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TOPOLOGICAL STRUCTURE OF THE SOLUTION SET FOR A FRACTIONAL p-LAPLACIAN PROBLEM WITH SINGULAR NONLINEARITY

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Abstract. We establish the existence of connected components of positive solutions for the equation $(-\Delta_p)^s u = \lambda f(u)$, under Dirichlet boundary conditions, where the domain is a bounded in $\mathbb{R}^N$ and has smooth boundary, $(-\Delta_p)^s$ is the fractional p-Laplacian operator, and $f : (0,\infty) \to \mathbb{R}$ is a continuous function which may blow up to $\pm\infty$ at the origin.

1. Introduction

We establish the existence of a continuum of positive solutions to the problem

$$\begin{gathered} (-\Delta_p)^s u = \lambda f(u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Omega^c, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N > 1$, is a bounded domain with smooth boundary $\partial\Omega$, $\Omega^c = \mathbb{R}^N \backslash \Omega$, $s \in (0,1)$, $\lambda > 0$ and $p \in (1,\infty)$ are real numbers and $f : (0,\infty) \to \mathbb{R}$ is a continuous function which may blow up to $\pm\infty$ at the origin.

We assume that the nonlinearity f satisfies

- (A1) $f : (0,\infty) \to \mathbb{R}$ is continuous and $\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0$,
- (A2) there are positive numbers $\beta < 1$, a and A such that $f(u) \geq \frac{a}{u^\beta}$ if $u > A$ and $\limsup_{u\to 0} u^\beta |f(u)| < \infty$.

The above hypotheses include nonlinearities such as

- (i) $f(u) = \frac{1}{u^\beta} - \frac{1}{u^\alpha}$ with $0 < \beta < \alpha < 1$;
- (ii) $f(u) = u^q - \frac{1}{u^\beta}$ with $0 < q < p - 1$ and $\beta > 0$;
- (iii) $f(u) = \ln u$.

There is a substantial literature on singular problems dealing with the fractional p-Laplacian operator; we refer the reader to Arora, Giacomoni and Warnault [1], Canino, Montoro, Sciunzi and Squassina [2], Diaz, Morel and Oswald [7], Giacomoni, Mukherjee and Sreenadh [9], Lazer and McKenna [13], Mukherjee and Sreenadh [14], Ho, Perera, Sim and Squassina [10], and the references therein. See also Cui and Sun [4] for other aspects of fractional p-Laplacian problems.

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In their fundamental work, Crandall, Rabinowitz and Tartar [3], employed topological methods, Schauder theory, and maximum principles to prove the existence of an unbounded connected subset in $\mathbb{R}\times C_0(\Omega)$ of positive solutions $u \in C^2(\Omega)\cap C(\Omega)$ of the problem

$$\begin{gathered} -Lu = g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where L is a second-order uniformly elliptical operator, g is a continuous function satisfying some hypotheses, and $C_0(\Omega) = \{u \in C(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$.

Our goal is to extend the results obtained by Crandall, Rabinowitz and Tartar[3] to the non-local fractional operator $(-\Delta_p)^s$. In contrast to that paper, we had to overcome the less regularity of this operator to obtain regularity up to the border of Ω.

To state our main result, we introduce some notation. For a measurable function $u\colon \mathbb{R}^N \to \mathbb{R}$, we introduce the Gagliardo semi-norm

$$[u]_{s,p} := \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \,\mathrm{d}x\,\mathrm{d}y \Big)^{1/p}$$

and consider the space

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

equipped with the norm

$$\|u\|_{s,p,\mathbb{R}^N} = \|u\|_{L^p(\mathbb{R}^N)} + [u]_{s,p},$$

where $\|\cdot\|_{L^p(\mathbb{R}^N)}$ denotes the $L^p(\mathbb{R}^N)$ norm. We also consider the space

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : [u]_{s,p} < \infty,\ u = 0 \text{ a.e. in } \Omega^c\},$$

which is a Banach space with respect to the norm $\|u\| = [u]_{s,p}$.

A weak solution $u \in W_0^{s,p}(\Omega)$ to the problem (1.1) satisfies

$$\iint_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y = \lambda \int_\Omega f(u)v \,\mathrm{d}x, \tag{1.2}$$

for every $v \in W_0^{s,p}(\Omega)$, where $[a-b]^{p-1}$ denotes $|a-b|^{p-2}(a-b)$.

Let p' and $*$ stand for the conjugate exponent of p and the dual Banach space respectively, we denote

$$W^{-s,p'(\Omega)} := \big(W_0^{s,p}(\Omega)\big)^*,$$

and its pairing with $W_0^{s,p}(\Omega)$ by $\langle\cdot,\cdot\rangle$. We observe that the expression

$$\langle(-\Delta_p)^s u, v\rangle := \iint_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y, \quad u,v \in W_0^{s,p}(\Omega),$$

defines a continuous, bounded and strictly monotone operator $(-\Delta_p)^s\colon W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ given by $u \mapsto (-\Delta_p)^s u$ as a consequence of Hölder's inequality. Observe further that $(-\Delta_p)^s$ is strictly monotone and coercive, that is

$$\langle(-\Delta_p)^s u - (-\Delta_p)^s v, u-v\rangle > 0, \quad u,v \in W_0^{s,p}(\Omega),\ u \neq v$$

and

$$\frac{\langle(-\Delta_p)^s u, u\rangle}{\|u\|} \to \infty \quad \text{as } \|u\| \to \infty.$$

For all $\alpha \in (0,1]$ and all $u : \overline{\Omega} \to \mathbb{R}$, we set

$$[u]_{C^\alpha(\overline{\Omega})} = \sup_{x,y\in\overline{\Omega},\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

and consider the Banach space

$$C^\alpha(\overline{\Omega}) = \big\{u \in C(\overline{\Omega}) : [u]_{C^\alpha(\overline{\Omega})} < \infty\big\},$$

endowed with the norm $\|u\|_{C^\alpha(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + [u]_{C^\alpha(\overline{\Omega})}$.

The solution set of problem (1.1) is

$$\mathcal{S} := \{(\lambda, u) \in (0,\infty) \times C(\overline{\Omega}) : \text{u is a solution of (1.1)}\}.$$

We now can state our main result.

Theorem 1.1. *Under assumptions* (A1) *and* (A2)*, there is a number $\lambda_0 > 0$ and a connected subset Σ of $[\lambda_0,\infty) \times C(\overline{\Omega})$ satisfying*

(i) $\Sigma \subset \mathcal{S}$;
(ii) $\Sigma \cap \big(\{\lambda\} \times C(\overline{\Omega})\big) \neq \emptyset$, $\lambda_0 \le \lambda < \infty$.

2. Auxiliary results

We start by introducing notation and recalling some results. Let $M = (M, d)$ be a metric space and $\{\Sigma_n\}$ a sequence of connected components of M. The *upper limit* of $\{\Sigma_n\}$ is defined by

$$\overline{\lim}\ \Sigma_n = \big\{u \in M : \text{there is } (u_{n_i}) \subseteq \cup\Sigma_n \text{ with } u_{n_i} \in \Sigma_{n_i} \text{ and } u_{n_i} \to u\big\}.$$

Remark 2.1 ([17])**.** $\overline{\lim}\ \Sigma_n$ is a closed subset of M.

In the proof of Theorem 1.1 we use topological arguments to construct a suitable connected component of the solution set $\mathcal{S}$ of (1.1). More precisely, we apply in a nontrivial way [16, Theorem 2.1], whose proof is based on the famous Whyburn's lemma [17, Theorem 9.3].

Theorem 2.2 (Sun and Song [16])**.** *Let M be a metric space and $\{\alpha_n\}, \{\beta_n\} \in \mathbb{R}$ be sequences satisfying*

$$\dots < \alpha_n < \dots < \alpha_1 < \beta_1 < \dots < \beta_n < \dots$$

with $\alpha_n \to -\infty$ and $\beta_n \to \infty$. Assume that $\{\Sigma_n^*\}$ is a sequence of connected subsets of $\mathbb{R} \times M$ satisfying

(i) *$\Sigma_n^* \cap (\{\alpha_n\} \times M) \neq \emptyset$ for each n;*
(ii) *$\Sigma_n^* \cap (\{\beta_n\} \times M) \neq \emptyset$ for each n;*
(iii) *for each $\alpha, \beta \in (-\infty,\infty)$ with $\alpha < \beta$, $\cup\Sigma_n^* \cap ([\alpha,\beta] \times M)$ is a relatively compact subset of $\mathbb{R} \times M$.*

Then there is a number $\lambda_0 > 0$ and a connected component Σ^* of $\overline{\lim}\ \Sigma_n^*$ such that

$$\Sigma^* \cap (\{\lambda\} \times M) \neq \emptyset \quad \textit{for each } \lambda \in (\lambda_0, \infty).$$

Lemma 2.3 ([15])**.** *Let $p > 1$. There exists a constant $C_p > 0$ such that*

$$\big(|x|^{p-2}x - |y|^{p-2}y, x-y\big) \ge \begin{cases} C_p|x-y|^p, & \text{if } p \ge 2 \\ C_p\frac{|x-y|^p}{(1+|x|+|y|)^{2-p}} & \text{if } p \le 2, \end{cases}$$

where $x, y \in \mathbb{R}^N$ and $(\cdot,\cdot)$ is the usual inner product of $\mathbb{R}^N$.

