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A SECOND ORDER CONVERGENT DIFFERENCE SCHEME FOR THE INITIAL-BOUNDARY VALUE PROBLEM OF ROSENAU-BURGERS EQUATION

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Abstract. We construct a two-level implicit nonlinear finite difference scheme for the initial boundary value problem of Rosenau-Burgers equation based on the method of order reduction. We discuss conservation, unique solvability, and convergence for the difference scheme. The new scheme is shown to be second-order convergent in time and space. Finally, numerical simulations illustrate our theoretical analysis.

1. Introduction

In the study of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be described using the well-known KDV equation which was suggested by Korteweg and de Vries in 1895 [1]. To overcome this shortcoming, Rosenau [2, 3] proposed an equation in the form

$$u_t + u_{xxxxt} + u_x + uu_x = 0. \tag{1.1}$$

The theoretical results on the existence, uniqueness and regularity of the solutions to (1.1) have been investigated by Park [4]. But it is difficult to find the analytical solution for (1.1). Much attention has devoted to numerical solutions of (1.1) by various numerical methods [5, 6, 7, 8].

On the other hand, for the further consideration of dissipation in space for dynamic systems, such as the phenomenon of bore propagation and water waves, the viscous term $-u_{xx}$ needs to be included

$$u_t + u_{xxxxt} - u_{xx} + u_x + uu_x = 0. \tag{1.2}$$

This equation is usually called the Rosenau–Burgers equation. Many scholars had proposed difference schemes for the Rosenau–Burgers Equation. Hu et al [9] considered a nonlinear Crank-Nicolson difference scheme for the Rosenau-Burgers equation by the Newton iterative algorithm. Pan and Zhang [10] discussed a three-level linear implicit difference scheme for the Rosenau-Burgers equation which is unconditionally stable. Ahmat and Abduwali [11] investigated two class of modified local Crank- Nicolson schemes for the Rosenau-Burgers equation,which has simple

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structure and high accuracy. Omrani [12] explored the Galerkin–Crank–Nicolson discrete method. Guo et al [13] considered a two-level implicit nonlinear discrete scheme, which preserves the energy decline property of the Rosenau–Burgers equation. Luo et al [14] used a three-level linear implicit finite difference scheme with energy dissipation which has second-order and fourth-order in time and space. For the two-dimensional case, Rouatbi et al [15] presented a linearized Crank–Nicolson difference scheme.

In this article, we consider the initial-boundary value problem of the Rosenau-Burgers equation by the finite difference method.

$$u_t + u_{xxxxt} - u_{xx} + u_x + uu_x = 0, \quad 0 < x < L,\ 0 < t \le T, \tag{1.3}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le L, \tag{1.4}$$

$$u(0,t) = 0, \quad u(L,t) = 0, u_{xx}(0,t) = 0, \quad u_{xx}(L,t) = 0, \quad 0 \le t \le T, \tag{1.5}$$

where $\varphi(x)$ is a given function. The initial-boundary value problem of the Rosenau–Burgers equation contains at least one derivative boundary condition. Therefore, a difference scheme must be established at the node adjacent to this boundary point that is consistent with the difference schemes on the other nodes. To achieve this, we employ the method of order reduction to get a difference scheme with convergence order two in both space and time. Additionally, we rigorously prove the convergence of this particular difference.

This article is structured as follows. In Section 2, we introduce some useful notation and lemmas. In Section 3, we describe a conservative two-level implicit nonlinear finite difference scheme for the Rosenau–Burgers equation. The scheme has second-order accuracy in space and time. In Section 4, we analize the unique solvability is analyzed. In Section 5, we prove the convergence and stability for the difference schem. Finally, a numerical example illustrates our theoretical results.

2. Preliminaries

In this section, we introduce notation and lemmas that will be used throughout this article. To partition the domain $[0,L] \times [0,T]$, we use positive integers M and N. Let $h = L/M$ and $\tau = T/N$. Denote $x_i = ih$, $0 \le i \le M$; $t_k = k\tau$, $0 \le k \le N$;$\Omega_h = \{x_i : 0 \le i \le M\}$, $\Omega_\tau = \{t_k : 0 \le k \le N\}$, $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. For each grid function $v = \{v_i^k : 0 \le i \le M, 0 \le k \le N\}$ defined on $\Omega_{h\tau}$, we denote

$$\delta_x^+ v_i^k = \frac{1}{h}(v_{i+1}^k - v_i^k), \delta_x^2 v_i^k = \frac{1}{h}(\delta_x^+ v_i^k - \delta_x^+ v_{i-1}^k), \Delta_x v_i^k = \frac{1}{2h}(v_{i+1}^k - v_{i-1}^k),$$

$$v_i^{k+\frac{1}{2}} = \frac{1}{2}(v_i^k + v_i^{k+1}), v_{i+\frac{1}{2}}^k = \frac{1}{2}(v_i^k + v_{i+1}^k), \delta_t v_i^{k+\frac{1}{2}} = \frac{1}{\tau}(v_i^{k+1} - v_i^k),$$

