R_0^{p^2(a+1-b)/(p-1)}}, & \text{if } R_0-1<|x|<R_0;\\ \frac{\Lambda}{|x|^{p^2(a+1-b)/(p-1)}}, & \text{if } R_0\leqslant|x|.\end{cases}$$

An example of a potential function Q can be obtained in a similar way with minor modifications.

This class of (p-q)-Laplacian operator generalizes several types of problems; see Alves, Assunção and Miyagaki [1], Figueiredo [12], and Figueiredo and Nascimento [13] and references therein for more information.

In problem (1.1) we allow the vanishig of the potential functions at infinity, that is, the particular conditions $\liminf_{|x|\to+\infty} P(x) = 0$ and $\liminf_{|x|\to+\infty} Q(x) = 0$, called the zero mass cases. The presence of the singularities, both on the differential operators and on the nonlinearities, and the use of vanishing potentials constitute the main features of our work.

Theorem 1.1. *Consider $a < (N-p)/p$, $a \leqslant b < a+1$, $c < (N-q)/q$, $c \leqslant d < c+1$, $2 \leqslant q \leqslant p < N$ with $N \geqslant 3$, and suppose that the potential functions P and Q verify the hypotheses (H5)–(H7), and that the nonlinearity f satisfies the hypotheses (H1)–(H4). Then there exists a constant $\Lambda^* = \Lambda^*(P_\infty, Q_\infty, \theta, \tau, c_0)$ such that problem* (1.1) *has a positive solution for every $\Lambda \geqslant \Lambda^*$.*

The main difficulty in proving the existence of solution to problem (1.1) resides in the fact that the embedding of the Sobolev space $D^{1,p}_{a,b}(\mathbb{R}^N)$ in the Lebesgue space $L^{p^*(a,b)}(\mathbb{R}^N)$ is not compact due to the action of a group of homoteties and translations. Besides, the Palais-Smale condition for the corresponding energy functional cannot be obtained directly. Adding to these difficulties, we have to consider the presence a sum of two differential operators and when $q < p$ the study of problem (1.1) does not allow the use of the Lagrange's multipliers method due to the lack of homogeneity; moreover, the first eigenvalue of one operator brings no valuable information on the eigenvalue of the other one; finally, the method of sub- and super-solutions cannot be applied. Therefore, to study problem (1.1) we are required to make a careful analysis of the energy level of the Palais-Smale sequences in order to obtain their boundedness and also to overcome the lack of compactness. Furthermore, we have to adapt the Moser iteration scheme to our setting, since this is a crucial step to obtain an estimate for the solution.

Inspired mainly by Wu and Yang [19], Benouhiba and Belyacine [7], and Alves, Assunção, and Miyagaki [1] regarding the (p,q)-Laplacian type operator, by Bastos, Miyagaki and Vieira [5] regarding the singularity in the operators, and by Alves and Souto [2], with respect to the set of hypotheses, we adapt the penalization method developed by del Pino and Felmer [10] to show our existence result. The basic idea can be described in the following way. In section 2 we consider an auxiliary problem and study its corresponding energy functional, showing that it satisfies the geometry of the mountain-pass lemma and that every Palais-Smale sequence is bounded in an appropriate Sobolev space. Using the standard theory, this implies that the auxiliary problem has a solution. In section 3 we show, using the Moser iteration scheme, that the solution of the auxiliary problem satisfies an estimate involving the $L^\infty(\mathbb{R}^N)$ norm. Finally, in section 4 we use this estimate to show that the solution of the auxiliary problem is also a solution of the original problem (1.1).

2. An auxiliary problem

To prove the existence of a positive solution to problem (1.1) we establish a variational setting and apply the mountain-pass lemma. Using hypothesis (H5) we define the space

$$
\begin{aligned}
E \coloneqq \Big\{u \in D^{1,p}_{a,b}(\mathbb{R}^N) \cap D^{1,q}_{c,d}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{P(x)|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x < +\infty \\
\text{and } \int_{\mathbb{R}^N} \frac{Q(x)|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x < +\infty\Big\},
\end{aligned}
$$

which can be endowed with the norm $\|u\| = \|u\|_{1,p} + \|u\|_{1,q}$, where

$$
\begin{aligned}
\|u\|_{1,p} &\coloneqq \Big(\int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{P(x)|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\Big)^{1/p}, \\
\|u\|_{1,q} &\coloneqq \Big(\int_{\mathbb{R}^N} \frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{Q(x)|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big)^{1/q}.
\end{aligned}
$$

The Euler-Lagrange energy functional $I\colon E \to \mathbb{R}$ associated with problem (1.1) is defined by

$$
\begin{aligned}
I(u) \coloneqq{}& \frac{1}{p}\int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} \frac{P(x)|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\
&+ \frac{1}{q}\int_{\mathbb{R}^N} \frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} \frac{Q(x)|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x.
\end{aligned}
$$

Using the hypotheses on the nonlinearity f we can deduce that $I \in C^1(E;\mathbb{R})$; moreover, for every $u, v \in E$ its Gâteaux derivative can be computed by

$$
\begin{aligned}
I'(u)v ={}& \int_{\mathbb{R}^N} \frac{|\nabla u|^{p-2}}{|x|^{ap}}\,\nabla u \cdot \nabla v\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{P(x)|u|^{p-2}uv}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^N} \frac{|\nabla u|^{q-2}}{|x|^{cq}}\,\nabla u \cdot \nabla v\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{Q(x)|u|^{q-2}uv}{|x|^{cq^*(c,d)}}\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u)v\,\mathrm{d}x.
\end{aligned}
$$

It is a well known fact that if u is a critical point of the energy functional I, then u is a weak solution to problem (1.1). This means that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|\nabla u|^{p-2}}{|x|^{ap}}\nabla u \cdot \nabla \phi\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{P(x)|u|^{p-2}u\phi}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^N} \frac{|\nabla u|^{q-2}}{|x|^{cq}}\nabla u \cdot \nabla \phi\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{Q(x)|u|^{q-2}u\phi}{|x|^{cq^*(c,d)}}\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u)\phi\,\mathrm{d}x = 0
\end{aligned}
$$

for every $v \in E$.

Now we define $D^{1,p}_0(B_1(0)) \coloneqq D^{1,p}_{0,a,b}(B_1(0))$ as the completion of the space $C^\infty_0(B_1(0))$ with respect to the norm $\|u\|_{D^{1,p}_{a,b}}(B_1(0))$; we also define $D^{1,q}_0(B_1(0)) \coloneqq D^{1,q}_{0,c,d}(B_1(0))$ as the completion of the space $C^\infty_0(B_1(0))$ with respect to the norm $\|u\|_{D^{1,q}_{c,d}}(B_1(0))$. In the intersection of these Sobolev spaces we define an auxiliary energy functional $I_\infty\colon D^{1,p}_0(B_1(0)) \cap D^{1,q}_0(B_1(0)) \to \mathbb{R}$ by

$$
\begin{aligned}
I_\infty(u) \coloneqq{}& \frac{1}{p}\int_{B_1(0)} \frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x + \frac{1}{p}\int_{B_1(0)} \frac{P_\infty|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\
&+ \frac{1}{q}\int_{B_1(0)} \frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x + \frac{1}{q}\int_{B_1(0)} \frac{Q_\infty|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x - \int_{B_1(0)} F(u)\,\mathrm{d}x.
\end{aligned}
$$

Using the hypotheses (H5) and (H6) it can be shown that this functional is well defined. Our first lemma concerns the geometry of this functional.

Lemma 2.1. *The functional I_∞ satisfies the geometry of the mountain-pass lemma. More precisely, the following claims are valid.*

(1) *There exist $r_0, \mu_0 \in \mathbb{R}^+$ such that $I_\infty(u) \geqslant \mu_0$ for $\|u\| = r_0$.*
(2) *There exists $e_0 \in \left[D_0^{1,p}(B_1(0)) \cap D_0^{1,q}(B_1(0))\right] \backslash \{0\}$ such that $\|e_0\| \geqslant r_0$ and $I_\infty(e_0) < 0$.*

Proof. By using hypotheses (H1)–(H3), it is standard to verify item (1).

By hypothesis (H3) it follows that there exist $\theta > p$ and $C_0 \in \mathbb{R}^+$ such that $F(s) \geqslant C_0|s|^\theta$. Now, if $u \in \left[D_0^{1,p}(B_1(0)) \cap D_0^{1,q}(B_1(0))\right] \backslash \{0\}$, then

$$\begin{aligned} I_\infty(tu) \leqslant{}& \frac{|t|^p}{p} \int_{B_1(0)} \frac{|\nabla u|^p}{|x|^{ap}} \,\mathrm{d}x + \frac{P_\infty |t|^p}{p} \int_{B_1(0)} \frac{|u|^p}{|x|^{ap^*(a,b)}} \,\mathrm{d}x \\ &+ \frac{|t|^q}{q} \int_{B_1(0)} \frac{|\nabla u|^q}{|x|^{cq}} \,\mathrm{d}x + \frac{Q_\infty |t|^q}{q} \int_{B_1(0)} \frac{|u|^q}{|x|^{cq^*(c,d)}} \,\mathrm{d}x \\ &- C_0 |t|^\theta \int_{B_1(0)} |u|^\theta \,\mathrm{d}x. \end{aligned}$$

Using this inequality we deduce that there exist $t_u \in \mathbb{R}^+$ large enough such that, taking $e_0 = t_u u$, we have $\|e_0\| \geqslant r_0$ and $I_\infty(e_0) < 0$. This concludes the proof of item (2). $\square$

We denote by d_∞ the mountain-pass level associated with the functional I_∞, that is,

$$d_\infty \coloneqq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\infty(\gamma(t)),$$

where

$$\Gamma \coloneqq \left\{\gamma \in C([0,1]; D_0^{1,p}(B_1(0)) \cap D_0^{1,q}(B_1(0)))\colon \gamma(0) = 0 \text{ and } \gamma(1) = e_0\right\}$$

and the function $e_0 \in [D_0^{1,p}(B_1(0)) \cap D_0^{1,q}(B_1(0))] \backslash \{0\}$ is given in Lemma 2.1. It is standard to verify that the mountain-pass level $d_\infty = d_\infty(P_\infty, Q_\infty, \theta, f)$.