We also recall the following Hardy-type inequality (see [10]).

Lemma 2.4. *For any $p \in (1,\infty)$ and $s \in (0,1)$,*
$$\int_\Omega \frac{|u(x)|^p}{\mathrm{dist}(x,\partial\Omega)^{sp}}\,\mathrm{d}x \le C\|u\|^p, \quad u \in W_0^{s,p}(\Omega).$$

The next lemma, which will be proved later, is an important technical result because it proves C^α-regularity up to the boundary for the weak solutions of a non-linear problem driven by the fractional p-Laplacian operator. We denote the Euclidean distance from x to $\partial\Omega$ by
$$\mathrm{d}(x) = \mathrm{dist}(x,\partial\Omega).$$

Proposition 2.5. *Let $f \in L^\infty_{\rm loc}(\Omega)$ be a nonnegative function. Assume that there are $\beta, s \in (0,1)$ and $C > 0$ such that*
$$|f(x)| \le \frac{C}{\mathrm{d}^{s\beta}(x)}, \quad x \in \Omega. \tag{2.1}$$
Then there exists a unique weak solution $u \in W_0^{s,p}(\Omega)$ to the problem
$$\begin{gathered} (-\Delta_p)^s u = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Omega^c. \end{gathered} \tag{2.2}$$
Furthermore

(i) $u \in L^\infty(\Omega)$.
(ii) *There exist constants $\alpha \in (0,1)$ and $\Lambda > 0$ (Λ depending only on C, β, Ω) such that $u \in C^\alpha(\overline{\Omega})$ and $\|u\|_{C^\alpha(\overline{\Omega})} \le \Lambda$.*

Proof. A weak solution u to (2.2) satisfies (1.2) for $\lambda = 1$. So, the Browder-Minty Theorem guarantees that $(-\Delta_p)^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a homeomorphism. We denote
$$F_f(u) = \int_\Omega f u \,\mathrm{d}x, \ u \in W_0^{s,p}(\Omega).$$

We now prove that $F_f \in W^{-s,p'}(\Omega)$. In fact, let V be an open neighborhood of $\partial\Omega$ such that $0 < \mathrm{d}(x) < 1$ for all $x \in V$. Thus,
$$1 < \frac{1}{\mathrm{d}^{s\beta}(x)} < \frac{1}{\mathrm{d}^s(x)} \quad \forall x \in V.$$
Now, if $v \in W_0^{s,p}(\Omega)$, for a positive constant C_1 it holds
$$|F_f(v)| \le \int_\Omega |f|\,|v|\,\mathrm{d}x = \int_{V^c} |f|\,|v|\,\mathrm{d}x + \int_V |f|\,|v|\,\mathrm{d}x \le C_1\|v\| + \int_\Omega \Big|\frac{v}{d^s}\Big|\,\mathrm{d}x.$$
Applying Hölder's inequality and Lemma 2.4 we obtain a constant $C > 0$ such that
$$|F_f(v)| \le C\|v\|,$$
showing that $F_f \in W^{-s,p'}(\Omega)$. It follows that there exists a unique $u \in W_0^{s,p}(\Omega)$ such that $(-\Delta_p)^s u = F_f$, that is, u is a weak solution to problem (2.2).

To prove that $u \in L^\infty(\Omega)$, we define, for each $k \in \mathbb{N}$,
$$A_k := \{x \in \Omega : u(x) \ge k\}.$$
Denoting $(u-k)^+ := \max\{u-k, 0\}$, we have $(u-k)^+ \in W_0^{s,p}(\Omega)$. Since the inequality
$$|v(x)-v(y)|^{p-2}(v(x)-v(y))(v^+(x)-v^+(y)) \ge |v^+(x)-v^+(y)|^p \tag{2.3}$$

is valid for any measurable v, almost everywhere for $x, y \in \mathbb{R}^N$, taking $v^+ = (u-k)^+$ as a test function in (1.2) (with $\lambda = 1$), (2.3) yields

$$\begin{aligned}\iint_{\mathbb{R}^N} \frac{|v^+(x)-v^+(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y &\le \iint_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{p-1}(v^+(x)-v^+(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega f(x)v^+\,\mathrm{d}x.\end{aligned}$$

Then, as a consequence of [12, Lemma 5.1, Chapter 2], we conclude that there exists $k_1 > 0$, independent of u, such that

$$u \le k_1 \quad \text{a.e. in } \Omega. \tag{2.4}$$

Now, observe that the function $-u$ satisfies

$$\begin{gathered}(-\Delta_p)^s(-u) = -f \quad \text{in } \Omega \\ u = 0 \text{on } \Omega^c.\end{gathered}$$

Repeating the argument above we obtain $k_2 > 0$, independent of u, such that

$$-u \le k_2 \quad \text{a.e. in } \Omega. \tag{2.5}$$

From this and (2.4) we conclude the existence of $M > 0$ (independent of u) such that

$$|u(x)| \le M \text{ a.e in } \Omega,$$

proving that $\|u\|_{L^\infty(\Omega)} \le M$.

We shall now prove the existence of $\alpha \in (0,1)$ such that $u \in C^\alpha(\overline{\Omega})$. For any $x_0 \in \Omega$, take $R_0 := \frac{d(x_0)}{2}$. Then $B_{R_0}(x_0) \subset B_{2R_0}(x_0) \subset \Omega$. Let $u \in W^{s,p}(B_{2R_0}(x_0)) \cap L^\infty(B_{2R_0}(x_0))$ be the weak solution of (2.2). We have

$$(-\Delta_p)^s u = f(x) \le \frac{C}{\mathrm{d}^{s\beta}(x)} \le \frac{C}{R_0^{s\beta}} \quad \text{in } B_{R_0}(x_0).$$

By applying [11, corollary 5.5], we infer the existence of a constant $M > 0$ and $\alpha \in (0,1)$ such that

$$\begin{aligned}[u]_{C^\alpha(B_{R_0}(x_0))} &\le M\Big[\big(R_0^{s(p-\beta)}\big)^{\frac{1}{p-1}} + \Big(R_0^{sp}\int_{(B_{R_0}(x_0))^c} \frac{|u(y)|}{|x-y|^{N+sp}}\,\mathrm{d}x\Big)^{\frac{1}{p-1}}\Big]R_0^{-\alpha} \\ &\le \tilde{C}.\end{aligned} \tag{2.6}$$

The constant $\tilde{C}$ is independent of the choice of the point x_0 (and R_0). Because $u \in L^\infty(\Omega)$, by a covering argument for any $\Omega' \subset\subset \Omega$ we conclude that

$$\|u\|_{C^\alpha(\Omega')} \le C_{\Omega'},$$

completing the proof of the interior regularity.

To handle regularity up to the border, we establish a result that will also be used later.

Claim 1: There exist positive constants C_1 and C_2 such that, for any $0 < \epsilon < s$, we have

$$C_1\,\mathrm{d}^s \le u \le C_2\,\mathrm{d}^{s-\epsilon}, \quad \text{in } \Omega.$$

Proof. Set $f_n := \min\{n, f\}$. Since $f_n \in L^\infty(\Omega)$, it is clear that $F_{f_n} \in W^{-s,p'}(\Omega)$. So, for each $n \in \mathbb{N}$ there exists $u_n \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\begin{gathered}(-\Delta_p)^s u_n = f_n \quad \text{in } \Omega \\ u_n = 0 \quad \text{on } \Omega^c.\end{gathered}$$

Note also that $f_n \to \infty$ as $n \to \infty$ a.e., and $f_n \le f$ in Ω.