Let $\mathcal{V}_h = \{v : v = (v_0, v_1, \dots, v_{M-1}, v_M)\}$ and $\mathring{\mathcal{V}}_h = \{v : v \in \mathcal{V}_h, v_0 = v_M = 0\}$ be the spaces of grid functions on Ω_h. For any grid functions $u, v \in \mathcal{V}_h$, we define the discrete inner product

$$(u,v) = h\sum_{i=1}^{M-1} u_i v_i,$$

and the corresponding norms (seminorm)

$$\|u\| = \sqrt{(u,u)}, \quad |u|_1 = \sqrt{(\delta_x^+ u, \delta_x^+ u)}, \quad \|u\|_\infty = \max_{1 \le i \le M} |u_i|$$

In addition, we define the function

$$\psi(u,v)_i = \frac{1}{3}[u_i\Delta_x v_i + \Delta_x(uv)_i], \quad 1 \le i \le M.$$

Lemma 2.1 ([16]). *For each grid function $u, v \in \mathring{\mathcal{V}}_h$, we have*

$$\|v\|_\infty \le \frac{\sqrt{L}}{2}|v|_1, \quad |v|_1 \le \frac{2}{h}\|v\|, \quad \|v\| \le \frac{L}{\sqrt{6}}|v|_1,$$
$$(\delta_x^2 u, v) = -(\delta_x^+ u, \delta_x^+ v).$$

And for an arbitrary $\varepsilon > 0$, we have

$$\|v\|_\infty^2 \le \varepsilon|v|_1^2 + \frac{1}{4\varepsilon}\|v\|^2.$$

Lemma 2.2 ([17]). *For each grid functions $u \in \mathcal{V}_h$ and $v \in \mathring{\mathcal{V}}_h$ we have*

$$(\psi(u,v),v) = 0, \quad (\Delta_x u, u) = 0.$$

Lemma 2.3 (Gronwall inequality [17]). *Let $\{F^k\}_{i=0}^\infty$ and $\{g^k\}_{i=0}^\infty$ be two non-negative sequences and satisfy*

$$F^{k+1} \le (1 + c\tau)F^k + \tau g^k, \quad k = 0, 1, 2, \dots,$$

then

$$F^k \le \exp(ck\tau)\Big(F^0 + \tau\sum_{l=0}^{k-1} g^l\Big), \quad k = 0, 1, 2, \dots$$

3. Nnonlinear conservative difference scheme

In this section, we use the method of reduction of order to establish a difference scheme for the problem (1.3)-(1.5), and illustrate the truncation errors in detail, then we analyze the conservation of the difference scheme.

3.1. **Construction of difference scheme.** Let $v = u_{xx}$, then problem (1.3)–(1.5) is equivalent to

$$u_t + v_{xxt} - v + u_x + uu_x = 0, \quad 0 < x < L,\ 0 < t \le T, \tag{3.1}$$
$$v = u_{xx}, \quad 0 < x < L,\ 0 < t \le T, \tag{3.2}$$
$$u(x,0) = \varphi(x), \quad 0 < x < L, \tag{3.3}$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad u_{xx}(0,t) = 0, \quad u_{xx}(L,t) = 0, \quad 0 \le t \le T\,. \tag{3.4}$$

According to (3.1) and (3.4),

$$v(0,t) = 0, \quad v(L,t) = 0, \quad 0 \le t \le T\,. \tag{3.5}$$

We define the grid functions

$$U = \{U_i^k : 0 \le i \le M, 0 \le k \le N\} \quad \text{and} \quad V = \{V_i^k : 0 \le i \le M, 0 \le k \le N\},$$

where $U_i^k = u(x_i, t_k)$ and $V_i^k = v(x_i, t_k)$.

Considering (3.1) at the point $(x_i, t^{k+\frac{1}{2}})$ and (3.4) at the point (x_i, t_k),

$$\begin{aligned} &u_t(x_i, t_{k+\frac12}) + v_{xxt}(x_i, t_{k+\frac12}) - v(x_i, t_{k+\frac12}) + u_x(x_i, t_{k+\frac12}) \\ &+ u(x_i, t_{k+\frac12})u_x(x_i, t_{k+\frac12}) = 0, \quad 1 \le i \le M-1,\ 1 \le k \le N-1, \end{aligned} \tag{3.6}$$
$$v(x_i, t_k) = u_{xx}(x_i, t_k), \quad 1 \le i \le M-1,\ 1 \le k \le N. \tag{3.7}$$

By using the Taylor expansion,

$$\begin{aligned}&\delta_t U_i^{k+\frac12}+\delta_t{\delta_x}^2 V_i^{k+\frac12}-V_i^{k+\frac12}+\Delta_x U_i^{k+\frac12}+\psi(U^{k+\frac12},U^{k+\frac12})_i=R_i^{k+\frac12},\\&1\le i\le M-1,\ 0\le k\le N-1,\end{aligned}\tag{3.8}$$

$$V_i^k=\delta_x^2U_i^k+Q_i^k,\quad 1\le i\le M-1,\ 0\le k\le N,\tag{3.9}$$

there exist constants c_1 and c_2 such that

$$\begin{gathered}|R_i^{k+\frac12}|\le c_1(\tau^2+h^2),\quad 1\le i\le M-1,\ 0\le k\le N-1,\\ |Q_i^k|\le c_2h^2,\quad 1\le i\le M-1,\ 0\le k\le N,\\ |\delta_tQ_i^{k+\frac12}|\le c_2(\tau^2+h^2),\quad 1\le i\le M-1,\ 0\le k\le N-1.\end{gathered}\tag{3.10}$$

