Differential Equations and Computational Simulations III
J. Graef, R. Shivaji, B. Soni, & J. Zhu (Editors)
Electronic Journal of Differential Equations, Conference 01, 1997.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp 147.26.103.110 or 129.120.3.113 (login: ftp)

TRAVELING WAVE FRONTS IN SPATIALLY DISCRETE REACTION-DIFFUSION EQUATIONS ON HIGHER DIMENSIONAL LATTICES

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Abstract. This paper deals with the existence of traveling wave fronts of spatially discrete reaction-diffusion equations with delay on lattices with general dimension. A monotone iteration starting from an upper solution is established, and the sequence generated from the iteration is shown to converge to a profile function. The main theorem is then applied to a particular equation arising from branching theory.

1. Introduction

Consider the spatially discrete reaction-diffusion equation

$$u'_\eta(t) = \alpha(\Delta_n u)_\eta + f(u_\eta), \quad \eta \in \Omega \subset \mathbb{Z}^n \tag{1.1}$$

where $\alpha > 0$ is a constant, and Δ_n is the standard n-dimensional discrete Laplacian,

$$(\Delta_n u)_\eta = \Big(\sum_{|\xi-\eta|=1} u_\xi\Big) - 2n\, u_\eta\,. \tag{1.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Systems of differential equations with an underlying lattice structure (referred as *lattice differential equations* in literature) occur in mathematical models in many scientific disciplines, and have attracted many mathematicians and scientists from other fields. We mention here, among the others, materials science [1], population biology [13,16], pattern recognition [3,4]. For additional references, see the excellent survey papers [5,6,17,20].

In addition to the above motivation for studying equation (1.1), there are also some theoretical reasons. As indicated in the title of this paper, Eq.(1.1) is a spatial discretization of the partial differential equation (reaction-diffusion equation)

$$\frac{\partial u(t,x)}{\partial t} = \alpha\Delta u(t,x) + f(u(t,x)), \quad x \in \Omega \in \mathbb{R}^n \tag{1.3}$$

1991 Mathematics Subject Classifications: 34B99,34C37, 34K99, 35K57.
Key words and phrases: spatially discrete, reaction-diffusion equation, delay, lattice, traveling wave front, upper-lower solution.

Published November 12, 1998.
Supported by an NSERC (Canada) Postdoctoral Fellowship.

where Δ is the usual Laplacian with respect to the spatial variable x. Therefore, it is interesting and worthwhile to compare the dynamics of (1.3) with that of (1.1). It has been noticed that an anisotropy in directional dependence is often introduced in discretizing the n-dimensional Laplacian for $n \geq 2$, and thus, spatially discrete equations often exhibit more complicated and richer dynamics than spatially continuous equations. See, for example, [1,6,7,22,27].

We all know that traveling wave solutions play an important role in understanding the dynamics of the PDE (1.3). Naturally, we expect that traveling wave solutions be also an important class of solutions for (1.1). For (1.3), a traveling wave solution takes the form $u(t,x) = \phi(\sigma \cdot x + ct)$ for some function $\phi : \mathbb{R} \to \mathbb{R}$ where $\sigma \in \mathbb{R}^n$ is a unit vector representing the direction of motion of the wave, and $c > 0$ is the wave speed. Note that both the wave profile function and the wave speed c are unknown. By substituting the traveling wave formula into (1.3), we arrive at a second order ordinary differential equation

$$c\phi'(\xi) = \alpha\phi''(\xi) + f(\phi(\xi)), \quad \xi \in \mathbb{R} \tag{1.4}$$

where $\xi = \sigma \cdot x + ct$ is the moving coordinate. Usually, one imposes the boundary conditions

$$\phi(-\infty) = q_-, \quad \phi(\infty) = q_+ \tag{1.5}$$

to obtain a traveling wave *front* that represents a transition from one equilibrium to another in applications. Observe that (1.4) is *independent* of the dimension n and the direction σ.