Let $\lambda_{s,p}$ be the first eigenvalue and $\varphi_{s,p}$ be a positive eigenfunction of the operator $(-\Delta_p)^s$. There exists a constant $c > 0$ such that

$$\frac{1}{c}\,\mathrm{d}^s(x) \le \varphi_{s,p}(x) \le c\,\mathrm{d}^s(x) \quad \text{for any } x \in \Omega.$$

Indeed, the upper estimate follows from [8, Theorem 3.2] and [11, Theorem 4.4], and the lower estimate from [11, Theorem 1.1] and [5, Theorem 1.5]. Hence, choosing a constant $a > 0$ small enough, for any $x \in \Omega$ it follows that

$$(-\Delta_p)^s(a\varphi_{s,p}) \le f_n(x) = (-\Delta_p)^s u_n \le f = (-\Delta_p)^s u$$

By applying [11, Proposition 2.10], we conclude the existence of $C_1 > 0$ such that

$$C_1\,\mathrm{d}^s(x) \le u_n(x) \le u(x) \quad \text{for any } x \in \Omega. \tag{2.7}$$

We now handle the upper estimate. Since $s\beta \in (0, s)$, we obtain

$$(-\Delta_p)^s u = f(x) \le K_{s\beta}(x) = (-\Delta_p)^s u_{s\beta},$$

where $u_{s\beta}$ is the solution obtained in [1, Theorem 4.2]. Therefore, $u \le u_{s\beta}$ in Ω. Another application of [1, Theorem 4.2 (ii)] yields

$$u \le C_2\,\mathrm{d}^{s-\epsilon} \quad \text{in } \Omega \text{ for any } \epsilon > 0,$$

completing the proof of our Claim. □

Now, since $u = 0$ in Ω^c, it is sufficient to prove the regularity in Ω_η for $\eta > 0$ small enough, where

$$\Omega_\eta := \{x \in \Omega : \mathrm{d}(x) < \eta\}.$$

Let $x, y \in \Omega_\eta$ and suppose, without loss of generality, $\mathrm{d}(x) \ge \mathrm{d}(y)$.

We consider two cases. If $|x-y| < \frac{\mathrm{d}(x)}{2}$, set $2R_0 = \mathrm{d}(x)$ and $y \in B_{R_0}(x)$. Hence we apply (2.6) in $B_{R_0}(x)$ and obtain the regularity. However, if $|x-y| \ge \frac{d(x)}{2} \ge \frac{d(y)}{2}$, since Claim 2 guarantees that $u \le C_2 d^\delta(x)$ for some $\delta, C_2 > 0$, we conclude that

$$\frac{|u(x)-u(y)|}{|x-y|^\delta} \le \frac{|u(x)|}{|x-y|^\delta} + \frac{|u(y)|}{|x-y|^\delta} \le C\Big(\frac{u(x)}{\mathrm{d}^\delta(y)} + \frac{u(y)}{\mathrm{d}^\delta(y)}\Big) \le C\,.$$

The proof is complete. □

Remark 2.6. Let us denote

$$\mathcal{M}_{\beta,\infty} = \big\{g \in L^\infty_{\rm loc}(\Omega) : |g(x)| \le \frac{C}{\mathrm{d}^{s\beta}(x)},\ x \in \Omega\big\}.$$

Then the solution operator associated with (2.2) is

$$S\colon \mathcal{M}_{\beta,\infty} \to W_0^{s,p}(\Omega) \cap C^\alpha(\overline{\Omega}), \quad S(g) = u.$$

Notice that

$$\|S(g)\|_{C^\alpha(\overline{\Omega})} \le M$$

for all $g \in \mathcal{M}_{\beta,\infty}$, with M depending only on C, β, Ω.

For each $s \in \mathbb{R}$ we consider $f_{\chi_I}(s)$, where χ_I is the characteristic function of the interval $I \subset \mathbb{R}$.

Corollary 2.7. *Let $f, \widetilde{f} \in L^\infty_{\rm loc}(\Omega)$ with $f \geq 0$, $f \neq 0$ satisfying (2.1). Then, for each $\epsilon > 0$, the problem*

$$\begin{gathered}(-\Delta_p)^s u_\epsilon = f\chi_{\{\,\mathrm{d}^s > \epsilon\}} + \widetilde{f}\chi_{\{\,\mathrm{d}^s < \epsilon\}} \quad \text{in } \Omega;\\ u_\epsilon = 0 \quad \text{on } \partial\Omega\end{gathered}$$

admits a unique solution $u_\epsilon \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$. In addition, for any solution u of (2.2) there exists $\epsilon_0 > 0$ such that

$$u_\epsilon \geq \frac{u}{2} \quad \text{in } \Omega \quad \text{for each } \epsilon \in (0, \epsilon_0).$$

Proof. Existence and uniqueness of u_ϵ follows directly from Proposition 2.5. If u is a the solution of (2.2), there exist $M > 0$ and $\alpha \in (0,1)$ such that

$$\|u\|_{C^\alpha(\overline{\Omega})}, \quad \|u_\epsilon\|_{C^\alpha(\overline{\Omega})} < M.$$

Claim 1 yields $u \geq C_1 \mathrm{d}^s$ in Ω. Multiplying the equation

$$(-\Delta_p)^s u - (-\Delta_p)^s u_\epsilon = f - \big(f\chi_{[\,\mathrm{d}^s(x) > \epsilon]} + \widetilde{f}\chi_{[\,\mathrm{d}^s(x) < \epsilon]}\big)$$

by $u - u_\epsilon$ and integrating we have

$$\begin{aligned}&\iint_{\mathbb{R}^N} \Big(\big(\frac{[u(x) - u(y)]^{p-1}}{|x-y|^{N+sp}} - \frac{[u_\epsilon(x) - u_\epsilon(y)]^{p-1}}{|x-y|^{N+sp}} \Big) \\ &\times \Big(\big(u(x) - u(y)\big) - \big(u_\epsilon(x) - u_\epsilon(y)\big) \Big) \,\mathrm{d}y\,\mathrm{d}x \\ &\leq 2M \int_{\mathrm{d}^s(x) < \epsilon} |f - \widetilde{f}|\,\mathrm{d}x.\end{aligned}$$

As a consequence of Lemma 2.3, we obtain $\|u - u_\epsilon\| \to 0$ as $\epsilon \to 0$.

If $\nu < \alpha$, the compact embedding $C^\alpha(\overline{\Omega}) \hookrightarrow C^\nu(\overline{\Omega})$ yields

$$\|u - u_\epsilon\|_{C^\nu(\overline{\Omega})} \leq \frac{C}{2}\,\mathrm{d}^s.$$

Therefore, for ϵ small enough, it follows from (2.7) that

$$u_\epsilon \geq u - \frac{C}{2}\,\mathrm{d}^s \;\geq\; u - \frac{u}{2} = \frac{u}{2} \quad \text{in } \Omega.$$

The proof is complete. ☐

The next result is crucial for this work.

Lemma 2.8. *Let $\beta \in (0,1)$. Then the problem*

$$\begin{gathered}(-\Delta_p)^s \phi = \frac{1}{\phi^\beta} \quad \text{in } \Omega,\\ \phi > 0 \quad \text{in } \Omega,\\ \phi = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{2.8}$$

admits a unique weak solution $\phi \in W^{s,p}_0(\Omega)$. Moreover $\phi \geq c\varphi_{s,p}$ in Ω for some constant $c > 0$. Here $\varphi_{s,p}$ is a positive eigenfunction for the operator $(-\Delta_p)^s$ associated with its first eigenvalue $\lambda_{s,p}$.

Proof. We consider the sequence of approximation problems

$$\begin{gathered}(-\Delta_p)^s\phi_n = \frac{1}{(\phi_n+\frac{1}{n})^\beta} \quad \text{in } \Omega,\\ \phi_n > 0 \quad \text{in } \Omega.\\ \phi_n = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{2.9}$$

As a consequence of [2, Proposition 2.3, Lemma 2.2, Lemma 3.1 and Lemma 3.4.], for any $n \geq 1$, there exists a weak solution $\phi_n \in W_0^{s,p}(\Omega)\cap L^\infty(\Omega)$ to problem (2.9), with $\{\phi_n\}$ bounded in $W_0^{s,p}(\Omega)$ and $\phi_n \leq \phi_{n+1}$.

Then, up to a subsequence, we have $\phi_n \rightharpoonup \phi$ in $W_0^{s,p}(\Omega)$, $\phi_n \to \phi$ in $L^r(\Omega)$ for $1 \leq r < p_s^*$ and $\phi_n \to \phi$ a.e. in Ω. By applying [2, Theorem 3.2.] we have that ϕ is a weak solution to problem (2.8).

Consider $c > 0$ such that $c^{p-1}\varphi_{s,p}^{p-1} \leq \frac{1}{(\|\phi_1\|_\infty+1)^\beta}$. We have

$$(-\Delta_p)^s(c\varphi_{s,p}) = c^{p-1}\varphi_{s,p}^{p-1} \leq \frac{1}{(\|\phi_1\|_\infty+1)^\beta} \leq \frac{1}{(\phi_1+1)^\beta} = (-\Delta_p)^s\phi_1.$$

Therefore, it follows from the comparison principle that

$$c\varphi_{s,p} \leq \phi_1 \leq \dots \leq \phi_n \leq \dots \leq \phi. \tag{2.10}$$

Combining the left-hand side of (2.9) with (2.10), we obtain $\phi \geq c\varphi_{s,p}$ in Ω for some constant $c > 0$. □

3. Lower and upper solutions

In this section we prove the existence of both a lower and an upper solutions to problem (1.1). For the convenience of the reader, we start by stating some definitions.