For $R > 1$ and for $\theta > p$ given in hypothesis (H3), we set $k \coloneqq \theta p/(\theta - p) > p$ and we define a new nonlinearity $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$g(x,t) \coloneqq \begin{cases} f(t), & \text{if } |x| \leqslant R \\ & \text{or if } |x| > R \text{ and } f(t) \leqslant \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}; \\ \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}, & \text{if } |x| > R \text{ and } f(t) > \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}. \end{cases}$$

Using the notation $G(x,t) \coloneqq \int_0^t g(x,s) \,\mathrm{d}s$, by direct computations we obtain the set of inequalities

$$g(x,t) \leqslant \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}, \quad \text{for all } |x| \geqslant R; \tag{2.1}$$

$$G(x,t) = F(t), \quad \text{if } |x| \leqslant R; \tag{2.2}$$

$$G(x,t) \leqslant \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}, \quad \text{if } |x| > R > 1. \tag{2.3}$$

Now we define the auxiliary problem

$$\begin{gathered} -\operatorname{div}\Big(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\Big)-\operatorname{div}\Big(\frac{|\nabla u|^{q-2}\nabla u}{|x|^{cq}}\Big) \\ +\frac{P(x)|u|^{p-2}u}{|x|^{ap^*(a,b)}}+\frac{Q(x)|u|^{q-2}u}{|x|^{cq^*(c,d)}}=g(x,u), \\ u(x)>0, \quad u\in D^{1,p}_{a,b}(\mathbb{R}^N)\cap D^{1,q}_{c,d}(\mathbb{R}^N), \end{gathered} \tag{2.4}$$

The Euler-Lagrange energy functional $J\colon E\to\mathbb{R}$ associated with the auxiliary problem (2.4) is

$$\begin{aligned} J(u)&:=\frac{1}{p}\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^N}\frac{P(x)|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\ &\quad+\frac{1}{q}\int_{\mathbb{R}^N}\frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}\frac{Q(x)|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x-\int_{\mathbb{R}^N}G(x,u)\,\mathrm{d}x. \end{aligned}$$

Using the hypotheses on the nonlinearity f and on the potential functions P and Q, we can show that $J\in C^1(E;\mathbb{R})$; moreover, for every $u,v\in E$ its Gâteaux derivative can be computed as

$$\begin{aligned} J'(u)v&=\int_{\mathbb{R}^N}\frac{|\nabla u|^{p-2}}{|x|^{ap}}\nabla u\cdot\nabla v\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{P(x)|u|^{p-2}uv}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\ &\quad+\int_{\mathbb{R}^N}\frac{|\nabla u|^{q-2}}{|x|^{cq}}\nabla u\cdot\nabla v\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{Q(x)|u|^{q-2}uv}{|x|^{cq^*(c,d)}}\,\mathrm{d}x-\int_{\mathbb{R}^N}g(x,u)v\,\mathrm{d}x. \end{aligned}$$

As before, critical points of the energy functional J are weak solutions to problem (2.4).

Our next goal is to apply the mountain-pass lemma to show that problem (2.4) has a positive solution.

Lemma 2.2. *The functional J satisfies the geometry of the mountain-pass lemma. More precisely, the following claims are valid.*

1. *There exist $r_1,\ \mu_1\in\mathbb{R}^+$ such that $J(u)\geqslant\mu_1$ for $\|u\|=r_1$.*
2. *There exists $e_1\in\big[D^{1,p}_0(B_1(0))\cap D^{1,q}_0(B_1(0))\big]\backslash\{0\}$ such that $\|e_1\|\geqslant r_1$ and $J(e_1)<0$.*

Proof. Using (2.2) and inequality (2.3), together with hypotheses (H1) and (H3) and the first inequality in (1.2), we obtain

$$\begin{aligned} J(u)&\geqslant\frac{1}{p}\|u\|^p_{1,p}+\frac{1}{q}\|u\|^q_{1,q}-\int_{|x|\leqslant R}F(u)\,\mathrm{d}x-\int_{|x|>R}\frac{P(x)|u|^{p-2}u}{k|x|^{ap^*(a,b)}}\,\mathrm{d}x \\ &\geqslant\frac{1}{p}\|u\|^p_{1,p}+\frac{1}{q}\|u\|^q_{1,q}-\frac{c_0}{\theta}\int_{\mathbb{R}^N}\frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}\,\mathrm{d}x-\frac{1}{kp}\|u\|^p_{1,p} \\ &=\Big(\frac{1}{p}-\frac{1}{kp}\Big)\|u\|^p_{1,p}+\frac{1}{q}\|u\|^q_{1,q}-\frac{c_0}{\theta}|u|^{p^*(a,b)}_{L_b^{p^*(a,b)}}. \end{aligned}$$

Recall the Caffarelli-Kohn-Nirenberg inequality

$$|u|^p_{L_b^{p^*(a,b)}(\mathbb{R}^N)}\leqslant S_{a,b,p}\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x,\quad\text{for all } u\in D^{1,p}_{a,b}(\mathbb{R}^N); \tag{2.5}$$

$$|u|^q_{L_d^{q^*(c,d)}(\mathbb{R}^N)}\leqslant S_{c,d,q}\int_{\mathbb{R}^N}\frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x,\quad\text{for all } u\in D^{1,q}_{c,d}(\mathbb{R}^N). \tag{2.6}$$

Setting $S \equiv \max\{S_{a,b,p}, S_{c,d,q}\}$ in the computations above, we obtain

$$\begin{aligned} J(u) &\geqslant \Big(\frac{1}{p} - \frac{1}{kp}\Big)\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q - \frac{c_0}{\theta} S^{p^*(a,b)/p}\Big(\int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{p^*(a,b)/p} \\ &\geqslant \min\Big\{\frac{1}{p} - \frac{1}{kp}, \frac{1}{q}\Big\}\Big(\|u\|_{1,p}^p + \|u\|_{1,q}^q\Big) \\ &\quad - \frac{c_0}{\theta} S^{p^*(a,b)/p}\Big(\|u\|_{1,p}^p + \|u\|_{1,q}^q\Big)^{p^*(a,b)/p}. \end{aligned}$$

If we take $\|u\|_{1,p}$ and $\|u\|_{1,q}$ small enough, it follows that $\|u\|_{1,p}^p$ and $\|u\|_{1,q}^q$ are also small enough. For this reason, we obtain the existence of r_1, $\mu_1 \in \mathbb{R}^+$ such that $J(u) \geqslant \mu_1$ for $\|u\| = r_1$. This concludes the proof of item (1).

By definition, $G(x,u) = F(u)$ for all $u \in \big[D_{a,b}^{1,p}(B_1(0)) \cap D_{c,d}^{1,q}(B_1(0))\big]\backslash\{0\}$. Arguing as in the proof of Lemma 2.1 we conclude that there exist $r_1, t_u \in \mathbb{R}^+$ such that $e_1 := t_u u$ verify the inequalities $\|e_1\| \leqslant r_1$ and $J(e_1) < 0$. This concludes the proof of item (2). □

Since the functional J has the geometry of the mountain-pass lemma, using Willem [18, Theorem 1.15] we obtain a Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E$ such that $J(u_n) \to \alpha$ and $J'(u_n) \to 0$ as $n \to +\infty$. Here $\alpha \in \mathbb{R}^+$ is the mountain-pass level associated with the energy functional J, that is,

$$\alpha := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma := \big\{\gamma \in C([0,1]; D_0^{1,p}(B_1(0)) \cap D_0^{1,q}(B_1(0)) \colon \gamma(0) = 0 \text{ and } \gamma(1) = e_1\big\}$$

and $e_1 \in \big[D_{a,b}^{1,p}(B_1(0)) \cap D_{c,d}^{1,q}(B_1(0))\big]\backslash\{0\}$ is the same function satisfying inequality $J(e_1) < 0$ in Lemma 2.2. Using the hypothesis (H4), without loss of generality we can suppose that the sequence $(u_n)_{n\in\mathbb{N}} \subset E$ consists of nonnegative functions.

We note that for all $u \in \big[D_0^{1,p}(B_1(0)) \cap D_0^{1,q}(B_1(0))\big]\backslash\{0\}$ the inequality $J(u) \leqslant I_\infty(u)$ is valid, and this implies that

$$\alpha \leqslant d_\infty. \tag{2.7}$$

Now we show that the Palais-Smale sequences for the functional J are bounded.

Lemma 2.3. *Suppose that the potential functions P, Q verify hypothesis (H5), and that the nonlinearity f satisfies the hypotheses (H1)–(H4). If $(u_n)_{n\in\mathbb{N}} \subset E$ is a Palais-Smale sequence for the energy functional J, then the sequence $(u_n)_{n\in\mathbb{N}} \subset E$ is bounded.*

Proof. To get our thesis it is sufficient to prove that both sequences $(\|u_n\|_{1,q}^q)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}} \subset \mathbb{R}$ are bounded, which we do in the two claims below.

Before that, however, we remark that there exist constants $\alpha_1 > 0$ and $n_0 \in \mathbb{N}$ such that $J(u_n) \leqslant \alpha_1$ and $|J'(u_n u_n)| \leqslant \min\big\{\|u_n\|_{1,q}, \|u_n\|_{1,p}\big\}$ for all $n \in \mathbb{N}$ such that $n \geqslant n_0$; and since $\theta > p > 1$, for all $n \geqslant n_0$ we have

$$J(u_n) - \frac{1}{\theta} J'(u_n)u_n \leqslant \alpha_1 + \frac{1}{\theta}\|u_n\| \leqslant \alpha_1 + \min\big\{\|u_n\|_{1,q}, \|u_n\|_{1,p}\big\}. \tag{2.8}$$

Claim 2.4. *The sequence $(\|u_n\|_{1,q}^q)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded.*

Proof. We divide our analysis into cases that mirror the definition of the nonlinearity g. If $|x| > R$ and $f(t) > P(x)|t|^{p-2}t/k|x|^{ap^*(a,b)}$, then

$$\int_{\mathbb{R}^N} G(x,u_n)\,\mathrm{d}x = \frac{1}{p}\int_{\mathbb{R}^N} g(x,u_n)u_n\,\mathrm{d}x,$$

and this implies that

$$J(u_n) - \frac{1}{p}J'(u_n)u_n = \Big(\frac{1}{q} - \frac{1}{p}\Big)\|u_n\|_{1,q}^q. \tag{2.9}$$

Combining inequality (2.8) with equality (2.9), we conclude that

$$\Big(\frac{1}{q} - \frac{1}{p}\Big)\|u_n\|_{1,q}^q \leqslant \alpha_1 + \|u_n\|_{1,q}.$$

So, in this case the sequence $(\|u_n\|_{1,q}^q)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded, say $\|u_n\|_{1,q}^q \leqslant \alpha_q$ for every $n \in \mathbb{N}$.