Notice that the initial-boundary conditions (3.3)–(3.5) become

$$\begin{gathered}U_i^0=\varphi(x_i),\quad 1\le i\le M-1,\\ U_0^k=0,\quad U_M^k=0,\quad 0\le k\le N,\\ V_0^k=0,\quad V_M^k=0,\quad 0\le k\le N.\end{gathered}\tag{3.11}$$

Omitting the small terms in (3.8) and (3.9) and combining with (3.11), we can derive the difference scheme for (3.1)-(3.4),

$$\begin{aligned}&\delta_t u_i^{k+\frac12}+\delta_t{\delta_x}^2 v_i^{k+\frac12}-v_i^{k+\frac12}+\Delta_x u_i^{k+\frac12}+\psi(u^{k+\frac12},u^{k+\frac12})_i=0,\\&1\le i\le M-1,\quad 0\le k\le N-1,\end{aligned}\tag{3.12}$$

$$v_i^k=\delta_x^2u_i^k,\quad 1\le i\le M-1,\ 0\le k\le N,\tag{3.13}$$

$$u_i^0=\varphi(x_i),\quad 1\le i\le M-1,\tag{3.14}$$

$$u_0^k=0,\quad u_M^k=0,\quad 0\le k\le N,\tag{3.15}$$

$$v_0^k=0,\quad v_M^k=0,\quad 0\le k\le N.\tag{3.16}$$

3.2. **Conservation law.**

Theorem 3.1. *Let* $\{u_i^k,v_i^k:0\le i\le M,0\le k\le N\}$ *be the solutions of* (3.12)–(3.16). *Denote*

$$E^k=\|u^k\|^2+\|v^k\|^2+2\tau\sum_{l=0}^{k-1}|u^{l+\frac12}|_1^2,\quad 0\le k\le N.$$

Then we have

$$E^k=E^0,\quad 1\le k\le N.$$

Proof. Take the inner product of (3.12) with $u^{k+\frac12}$,

$$\begin{aligned}&(\delta_tu^{k+\frac12},u^{k+\frac12})+(\delta_t\delta_x^2v^{k+\frac12},u^{k+\frac12})-(v^{k+\frac12},u^{k+\frac12})+(\varDelta_xu^{k+\frac12},u^{k+\frac12})\\&+\frac13(\psi(u^{k+\frac12},u^{k+\frac12}),u^{k+\frac12})=0,\quad 0\le k\le N-1.\end{aligned}$$

From Lemma 2.2 it follows that

$$\frac{1}{2\tau}(\|u^{k+1}\|^2-\|u^k\|^2)+\frac{1}{2\tau}(\|v^{k+1}\|^2-\|v^k\|^2)+|u^{k+\frac12}|_1^2=0.$$

This equation can be written as

$$\|u^{k+1}\|^2+\|v^{k+1}\|^2+2\tau|u^{k+\frac12}|_1^2=\|u^k\|^2+\|v^k\|^2.$$

Therefore, replacing the superscript k with l and summing for l from 0 to $k-1$ yields

$$E^k = \|u^k\|^2 + \|v^k\|^2 + 2\tau \sum_{l=0}^{k-1} |u^{l+\frac{1}{2}}|_1^2 = E^{k-1} = \cdots = E^0. \qquad \square$$

4. Solvability

In this section, we analyze the unique solvability of the difference scheme established in (3.12)-(3.16).

Theorem 4.1 (Browder theorem [16])**.** *Let $(H, (\cdot,\cdot))$ be a finite dimensional inner product space, $\|\cdot\|$ be the associated norm, and $\Pi : H \to H$ be continuous operator,*

$$\exists \alpha > 0, \quad \forall z \in H, \quad \|z\| = \alpha, \quad \mathrm{Re}(\Pi(z), z) \geq 0.$$

Then there exists a $z^* \in H$ satisfying $\|z^*\| \leq \alpha$ such that $\Pi(z^*) = 0$.

Theorem 4.2. *The difference scheme* (3.12)-(3.16) *has at least one solution.*

Proof. Let

$$u^k = (u_0^k, u_1^k, \dots, u_M^k), \quad v^k = (v_0^k, v_1^k, \dots, v_M^k).$$

It is easy to obtain u^0 from (3.13) and (3.14). From (3.13)-(3.16), v^0 can be found by computing an associated system of linear equations. Suppose that $\{u^k, v^k\}$ have been determined, then we may regard $\{u^{k+\frac{1}{2}}, v^{k+\frac{1}{2}}\}$ as unknowns and

$$u_i^{k+1} = 2u_i^{k+\frac{1}{2}} - u_i^k, \quad v_i^{k+1} = 2v_i^{k+\frac{1}{2}} - v_i^k, \quad 0 \leq i \leq M.$$

We denote

$$\omega_i = u_i^{k+\frac{1}{2}}, \quad z_i = v_i^{k+\frac{1}{2}}, \quad 0 \leq i \leq M,$$