Definition 3.1. A function $\underline{u} \in W_0^{s,p}(\Omega)$ with $\underline{u} > 0$ in Ω such that

$$\iint_{\mathbb{R}^N} \frac{[\underline{u}(x)-\underline{u}(y)]^{p-1}\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,\mathrm{d}y\,\mathrm{d}x \leq \lambda\int_\Omega f(\underline{u})\varphi\,\mathrm{d}x,$$

for all $\varphi \in W_0^{s,p}(\Omega)$, $\varphi \geq 0$ is a lower solution of (1.1).

A function $\overline{u} \in W_0^{s,p}(\Omega)$ with $\overline{u} > 0$ in Ω such that

$$\iint_{\mathbb{R}^N} \frac{[\overline{u}(x)-\overline{u}(y)]^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,\mathrm{d}y\,\mathrm{d}x \geq \lambda\int_\Omega f(\overline{u})\varphi\,\mathrm{d}x,$$

for all $\varphi \in W_0^{s,p}(\Omega)$, $\varphi \geq 0$ is called an upper solution of (1.1).

Theorem 3.2. *Assume* (A1) *and* (A2). *Then there exist $\lambda_0 > 0$ and a non-negative function $\psi \in C^\alpha(\overline{\Omega})$, with $\psi > 0$ in Ω, $\psi = 0$ in Ω^c, $\alpha \in (0,1)$ such that for each $\lambda \in [\lambda_0,\infty)$, $\underline{u} = \lambda^r\psi$ is a lower solution of* (1.1)*, where $r = 1/(p+\beta-1)$.*

Proof. According to (A2), there exists $b > 0$ such that

$$f(t) > -\frac{b}{t^\beta} \quad \text{if } t > 0. \tag{3.1}$$

Applying Lemma 2.8 there exist both a function $\phi \in W_0^{s,p}(\Omega)$ such that

$$\begin{gathered}(-\Delta_p)^s\phi = \frac{1}{\phi^\beta} \quad \text{in } \Omega,\\ \phi > 0 \quad \text{in } \Omega,\\ \phi = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{3.2}$$

and a constant $c > 0$ such that $\phi \geq c\varphi_{s,p}$ in Ω. Thus by (2.9) we obtain

$$\phi \geq c\,\mathrm{d}^s \quad \text{in } \Omega. \tag{3.3}$$

Now, take $\delta = a^{\frac{p-1}{\beta-1+p}}$ and $\gamma = 2^\beta b \delta^{-\frac{\beta}{p-1}}$, where a is the constant given in (A2). According to Corollary 2.7, there exists a constant $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, the problem

$$\begin{gathered} (-\Delta_p)^s \psi = \delta\phi^{-\beta}\chi_{[\,\mathrm{d}^s>\epsilon]} - \gamma\phi^{-\beta}\chi_{[\,\mathrm{d}^s<\epsilon]} \quad \text{in } \Omega, \\ \psi > 0 \quad \text{in } \Omega, \\ \psi = 0 \quad \text{in } \Omega^c \end{gathered} \tag{3.4}$$

admits a solution $\psi \in C^\alpha(\overline{\Omega})$ satisfying

$$\psi \geq \Big(\frac{\delta^{1/(p-1)}}{2}\Big)\phi. \tag{3.5}$$

If $\lambda > 0$ and $r = 1/(p+\beta-1)$, we define $\underline{u} = \lambda^r\psi$.

Now, take $\lambda_0 = \big[\frac{2A}{(C_1\epsilon\delta^{\frac{1}{p-1}})}\big]^{1/r}$, where $\epsilon \in (0, \epsilon_0)$ and A is given by (A2).

Claim 2: $\underline{u}$ is a lower solution of (1.1) for any $\lambda \geq \lambda_0$.

Indeed, take $\xi \in W_0^{s,p}(\Omega)$, $\xi \geq 0$. As a consequence of (3.4), we have

$$\begin{aligned} &\iint_{\mathbb{R}^N} \frac{[\underline{u}(x) - \underline{u}(y)]^{p-1}\,(\xi(x) - \xi(y))}{|x-y|^{N+sp}}\,\mathrm{d}y\,\mathrm{d}x \\ &= \lambda^{r(p-1)}\delta \int_{\{\,\mathrm{d}^s>\epsilon\}} \frac{\xi}{\phi^\beta}\,\mathrm{d}x - \lambda^{r(p-1)}\gamma \int_{\{\,\mathrm{d}^s<\epsilon\}} \frac{\xi}{\phi^\beta}\,\mathrm{d}x. \end{aligned}$$

We consider two cases.

Case 1: $\mathrm{d}^s > \epsilon$. For each $\lambda \geq \lambda_0$, by using (3.3) and (3.4), we obtain

$$\underline{u} = \lambda^r\psi \geq \lambda^r \frac{\delta^{\frac{1}{p-1}}}{2}\phi \geq \lambda^r \frac{\delta^{\frac{1}{p-1}}}{2} C_1 d^s > \lambda^r \frac{\delta^{\frac{1}{p-1}}}{2} C_1 \epsilon > A.$$

So, $\underline{u}(x) > A$ for each $\lambda \geq \lambda_0$ with $d^s(x) > \epsilon$. According to (3.2) and (3.3), we have

$$(-\Delta_p)^s \delta^{\frac{1}{p-1}}\phi = \frac{\delta}{\phi^\beta} \geq (-\Delta_p)^s\psi.$$

Thus, the weak comparison principle implies that

$$\delta^{\frac{1}{p-1}}\phi \geq \psi \quad \text{in } \Omega. \tag{3.6}$$

It follows from (A2) and (3.6) that

$$\begin{aligned} \lambda \int_{\mathrm{d}^s>\epsilon} f(\underline{u})\xi\,\mathrm{d}x &\geq \lambda a \int_{\mathrm{d}^s>\epsilon} \frac{\xi}{\underline{u}^\beta}\,\mathrm{d}x \\ &= \lambda^{1-r\beta} a \int_{\mathrm{d}^s>\epsilon} \frac{\xi}{\psi^\beta}\,\mathrm{d}x \\ &\geq \lambda^{\frac{p-1}{p+\beta-1}} \frac{a}{\delta^{\frac{\beta}{p-1}}} \int_{\mathrm{d}^s>\epsilon} \frac{\xi}{\phi^\beta}\,\mathrm{d}x \\ &= \lambda^{r(p-1)}\delta \int_{\mathrm{d}^s>\epsilon} \frac{\xi}{\phi^\beta}\,\mathrm{d}x. \end{aligned} \tag{3.7}$$

Case 2: $\mathrm{d}^s < \epsilon$. Applying (3.1) and (3.5) we obtain

$$
\begin{aligned}
\lambda \int_{\{\,\mathrm{d}<\epsilon\}} f(\underline{u})\xi \,\mathrm{d}x &\geq -\lambda b \int_{\{\,\mathrm{d}<\epsilon\}} \frac{\xi}{\underline{u}^\beta}\,\mathrm{d}x \\
&= -\lambda^{1-r\beta} b \int_{\mathrm{d}<\epsilon} \frac{\xi}{\psi^\beta}\,\mathrm{d}x \\
&\geq -\lambda^{r(p-1)} b \frac{2^\beta}{\delta^{\frac{\beta}{p-1}}} \int_{\mathrm{d}<\epsilon} \frac{\xi}{\phi^\beta}\,\mathrm{d}x \\
&= -\lambda^{r(p-1)} \gamma \int_{\mathrm{d}<\epsilon} \frac{\xi}{\phi^\beta}\,\mathrm{d}x.
\end{aligned} \tag{3.8}
$$

It follows from (3.7) and (3.8) that

$$
\lambda \int_\Omega f(\underline{u})\xi \,\mathrm{d}x \geq \iint_{\mathbb{R}^N} \frac{[\underline{u}(x)-\underline{u}(y)]^{p-1}(\xi(x)-\xi(y))}{|x-y|^{N+sp}}\,\mathrm{d}y\,\mathrm{d}x\,.
$$

The proof is complete. □

Next, we show the existence of an upper solution.

Theorem 3.3. *Assume* (A1) *and* (A2) *and let* $\Lambda > \lambda_0$ *with* λ_0 *be as in Theorem 3.2. Then for each* $\lambda \in [\lambda_0, \Lambda]$, (1.1) *admits an upper solution* $\overline{u} = \overline{u}_\lambda = M\phi$ *where* $M > 0$ *is a constant and* ϕ *is given by* (3.2).