Analogously, for the discrete reaction-diffusion equation (1.1) we can also look for traveling wave solutions of the form $u_\eta(t) = y(\sigma\cdot\eta + ct)$, where $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \mathbb{R}^n$ and $c > 0$ are as before. Now substitution of $u_\eta(t) = y(\sigma \cdot \eta + ct)$ into (1) yields the difference-differential equation

$$cy'(s) = \alpha \sum_{k=1}^{n} [y(s+\sigma_k) + y(s-\sigma_k)] - 2\alpha n y(s) + f(y(s)), \quad s \in \mathbb{R} \tag{1.6}$$

where $s = \sigma\cdot\eta + ct$. Just as for PDE case, one also imposes the boundary conditions

$$y(-\infty) = q_-, \quad y(\infty) = q_+ \tag{1.7}$$

for Eq.(1.6). One notices that in contrast to the second order ordinary differential equation (1.4), the difference-differential equation (1.6) is a genuinely infinite dimensional problem. Moreover, it depends on the dimension n as well as the direction σ and involves not only retarded but also advanced arguments. While a great deal is known [10] about differential equations with retarded arguments, very little of any general theory has addressed the so-called "mixed" type equation (1.6) in which both forward $s + \sigma_k$ and backward $s - \sigma_k$ shifts of the argument s appear. It was not until recently, a systematic study of the general theory of such mixed equations and of the global structure of the solutions was initiated in [18,19].

There have been many arguments and evidences that time delay always exists in reality and should be taken into consideration in modeling. See, for example, [8,9,10,15,21]. For this reason, we incorporate a discrete delay into (1.1) and (1.3) to consider, respectively,

$$u'_\eta(t) = \alpha(\Delta_n u)_\eta + f(u_\eta(t), u_\eta(t-\tau)), \quad \eta \in \mathbb{Z}^n \tag{1.8}$$

and

$$\frac{\partial u(t,x)}{\partial t} = \alpha\Delta u(t,x) + f(u(t,x), u(t-\tau,x)), \quad x \in \mathbb{R}^n. \tag{1.9}$$

The corresponding wave equations for (1.8) and (1.9) become, respectively,

$$cy'(s) = \alpha \sum_{k=1}^{n} [y(s+\sigma_k) + y(s-\sigma_k)] - 2\alpha n y(s) + f(y(s), y(s-c\tau)), \quad s \in \mathbb{R} \tag{1.10}$$

and

$$c\phi'(\xi) = \alpha\phi''(\xi) + f(\phi(\xi), \phi(\xi - c\tau)), \quad \xi \in \mathbb{R}. \tag{1.11}$$

Here, (1.11) is an ordinary differential equation with only retarded argument, but (1.10) is again a *mixed* equation.

In this paper, we deal with the existence of traveling wave fronts of the delayed lattice differential equations (1.8). We mention that for $\tau = 0$, existence results are established in [11,12, 24-26], for one dimensional lattice ($n = 1$) using comparison and continuation methods. In [2] the existence of traveling wave fronts is explored, for two dimensional ($n = 2$) lattice differential equations with some idealized nonlinearities by considering differential inclusion. Recently (1.8) was studied, [22], with $n = 1$ but with general delay. In the remainder of this paper, we follow the idea of upper and lower solutions in [22] to study the existence of traveling wave fronts of (1.8) with general dimension n. The rest of this paper is organized as follows. In section 2, we establish an iteration scheme starting from an upper solution, and prove that the iteration converges to a solution of (1.10) and (1.7) provided that the upper solution is properly chosen. In Section 3, we apply the main theorem established in Section 2 to a particular equation arising from branching theory. By analyzing the corresponding characteristic equation, we are able to construct the required ordered pair of upper and lower solutions, and thus, claim the existence of traveling wave fronts with large velocity.

2. Monotone Iteration

We have seen in Section 1 that the existence of traveling wave fronts of (1.8) is equivalent to the existence of solutions of (1.10) and (1.7). Without loss of generality, we can assume $q_- = 0$ and $q_+ = q > 0$, because other cases can be reduced to such a case simply by a translation. So, in what follows, we look for solutions of (1.10) and (1.7) with $q_- = 0$ and $q_+ = q > 0$, i.e., solutions of

$$cy'(s) = \alpha \sum_{k=1}^{n} [y(s+\sigma_k) + y(s-\sigma_k)] - 2\alpha n y(s) + f(y(s), y(s-c\tau)), \quad s \in \mathbb{R} \tag{2.1}$$

and

$$y(-\infty) = 0, \quad y(\infty) = q. \tag{2.2}$$

It is obvious that if (2.1)-(2.2) has a solution, then 0 and q must be zeros of the nonlinear function f. Thus, it is natural to make the following assumption.