Then

$$u_i^{k+1} = 2w_i - u_i^k, \quad v_i^{k+1} = 2z_i - v_i^k, \quad 0 \leq i \leq M.$$

From (3.12), (3.13), (3.15) and (3.16), the system of equations can be considered with respect to $\{w_i\}_{i=0}^M$ and $\{z_i\}_{i=0}^M$:

$$\begin{aligned} &\frac{2}{\tau}(\omega_i - u_i^k) + \frac{2}{\tau}(\delta_x^2 z_i - \delta_x^2 v_i^k) - z_i + \varDelta_x \omega_i + \psi(\omega,\omega)_i = 0, \\ &1 \leq i \leq M-1, \end{aligned} \tag{4.1}$$

$$z_i = \delta_x^2 \omega_i, \quad 1 \leq i \leq M-1, \tag{4.2}$$

$$\omega_0 = 0, \omega_M = 0, \tag{4.3}$$

$$z_0 = 0, z_M = 0. \tag{4.4}$$

We define $\Pi(\omega) : \mathring{\mathcal{V}}_h \to \mathring{\mathcal{V}}_h$ by

$$\varPi(\omega)_i = \begin{cases} \frac{2}{\tau}(\omega_i - u_i^k) + \frac{2}{\tau}(\delta_x^2 z_i - \delta_x^2 v_i^k) - z_i + \varDelta_x \omega_i + \psi(\omega,\omega)_i, & 1 \leq i \leq M-1, \\ 0, & i = 0, M. \end{cases}$$

where $(v_0, v_1, \dots, v_M)$ is determined by (3.13) and (3.15). Then $\Pi(\omega)$ is a continuous function in $\mathring{\mathcal{V}}_h$. Taking the inner product of $\Pi(\omega)$ with ω, using Lemma 2.1 yields

$$\begin{aligned} (\Pi(\omega), \omega) &= \frac{2}{\tau}[\|\omega\|^2 - (u^k, \omega)] + \frac{2}{\tau}[\|z\|^2 - (\delta_x^2 u^k, \delta_x^2 \omega)] + |\omega|_1^2 \\ &\geq \frac{2}{\tau}[(\|\omega\|^2 - \|u^k\|\|\omega\|) + (\|z\|^2 - \|v^k\|\|z\|)] \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{2}{\tau}[(\|\omega\|^2 - \|u^k\|\|\omega\|) + \big(\|z\|^2 - \|v^k\| \cdot \frac{6}{L^2}\|\omega\|\big)] \\
&= \frac{2}{\tau}(\|\omega\| - \|u^k\| - \frac{6}{L^2}\|v^k\|)\|\omega\|.
\end{aligned}
$$

Thus, when $\|\omega\| = \|u^k\| + \frac{6}{L^2}\|v^k\|$, $(\Pi(\omega), \omega) \geq 0$. By Theorem 4.1, there exists an $\omega^* \in \mathring{\mathcal{V}}_h$ satisfying $\|\omega\| \leq \|u^k\| + \frac{6}{L^2}\|v^k\|$ such that $(\Pi(\omega^*)) = 0$. Consequently, the difference scheme (3.12), (3.13), (3.15) and (3.16) has at least one solution $u^{k+1} = 2\omega^* - u^k$.

Observing (4.2) and (4.4), when $(\omega_1^*, \omega_2^*, \dots, \omega_{M-1}^*)$ is known, $(z_1^*, z_2^*, \dots, z_{M-1}^*)$ can be determined by (4.2) and (4.4) uniquely. Thus

$$
v_i^{k+1} = 2z_i - v_i^k, 1 \leq i \leq M-1\,. \qquad \square
$$

Theorem 4.3. *When $\tau < 4/c_3^4$, the solution of the difference scheme* (3.12)–(3.16) *is unique.*

Proof. Suppose that both $\{u^{(1)}, v^{(1)} \in \mathring{\mathcal{V}}_h\}$ and $\{u^{(2)}, v^{(2)} \in \mathring{\mathcal{V}}_h\}$ are two solutions of (4.1)-(4.4). Then

$$
\begin{aligned}
&\frac{2}{\tau}(u_i^{(1)} - u_i^k) + \frac{2}{\tau}(\delta_x^2 v_i^{(1)} - \delta_x^2 v_i^k) - v_i^{(1)} + \Delta_x u_i^{(1)} + \psi(u^{(1)}, u^{(1)})_i = 0, \\
&1 \leq i \leq M-1,
\end{aligned} \tag{4.5}
$$

$$
v_i^{(1)} = \delta_x^2 u_i^{(1)}, \quad 1 \leq i \leq M-1, \tag{4.6}
$$

$$
u_0^{(1)} = 0, \quad u_M^{(1)} = 0, \tag{4.7}
$$

$$
v_0^{(1)} = 0, \quad v_M^{(1)} = 0. \tag{4.8}
$$

and

$$
\begin{aligned}
&\frac{2}{\tau}(u_i^{(2)} - u_i^k) + \frac{2}{\tau}(\delta_x^2 v_i^{(2)} - \delta_x^2 v_i^k) - v_i^{(2)} + \Delta_x u_i^{(2)} + \psi(u^{(2)}, u^{(2)})_i = 0, \\
&1 \leq i \leq M-1,
\end{aligned} \tag{4.9}
$$

$$
v_i^{(2)} = \delta_x^2 u_i^{(2)}, \quad 1 \leq i \leq M-1, \tag{4.10}
$$

$$
u_0^{(2)} = 0, u_M^{(2)} = 0, \tag{4.11}
$$

$$
v_0^{(2)} = 0, v_M^{(2)} = 0. \tag{4.12}
$$

Let

$$
u_i = u_i^{(1)} - u_i^{(2)}, \quad v_i = v_i^{(1)} - v_i^{(2)}, \quad 0 \leq i \leq M.
$$