Proof. Choose $\overline{\epsilon} > 0$ such that

$$
\Lambda\overline{\epsilon}\,||\phi||_\infty^{p-1+\beta} < \frac{1}{2}. \tag{3.9}
$$

According to (A1) and (A2), there exist $A_1 > 0$ and $C > 0$ such that

$$
|f(u)| \leq \overline{\epsilon} u^{p-1} \quad \text{for } u > A_1, \tag{3.10}
$$

$$
|f(u)| \leq \frac{C}{u^\beta} \quad \text{for } u \leq A_1. \tag{3.11}
$$

Choose

$$
M \geq \max\big\{\Lambda^r \delta^{\frac{1}{p-1}},\, (2\Lambda C)^{\frac{1}{p+\beta-1}}\big\}. \tag{3.12}
$$

Now, (3.9) and (3.12) yield

$$
\Lambda\overline{\epsilon}\,(M\|\phi\|_\infty)^{p+\beta-1} + \Lambda C \leq \frac{M^{p+\beta-1}}{2} + \frac{M^{p+\beta-1}}{2} = M^{p+\beta-1}. \tag{3.13}
$$

Let $\overline{u} = M\phi$. By taking $\lambda \leq \Lambda$, it follows from (3.10)-(3.11) that

$$
\begin{aligned}
\lambda f(\overline{u}) &\leq \lambda |f(\overline{u})| \leq \lambda\big[\overline{\epsilon}\,\overline{u}^{p-1}\chi_{\{\overline{u}>A_1\}} + \frac{C}{\overline{u}^\beta}\chi_{\{\overline{u}\leq A_1\}}\big] \\
&\leq \lambda\big[\overline{\epsilon}\,\overline{u}^{p-1}\chi_{\{\overline{u}>A_1\}} + \overline{\epsilon}\,\overline{u}^{p-1}\chi_{\{\overline{u}\leq A_1\}} + \frac{C}{\overline{u}^\beta}\chi_{\{\overline{u}\leq A_1\}} + \frac{C}{\overline{u}^\beta}\chi_{\{\overline{u}>A_1\}}\big] \\
&= \lambda[\overline{\epsilon u}^{p-1} + \frac{C}{\overline{u}^\beta}].
\end{aligned} \tag{3.14}
$$

We conclude that

$$
\begin{aligned}
\lambda f(M\phi) &\leq \lambda\big[\frac{\overline{\epsilon}(M\|\phi\|_\infty)^{p+\beta-1} + C}{[M\phi]^\beta}\big] \\
&\leq \Lambda\frac{\overline{\epsilon}(M\|\phi\|_\infty)^{p+\beta-1}}{[M\phi]^\beta} + \Lambda\frac{C}{[M\phi]^\beta}.
\end{aligned} \tag{3.15}
$$

Replacing (3.13) and (3.14) into (3.15), we obtain

$$\lambda f(M\phi) \leq \frac{M^{p+\beta-1}}{[M\phi]^\beta} = \frac{M^{p-1}}{\phi^\beta}.$$

Thus

$$\lambda f(\overline{u}) \leq \frac{M^{p-1}}{\phi^\beta}.$$

Now, taking a non-negative $\eta \in W_0^{s,p}(\Omega)$, it follows from (3.2) that

$$\begin{aligned}
\lambda \int_\Omega f(\overline{u})\eta \,\mathrm{d}x &\leq M^{p-1} \int_\Omega \frac{\eta}{\phi^\beta}\,\mathrm{d}x \\
&= M^{p-1} \iint_{\mathbb{R}^N} \frac{[\phi(x)-\phi(y)]^{p-1}(\eta(x)-\eta(y))}{|x-y|^{N+sp}}\,\mathrm{d}x \\
&= \iint_{\mathbb{R}^N} \frac{[M\phi(x)-M\phi(y)]^{p-1}(\eta(x)-\eta(y))}{|x-y|^{N+sp}}\,\mathrm{d}x \\
&= \iint_{\mathbb{R}^N} \frac{[\overline{u}(x)-\overline{u}(y)]^{p-1}(\eta(x)-\eta(y))}{|x-y|^{N+sp}}\,\mathrm{d}x,
\end{aligned}$$

showing that $\overline{u} = M\phi$ is an upper solution of (1.1) for $\lambda \in [\lambda_0, \Lambda]$. □

Lemma 3.4. *If $u \in W_0^{s,p}(\Omega)$ be a weak solution of problem* (1.1)*. Then $u \in L^\infty(\Omega)$.*

Proof. If $u \in W_0^{s,p}(\Omega)$ solves (1.1), then

$$\langle(-\Delta_p)^s, \phi\rangle = \iint_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(u)v\,\mathrm{d}x \tag{3.16}$$

for any $v \in W_0^{s,p}(\Omega)$.

For each $k \in \mathbb{N}$, set $A_k := \{x \in \Omega : u(x) > k\}$. Since $u \in W_0^{s,p}(\Omega)$ and $u > 0$ in Ω, we have that $(u-k)^+ \in W_0^{s,p}(\Omega)$. Taking $v = (u-k)^+$ in (3.16), we obtain

$$\langle(-\Delta_p)^s, (u-k)^+\rangle = \int_\Omega f(u)(u-k)^+\,\mathrm{d}x. \tag{3.17}$$

Applying the algebraic inequality $|a-b|^{p-2}(a-b)(a^+-b^+) \geq |a^+-b^+|^p$ to estimate the left-hand side of (3.17), we obtain

$$\begin{aligned}
\Big(\int_{A_k}(u-k)^{p_s^*}\,\mathrm{d}x\Big)^{\frac{p}{p_s^*}} &\leq C\iint_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq C\langle(-\Delta_p)^s, (u-k)^+\rangle \\
&= C\int_{A_k} f(u)(u-k)^+\,\mathrm{d}x.
\end{aligned}$$

Now we estimate the right hand side of (3.17). It follows from (A1) and (A2) the existence of a number $M > 0$ such that

$$|f(t)| \leq M\big(\frac{1}{t^\beta} + t^{p-1}\big), \quad \forall t > 0.$$

Therefore, if $k > 1$, we have

$$\int_{A_k} f(u)(u-k)^+\,\mathrm{d}x \leq 2M\int_{A_k} u^{p-1}(u-k)\,\mathrm{d}x. \tag{3.18}$$

Since $u^{p-1}(u-k) \le 2^{p-1}(u-k)^p + 2^{p-1}k^{p-1}(u-k)$, it follows that

$$\int_{A_k} u^{p-1}(u-k)\,\mathrm{d}x \le 2^{p-1}\int_{A_k}(u-k)^p\,\mathrm{d}x + 2^{p-1}k^{p-1}\int_{A_k}(u-k)\,\mathrm{d}x.$$

Applying Hölder's inequality, we obtain

$$\int_{A_k}(u-k)^p\,\mathrm{d}x \le |A_k|^{\frac{p_s^*-p}{p_s^*}}\Big(\int_{A_k}(u-k)^{p_s^*}\,\mathrm{d}x\Big)^{\frac{p}{p_s^*}}. \tag{3.19}$$

So, as a consequence of (3.18)-(3.19), we have

$$\int_{A_k}(u-k)^p\,\mathrm{d}x \le |A_k|^{\frac{p_s^*-p}{p_s^*}}2MC\Big[2^{p-1}\int_{A_k}(u-k)^p\,\mathrm{d}x + 2^{p-1}k^{p-1}\int_{A_k}(u-k)\,\mathrm{d}x\Big].$$

Denoting $L = 2MC$ yields

$$[1-2^{p-1}L|A_k|^{\frac{p_s^*-p}{p_s^*}}]\int_{A_k}(u-k)^p\,\mathrm{d}x \le 2^{p-1}k^{p-1}L|A_k|^{\frac{(p_s^*-p)}{p_s^*}}\int_{A_k}(u-k)\,\mathrm{d}x.$$

If $k\to\infty$, then $|A_k|\to 0$. Therefore, there exists $k_0>0$ such that

$$1-2^{p-1}L|A_k|^{\frac{p_s^*-p}{p_s^*}} \ge \frac{1}{2} \quad \text{if } k\ge k_0>1.$$

Thus, for such k, we conclude that

$$\frac{1}{2}\int_{A_k}(u-k)^p\,\mathrm{d}x \le 2^{p-1}k^{p-1}L|A_k|^{\frac{p_s^*-p}{p_s^*}}\int_{A_k}(u-k)\,\mathrm{d}x. \tag{3.20}$$

Hölder's inequality and (3.20) yield

$$\int_{A_k}(u-k)^p\,\mathrm{d}x \le |A_k|^{p-1}\int_{A_k}(u-k)^p\,\mathrm{d}x \le |A_k|^{p-1}2^{p-1}k^{p-1}L|A_k|^{\frac{p_s^*-p}{p_s^*}}\int_{A_k}(u-k)\,\mathrm{d}x.$$