If $|x| \leqslant R$ or if $|x| > R$ and $f(t) \leqslant P(x)|t|^{p-2}t/k|x|^{ap^*(a,b)}$, the boundedness of the sequence can be proved using the same ideas as that of the previous case with some minor changes. This concludes the proof of the first claim. □

Claim 2.5. *The sequence $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded.*

Proof of Claim 2.5. We divide our analysis into the same cases. If $|x| > R$ and $f(t) > P(x)|t|^{p-2}t/k|x|^{ap^*(a,b)}$, then we have

$$\begin{aligned}
&J(u_n) - \frac{1}{\theta}J'(u_n)u_n \\
&\geqslant \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|u_n\|_{1,p}^p + \Big(\frac{1}{q} - \frac{1}{\theta}\Big)\|u_n\|_{1,q}^q - \frac{1}{kp}\Big\{\int_{\mathbb{R}^N} \frac{P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\Big\} \\
&\geqslant \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\left\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\right\} - \frac{1}{kp}\left\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\right\} \\
&= \frac{(p-1)}{kp}\left\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\right\}.
\end{aligned} \tag{2.10}$$

Combining inequalities (2.8) and (2.10) and using Claim 2.4 we obtain

$$\frac{(p-1)}{kp}\|u_n\|_{1,p}^p \leqslant \alpha_1 + \|u_n\|_{1,p}.$$

This means that in this case the sequence $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded.

If $|x| \leqslant R$ or if $|x| > R$ and $f(t) \leqslant a(x)|t|^{p-2}t/k$, then

$$\int_{\mathbb{R}^N} G(x,u_n)\,\mathrm{d}x + \frac{1}{\theta}\int_{\mathbb{R}^N} g(x,u_n)u_n\,\mathrm{d}x \geqslant 0.$$

Hence,

$$
\begin{aligned}
& J(u_n) - \frac{1}{\theta} J'(u_n)u_n \\
& \geqslant \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|u_n\|_{1,p}^p + \Big(\frac{1}{q} - \frac{1}{\theta}\Big)\|u_n\|_{1,q}^q - \int_{\mathbb{R}^N} G(x,u_n)\,\mathrm{d}x \\
& \quad + \frac{1}{\theta}\int_{\mathbb{R}^N} g(x,u_n)u_n\,\mathrm{d}x \\
& \geqslant \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\left\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\right\} - \int_{\mathbb{R}^N} G(x,u_n)\,\mathrm{d}x \\
& \quad + \frac{1}{\theta}\int_{\mathbb{R}^N} g(x,u_n)u_n\,\mathrm{d}x \\
& \geqslant \frac{1}{k}\left\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\right\} \\
& \geqslant \frac{(p-1)}{kp}\left\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\right\}.
\end{aligned} \tag{2.11}
$$

Combining inequalities (2.8) and (2.11) we obtain

$$
\frac{1}{k}\|u_n\|_{1,p}^p \leqslant \alpha_1 + \|u_n\|_{1,p}.
$$

This means that also in this case the sequence $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded. This concludes the proof of the second claim. $\square$

The proof of the lemma follows immediately from Claims 2.4 and 2.5. $\square$

The following result shows that the functional J satisfies the Palais-Smale condition.

Lemma 2.6. *Suppose that the potential functions P, Q satisfy (H5)–(H7), and that the nonlinearity f satisfies the (H1)–(H4). Then the Palais-Smale condition is valid for the energy functional J.*

Proof. Let $(u_n)_{n\in N} \subset E$ be a Palais-Smale sequence at the level α. This means that $J(u_n) \to \alpha$ and $J'(u_n) \to 0$ as $n \to \infty$, and by Lemma 2.3 this sequence is bounded. Then there exist a subsequence of $(u_n)_{n\in N} \subset E$, which we still denote in the same way, and there exists a function $u \in E$ such that $u_n \rightharpoonup u$ weakly in E as $n \to +\infty$.

Now we set $m := \max\{a+1-b, c+1-d\}$ and $m_1 := \max\{b-a, d-c\}$ and we denote the volume of the unitary ball by $|B_1(0)| = \omega_N$. For each $\epsilon > 0$, there exist $r > R > 1$ such that

$$
\begin{aligned}
& 2^{m_1+1}\omega_N^{m/N}\big(1 - \frac{1}{k}\big)^{-1}\Big\{\Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}\mathrm{d}x\Big)^{1/p^*(a,b)}\|u\|^{p-1} \\
& + \Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u|^{q^*(c,d)}}{|x|^{dq^*(c,d)}}\mathrm{d}x\Big)^{1/q^*(c,d)}\|u\|^{q-1}\Big\} < \epsilon.
\end{aligned} \tag{2.12}
$$

Let $\eta = \eta_r \in C^\infty(B_r^c(0))$ be a cut off function such that $0 \leqslant \eta \leqslant 1$, with $\eta = 1$ in $B_{2r}^c(0)$ and $|\nabla\eta| \leqslant 2/r^{m_1}$ for all $x \in \mathbb{R}^N$. Since the sequence $(u_n)_{n\in N} \subset E$ is bounded, it follows that the sequence $(\eta u_n)_{n\in N} \subset E$ is also bounded. Therefore,

$J'(u_n)(\eta u_n) = o_n(1)$, that is,

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{|\nabla u_n|^{p-2}}{|x|^{ap}} \nabla u_n \cdot \nabla(\eta u_n)\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{P(x)|u_n|^{p-2}u_n(\eta u_n)}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \frac{|\nabla u_n|^{q-2}}{|x|^{cq}} \nabla u_n \cdot \nabla(\eta u_n)\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{Q(x)|u|_n^{q-2}u_n(\eta u_n)}{|x|^{cq^*(c,d)}}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} g(x,u_n)(\eta u_n)\,\mathrm{d}x + o(1). \end{aligned} \tag{2.13}$$

This expression and the properties of the cut off function η imply that

$$\begin{aligned} &\int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{|\nabla u_n|^{p-2}u_n}{|x|^{ap}} \nabla u_n \cdot \nabla\eta\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{\eta P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\ &+ \int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{|\nabla u_n|^{q-2}u_n}{|x|^{cq}} \nabla u_n \cdot \nabla\eta\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{\eta\, Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x \\ &= \int_{|x|\geqslant r} \eta g(x,u_n)u_n\,\mathrm{d}x + o(1). \end{aligned}$$

From (2.1), it follows that

$$\int_{|x|\geqslant r} \eta g(x,u_n)u_n\,\mathrm{d}x \leqslant \int_{|x|\geqslant r} \frac{\eta P(x)|u_n|^p}{k|x|^{ap^*(a,b)}}\,\mathrm{d}x;$$

thus, we obtain

$$\begin{aligned} &\int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{\eta P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\ &+ \int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{\eta\, Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x - \int_{|x|\geqslant r} \frac{\eta P(x)|u_n|^p}{k|x|^{ap^*(a,b)}}\,\mathrm{d}x \\ &\leqslant \int_{|x|\geqslant r} \frac{|\nabla u_n|^{p-1}|u_n||\nabla\eta|}{|x|^{ap}}\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{|\nabla u_n|^{q-1}|u_n||\nabla\eta|}{|x|^{cq}}\,\mathrm{d}x + o(1) \\ &\leqslant \frac{2}{r^{m_1}}\Big\{ \int_{r\leqslant|x|\leqslant 2r} \frac{|\nabla u_n|^{p-1}|u_n|}{|x|^{ap}}\,\mathrm{d}x + \int_{r\leqslant|x|\leqslant 2r} \frac{|\nabla u_n|^{q-1}|u_n|}{|x|^{cq}}\,\mathrm{d}x\Big\} + o(1). \end{aligned}$$

Subtracting the terms

$$\frac{1}{k}\int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x + \frac{1}{k}\int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x + \frac{1}{k}\int_{|x|\geqslant r} \frac{\eta Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x$$

from the left-hand side of the previous inequality and grouping the several integrals, we deduce that

$$\begin{aligned} &(1-\frac{1}{k})\Big\{ \int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{\eta P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\ &+ \int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{\eta Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big\} \\ &\leqslant \frac{2}{r^{m_1}}\Big\{ \int_{r\leqslant|x|\leqslant 2r} \frac{|\nabla u_n|^{p-1}|u_n|}{|x|^{ap}}\,\mathrm{d}x + \int_{r\leqslant|x|\leqslant 2r} \frac{|\nabla u_n|^{q-1}|u_n|}{|x|^{cq}}\,\mathrm{d}x\Big\} + o(1). \end{aligned}$$

Now we use Hölder's inequality to obtain

$$\int_{r\leqslant|x|\leqslant 2r} \frac{|u_n||\nabla u_n|^{p-1}}{|x|^{ap}}\,\mathrm{d}x$$

$$
\begin{aligned}
&\leqslant \Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u_n|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{1/p}\Big\{\Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{1/p}\Big\}^{p-1}\\
&\leqslant \Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u_n|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{1/p}\|u_n\|_{1,p}^{p-1};
\end{aligned}
$$

and in a similar way, we obtain

$$
\int_{r\leqslant|x|\leqslant 2r} \frac{|u_n||\nabla u_n|^{q-1}}{|x|^{cq}}\,\mathrm{d}x \leqslant \Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u_n|^q}{|x|^{cq}}\,\mathrm{d}x\Big)^{1/q}\|u_n\|_{1,q}^{q-1}.
$$

By the compactness of the embedding $D_{a,b}^{1,p}(\overline{B}_{2r}\backslash B_r) \hookrightarrow L_a^p(\overline{B}_{2r}\backslash B_r)$, we infer that $u_n \to u$ strongly in $L_a^p(\overline{B}_{2r}\backslash B_r)$ as $n\to\infty$. Since $(\eta u_n)_{n\in\mathbb{N}} \subset D_{a,b}^{1,p}(\mathbb{R}^N)\cap D_{c,d}^{1,q}(\mathbb{R}^N)$, it follows that

$$
\begin{aligned}
&\limsup_{n\to\infty}\big(1-\frac{1}{k}\big)\Big\{\int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{\eta P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\
&+ \int_{|x|\geqslant r} \frac{\eta|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x + \int_{|x|\geqslant r} \frac{\eta Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big\}\\
&\leqslant \frac{2}{r^{m_1}}\limsup_{n\to\infty}\Big\{\Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u_n|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{1/p}\|u_n\|_{1,p}^{p-1}\\
&\quad + \Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u_n|^q}{|x|^{cq}}\,\mathrm{d}x\Big)^{1/q}\|u_n\|_{1,q}^{q-1}\Big\}\\
&= \frac{2}{r^{m_1}}\Big\{\Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{1/p}\|u\|_{1,p}^{p-1}\\
&\quad + \Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u|^q}{|x|^{cq}}\,\mathrm{d}x\Big)^{1/q}\|u\|_{1,q}^{q-1}\Big\}.
\end{aligned}
\tag{2.14}
$$

Applying Hölder's inequality once more, we obtain

$$
\begin{aligned}
\Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{1/p} &\leqslant \big(2^{N(b-a)/(a+1-b)}\omega_N r^{N(b-a)/(a+1-b)}\big)^{(a+1-b)/N}\\
&\quad\times \Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}\,\mathrm{d}x\Big)^{1/p^*(a,b)};
\end{aligned}
\tag{2.15}
$$

and in a similar way, we obtain

$$
\begin{aligned}
\Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u|^q}{|x|^{cq}}\,\mathrm{d}x\Big)^{1/q} &\leqslant \big(2^{N(d-c)/(c+1-d)}\omega_N r^{N(d-c)/(c+1-d)}\big)^{(c+1-d)/N}\\
&\quad\times \Big(\int_{r\leqslant|x|\leqslant 2r} \frac{|u|^{q^*(c,d)}}{|x|^{dq^*(c,d)}}\,\mathrm{d}x\Big)^{1/q^*(c,d)}.
\end{aligned}
\tag{2.16}
$$