Subtracting (4.9)-(4.12) from (4.5)–(4.8) leads to

$$
\begin{aligned}
&\frac{2}{\tau}u_i + \frac{2}{\tau}\delta_x^2 v_i - v_i + \Delta_x u_i + [\psi(u^{(1)}, u^{(1)})_i - \psi(u^{(2)}, u^{(2)})_i] = 0, \\
&1 \leq i \leq M-1,
\end{aligned} \tag{4.13}
$$

$$
v_i = \delta_x^2 u_i, \quad 1 \leq i \leq M-1, \tag{4.14}
$$

$$
u_0 = 0, \quad u_M = 0, \tag{4.15}
$$

$$
v_0 = 0, \quad v_M = 0. \tag{4.16}
$$

Taking the inner product of (4.13) with u yields

$$
\frac{2}{\tau}\|u\|^2 + \frac{2}{\tau}(\delta_x^2 v, u) - (v, u) + (\mathit{\Delta}_x u, u) = -(\psi(u^{(1)}, u^{(1)}) - \psi(u^{(2)}, u^{(2)}), u) \tag{4.17}
$$

In view of the definition of u and Lemma 2.2, we have

$$\begin{aligned}
-(\psi(u^{(1)},u^{(1)})-\psi(u^{(2)},u^{(2)}),u) &= -(\psi(u,u^{(1)}),u)\\
&= -\frac{h}{3}\sum_{i=1}^{M-1}[u_i\Delta_x u_i^{(1)}+\Delta_x(uu^{(1)})_i]u_i\\
&= \frac{h}{3}\sum_{i=1}^{M-1}[u_i^{(1)}\Delta_x(u^2)_i+(uu^{(1)})_i\Delta_x u_i]\\
&\le \frac{1}{3}\big(2\|u\|_\infty\|u^{(1)}\|\,|u|_1+\|u\|_\infty\|u^{(1)}\|\,|u|_1\big)\\
&\le c_3\|u\|_\infty|u|_1\,.
\end{aligned}$$

Then it follows from (4.17) that

$$\frac{2}{\tau}\|u\|^2+\frac{2}{\tau}\|v\|^2+|u|_1^2\le c_3\|u\|_\infty|u|_1.$$

Using Lemma 2.1 gives

$$\begin{aligned}
\frac{2}{\tau}\|u\|^2+|u|_1^2 &\le c_3\big(\varepsilon|u|_1+\frac{1}{2\varepsilon}\|u\|\big)|u|_1\\
&\le c_3\varepsilon|u|_1^2+c_3\varepsilon|u|_1^2+\frac{1}{4c_3\varepsilon}(\frac{c_3}{2\varepsilon})^2\|u\|^2\\
&= 2c_3\varepsilon|u|_1^2+\frac{c_3}{16\varepsilon^3}\|u\|\,.
\end{aligned}$$

Let $\varepsilon=\frac{1}{2c_3}$. Then $\frac{2}{\tau}\|u\|^2\le\frac{{c_3}^4}{2}\|u\|^2$. When $\tau<\frac{4}{{c_3}^4}$, we obtain $\|u\|=0$. □

5. Convergence

In this section, we analyze the convergence of the difference scheme (3.12)-(3.16). Let

$$c_0=\max_{(x,t)\in[0,L]\times[0,T]}\{|u(x,t)|,|u_x(x,t)|\}.$$

The convergent result is summarized as follows.

Theorem 5.1. *Let $\{u(x,t),v(x,t)\}$ be the solutions of (3.1)-(3.5) and $\{u_i^k,v_i^k : 0\le i\le M, 0\le k\le N\}$ be the solutions of the difference scheme (3.12)-(3.16). Also let*

$$e_i^k=U_i^k-u_i^k,\quad f_i^k=V_i^k-v_i^k,\quad 0\le i\le M,\ 0\le k\le N,$$

and

$$c_4=3{c_0}^2+\frac{{c_0}^2L^2}{2}+\frac{3L}{4},\quad c_5=e^{\frac{3}{2}MT}\sqrt{6\tau M_1},$$
$$M=(c_4+1),\quad M_1={c_2}^2+{c_1}^2.$$