Therefore,

$$\int_{A_k}(u-k)\,\mathrm{d}x \le \gamma k|A_k|^{1+\epsilon}, \quad \forall k\ge k_0, \tag{3.21}$$

where $\gamma^{p-1} = 2^2L$ and $\epsilon = \frac{p_s^*-p}{p_s^*(p-1)} > 0$. Set

$$g(k) := \int_{A_k}(u-k)\,\mathrm{d}x = \int_k^\infty |A_t|dt,$$

where the equality between integrals is a consequence of Cavaliere's Principle. By (3.21) it follows that

$$g(k) \le \gamma k[-g'(k)]^{1+\epsilon}. \tag{3.22}$$

Taking $k>k_0$ and integrating (3.22) from k_0 to k, since $g(k)>0$ it follows that

$$\frac{1}{\gamma^{\frac{1}{1+\epsilon}}}\big[k^{\frac{\epsilon}{1+\epsilon}}\big] \le \big\{[g(k_0)]^{\frac{\epsilon}{1+\epsilon}} - [g(k)]^{\frac{\epsilon}{1+\epsilon}}\big\} \le [g(k_0)]^{\frac{\epsilon}{1+\epsilon}}.$$

Thus

$$k \le \gamma^{\frac{1}{1+\gamma}}[g(k_0)]^{\frac{\epsilon}{1+\epsilon}} - k_0^{\frac{\epsilon}{1+\epsilon}}.$$

We denote $\Lambda = \frac{1}{1+\gamma}[g(k_0)]^{\frac{\epsilon}{1+\epsilon}} - k_0^{\frac{\epsilon}{1+\epsilon}}$. Note that $k\le\Lambda$, if $|A_k|>0$. Since Λ does not depend on k, we conclude that $|A_k| = 0$ for all $k>\Lambda$, that is, $u\in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \le \gamma^{\frac{1}{1+\gamma}}[g(k_0)]^{\frac{\epsilon}{1+\epsilon}} - k_0^{\frac{\epsilon}{1+\epsilon}}.$$

□

4. Finding a solution for (1.1)

Take $\Lambda > \lambda_0$ and set $I_\Lambda := [\lambda_0, \Lambda]$. For each $\lambda \in I_\Lambda$, according to Theorem 3.2,

$$\underline{u} = \underline{u}_\lambda = \lambda^r \psi$$

is a lower solution of (1.1). Let $M = M_\Lambda \geq \Lambda^r \delta^{\frac{1}{p-1}}$. By Theorem 3.3 we have that

$$\overline{u} = \overline{u}_\lambda = M_\Lambda \phi$$

is an upper solution of (1.1). It follows from (3.6) that

$$\underline{u} = \lambda^r \psi \leq \Lambda^r \delta^{\frac{1}{p-1}} \phi \leq M\phi = \overline{u}. \tag{4.1}$$

We consider the convex, closed subset of $I_\Lambda \times C(\overline{\Omega})$ given by

$$\mathcal{G}_\Lambda := \left\{(\lambda, u) \in I_\Lambda \times C(\overline{\Omega}) : \lambda \in I_\Lambda,\ \underline{u} \leq u \leq \overline{u} \text{ and } u = 0 \text{ on } \Omega^c\right\}.$$

For each $u \in C(\overline{\Omega})$, set

$$f_\Lambda(u) = \chi_{S_1} f(\underline{u}) + \chi_{S_2} f(u) + \chi_{S_3} f(\overline{u}), \quad x \in \Omega,$$

where χ_{S_i} denotes the characteristic function of S_i, which are defined by

$$\begin{gathered}
S_1 = \{x \in \Omega : u(x) < \underline{u}(x)\}, \\
S_2 = \{x \in \Omega : \underline{u}(x) \leq u(x) \leq \overline{u}(x)\}, \\
S_3 = \{x \in \Omega : \overline{u}(x) < u(x)\}.
\end{gathered}$$

Lemma 4.1. *For each $u \in C(\overline{\Omega})$, $f_\Lambda(u) \in L^\infty_{\rm loc}(\Omega)$ and there exist $C > 0$ and $\beta \in (0,1)$ such that*

$$|f_\Lambda(u)(x)| \leq \frac{C}{\mathrm{d}^{s\beta}(x)}, \quad x \in \Omega. \tag{4.2}$$

Proof. Let $\mathcal{K} \subset \Omega$ be a compact subset. Then both $\underline{u}$ and $\overline{u}$ achieve a positive maximum and a positive minimum on $\mathcal{K}$. Since f is continuous in $(0, \infty)$, we conclude that $f_\Lambda(u) \in L^\infty_{\rm loc}(\Omega)$.

Since $\Omega = \cup_{i=1}^3 S_i$, to prove (4.2) it suffices to show that

$$|f(u(x))| \leq \frac{C}{\mathrm{d}^{s\beta}(x)}, \quad x \in S_i,\ i = 1, 2, 3.$$

According to hypothesis (A2), there are $C, \delta > 0$ such that

$$|f(s)| \leq \frac{C}{s^\beta}, \quad 0 < s < \delta.$$

Let

$$\Omega_\delta = \{x \in \Omega :\ \mathrm{d}^s(x) < \delta\}.$$

Recalling that $\underline{u} \in C^\alpha(\overline{\Omega})$ if $\alpha \in (0,1)$, we denote

$$D = \max_{\overline{\Omega}} \mathrm{d}^s(x), \quad \nu_\delta := \min_{\overline{\Omega^c_\delta}} \mathrm{d}^s(x), \quad \nu^\delta := \max_{\overline{\Omega^c_\delta}} \mathrm{d}^s(x)$$

and observe that $0 < \nu_\delta \leq \nu^\delta \leq D < \infty$ and also that $f([\nu_\delta, \nu^\delta])$ is compact.

Applying Theorems 3.2 and 3.3, Lemma 2.8 and inequalities (2.5) and (4.1), we infer that

$$0 < \lambda_0^r \psi \leq \lambda^r \psi = \underline{u} \leq \overline{u} = M\phi \quad \text{in } \Omega$$

and

$$\frac{1}{\underline{u}^\beta},\ \frac{1}{\overline{u}^\beta} \leq \frac{1}{(\lambda_0^r \psi(x))^\beta} \leq \frac{C}{\mathrm{d}^{s\beta}(x)}, \quad x \in \Omega_\delta.$$

To complete the proof, we consider three cases:
(i) $x \in S_1$. In this case, $f_\Lambda(u(x)) = f(\underline{u}(x))$. If $x \in S_1 \cap \Omega_\delta$, we infer that

$$|f_\Lambda(u(x))| \le \frac{C}{\underline{u}^\beta(x)} \le \frac{C}{\mathrm{d}^{s\beta}(x)}.$$

However, if $x \in S_1 \cap \Omega_\delta^c$, take positive numbers d_i $(i = 1, 2)$ such that

$$d_1 \le \underline{u}(x) \le d_2, \quad x \in \Omega_\delta^c.$$

Hence

$$|f_\Lambda(u(x))| \le \frac{C}{\mathrm{d}^{s\beta}(x)}, \quad x \in S_1.$$

(ii) $x \in S_2$. In this case

$$0 < \lambda_0^r \psi \le u \le M\phi,$$

and, as a consequence,

$$|f(u(x))| \le \frac{C}{u^\beta(x)}, \quad x \in \Omega_\delta.$$

Hence, there is a positive constant $\widetilde{C}$ such that

$$|f(u(x))| \le \widetilde{C}, \quad x \in \overline{\Omega_\delta^c}.$$

Thus

$$|f(u(x))| \le \begin{cases} \widetilde{C} & \text{if } x \in \overline{\Omega_\delta^c}, \\ \frac{C}{\mathrm{d}^{s\beta}(x)} & \text{if } x \in \Omega_\delta. \end{cases}$$

We also have

$$\frac{1}{D^\beta} \le \frac{1}{\mathrm{d}^{s\beta}(x)}, \quad x \in \overline{\Omega_\delta^c},$$

and therefore there exist a constant $C > 0$ such that

$$|f(u(x))| \le \begin{cases} \frac{C}{D^\beta} & \text{if } x \in \overline{\Omega_\delta^c}, \\ \frac{C}{\mathrm{d}^{s\beta}(x)} & \text{if } x \in \Omega_\delta. \end{cases}$$

Thus,

$$|f(u(x))| \le \frac{C}{\mathrm{d}^{s\beta}(x)}, \quad x \in S_2.$$

(iii) $x \in S_3$. In this case $f_\Lambda(u(x)) = f(\overline{u}(x))$. The proof is similar to the case (i). □

Remark 4.2. According to Proposition 2.5, Lemma 4.1 and Remark 2.6, for each $v \in C(\overline{\Omega})$ and $\lambda \in I_\Lambda$, we have

$$\lambda f_\Lambda(v) \in L^\infty_{\rm loc}(\Omega) \quad \text{and} \quad |\lambda f_\Lambda(v)| \le \frac{C_\Lambda}{\mathrm{d}^{s\beta}(x)} \quad \text{in } \Omega, \tag{4.3}$$

where $C_\Lambda > 0$ is a constant independent of v and $\beta \in (0, 1)$. So, for each v,

$$\begin{gathered} (-\Delta_p)^s u = \lambda f_\Lambda(v) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Omega^c \end{gathered}$$

admits a unique solution $u = S(\lambda f_\Lambda(v)) \in W_0^{s,p}(\Omega) \cap C^\alpha(\overline{\Omega})$.