Substituting inequalities (2.15) and (2.16) in (2.14) and recalling the definitions of m and m_1, we obtain

$$\begin{aligned} &\limsup_{n\to\infty}\big(1-\frac{1}{k}\big)\Big\{\int_{|x|\geqslant r}\frac{\eta|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x+\int_{|x|\geqslant r}\frac{\eta P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\ &+\int_{|x|\geqslant r}\frac{\eta|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x+\int_{|x|\geqslant r}\frac{\eta Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big\}\\ &\leqslant 2^{m_1+1}\omega_{N}^{m/N}\Big\{\Big(\int_{r\leqslant|x|\leqslant 2r}\frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}\,\mathrm{d}x\Big)^{1/p^*(a,b)}\|u\|^{p-1}\\ &\quad+\Big(\int_{r\leqslant|x|\leqslant 2r}\frac{|u|^{q^*(c,d)}}{|x|^{dq^*(c,d)}}\,\mathrm{d}x\Big)^{1/q^*(c,d)}\|u\|^{q-1}\Big\}. \end{aligned} \tag{2.17}$$

In particular, since $\eta=1$ outside the ball of radius $2r$, by inequalities (2.14) and (2.17) we obtain

$$\begin{aligned} &\limsup_{n\to\infty}\big(1-\frac{1}{k}\big)\Big\{\int_{|x|\geqslant 2r}\frac{|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x+\int_{|x|\geqslant 2r}\frac{P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\ &+\int_{|x|\geqslant 2r}\frac{|\nabla u_n|^q}{|x|^{cd}}\,\mathrm{d}x+\int_{|x|\geqslant 2r}\frac{Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big\}\\ &\leqslant 2^{m_1+1}\omega_{N}^{m/N}\Big\{\Big(\int_{r\leqslant|x|\leqslant 2r}\frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}\,\mathrm{d}x\Big)^{1/p^*}\|u\|^{p-1}\\ &\quad+\Big(\int_{r\leqslant|x|\leqslant 2r}\frac{|u|^{q^*(c,d)}}{|x|^{dq^*(c,d)}}\,\mathrm{d}x\Big)^{1/q^*(c,d)}\|u\|_{1,q}^{q-1}\Big\}. \end{aligned} \tag{2.18}$$

Therefore, by inequalities (2.12) and (2.18) it follows that

$$\begin{aligned} &\limsup_{n\to\infty}\Big\{\int_{|x|\geqslant 2r}\frac{|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x+\int_{|x|\geqslant 2r}\frac{P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\ &+\int_{|x|\geqslant 2r}\frac{|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x+\int_{|x|\geqslant 2r}\frac{Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big\}<\epsilon. \end{aligned} \tag{2.19}$$

Combining inequalities (2.13) and (2.19), we deduce that

$$\limsup_{n\to\infty}\int_{|x|\geqslant 2r}g(x,u_n)u_n\,\mathrm{d}x=0. \tag{2.20}$$

Now we use the dominated convergence theorem together with the fact that g has subcritical growth to infer that

$$\limsup_{n\to\infty}\int_{|x|\leqslant 2r}g(x,u_n)u_n\,\mathrm{d}x=\int_{|x|\leqslant 2r}g(x,u)u\,\mathrm{d}x; \tag{2.21}$$

and since $\int_{\mathbb{R}^N}g(x,u_n)u_n\,\mathrm{d}x<\infty$, by the choice of $r>R>1$ and from equalities (2.20) and (2.21), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}g(x,u_n)u_n\,\mathrm{d}x=\int_{\mathbb{R}^N}g(x,u)u\,\mathrm{d}x. \tag{2.22}$$

It remains to show that the norm sequence $(\|u_n\|)_{n\in\mathbb{N}}\subset\mathbb{R}$ is such that $\|u_n\|\to\|u\|\in\mathbb{R}$ as $n\to\infty$. Using Hölder's inequality and making some computations, it follows that

$$o(1)=\left(J'(u_n)-J'(u)\right)(u_n-u)$$

$$
\begin{aligned}
&\geqslant \Big\{\Big(\int_{\mathbb{R}^N}\frac{|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{(p-1)/p}-\Big(\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{(p-1)/p}\Big\}\\
&\quad\times\Big\{\Big(\int_{\mathbb{R}^N}\frac{|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{1/p}-\Big(\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{1/p}\Big\}\\
&\quad+\Big\{\Big(\int_{\mathbb{R}^N}\frac{P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\Big)^{(p-1)/p}-\Big(\int_{\mathbb{R}^N}\frac{P(x)|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\Big)^{(p-1)/p}\Big\}\\
&\quad\times\Big\{\Big(\int_{\mathbb{R}^N}\frac{P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\Big)^{1/p}-\Big(\int_{\mathbb{R}^N}\frac{P(x)|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\Big)^{1/p}\Big\}\\
&\quad+\Big\{\Big(\int_{\mathbb{R}^N}\frac{|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x\Big)^{(q-1)/q}-\Big(\int_{\mathbb{R}^N}\frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x\Big)^{(q-1)/q}\Big\}\\
&\quad\times\Big\{\Big(\int_{\mathbb{R}^N}\frac{|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x\Big)^{1/q}-\Big(\int_{\mathbb{R}^N}\frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x\Big)^{1/q}\Big\}\\
&\quad+\Big\{\Big(\int_{\mathbb{R}^N}\frac{Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big)^{(q-1)/q}-\Big(\int_{\mathbb{R}^N}\frac{Q(x)|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big)^{(q-1)/q}\Big\}\\
&\quad\times\Big\{\Big(\int_{\mathbb{R}^N}\frac{Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big)^{1/q}-\Big(\int_{\mathbb{R}^N}\frac{Q(x)|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big)^{1/q}\Big\}\\
&\quad-\int_{\mathbb{R}^N}\left(g(x,u_n)-g(x,u)\right)(u_n-u)\,\mathrm{d}x.
\end{aligned}
$$

We remark that all the terms between curly brackets in the previous expression have the same signs; therefore, by the limit (2.22) we obtain

$$
\begin{gathered}
\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|\nabla u_n|^p}{|x|^{ap}}\,\mathrm{d}x=\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x,\\
\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{P(x)|u_n|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x=\int_{\mathbb{R}^N}\frac{P(x)|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x,\\
\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|\nabla u_n|^q}{|x|^{cq}}\,\mathrm{d}x=\int_{\mathbb{R}^N}\frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x,\\
\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{Q(x)|u_n|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x=\int_{\mathbb{R}^N}\frac{Q(x)|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x.
\end{gathered}
$$

This implies that

$$
\lim_{n\to\infty}\|u_n\|_{1,p}^p=\|u\|_{1,p}^p \quad\text{and}\quad \lim_{n\to\infty}\|u_n\|_{1,q}^q=\|u\|_{1,q}^q.
$$

Moreover, $u_n \rightharpoonup u$ weakly in E as $n\to\infty$; and finally, $u_n\to u$ strongly in E as $n\to\infty$. For the details, see DiBenedetto [11, Proposition V.11.1]. □

Lemma 2.7. *Suppose that there exists a sequence $(u_n)_{n\in\mathbb{N}}\subset E$ and a function $u\in E$ such that $u_n\to u$ in E and $J'(u_n)\to 0$ as $n\to\infty$. Then there exists a subsequence, still denoted the same way, such that $\nabla u_n\to\nabla u$ a.e. in $\mathbb{R}^N$*

For a proof the above, see Assunção, Carrião, and Miyagaki [4] or Benmouloud, Echarghaoui, and Sbaï [6].

Using Lemmas 2.1, 2.2, 2.3, 2.6, and 2.7 we conclude that there exists $u\in E$ which is a critical point for the functional J. Moreover, this critical point is a positive ground state solution to the auxiliary problem (2.4), that is, $J(u)=\alpha>0$ and $J'(u)=0$.

3. Estimate for the solution to the auxiliary problem

In this section we show that the solution to (2.4) obtained in the previous section satisfies an important estimate. To do this we use several lemmas.

Lemma 3.1. *For $R > 1$, every positive ground state solution u to problem (2.4) satisfies the estimate*

$$\|u\|_{1,p}^p + \|u\|_{1,q}^q \leqslant \frac{kpd_\infty}{p-1}.$$

Proof. Combining inequalities (2.7), (2.10) and (2.11), it follows that

$$\frac{(p-1)}{kp}\left\{\|u\|_{1,p}^p + \|u\|_{1,q}^q\right\} \leqslant J(u) - \frac{1}{\theta}J'(u)u = J(u) = \alpha \leqslant d_\infty.$$

The conclusion of the lemma follows immediately. □

We remark that the boundedness of the norm of the ground state solution to problem (2.4) shown in Lemma 3.1 depends only on the potential functions P_∞ and Q_∞, on the nonlinearity f and on the constant θ; it is independent of the constant $R > 1$.

The next lemma is a crucial step to establish an important estimate involving the norm of the solution to the auxiliary problem (2.4) in the space $L^\infty(\mathbb{R}^N)$. To prove it, we adapt the arguments by Alves and Souto [2]; see also Gilbarg and Trudinger [14, Section 8.6], Brézis and Kato [8], Pucci and Servadei [16], and Bastos, Miyagaki, and Vieira [5].

Lemma 3.2. *Consider $a < (N-p)/p$, $a \leqslant b < a+1$, $c < (N-q)/q$, $c \leqslant d < c+1$, $2 \leqslant q \leqslant p < N$, and $r \in \mathbb{R}$ such that $p(a+1-b)r > N$. Suppose that $A, B\colon \mathbb{R}^N \to \mathbb{R}$ are nonnegative potential functions. Let $H\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|H(x,s)| \leqslant h(x)|s|^{p-1}s/|x|^{bp^*(a,b)}$ for all $s > 0$ where the function $h\colon \mathbb{R}^N \to \mathbb{R}$ is such that $h \in L^r_{ap^*(a,b)/r}(\mathbb{R}^N)$. Suppose also that $v \in E \subset D^{1,p}_{a,b}(\mathbb{R}^N) \cap D^{1,q}_{c,d}(\mathbb{R}^N)$ is a weak solution to the problem*

$$\begin{aligned}
&-\operatorname{div}\Big(\frac{|\nabla v|^{p-2}\nabla v}{|x|^{ap}}\Big) - \operatorname{div}\Big(\frac{|\nabla v|^{q-2}\nabla v}{|x|^{cq}}\Big)\\
&+\frac{A(x)|v|^{p-2}v}{|x|^{ap^*(a,b)}} + \frac{B(x)|v|^{q-2}v}{|x|^{cq^*(c,d)}} = H(x,v).
\end{aligned} \tag{3.1}$$

Then there exists a constant $M_1 = M_1(\|h\|_{L^r_{ap^*(a,b)/r}(\mathbb{R}^N)}) > 0$ that does not depend on the functions A and B, such that

$$\|v\|_{L^\infty(\mathbb{R}^N)} \leqslant M_1 \max\left\{\|v\|_{L^{p^*(a,b)}(\mathbb{R}^N)}, K, KL_v, 1\right\},$$

where K and L_v are defined by (3.6) and by (3.7), respectively.