If τ and h satisfy $\tau^2+h^2\le 1/c_5$ and $2c_4\tau\le 1/3$, then the error estimate is

$$|e^k|_1\le c_5(\tau^2+h^2),\quad 0\le k\le N.$$

Proof. Subtracting (3.12)-(3.16) from (3.8), (3.9) and (3.11), a system of error equations is

$$\delta_t e_i^{k+\frac{1}{2}} + \delta_t \delta_x^2 f_i^{k+\frac{1}{2}} - f_i^{k+\frac{1}{2}} + \Delta_x e_i^{k+\frac{1}{2}} + [\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i]$$
$$= R_i^{k+\frac{1}{2}}, \quad 1 \le i \le M-1,\ 0 \le k \le N-1, \tag{5.1}$$

$$f_i^k = \delta_x^2 e_i^k + Q_i^k, \quad 1 \le i \le M-1,\ 0 \le k \le N, \tag{5.2}$$

$$e_i^0 = 0, \quad 1 \le i \le M-1, \tag{5.3}$$

$$e_0^k = 0, \quad e_M^k = 0, \quad 0 \le k \le N, \tag{5.4}$$

$$f_0^k = 0, \quad f_M^k = 0, \quad 0 \le k \le N. \tag{5.5}$$

It follows from (5.2) that

$$f_i^{k+\frac{1}{2}} = \delta_x^2 e_i^{k+\frac{1}{2}} + Q_i^{k+\frac{1}{2}}, \quad 1 \le i \le M-1,\ 0 \le k \le N-1.$$

Taking the inner product of (5.1) with $\delta_t e_i^{k+\frac{1}{2}}$ yields

$$\|\delta_t e^{k+\frac{1}{2}}\|^2 + (\delta_t \delta_x^2 f^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) - (f^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) + (\Delta_x e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}})$$
$$+ (\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}) - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}), \delta_t e^{k+\frac{1}{2}}) = (R^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}),$$

for $1 \le k \le N-1$.

From Lemma 2.1, it follows that

$$\begin{aligned}
-(\delta_t \delta_x^2 f^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) &= -(\delta_t f^{k+\frac{1}{2}}, \delta_t \delta_x^2 e^{k+\frac{1}{2}}) = -(\delta_t f^{k+\frac{1}{2}}, \delta_t (f^{k+\frac{1}{2}} - Q^{k+\frac{1}{2}})) \\
&= -(\delta_t f^{k+\frac{1}{2}}, \delta_t f^{k+\frac{1}{2}}) + (\delta_t f^{k+\frac{1}{2}}, \delta_t Q^{k+\frac{1}{2}}) \\
&\le -\|\delta_t f^{k+\frac{1}{2}}\|^2 + \|\delta_t f^{k+\frac{1}{2}}\| t \|\delta_t Q^{k+\frac{1}{2}}\|,
\end{aligned}$$

$$\begin{aligned}
(f^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) &= (\delta_x^2 e^{k+\frac{1}{2}} + Q^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) \\
&\le -\frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2) + \|Q^{k+\frac{1}{2}}\| \, \|\delta_t e^{k+\frac{1}{2}}\|,
\end{aligned}$$

and

$$-(\Delta_x e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) \le \|\Delta_x e^{k+\frac{1}{2}}\| \, \|\delta_t e^{k+\frac{1}{2}}\|,$$

From the definition of $\psi(u,v)_i$ and applying Lemma 2.2, we obtain

$$\begin{aligned}
&-(\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}) - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}), \delta_t e^{k+\frac{1}{2}}) \\
&= -(\psi(U^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}) + \psi(e^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}) - \psi(e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}), \delta_t e^{k+\frac{1}{2}}).
\end{aligned}$$

Note that

$$\begin{aligned}
(\psi(e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}), \delta_t e^{k+\frac{1}{2}}) &= (\psi(e^{k+\frac{1}{2}}, e^k + \frac{\tau}{2}\delta_t e^{k+\frac{1}{2}}), \delta_t e^{k+\frac{1}{2}}) \\
&= (\psi(e^{k+\frac{1}{2}}, e^k), \delta_t e^{k+\frac{1}{2}}) + \frac{\tau}{2}(\psi(e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}), \delta_t e^{k+\frac{1}{2}}) \\
&= (\psi(e^{k+\frac{1}{2}}, e^k), \delta_t e^{k+\frac{1}{2}}).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&-(\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}) - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}), \delta_t e^{k+\frac{1}{2}}) \\
&= -(\psi(U^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}) + \psi(e^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}) - \psi(e^{k+\frac{1}{2}}, e^k), \delta_t e^{k+\frac{1}{2}})
\end{aligned}$$