Set

$$F_\Lambda(u)(x) = f_\Lambda(u(x)),\ u \in C(\overline{\Omega}).$$

and consider the operator $T : I_\Lambda \times C(\overline{\Omega}) \to W_0^{s,p}(\Omega) \cap C^\alpha(\overline{\Omega})$, defined by

$$T(\lambda, u) = S(\lambda F_\Lambda(u)) \quad \text{if } \lambda_0 \le \lambda \le \Lambda,\ u \in C(\overline{\Omega}).$$

Observe that, if $(\lambda, u) \in I_\Lambda \times C(\overline{\Omega})$ is such that $u = T(\lambda, u)$, then u is a solution to the problem

$$\begin{gathered}(-\Delta_p)^s u = \lambda f_\Lambda(u) \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \Omega^c.\end{gathered}$$

Lemma 4.3. *If $(\lambda, u) \in I_\Lambda \times C(\overline{\Omega})$ and $u = T(\lambda, u)$, then $(\lambda, u) \in \mathcal{G}_\Lambda$.*

Proof. Suppose that $(\lambda, u) \in I_\Lambda \times C(\overline{\Omega})$ satisfies $T(\lambda, u) = u$. Then

$$\iint_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y = \lambda \int_\Omega f_\Lambda(u) v\,\mathrm{d}x, \quad \forall v \in W_0^{s,p}(\Omega).$$

We claim that $u \ge \underline{u}$. Assume, by contradiction, that $v := (\underline{u} - u)^+ \not\equiv 0$. Then

$$\begin{aligned}
&\iint_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{u<\underline{u}} \frac{[u(x)-u(y)]^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&= \lambda \int_{u<\underline{u}} f_\Lambda(u) v\,\mathrm{d}x = \lambda \int_{u<\underline{u}} f(\underline{u}) v\,\mathrm{d}x\\
&\ge \iint_{u<\underline{u}} \frac{[\underline{u}(x)-\underline{u}(y)]^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{\mathbb{R}^N} \frac{[\underline{u}(x)-\underline{u}(y)]^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Hence

$$\iint_{\mathbb{R}^N} \Big[\frac{[\underline{u}(x)-\underline{u}(y)]^{p-1}}{|x-y|^{N+sp}} - \frac{[u(x)-u(y)]^{p-1}}{|x-y|^{N+sp}}\Big](v(x)-v(y))\,\mathrm{d}x\,\mathrm{d}y \le 0.$$

It follows that

$$\iint_{\mathbb{R}^N} \frac{|(\underline{u}(x)-u(x)) - (\underline{u}(y)-u(y))|^p}{|x-y|^{N+sp}}\,\mathrm{d}y\,\mathrm{d}x \le 0,$$

contradicting $\varphi \not\equiv 0$. Thus, $(\underline{u} - u)^+ = 0$, that is, $\underline{u} - u \le 0$, and so $\underline{u} \le T(\lambda, u)$.

Similarly, we obtain $u \le \overline{u}$ in Ω, which gives $\overline{u} \ge T(\lambda, u)$. the proof is complete. □

Remark 4.4. Observe that the definitions of f_Λ and $\mathcal{G}_\Lambda$ imply that, for each $(\lambda, u) \in \mathcal{G}_\Lambda$, we have $f_\Lambda(u) = f(u)$ for $x \in \Omega$.

Remark 4.5. According to Remark 2.6, there exists $R_\Lambda > 0$ such that $\mathcal{G}_\Lambda \subset B(0, R_\Lambda) \subset C(\overline{\Omega})$ and

$$T\big(I_\Lambda \times \overline{B(0, R_\Lambda)}\big) \subseteq B(0, R_\Lambda).$$

Note that, by (4.3) and Lemma 4.3, if $(\lambda, u) \in I_\Lambda \times C(\overline{\Omega})$ satisfies $u = T(\lambda, u)$ then (λ, u) is a solution of (P_λ). So, Remark 4.2 shows that it suffices to find a fixed point of T in order to solve (1.1).

Lemma 4.6. *The mapping* $T\colon I_\Lambda \times \overline{B(0,R_\Lambda)} \to \overline{B(0,R_\Lambda)}$ *is continuous and compact.*

Proof. Let $\{(\lambda_n,u_n)\} \subseteq I_\Lambda \times \overline{B(0,R_\Lambda)}$ be a sequence such that $\lambda_n \to \lambda$ and $u_n \to u$ in $C(\overline{\Omega})$, as $n \to \infty$. Set

$$v_n = T(\lambda_n,u_n) \quad \text{and} \quad v = T(\lambda,u)$$

so that

$$v_n = S(\lambda_n F_\Lambda(u_n)) \quad \text{and} \quad v = S(\lambda F_\Lambda(u)).$$

It follows that

$$\begin{aligned}
&\iint_{\mathbb{R}^N} \Big[\frac{[v_n(x)-v_n(y)]^{p-1}}{|x-y|^{N+sp}} - \frac{[v(x)-v(y)]^{p-1}}{|x-y|^{N+sp}}\Big](v_n(x)-v(y))\,\mathrm{d}x\,\mathrm{d}y \\
&= \lambda_n \int_\Omega (f_\Lambda(u_n)-f_\Lambda(u))\,(v_n-v)\,\mathrm{d}x \\
&\leq C\int_\Omega |f_\Lambda(u_n)-f_\Lambda(u)|\,\mathrm{d}x.
\end{aligned}$$

Since

$$|f_\Lambda(u_n)-f_\Lambda(u)| \leq \frac{C}{\mathrm{d}^{s\beta}(x)} \in L^1(\Omega)$$

and $f_\Lambda(u_n(x)) \to f_\Lambda(u(x))$ a.e. $x\in\Omega$, as $n\to\infty$, it follows that

$$\int_\Omega |f_\Lambda(u_n)-f_\Lambda(u)|\,\mathrm{d}x \to 0, \quad \text{as } n\to\infty.$$

Therefore $v_n \to v$ as $n \to \infty$ in $W_0^{1,p}(\Omega)$.

On the other hand, since $u_n \to u$ in $C(\overline{\Omega})$, as $n\to\infty$, the proof of Lemma 4.1 shows that

$$\lambda_n f_\Lambda(u_n) \in L^\infty_{\rm loc}(\Omega) \quad \text{and} \quad |\lambda_n f_\Lambda(u_n)| \leq \frac{C_\Lambda}{\mathrm{d}^{s\beta}(x)} \quad \text{in } \Omega.$$

Proposition 2.5 guarantees the existence of a constant $M>0$ such that

$$\|v_n\|_{C^\alpha(\overline{\Omega})} \leq M,$$

so that $v_n \to v$ in $C(\overline{\Omega})$. This shows that $T\colon I_\Lambda \times \overline{B(0,R_\Lambda)} \to \overline{B(0,R_\Lambda)}$ is continuous. The compactness of T is a consequence. □

5. Bounded connected sets of solutions of (1.1)

We recall the Leray-Schauder Continuation Theorem (see [6]) for the convenience of the reader.

Theorem 5.1. *Let D be an open bounded subset of the Banach space X. Let $a,b\in\mathbb{R}$ with $a<b$ and assume that $T:[a,b]\times\overline{D}\to X$ is compact and continuous. Consider $\Phi\colon[a,b]\times\overline{D}\to X$ defined by $\Phi(t,u)=u-T(t,u)$. Assume that*

- (i) $\Phi(t,u)\neq 0$ *for all* $t\in[a,b]$ *and all* $u\in\partial D$;
- (ii) $\deg\big(\Phi(t,.),D,0\big)\neq 0$ *for some* $t\in[a,b]$

and set

$$\mathcal{S}_{a,b} = \{(t,u)\in[a,b]\times\overline{D} : \Phi(t,u)=0\}.$$

Then, there exists a connected compact subset $\Sigma_{a,b}$ of $\mathcal{S}_{a,b}$ such that

$$\Sigma_{a,b}\cap(\{a\}\times D)\neq\emptyset \quad \textit{and} \quad \Sigma_{a,b}\cap(\{b\}\times D)\neq\emptyset.$$

Consider $\Phi : I_\Lambda \times \overline{B(0,R)} \to \overline{B(0,R)})$ defined by

$$\Phi(\lambda, u) = u - T(\lambda, u).$$

Lemma 5.2. Φ *satisfies:*

(i) $\Phi(\lambda, u) \neq 0 \ \forall (\lambda, u) \in I_\Lambda \times \partial B(0, R_\Lambda)$,
(ii) $\deg(\Phi(\lambda, .), B(0, R_\Lambda), 0) \neq 0$ *for each* $\lambda \in I_\Lambda$,

Proof. The verification of (i) is straightforward, since $T\big(I_\Lambda \times \overline{B(0, R_\Lambda)}\big) \subset B(0, R_\Lambda)$.