Proof. Consider $\beta > 1$ and, for every $m \in \mathbb{N}$, let us define the subsets

$$\begin{aligned}
A_m &:= \{x \in \mathbb{R}^N : 1 < |v(x)|^{\beta-1} \leqslant m\};\\
B_m &:= \{x \in \mathbb{R}^N : |v(x)|^{\beta-1} > m\};\\
C_m &:= \{x \in \mathbb{R}^N : |v(x)|^{\beta-1} \leqslant 1\}.
\end{aligned}$$

We also define the sequence of functions $(v_m)_{m\in\mathbb{N}} \subset D^{1,p}_{a,b}(\mathbb{R}^N) \cap D^{1,q}_{c,d}(\mathbb{R}^N)$ by

$$v_m(x) \coloneqq \begin{cases} |v(x)|^{p(\beta-1)}v(x), & \text{if } x \in A_m;\\ m^p v(x), & \text{if } x \in B_m;\\ |v(x)|^{q(\beta-1)}v(x), & \text{if } x \in C_m. \end{cases}$$

It is easy to verify that for every $x \in \mathbb{R}^N$ we have

$$v_m(x) \leqslant \max\left\{|v(x)|^{p(\beta-1)+1}, |v(x)|^{q(\beta-1)+1}\right\}.$$

Additionally, simple computations show that

$$\nabla v_m(x) = \begin{cases} (p(\beta-1)+1)\,|v(x)|^{p(\beta-1)}\nabla v(x), & \text{if } x \in A_m;\\ m^p \nabla v(x), & \text{if } x \in B_m;\\ (q(\beta-1)+1)\,|v(x)|^{q(\beta-1)}\nabla v(x), & \text{if } x \in C_m. \end{cases}$$

Furthermore, $(v_m)_{m\in\mathbb{N}} \subset E$. Indeed,

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{P(x)|v_m|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\ &\leqslant \int_{A_m} \frac{P(x)\left(|v|^{p-1}v\right) m^{p(p-1)+p}}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\ &\quad + \int_{B_m} \frac{P(x)|v|^{p-1}v m^{p(p-1)+p}}{|x|^{ap^*(a,b)}}\,\mathrm{d}x + \int_{C_m} \frac{P(x)\left(|v|^{p-1}v\right)}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\ &\leqslant m^{p^2} \int_{\mathbb{R}^N} \frac{P(x)|v|^{p-1}v}{|x|^{ap^*(a,b)}}\,\mathrm{d}x < +\infty. \end{aligned}$$

And in a similar way, we have

$$\int_{\mathbb{R}^N} \frac{Q(x)|v_m|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x \leqslant m^{pq} \int_{\mathbb{R}^N} \frac{Q(x)|v|^{q-1}v}{|x|^{cq^*(c,d)}}\,\mathrm{d}x < +\infty.$$

Multiplying both sides of the differential equation (3.1) by the test function v_m and integrating with the help of the divergence theorem, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^N} H(x,v)v_m\,\mathrm{d}x &= \int_{\mathbb{R}^N} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{|\nabla v|^{q-2}}{|x|^{ap}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x\\ &\quad + \int_{\mathbb{R}^N} \frac{A(x)|v|^{p-2}vv_m}{|x|^{ap^*(a,b)}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{B(x)|v|^{q-2}vv_m}{|x|^{cq^*(c,d)}}\,\mathrm{d}x \end{aligned} \tag{3.2}$$

By the definition of the function v_m, the first two terms on the right-hand side of equality (3.2) can be written in the form

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{|\nabla v|^{q-2}}{|x|^{cq}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x\\ &= (p(\beta-1)+1)\Big\{\int_{A_m} \frac{|\nabla v|^p|v|^{p(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x + \int_{A_m} \frac{|\nabla v|^q|v|^{p(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x\Big\}\\ &\quad + (q(\beta-1)+1)\Big\{\int_{C_m} \frac{|\nabla v|^p|v|^{q(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x + \int_{C_m} \frac{|\nabla v|^q|v|^{q(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x\Big\}\\ &\quad + m^p\Big\{\int_{B_m} \frac{|\nabla v|^p}{|x|^{ap}}\,\mathrm{d}x + \int_{B_m} \frac{|\nabla v|^q}{|x|^{cq}}\,\mathrm{d}x\Big\}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&(p(\beta-1)+1)\Big\{\int_{A_m}\frac{|\nabla v|^p|v|^{p(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x+\int_{A_m}\frac{|\nabla v|^q|v|^{p(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x\Big\}\\
&+(q(\beta-1)+1)\Big\{\int_{C_m}\frac{|\nabla v|^p|v|^{q(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x+\int_{C_m}\frac{|\nabla v|^q|v|^{q(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x\Big\}\\
&=\int_{\mathbb{R}^N}\frac{|\nabla v|^{p-2}}{|x|^{ap}}\nabla v\cdot\nabla v_m\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{|\nabla v|^{q-2}}{|x|^{cq}}\nabla v\cdot\nabla v_m\,\mathrm{d}x\\
&\quad-m^p\Big\{\int_{B_m}\frac{|\nabla v|^p}{|x|^{ap}}\,\mathrm{d}x+\int_{B_m}\frac{|\nabla v|^q}{|x|^{cq}}\,\mathrm{d}x\Big\}\\
&\leqslant\int_{\mathbb{R}^N}\frac{|\nabla v|^{p-2}}{|x|^{ap}}\nabla v\cdot\nabla v_m\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{A(x)|v|^{p-2}vv_m}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}\frac{|\nabla v|^{q-2}}{|x|^{cq}}\nabla v\cdot\nabla v_m\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{B(x)|v|^{q-2}vv_m}{|x|^{cq^*(c,d)}}\,\mathrm{d}x.
\end{aligned}
\tag{3.3}
$$

Now we define another sequence of functions $(w_m)_{m\in\mathbb{N}}\subset E$ by

$$
w_m(x)=\begin{cases}|v(x)|^{\beta-1}v(x), & \text{if } x\in A_m\cup C_m;\\ mv(x), & \text{if } x\in B_m.\end{cases}
$$

Direct computations show that

$$
\nabla w_m(x)=\begin{cases}\beta|v(x)|^{\beta-1}\nabla v(x), & \text{if } x\in A_m\cup C_m;\\ m\nabla v(x), & \text{if } x\in B_m.\end{cases}
$$

Using the definitions of the sets A_m, B_m, C_m, the definition of the function v_m, as well as the fact that $2\leqslant q\leqslant p<N$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\frac{|\nabla w_m|^p}{|x|^{ap}}\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{A(x)|w_m|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\frac{|\nabla v|^{p-2}}{|x|^{ap}}\nabla v\cdot\nabla v_m\,\mathrm{d}x-\int_{\mathbb{R}^N}\frac{A(x)|v|^{p-2}vv_m}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\
&+\int_{\mathbb{R}^N}\frac{|\nabla w_m|^q}{|x|^{cq}}\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{B(x)|w_m|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\frac{|\nabla v|^{q-2}}{|x|^{cq}}\nabla v\cdot\nabla v_m\,\mathrm{d}x-\int_{\mathbb{R}^N}\frac{B(x)|v|^{q-2}vv_m}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\\
&=\beta^p\int_{A_m\cup C_m}\frac{|\nabla v|^p|v|^{p(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x+\beta^q\int_{A_m\cup C_m}\frac{|\nabla v|^q|v|^{q(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x\\
&\quad-(p(\beta-1)+1)\Big\{\int_{A_m}\frac{|\nabla v|^p|v|^{p(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x+\int_{A_m}\frac{|\nabla v|^q|v|^{p(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x\Big\}\\
&\quad+\int_{A_m}\frac{B(x)\left(|v|^{q\beta}-|v|^{p(\beta-1)+q}\right)}{|x|^{cq^*(c,d)}}\,\mathrm{d}x+\int_{C_m}\frac{A(x)\left(|v|^{p\beta}-|v|^{p+q(\beta-1)}\right)}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\
&\quad+(m^q-m^p)\int_{B_m}\frac{B(x)|v|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\\
&\quad-(q(\beta-1)+1)\Big\{\int_{C_m}\frac{|\nabla v|^p|v|^{q(\beta-1)}}{|x|^{cq^*(c,d)}}\,\mathrm{d}x+\int_{C_m}\frac{|\nabla v|^q|v|^{q(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x\Big\}
\end{aligned}
$$

$$+ (m^q - m^p) \int_{B_m} \frac{|\nabla v|^q}{|x|^{cq}}\,\mathrm{d}x.$$

This implies that

$$\begin{aligned}
&\int_{R^N} \frac{|\nabla w_m|^p}{|x|^{ap}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{A(x)|w_m|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x + \int_{R^N} \frac{|\nabla w_m|^q}{|x|^{cq}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{B(x)|w_m|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x \\
&\leqslant (\beta^p - (p(\beta-1)+1)) \int_{A_m} \frac{|\nabla v|^p |v|^{p(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x + \beta^p \int_{C_m} \frac{|\nabla v|^p |v|^{p(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x \\
&\quad + (\beta^q - (q(\beta-1)+1)) \int_{C_m} \frac{|\nabla v|^q |v|^{q(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x + \beta^q \int_{A_m} \frac{|\nabla v|^q |v|^{q(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{A(x)|v|^{p-2} v v_m}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \frac{|\nabla v|^{q-2}}{|x|^{cq}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{B(x)|v|^{q-2} v v_m}{|x|^{cq^*(c,d)}}\,\mathrm{d}x.
\end{aligned}$$

So, using inequality (3.3) and some estimates we deduce that

$$\begin{aligned}
&\int_{R^N} \frac{|\nabla w_m|^p}{|x|^{ap}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{A(x)|w_m|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x + \int_{R^N} \frac{|\nabla w_m|^q}{|x|^{cq}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{B(x)|w_m|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x \\
&\leqslant \Big(\frac{\beta^p}{q(\beta-1)+1}\Big)\Big\{ \int_{\mathbb{R}^N} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{A(x)|v|^{p-2} v v_m}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \frac{|\nabla v|^{q-2}}{|x|^{cq}}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{B(x)|v|^{q-2} v v_m}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big\} \\
&\quad + \beta^p \int_{C_m} \frac{|\nabla v|^p |v|^{p(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x + \beta^q \int_{A_m} \frac{|\nabla v|^q |v|^{q(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x.
\end{aligned}$$