$$\begin{aligned}
&= -\frac{h}{3}\sum_{i=1}^{M-1}[U_i^{k+\frac12}\Delta_x e_i^{k+\frac12} + 2\Delta_x(U^{k+\frac12}e^{k+\frac12})_i + e_i^{k+\frac12}\Delta_x U_i^{k+\frac12}]\delta_t e_i^{k+\frac12} \\
&\quad + \frac{h}{3}\sum_{i=1}^{M-1}[e_i^{k+\frac12}\Delta_x e_i^k + \Delta_x(e^{k+\frac12}e^k)_i]\delta_t e_i^{k+\frac12} \\
&= -\frac{h}{3}\sum_{i=1}^{M-1}(3U_i^{k+\frac12}\Delta_x e_i^{k+\frac12} + e_{i+1}^{k+\frac12}\delta_x U_{i+\frac12}^{k+\frac12} + e_{i-1}^{k+\frac12}\delta_x U_{i-\frac12}^{k+\frac12} + e_i^{k+\frac12}\Delta_x U_i^{k+\frac12})\delta_t e_i^{k+\frac12} \\
&\quad + \frac{h}{3}\sum_{i=1}^{M-1}(2e_i^{k+\frac12}\Delta_x e_i^k + \frac12 e_{i+1}^k\delta_x e_{i+\frac12}^{k+\frac12} + \frac12 e_{i-1}^k\delta_x e_{i-\frac12}^{k+\frac12})\delta_t e_i^{k+\frac12} \\
&\le c_0|e^{k+\frac12}|_1\cdot\|\delta_t e^{k+\frac12}\| + c_0\|e^{k+\frac12}\|\cdot\|\delta_t e^{k+\frac12}\| \\
&\quad + \frac13(2\|e^{k+\frac12}\|_\infty\cdot|e^k|_1 + \|e^k\|_\infty\cdot|e^{k+\frac12}|_1)\|\delta_t e^{k+\frac12}\| \\
&\le c_0|e^{k+\frac12}|_1\cdot\|\delta_t e^{k+\frac12}\| + c_0\|e^{k+\frac12}\|\cdot\|\delta_t e^{k+\frac12}\| + \frac{\sqrt{L}}{2}|e^{k+\frac12}|_1\cdot\|\delta_t e^{k+\frac12}\| \\
&\le \varepsilon_1\|\delta_t e^{k+\frac12}\|^2 + \frac{{c_0}^2}{4\varepsilon_1}|e^{k+\frac12}|_1^2 + \varepsilon_2\|\delta_t e^{k+\frac12}\|^2 + \frac{{c_0}^2}{4\varepsilon_2}\|e^{k+\frac12}\|_1^2 \\
&\quad + \varepsilon_3\|\delta_t e^{k+\frac12}\|^2 + \frac{1}{4\varepsilon_3}\frac{L}{4}|e^{k+\frac12}|_1^2 \\
&\le (\varepsilon_1+\varepsilon_2+\varepsilon_2)\|\delta_t e^{k+\frac12}\|^2 + (\frac{{c_0}^2}{4\varepsilon_1} + \frac{{c_0}^2}{4\varepsilon_2}\frac{L^2}{6} + \frac{L}{16\varepsilon_3})|e^{k+\frac12}|_1^2.
\end{aligned}$$

Let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{12}$. Then

$$-(\psi(U^{k+\frac12},U^{k+\frac12}) - \psi(u^{k+\frac12},u^{k+\frac12}),\delta_t e^{k+\frac12}) \le \frac14\|\delta_t e^{k+\frac12}\|^2 + c_4|e^{k+\frac12}|_1^2.$$

Substituting above equations into equation (5.1) gives

$$\begin{aligned}
&\|\delta_t e^{k+\frac12}\|^2 \\
&\le -\|\delta_t f^{k+\frac12}\|^2 + \|\delta_t f^{k+\frac12}\|\,\|\delta_t Q^{k+\frac12}\| - \frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2) \\
&\quad + \|Q^{k+\frac12}\|\cdot\|\delta_t e^{k+\frac12}\| + \|\Delta_x e^{k+\frac12}\|\cdot\|\delta_t e^{k+\frac12}\| + \frac14\|\delta_t e^{k+\frac12}\|^2 + c_4|e^{k+\frac12}|_1^2 \qquad (5.6) \\
&\le \frac14\|\delta_t Q^{k+\frac12}\| - \frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2) + \|Q^{k+\frac12}\|^2 + \frac14\|\delta_t e^{k+\frac12}\|^2 + |e^{k+\frac12}|_1^2 \\
&\quad + \frac14\|\delta_t e^{k+\frac12}\|^2 + \frac14\|\delta_t e^{k+\frac12}\|^2 + c_4|^{k+\frac12}|_1^2 + \|R^{k+\frac12}\|^2 + \frac14\|\delta_t e^{k+\frac12}\|^2.
\end{aligned}$$

From (3.10), simplifying and rearranging (5.6) leads to

$$\begin{aligned}
|e^{k+1}|_1{}^2 &\le |e^k|_1{}^2 + 2\tau\frac{c_2^2}{4}(\tau^2+h^2)^2 + 2\tau c_2^2(h^2)^2 + 2\tau {c_1}^2(\tau^2+h^2)^2 \\
&\quad + \tau(c_4+1)|e^{k+1}|_1^2 + \tau(c_4+1)|e^k|_1^2.
\end{aligned}$$

Then

$$(1-\tau M)|e^{k+1}|_1{}^2 \le (1+\tau M)|e^k|_1{}^2 + 2\tau M_1(\tau^2+h^2)^2, \quad 0\le k\le N-1.$$

If $\tau M \le \frac13$, then

$$|e^{k+1}|_1{}^2 \le (1+3\tau M)|e^k|_1{}^2 + 6\tau M_1(\tau^2+h^2)^2, \quad 0\le k\le N-1.$$

Using Gronwall's inequality we obtain

$$|e^k|_1^2 \le e^{3Mk\tau} 6\tau M_1(\tau^2+h^2)^2 \le e^{3MT} 6\tau M_1(\tau^2+h^2)^2, \quad 0 \le k \le N. \qquad \square$$