To prove (ii), set $R = R_\Lambda$, take $\lambda \in I_\Lambda$ and consider the homotopy

$$\Psi_\lambda(t, u) = u - tT(\lambda, u), \quad (t, u) \in [0, 1] \times \overline{B(0, R)}.$$

It follows that $0 \notin \Psi_\lambda(I \times \partial B(0, R))$. In fact, if $0 \in H_\lambda(I_\Lambda \times \partial B(0, R))$, then there exist $t_0 \in [0, 1]$ and $u_0 \in \partial B(0, R)$ such that $u_0 = t_0 T(\lambda, u_0)$. Since $u_0 \in \partial B(0, R)$, we have $t_0 \neq 0$. And $t_0 \neq 1$ because $u_0 \neq T(\lambda, u_0)$. Therefore

$$\frac{\|u_0\|}{t_0} = \|T(\lambda, u_0)\| < \|u_0\|,$$

which is a contradiction.

The homotopy invariance of the Leray-Schauder degree guarantees that

$$\deg(\Psi_\lambda(t, .), B(0, R), 0) = \deg(\Psi_\lambda(0, .), B(0, R), 0) = 1, \quad t \in [0, 1].$$

Thus,

$$\deg(\Phi(\lambda, .), B(0, R), 0) = 1, \quad \lambda \in I_\Lambda,$$

completing the proof. □

Theorem 5.3. *There exist a number* $\lambda_0 > 0$ *and a connected set* $\Sigma_\Lambda \subset [\lambda_0, \Lambda] \times C(\overline{\Omega})$ *satisfying*

(i) $\Sigma_\Lambda \subset \mathcal{S}$;
(ii) $\Sigma_\Lambda \cap \big(\{\lambda_0\} \times C(\overline{\Omega})\big) \neq \emptyset$;
(iii) $\Sigma_\Lambda \cap \big(\{\Lambda\} \times C(\overline{\Omega})\big) \neq \emptyset$

for each $\Lambda > \lambda_0$.

Proof. Maintaining the notation of Lemma 5.2, we apply Theorem 5.1 to the operator T. We have already proved that T is continuous, compact and $T\big(I_\Lambda \times \overline{B(0, R_\Lambda)}\big) \subset B(0, R_\Lambda)$. Set

$$\mathcal{S}_\Lambda = \big\{(\lambda, u) \in I_\Lambda \times \overline{B(0, R)} : \Phi(\lambda, u) = 0\big\} \subset \mathcal{G}_\Lambda.$$

By Theorem 5.1 there is a connected component $\Sigma_\Lambda \subset \mathcal{S}_\Lambda$ such that

$$\Sigma_\Lambda \cap (\{\lambda_*\} \times \overline{B(0, R)}) \neq \emptyset \quad \text{and} \quad \Sigma_\Lambda \cap (\{\Lambda\} \times \overline{B(0, R)}) \neq \emptyset.$$

We point out that $\mathcal{S}_\Lambda$ is the solution set of the auxiliary problem

$$\begin{gathered} (-\Delta_p)^s u = \lambda f_\Lambda(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Omega^c \end{gathered}$$

and, since $\Sigma_\Lambda \subset \mathcal{S}_\Lambda \subset \mathcal{G}_\Lambda$, it follows from the definition of f_Λ that

$$\begin{gathered} (-\Delta_p)^s u = \lambda f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Omega^c \end{gathered}$$

for $(\lambda, u) \in \Sigma_\Lambda$, showing that $\Sigma_\Lambda \subset \mathcal{S}$. This completes the proof. □

6. Proof of Theorem 1.1

Proof. Consider Λ as introduced in Section 4 and take a sequence $\{\Lambda_n\}$ such that $\lambda_0 < \Lambda_1 < \Lambda_2 < \dots$ with $\Lambda_n \to \infty$. Set $\beta_n = \Lambda_n$ and take a sequence $\{\alpha_n\} \subset \mathbb{R}$ such that $\alpha_n \to -\infty$ and $\dots < \alpha_n < \dots < \alpha_1 < \lambda_0$.

Keeping up the notation of Section 4, consider the sequence of intervals $I_n = [\lambda_0, \Lambda_n]$. Set $M = C(\overline{\Omega})$ and

$$\mathcal{G}_{\Lambda_n} := \big\{(\lambda, u) \in I_n \times \overline{B}_{R_n} : \underline{u} \le u \le \overline{u},\ u = 0 \text{ on } \partial\Omega\big\},$$

where $R_n = R_{\Lambda_n}$. Look at the sequence of compact operators

$$T_n \colon [\lambda_0, \Lambda_n] \times \overline{B}_{R_n} \to \overline{B}_{R_n}$$

defined by

$$T_n(\lambda, u) = S(\lambda F_{\Lambda_n}(u))) \quad \text{if } \lambda_0 \le \lambda \le \Lambda_n,\ u \in \overline{B}_{R_n}.$$

Next, we consider the extension $\widetilde{T}_n \colon \mathbb{R} \times \overline{B}_{R_n} \to \overline{B}_{R_n}$ of T_n, defined by

$$\widetilde{T}_n(\lambda, u) = \begin{cases} T_n(\lambda_0, u) & \text{if } \lambda \le \lambda_0, \\ T_n(\lambda, u) & \text{if } \lambda_0 \le \lambda \le \Lambda_n, \\ T_n(\Lambda_n, u) & \text{if } \lambda \ge \Lambda_n. \end{cases}$$

Observe that $\widetilde{T}_n$ is continuous and compact.

Applying Theorem 5.1 to $\widetilde{T}_n \colon [\alpha_n, \beta_n] \times \overline{B}_{R_n} \to \overline{B}_{R_n}$ we obtain a compact connected component Σ_n^* of

$$\mathcal{S}_n = \big\{(\lambda, u) \in [\alpha_n, \beta_n] \times \overline{B}_{R_n} : \Phi_n(\lambda, u) = 0\big\},$$

where $\Phi_n(\lambda, u) = u - \widetilde{T}_n(\lambda, u)$.

Note that Σ_n^* is also a connected subset of $\mathbb{R} \times M$. According to Theorem 2.2, there exists a connected component Σ^* of $\overline{\lim}\ \Sigma_n^*$ such that

$$\Sigma^* \cap (\{\lambda\} \times M) \neq \emptyset \quad \text{for each } \lambda \in \mathbb{R}.$$

Set $\Sigma = ([\lambda_*, \infty) \times M) \cap \Sigma^*$. Then $\Sigma \subset \mathbb{R} \times M$ is connected and

$$\Sigma \cap (\{\lambda\} \times M) \neq \emptyset, \quad \lambda_0 \le \lambda < \infty.$$

We claim that $\Sigma \subset \mathcal{S}$. Indeed, note that

$$\widetilde{T}_{n+1}\big|_{[\lambda_0, \Lambda_n] \times \overline{B}_{R_n}} = \widetilde{T}_n\big|_{[\lambda_0, \Lambda_n] \times \overline{B}_{R_n}} = T_n. \tag{6.1}$$

If $(\lambda, u) \in \Sigma$ and $\lambda > \lambda_0$, there is a sequence $(\lambda_{n_i}, u_{n_i}) \in \cup \Sigma_n^*$ with $(\lambda_{n_i}, u_{n_i}) \in \Sigma_{n_i}^*$ such that $\lambda_{n_i} \to \lambda$ and $u_{n_i} \to u$ as$n_i \to \infty$. Then $u \in B_{R_N}$ for some integer $N > 1$.

We can assume that $(\lambda_{n_i}, u_{n_i}) \in [\lambda_0, \Lambda_N] \times B_{R_N}$. Equality (6.1) guarantees that

$$u_{n_i} = T_{n_i}(\lambda_{n_i}, u_{n_i}) = T_N(\lambda_{n_i}, u_{n_i})$$

and passing to the limit we obtain $u = T_N(\lambda, u)$ which shows that $(\lambda, u) \in \Sigma_N$ and so

$$(\lambda, u) \in \mathcal{S} := \big\{(\lambda, u) \in (0, \infty) \times C(\overline{\Omega}) : \text{u is a solution of } (P_\lambda)\big\}.$$

This completes the proof. □

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The authors want to insert the following lines at the end of Remark 2.1, and to add 3 references.

These arguments were already used in [19, 20], one of them involving the p-Laplacian operator with singular term. Also [18] studied a nonlinear fourth-order operator with Navier boundary conditions.

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End of addendum.

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