Now we estimate some of the integrals that appear in the previous inequality. First, by definition of A_m we have

$$\begin{aligned}
\int_{A_m} \frac{|\nabla v|^q |v|^{q(\beta-1)}}{|x|^{cq}}\,\mathrm{d}x &= \int_{A_m} \frac{|\nabla v|^{q-2}}{[p(\beta-1)+1]|x|^{cq}|v|^{(p-q)(\beta-1)}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^N} \frac{|\nabla v|^{q-2}}{|x|^{cq}}\nabla v\cdot\nabla v_m\,\mathrm{d}x.
\end{aligned}$$

In a similar way, by the definition of C_m we have

$$\int_{C_m} \frac{|\nabla v|^p |v|^{p(\beta-1)}}{|x|^{ap}}\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x.$$

Using these inequalities we deduce that

$$\begin{aligned}
&\int_{R^N} \frac{|\nabla w_m|^p}{|x|^{ap}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{A(x)|w_m|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x + \int_{R^N} \frac{|\nabla w_m|^q}{|x|^{cq}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{B(x)|w_m|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x \\
&\leqslant \Big(\beta^p + \frac{\beta^p}{q(\beta-1)+1}\Big)\Big\{ \int_{\mathbb{R}^N} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{A(x)|v|^{p-2} v v_m}{|x|^{ap^*(a,b)}}\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \frac{|\nabla v|^{q-2}}{|x|^{cq}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{B(x)|v|^{q-2} v v_m}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big\} \\
&\leqslant \beta^p\Big\{ \int_{\mathbb{R}^N} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{A(x)|v|^{p-2} v v_m}{|x|^{ap^*(a,b)}}\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} \frac{|\nabla v|^{q-2}}{|x|^{cq}} \nabla v \cdot \nabla v_m \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{B(x)|v|^{q-2} v v_m}{|x|^{cq^*(c,d)}} \,\mathrm{d}x \Big\} \\
&= \beta^p \int_{\mathbb{R}^N} H(x,v) v_m \,\mathrm{d}x.
\end{aligned}
$$

Using the Caffarelli-Kohn-Nirenberg inequalities (2.5) and (2.6), the definition of the constant S and the hypothesis $H(x,s) \leqslant h(x)|s|^{p-2}s/|x|^{bp^*(a,b)}$, we obtain

$$
\begin{aligned}
&\Big(\int_{A_m \cup C_m} \frac{|w_m|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \Big)^{p/p^*(a,b)} \\
&\leqslant \Big(\int_{\mathbb{R}^N} \frac{|w_m|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \Big)^{p/p^*(a,b)} \\
&\leqslant S \int_{\mathbb{R}^N} \frac{|\nabla w_m|^p}{|x|^{ap}} \,\mathrm{d}x \\
&\leqslant S \Big\{ \int_{\mathbb{R}^N} \frac{|\nabla w_m|^p}{|x|^{ap}} \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{P(x)|w_m|^p}{|x|^{ap^*(a,b)}} \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \frac{|\nabla w_m|^q}{|x|^{cq}} \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{Q(x)|w_m|^q}{|x|^{cq^*(c,d)}} \,\mathrm{d}x \Big\} \\
&\leqslant S\beta^p \int_{\mathbb{R}^N} H(x,v) v_m \,\mathrm{d}x \\
&\leqslant S\beta^p \int_{\mathbb{R}^N} \frac{h(x)|v|^{p-2} v v_m}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \\
&= S\beta^p \Big\{ \int_{A_m} \frac{h(x)|v|^{p-2} v |v|^{p(\beta-1)} v}{|x|^{bp^*(a,b)}} \,\mathrm{d}x + \int_{B_m} \frac{h(x)|v|^{p-2} v\, m^p v}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \\
&\quad + \int_{C_m} \frac{h(x)|v|^{p-2} v |v|^{q(\beta-1)} v}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \Big\} \\
&= S\beta^p \Big\{ \int_{A_m} \frac{h(x)|v|^{p\beta}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x + \int_{B_m} \frac{h(x)|v|^{p}\, m^p}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \\
&\quad + \int_{C_m} \frac{h(x)|v|^{p+q(\beta-1)}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \Big\} \\
&\leqslant S\beta^p \Big\{ \int_{\mathbb{R}^N} \frac{h(x)|v|^{p\beta}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{h(x)|v|^{p}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \Big\},
\end{aligned}
$$

where in the last two passages we used the definitions of the functions v_m and w_m, together with the facts that in B_m we have $|w_m|^p \leqslant |v|^{p\beta}$ and in C_m we have $|v|^{p+q(\beta-1)} \leqslant |v|^p$.

Passing to the limit as $m \to \infty$ and using Lebesgue's dominated convergence theorem, we deduce that

$$
\Big(\int_{\mathbb{R}^N} \frac{|v|^{p^*(a,b)\beta}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \Big)^{p/p^*} \leqslant S\beta^p \Big\{ \int_{\mathbb{R}^N} \frac{h(x)|v|^{p\beta}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x + \int_{\mathbb{R}_N} \frac{h(x)|v|^{p}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \Big\}.
$$

Applying Hölder's inequality to both terms on the right-hand side of the previous inequality, we obtain

$$
\int_{\mathbb{R}^N} \frac{h(x)|v|^{p\beta}}{|x|^{bp^*(a,b)}} \,\mathrm{d}x \leqslant \|h\|_{L^r_{bp^*(a,b)/r}(\mathbb{R}^N)} \|v\|^{p\beta}_{L^{p\beta r'}_{bp^*(a,b)/p\beta r'}(\mathbb{R}^N)}
$$

$$\int_{\mathbb{R}^N} \frac{h(x)|v|^p}{|x|^{bp^*(a,b)}}\,\mathrm{d}x \leqslant \|h\|_{L^r_{bp^*(a,b)/r}(\mathbb{R}^N)} \|v\|^p_{L^{pr'}_{bp^*(a,b)/pr'}(\mathbb{R}^N)},$$

where $1/r + 1/r' = 1$. Hence

$$\begin{aligned}
\|v\|^{p\beta}_{L^{p^*(a,b)\beta}_{b/\beta}(\mathbb{R}^N)} &\leqslant S\|h\|_{L^r_{bp^*(a,b)/r}(\mathbb{R}^N)} \beta^p \Big\{ \|v\|^{p\beta}_{L^{p\beta r'}_{bp^*(a,b)/p\beta r'}(\mathbb{R}^N)} + \|v\|^p_{L^{pr'}_{bp^*(a,b)/r'}(\mathbb{R}^N)} \Big\} \\
&\leqslant S\|h\|_{L^r_{bp^*(a,b)/r}(\mathbb{R}^N)} \beta^p \Big\{ \max\big\{ \|v\|^{p\beta}_{L^{p\beta r'}_{bp^*(a,b)/p\beta r'}(\mathbb{R}^N)}, 1 \big\} \\
&\quad + \max\big\{ \|v\|^p_{L^{pr'}_{bp^*(a,b)/pr'}(\mathbb{R}^N)}, 1 \big\} \Big\} \\
&= C_1^p \beta^p \max\Big\{ \|v\|^{p\beta}_{L^{p\beta r'}_{bp^*(a,b)/p\beta r'}(\mathbb{R}^N)}, \max\big\{ \|v\|^p_{L^{pr'}_{bp^*(a,b)/pr'}(\mathbb{R}^N)}, 1 \big\} \Big\},
\end{aligned}$$

where we used $C_1^p \coloneqq S\|h\|_{L^r_{bp^*(a,b)/r}(\mathbb{R}^N)} > 0$ to denote a constant that depends on the parameters of problem (1.1) but does not depend on the function $v \in E \subset D^{1,p}_{a,b}(\mathbb{R}^N) \cap D^{1,q}_{c,d}(\mathbb{R}^N)$. Writing $\beta = \sigma^j$ for $j \in \mathbb{N}$ we deduce that

$$\begin{aligned}
&\|v\|_{L^{p^*(a,b)\sigma^j}_{b/\sigma^j}(\mathbb{R}^N)} \\
&\leqslant C_1^{1/\sigma^j} \sigma^{j/\sigma^j} \max\Big\{ \|v\|_{L^{p\sigma^j r'}_{bp^*(a,b)/p\sigma^j r'}(\mathbb{R}^N)}, \max\big\{ \|v\|^{1/\sigma^j}_{L^{pr'}_{bp^*(a,b)/pr'}(\mathbb{R}^N)}, 1 \big\} \Big\}.
\end{aligned} \tag{3.4}$$

Choosing $\sigma = p^*(a,b)/pr' > 1$, from inequality (3.4) with $j = 1$ we obtain

$$\|v\|_{L^{p^*(a,b)\sigma}_{b/\sigma}(\mathbb{R}^N)} \leqslant C_1^{1/\sigma} \sigma^{1/\sigma} \max\Big\{ \|v\|_{L^{p^*(a,b)}_b(\mathbb{R}^N)}, \max\big\{ \|v\|^{1/\sigma}_{L^{pr'}_{b\sigma}(\mathbb{R}^N)}, 1 \big\} \Big\}.$$

From inequality (3.4) with $j = 2$ together with the previous inequality we obtain

$$\begin{aligned}
&\|v\|_{L^{p^*(a,b)\sigma^2}_{b/\sigma^2}(\mathbb{R}^N)} \\
&\leqslant C_1^{1/\sigma^2} \sigma^{2/\sigma^2} \max\Big\{ \|v\|_{L^{p^*(a,b)\sigma}_{b/\sigma}(\mathbb{R}^N)}, \max\big\{ \|v\|^{1/\sigma^2}_{L^{pr'}_{b\sigma}(\mathbb{R}^N)}, 1 \big\} \Big\} \\
&\leqslant C_1^{1/\sigma^2} \sigma^{2/\sigma^2} \max\Big\{ C_1^{1/\sigma} \sigma^{1/\sigma} \max\Big\{ \|v\|_{L^{p^*(a,b)}_b(\mathbb{R}^N)}, \max\big\{ \|v\|^{1/\sigma}_{L^{pr'}_{b\sigma}(\mathbb{R}^N)}, 1 \big\} \Big\}, \\
&\quad \max\big\{ \|v\|^{1/\sigma^2}_{L^{pr'}_{b\sigma}(\mathbb{R}^N)}, 1 \big\} \Big\} \\
&\leqslant C_1^{1/\sigma + 1/\sigma^2} \sigma^{1/\sigma + 2/\sigma^2} \max\Big\{ \|v\|_{L^{p^*(a,b)}_b(\mathbb{R}^N)}, \max\big\{ \big(C_1^{1/\sigma} \sigma^{1/\sigma}\big)^{-1}, 1 \big\}, \\
&\quad \max\big\{ \big(C_1^{1/\sigma} \sigma^{1/\sigma}\big)^{-1}, 1 \big\} \max\big\{ \|v\|^{1/\sigma}_{L^{pr'}_{b\sigma}(\mathbb{R}^N)}, \|v\|^{1/\sigma^2}_{L^{pr'}_{b\sigma}(\mathbb{R}^N)}, 1 \big\} \Big\}.
\end{aligned}$$