From Theorem 5.1 and Lemma 2.1, we find that

$$\|e^k\|_\infty \le \frac{\sqrt{L}}{2}|e^k|_1 \le \frac{c_5\sqrt{L}}{2}(\tau^2+h^2), \quad 0 \le k \le N.$$

To demonstrate the results in the previous sections and confirm the accuracy of our method, we consider the Rosenau-Burgers equation with the following initial-boundary value problem.

$$u_t + u_{xxxxt} - u_{xx} + u_x + uu_x = 0, \quad x \in [0,1],\ t \in [0,1], \tag{5.7}$$

$$u(x,0) = u_0(x) = x^4(1-x)^4, \quad x \in [0,1], \tag{5.8}$$

$$u(0,t) = u(1,t) = 0, \quad u_{xx}(0,t) = u_{xx}(1,t) = 0. \tag{5.9}$$

Since we do not know the exact solution of (5.7)-(5.9), to obtain the error estimate, we consider the solution on mesh $h = 1/256$ as the reference solution, which is fine enough as a referenced exact solution for obtaining the error estimation.

When the exact solution is known, we define the discrete error in the L^∞-norm as follows

$$\mathrm{E}_\infty(h,\tau) = \max_{1\le i\le M, 0\le k\le N} |U_i^k - u_i^k|,$$

where U_i^k and u_i^k represent the exact solution and the numerical solution, respectively. Furthermore, denote the spatial and temporal convergence orders, respectively, as

$$\mathrm{Order}_\infty^h = \log_2 \frac{\mathrm{E}_\infty(2h,\tau)}{\mathrm{E}_\infty(h,\tau)}, \quad \mathrm{Order}_\infty^\tau = \log_2 \frac{\mathrm{E}_\infty(h,2\tau)}{\mathrm{E}_\infty(h,\tau)}.$$

Figure 1 shows the three dimensional image of numerical solutions. From Figure 2, we can know that with the time going by, the waveform changes and the peak value decreases. It shows that our scheme is stable.

Table 1 shows that the numerical results are conservative at different t where E^k is defined in Theorem 3.1. Table 2 and Table 3 list the errors and corresponding convergence orders. It shows that the maximal errors reduce with the decrease of the spatial step h and time step τ. The convergence orders are both two in space and time in L^∞-norm, which are consistent with our theoretical results. It reveals that the numerical method in this paper is accurate and efficient.

TABLE 1. Discrete energy E^k at h=1/256 and $\tau = 0.01$.

t	E^k
t=0.1	0.004792856601
t=0.3	0.004792856601
t=0.5	0.004792856601
t=0.7	0.004792856601
t=0.9	0.004792856601

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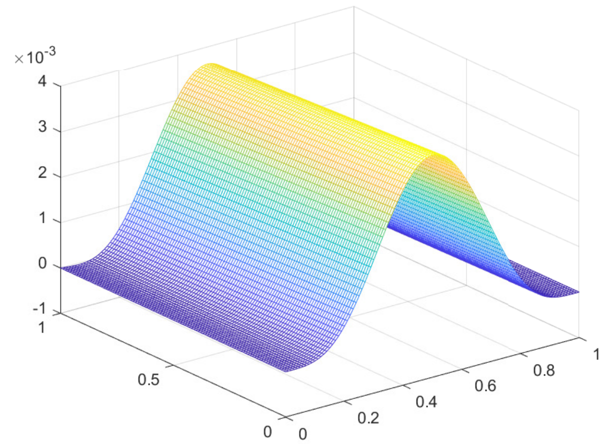

FIGURE 1. Three dimensional image of numerical solutions at h=1/256 and τ=0.1.

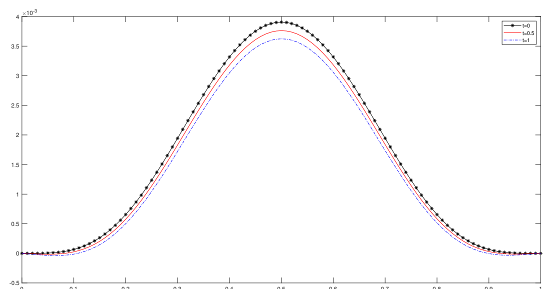

FIGURE 2. Numerical solutions with h=0.01 and τ=0.1 at various times.

TABLE 2. Errors and temporal convergence orders when $t = 1.0$.

h	τ	$\mathrm{E}_\infty(h,\tau)$	$\mathrm{Order}_\infty^\tau$
h=1/128	$\tau = 1/8$	$3.3233e-09$	–
h=1/128	$\tau = 1/16$	$8.2855e-10$	2.0040
h=1/128	$\tau = 1/32$	$2.0475e-10$	2.0168
h=1/128	$\tau = 1/64$	$4.9024e-11$	2.0623

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TABLE 3. Errors and spatial convergence orders when $t = 1.0$.

h	τ	$E_\infty(h,\tau)$	Order_∞^h
$h = 1/8$	τ=0.1	$4.7197e-06$	–
$h = 1/16$	τ=0.1	$1.1560e-06$	2.0295
$h = 1/32$	τ=0.1	$2.8428e-07$	2.0238
$h = 1/64$	τ=0.1	$6.7604e-08$	2.0721

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