Proceeding in this way, for $j \in \mathbb{N}$ we obtain

$$\|v\|_{L^{p^*(a,b)\sigma^j}_{b/\sigma^j}(\mathbb{R}^N)} \leqslant C_1^{s_j} \sigma^{t_j} \max\Big\{ \|v\|_{L^{p^*(a,b)}_b(\mathbb{R}^N)}, K_j, K_j L_j \Big\}, \tag{3.5}$$

where $s_j \coloneqq 1/\sigma + 1/\sigma^2 + \dots + 1/\sigma^j$; $t_j \coloneqq 1/\sigma + 2/\sigma^2 + \dots + j/\sigma^j$;

$$K_j \coloneqq \begin{cases} 1, & \text{if } j = 1; \\ \max_{1 \leqslant i \leqslant j-1} \big\{ C_1^{-s_i} \sigma^{-t_i}, 1 \big\}, & \text{if } j \geqslant 2; \end{cases}$$

and

$$L_j \coloneqq \max_{1 \leqslant i \leqslant j} \Big\{ \|v\|^{1/\sigma^i}_{L^{pr'}_{b\sigma}(\mathbb{R}^N)}, 1 \Big\}.$$

Since $\sigma > 1$, we have $\lim_{j\to\infty} s_j = 1/(\sigma-1)$ and $\lim_{j\to\infty} t_j = \sigma/(\sigma-1)^2$; hence,

$$\lim_{j\to\infty} K_j := K = \begin{cases} \left(C_1^{1/(\sigma-1)}\sigma^{\sigma/(\sigma-1)^2}\right)^{-1} & \text{if } C_1 \leqslant 1; \\ \left(C_1^{1/\sigma}\sigma^{1/\sigma}\right)^{-1} & \text{if } C_1 > 1; \end{cases} \tag{3.6}$$

and

$$\lim_{j\to\infty} L_j := L_v = \begin{cases} 1, & \text{if } \|v\|_{L^{pr'}_{b\sigma}(\mathbb{R}^N)} \leqslant 1; \\ \|v\|^{1/(\sigma-1)}_{L^{pr'}_{b\sigma}(\mathbb{R}^N)}, & \text{if } \|v\|_{L^{pr'}_{b\sigma}(\mathbb{R}^N)} > 1\ . \end{cases} \tag{3.7}$$

Using the fact that $v \in E \subset D^{1,p}_{a,b}(\mathbb{R}^N) \cap D^{1,q}_{c,d}(\mathbb{R}^N)$, applying Hölder's inequality we deduce that $L_v < +\infty$.

Finally, passing to the limit as $j \to \infty$ and using inequality (3.5) we obtain

$$\begin{aligned} \|v\|_{L^\infty(\mathbb{R}^N)} &= \lim_{j\to\infty} \|v\|_{L^{p^*(a,b)\sigma^j}_{b/\sigma^j}(\mathbb{R}^N)} \\ &\leqslant C_1^{1/(\sigma-1)}\sigma^{\sigma/(\sigma-1)^2} \max\left\{\|v\|_{L^{p^*(a,b)}_b(\mathbb{R}^N)}, K, KL_v, 1\right\} \\ &:= M_1 \max\left\{\|v\|_{L^{p^*(a,b)}_b(\mathbb{R}^N)}, K, KL_v, 1\right\}, \end{aligned} \tag{3.8}$$

where $M_1 = M_1(\|h\|_{L^r_{bp^*(a,b)/r}(\mathbb{R}^N)})$ depends on the parameters of problem (1.1) but does not depend on the function $v \in E \subset D^{1,p}_{a,b}(\mathbb{R}^N) \cap D^{1,q}_{c,d}(\mathbb{R}^N)$. This concludes the proof of the lemma. $\square$

Lemma 3.3. *For every $R > 1$ there exist a constant $M_2 = M_2(P_\infty, P_\infty)$ such that any positive ground state solution $u \in D^{1,p}_{a,b}(\mathbb{R}^N) \cap D^{1,q}_{c,d}(\mathbb{R}^N)$ to the auxiliary problem* (2.4) *satisfies*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant M_2.$$

Proof. Consider $R > 1$ and let $u \in D^{1,p}_{a,b}(\mathbb{R}^N) \cap D^{1,q}_{c,d}(\mathbb{R}^N)$ be a positive ground state solution to the auxiliary problem (2.4). Now we define the function $H\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$H(x,t) := \begin{cases} f(t), & \text{if } |x| \leqslant R \\ & \text{or if } |x| > R \text{ and } f(t) \leqslant \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}; \\ 0, & \text{if } |x| > R \text{ and } f(t) > \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}. \end{cases}$$

We also define the functions $A, B\colon \mathbb{R}^N \to \mathbb{R}$ by

$$A(x) = \begin{cases} P(x), & \text{if } |x| \leqslant R \\ & \text{or if } |x| > R \text{ and } f(u(x)) \leqslant \frac{P(x)|u(x)|^{p-2}u(x)}{k|x|^{ap^*(a,b)}}; \\ \left(1-\frac{1}{k}\right)P(x), & \text{if } |x| > R \text{ and } f(u(x)) > \frac{P(x)|u(x)|^{p-2}u(x)}{k|x|^{ap^*(a,b)}}, \end{cases}$$

and $B(x) = Q(x)$.

Considering these functions and using $v \in E$ as a test function, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \frac{|\nabla u|^{p-2}}{|x|^{ap}} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{A(x)|u|^{p-2}uv}{|x|^{ap^*(a,b)}} \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \frac{|\nabla u|^{q-2}}{|x|^{cq}} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{B(x)|u|^{q-2}uv}{|x|^{cq^*(c,d)}} \,\mathrm{d}x - \int_{\mathbb{R}^N} H(x,u)v \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \frac{|\nabla u|^{p-2}}{|x|^{ap}} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{P(x)|u|^{p-2}uv}{|x|^{ap^*(a,b)}} \,\mathrm{d}x \end{aligned}$$

$$+\int_{\mathbb{R}^N}\frac{|\nabla u|^{q-2}}{|x|^{cq}}\nabla u\cdot\nabla v\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{Q(x)|u|^{q-2}uv}{|x|^{cq^*(c,d)}}\,\mathrm{d}x-\int_{\mathbb{R}^N}g(x,u)v\,\mathrm{d}x.$$

From hypothesis (H1), for $|t|$ small enough we have

$$|H(x,t)|\leqslant|f(t)|\leqslant\frac{c_1|t|^{p^*(a,b)-1}}{|x|^{bp^*(a,b)}};$$

and from hypothesis (H2), for $|t|$ big enough we have

$$|H(x,t)|\leqslant|f(t)|\leqslant\frac{c_2|t|^{\tau-1}}{|x|^{bp^*(a,b)}}$$

with $\tau\in(p,p^*(a,b))$. Combining both cases we obtain $|H(x,t)|\leqslant|f(t)|\leqslant c_0|t|^{\tau-1}/|x|^{bp^*(a,b)}$ for every $t\in\mathbb{R}^+$ and for every $\tau\in(p,p^*(a,b))$. It follows that $|H(x,u)|\leqslant c_0|u(x)|^{\tau-p}|u(x)|^{p-1}/|x|^{bp^*(a,b)}=h(x)|u(x)|^{p-1}/|x|^{bp^*(a,b)}$, where we set $h(x)\coloneqq c_0|u(x)|^{\tau-p}$.

Direct computations show that $h\in L^r_{bp^*(a,b)/r}(\mathbb{R}^N)$ for $r=p^*(a,b)/(\tau-p)$. Indeed, recalling the definition of S we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Big|\frac{h(x)}{|x|^{bp^*(a,b)/r}}\Big|^r\,\mathrm{d}x\\
&=\int_{\mathbb{R}^N}\frac{|c_0|u(x)|^{\tau-p}|^{p^*(a,b)/(\tau-p)}}{|x|^{bp^*(a,b)}}\,\mathrm{d}x\\
&\leqslant c_0^{p^*(a,b)/(\tau-p)}\int_{\mathbb{R}^N}\frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}\,\mathrm{d}x\\
&\leqslant c_0^{p^*(a,b)/(\tau-p)}S^{p^*(a,b)/p}\Big(\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x\Big)^{p^*(a,b)/p}\\
&\leqslant c_0^{p^*(a,b)/(\tau-p)}S^{p^*(a,b)/p}\Big\{\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{|x|^{ap}}\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{P(x)|u|^p}{|x|^{ap^*(a,b)}}\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}\frac{|\nabla u|^q}{|x|^{cq}}\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{Q(x)|u|^q}{|x|^{cq^*(c,d)}}\,\mathrm{d}x\Big\}^{p^*(a,b)/p}\\
&\leqslant c_0^{p^*(a,b)/(\tau-p)}S^{p^*(a,b)/p}\left\{\|u\|_{1,p}^p+\|u\|_{1,q}^q\right\}^{p^*(a,b)/p}<+\infty.
\end{aligned}$$

In this way, any positive ground state solution $u\in D^{1,p}_{a,b}(\mathbb{R}^N)\cap D^{1,q}_{c,d}(\mathbb{R}^N)$ to the auxiliary problem (2.4) satisfies the hypothesis of Lemma 3.2. Concluding the argument, from Lemma 3.1 we have

$$\|u\|_{L_b^{p^*(a,b)}(\mathbb{R}^N)}\leqslant S^{1/p}\left\{\|u\|_{1,p}^p+\|u\|_{1,q}^q\right\}^{1/p}\leqslant\Big(\frac{Skpd_\infty}{p-1}\Big)^{1/p}.$$

Finally, combining estimate (3.8) with the previous inequality we obtain

$$\begin{aligned}
\|u\|_{L^\infty(\mathbb{R}^N)}&\leqslant M_1\max\Big\{\|u\|_{L_b^{p^*(a,b)}(\mathbb{R}^N)},K,KL_u,1\Big\}\\
&\leqslant M_1\max\Big\{\Big(\frac{Skpd_\infty}{p-1}\Big)^{1/p},K,KL_u,1\Big\}\coloneqq M_2,
\end{aligned}$$

where $M_2=M_2(N,p,q,r,a_\infty,b_\infty,\theta,c_0)$. The proof is complete. □

Lemma 3.4. *Suppose that $R_0\geqslant R>1$ and let $u\in D^{1,p}_{a,b}(\mathbb{R}^N)\cap D^{1,q}_{c,d}(\mathbb{R}^N)$ be a positive ground state solution to the auxiliary problem* (2.4). *Then u satisfies the* *inequality*

$$u(x) \leqslant M_2 \, \frac{R^{[N-p(a+1-b)]/(p-1)}}{|x|^{[N-p(a+1-b)]/(p-1)}}$$

for every $|x| \geqslant R > 1$*.*

Proof. Given $R_0 \geqslant R > 1$, we define the function $v \colon \mathbb{R}^N \backslash \{0\} \to \mathbb{R}$ by

$$v(x) \coloneqq M_2 \, \frac{R_0^{[N-p(a+1-b)]/(p-1)}}{|x|^{[N-p(a+1-b)]/(p-1)}}.$$

By hypothesis, $u \in D_{a,b}^{1,p}(\mathbb{R}^N) \cap D_{c,d}^{1,q}(\mathbb{R}^N)$ is a positive ground state solution to the auxiliary problem (2.4); therefore, we can apply Lemma 3.3 to deduce that $\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant M_2$. This implies that if $|x| = R_0$, then $\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant v(x)$. Now we define the function $w \colon \mathbb{R}^N \backslash \{0\} \to \mathbb{R}$ by

$$w(x) = \begin{cases} 0, & \text{if } |x| \leqslant R_0; \\ (u-v)^+, & \text{if } |x| \geqslant R_0. \end{cases}$$

In this way, $w \in D_{a,b}^{1,p}(\mathbb{R}^N) \cap D_{c,d}^{1,q}(\mathbb{R}^N)$; moreover, $w \in E$ because $u, v \in E$.

To complete the proof of the lemma we will show that $(u-v)^+ = 0$ for $|x| \geqslant R_0$. To accomplish this goal we use the hypotheses on the potential functions P and Q; we will also use the function $w \in E$ as a test function to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|\nabla u|^{p-2}}{|x|^{ap}} \nabla u \cdot \nabla w \, \mathrm{d}x + \int_{\mathbb{R}^N} \frac{|\nabla u|^{q-2}}{|x|^{cq}} \nabla u \cdot \nabla w \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} g(x,u) w \, \mathrm{d}x - \int_{\mathbb{R}^N} \frac{P(x)|u|^{p-2}uw}{|x|^{ap^*(a,b)}} \, \mathrm{d}x - \int_{\mathbb{R}^N} \frac{Q(x)|u|^{q-2}uw}{|x|^{cq^*(c,d)}} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N \backslash B_{R_0}(0) \wedge f(t) \leqslant \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}} f(u) w \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N \backslash B_{R_0}(0) \wedge f(t) > \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}} \frac{P(x)|u|^{p-2}uw}{k|x|^{ap^*(a,b)}} \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} \frac{P(x)|u|^{p-2}uw}{|x|^{ap^*(a,b)}} \, \mathrm{d}x - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} \frac{Q(x)|u|^{q-2}uw}{|x|^{cq^*(c,d)}} \, \mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^N \backslash B_{R_0}(0) \wedge f(t) \leqslant \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}} \frac{P(x)|u|^{p-2}uw}{k|x|^{ap^*(a,b)}} \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N \backslash B_{R_0}(0) \wedge f(t) > \frac{P(x)|t|^{p-2}t}{k|x|^{ap^*(a,b)}}} \frac{P(x)|u|^{p-2}uw}{k|x|^{ap^*(a,b)}} \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} \frac{P(x)|u|^{p-2}uw}{|x|^{ap^*(a,b)}} \, \mathrm{d}x - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} \frac{Q(x)|u|^{q-2}uw}{|x|^{cq^*(c,d)}} \, \mathrm{d}x \\
&= \big(\frac{1}{k} - 1\big) \int_{\mathbb{R}^N \backslash B_{R_0}(0)} \frac{P(x)|u|^{p-2}uw}{|x|^{ap^*(a,b)}} \, \mathrm{d}x - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} \frac{Q(x)|u|^{q-2}uw}{|x|^{cq^*(c,d)}} \, \mathrm{d}x \\
&\leqslant 0
\end{aligned} \tag{3.9}$$

because u is a positive function and w is a nonnegative function, while $k > 1$.

Using the radially symmetric form of the operator $-\operatorname{div}(|\nabla u|^{m-2}\nabla u/|x|^{am})$, we have

$$\int_{\mathbb{R}^N\setminus B_{R_0}(0)} \frac{|\nabla v|^{m-2}}{|x|^{em}}\,\nabla v\cdot\nabla\phi\,\mathrm{d}x = 0$$

for $m\in\{p,q\}$, for $e\in\{a,c\}$, and for every function $\phi\in E$; see Calzolari, Filippucci and Pucci [9] and Lindqvist [15]. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\cdot\nabla w\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{|\nabla v|^{q-2}}{|x|^{cq}}\,\nabla v\cdot\nabla w\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N\setminus B_{R_0}(0)} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\cdot\nabla w\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus B_{R_0}(0)} \frac{|\nabla v|^{q-2}}{|x|^{cq}}\,\nabla v\cdot\nabla w\,\mathrm{d}x \\
&= 0.
\end{aligned}\tag{3.10}$$

Defining the subsets

$$\begin{gathered}
\widetilde{A} \coloneqq \{x\in\mathbb{R}^N\colon |x|\geqslant R_0 \text{ and } u(x)>v(x)\}\\
\widetilde{B} \coloneqq \{x\in\mathbb{R}^N\colon |x|< R_0 \text{ or } u(x)\leqslant v(x)\},
\end{gathered}$$

we have $w(x)=u(x)-v(x)$ for $x\in\tilde{A}$ and $w(x)=0$ for $x\in\tilde{B}$. Using inequality (3.9) and equality (3.10) we obtain

$$\begin{aligned}
0 &\geqslant \int_{\mathbb{R}^N} \frac{|\nabla u|^{p-2}}{|x|^{ap}}\,\nabla u\cdot\nabla w\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{|\nabla u|^{q-2}}{|x|^{cq}}\,\nabla u\cdot\nabla w\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\cdot\nabla w\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{|\nabla v|^{q-2}}{|x|^{cq}}\,\nabla v\cdot\nabla w\,\mathrm{d}x \\
&= \int_{\widetilde{A}} \Big[\frac{|\nabla u|^{p-2}}{|x|^{ap}}\,\nabla u - \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\Big]\cdot(\nabla u-\nabla v)\,\mathrm{d}x \\
&\quad + \int_{\widetilde{A}} \Big[\frac{|\nabla u|^{q-2}}{|x|^{cq}}\,\nabla u - \frac{|\nabla v|^{q-2}}{|x|^{cq}}\,\nabla v\Big]\cdot(\nabla u-\nabla v)\,\mathrm{d}x.
\end{aligned}\tag{3.11}$$

Denoting by $\langle\cdot\,,\cdot\rangle\colon \mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ the standard scalar product, given $m\geqslant 2$ there exists a positive constant $c_m\in\mathbb{R}^+$ such that for every $x,y\in\mathbb{R}^N$ it is valid the inequality

$$\langle |x|^{p-2}x-|y|^{p-2}y, x-y\rangle \geqslant c_m\|x-y\|^p. \tag{3.12}$$

For the proof, we refer the reader to Simon [17]. From inequalities (3.11) and (3.12) it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|\nabla w|^{p}}{|x|^{ap}}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{|\nabla w|^{q}}{|x|^{cq}}\,\mathrm{d}x \\
&= \int_{\widetilde{A}} \frac{|\nabla u-\nabla v|^{p}}{|x|^{ap}}\,\mathrm{d}x + \int_{\widetilde{A}} \frac{|\nabla u-\nabla v|^{q}}{|x|^{cq}}\,\mathrm{d}x \\
&\leqslant c_p^{-1}\int_{\widetilde{A}} \Big[\frac{|\nabla u|^{p-2}}{|x|^{ap}}\,\nabla u - \frac{|\nabla v|^{p-2}}{|x|^{ap}}\,\nabla v\Big]\cdot(\nabla u-\nabla v)\,\mathrm{d}x \\
&\quad + c_q^{-1}\int_{\widetilde{A}} \Big[\frac{|\nabla u|^{q-2}}{|x|^{cq}}\,\nabla u - \frac{|\nabla v|^{q-2}}{|x|^{cq}}\,\nabla v\Big]\cdot(\nabla u-\nabla v)\,\mathrm{d}x \leqslant 0.
\end{aligned}$$

From this inequality we deduce that each term on the left-hand side of the previous inequality must be zero, that is, w is constant in $\mathbb{R}^N$. But we already know that $w(x)=0$ in the ball $B_{R_0}(0)$; therefore, $w(x)=0$ for every $x\in\mathbb{R}^N$. This implies that $(u-v)^+ = 0$ for $|x| \geqslant R_0$ and $u(x) \leqslant v(x)$ for every $x \in \mathbb{R}^N$. The proof of the lemma is complete. $\square$

4. Solution of the original problem

In this section we finally show that the solution to the auxiliary problem (2.4) obtained in section 2 is in fact a solution to problem (1.1).

Proof of Theorem 1.1. From Lemmas 2.3 and 2.6, the auxiliary problem (2.4) has a positive ground state solution $u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$. To accomplish our goal we need to show that for every $x \in B_R^c(0)$ the function u satisfies the inequality

$$f(u) \leqslant \frac{P(x)|u|^{p-2}u}{k|x|^{ap^*(a,b)}}.$$

From Lemma 3.4 and by the first inequality in (1.2), if $|x| \geqslant R$, then

$$\begin{aligned}\frac{f(u)}{|u|^{p-2}u} &\leqslant \frac{c_0}{|x|^{bp^*(a,b)}}\,|u|^{p^*(a,b)-p} \\ &\leqslant \frac{c_0}{|x|^{ap^*(a,b)}}\,M_2^{p^*(a,b)-p}\,\frac{R^{p^2(a+1-b)/(p-1)}}{|x|^{p^2(a+1-b)/(p-1)}}.\end{aligned}$$

Now we define the constant

$$\Lambda^* := c_0 k M_2^{p^*(a,b)-p}.$$

Considering $\Lambda \geqslant \Lambda^*$, it follows from the hypothesis (H6) that

$$\begin{aligned}\frac{f(u)}{|u|^{p-2}u} &\leqslant \frac{\Lambda^*}{k|x|^{ap^*(a,b)}}\,\frac{R^{p^2(a+1-b)/(p-1)}}{|x|^{p^2(a+1-b)/(p-1)}} \\ &\leqslant \frac{\Lambda}{k|x|^{ap^*(a,b)}}\,\frac{R^{p^2(a+1-b)/(p-1)}}{|x|^{p^2(a+1-b)/(p-1)}} \\ &\leqslant \frac{P(x)}{k|x|^{ap^*(a,b)}}.\end{aligned}$$

The proof is complete. $\square$

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