(A1) f is continuous and $f(0,0) = 0 = f(q,q)$ and $f(u,u) \neq 0$ for $u \in (0,q)$.

Moreover, in order to get the monotonicity of our iteration, we need the following quasi-monotonicity condition for f.

(A2) There exists a $\beta > 0$ such that for any u_1, u_2, v_1 and v_2 with $0 \le u_1 \le u_2 \le q$ and $0 \le v_1 \le v_2 \le q$, one has

$$f(u_2, v_2) - f(u_1, v_1) + \beta(u_2 - u_1) \ge 0\,.$$

Define the set of profiles by

$$\Gamma = \left\{ \begin{array}{l} \rho : \mathbb{R} \to [0, q], \quad \rho \quad \text{is continuous and nondecreasing,} \\ \lim_{t\to-\infty} \rho(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} \rho(t) = q. \end{array} \right\}$$

Also define $H_\beta : C(\mathbb{R};\mathbb{R}) \to C(\mathbb{R};\mathbb{R})$ by

$$H_\beta(\rho)(t) = f(\rho(t), \rho(t - c\tau)) + \beta\rho(t) + \alpha\sum_{k=1}^{n}[\rho(t + \sigma_k) + \rho(t - \sigma_k)], \quad t \in \mathbb{R}. \tag{2.3}$$

Then H_β has the following properties.

Proposition 2.1. *Assume (A1) and (A2) are satisfied.*

(i) If ρ is in Γ, then $H_\beta(\rho)(t)$ is nondecreasing, and $\lim_{t\to-\infty} H_\beta(\rho)(t) = 0$ and $\lim_{t\to\infty} H_\beta(\rho)(t) = (\beta + 2n\alpha)q$;

(ii) $H_\beta(\psi)(t) \le H_\beta(\phi)(t)$ for $t \in \mathbb{R}$, if ψ, $\phi \in C(\mathbb{R},\mathbb{R})$ with $0 \le \psi \le \phi \le q$.

Proof. The two limits in (i) are obvious. Fix $t \in \mathbb{R}$ and $s > 0$. Using (A2), we get

$$\begin{aligned} &H_\beta(\rho)(t + s) - H_\beta(\rho)(t) \\ &= f(\rho(t + s), \rho(t + s - c\tau)) - f(\rho(t), \rho(t - c\tau)) + \beta[\rho(t + s) - \rho(t)] \\ &\quad + \alpha\sum_{k=1}^{n}[\rho(t + s + \sigma_k) - \rho(t + \sigma_k)] + \alpha\sum_{k=1}^{n}[\rho(t + s - \sigma_k) - \rho(t - \sigma_k)] \\ &\ge 0. \end{aligned}$$

This proves (i). As for (ii), it is just an immediate consequence of (A2). This completes the proof.

Denote $\mu = \beta + 2n\alpha$ and rewrite (2.1) as

$$c\frac{d}{dt}y(t) = -\mu y(t) + H_\beta(y)(t). \tag{2.4}$$

It is easy to verify that $y : \mathbb{R} \to [0, q]$ is a solution of (2.4) with $\lim_{t\to-\infty} y(t) = 0$ if and only if it solves the following integral equation

$$y(t) = e^{-\frac{\mu t}{c}} \int_{-\infty}^{t} \frac{1}{c} e^{\frac{\mu s}{c}} H_\beta(x)(s)\, ds. \tag{2.5}$$

Definition 2.1. $\rho \in C(\mathbb{R},\mathbb{R})$ is called an *upper solution* of (2.1) if it is differentiable almost everywhere, and satisfies

$$c\frac{d}{dt}\rho(t) \ge \alpha\sum_{k=1}^{n}[\rho(t + \sigma_k) + \rho(t - \sigma_k)] - 2n\alpha\rho(t) + f(\rho(t), \rho(t - c\tau)) \tag{2.6}$$

a.e. on $\mathbb{R}$. *Lower solutions* of (2.1) can be similarly defined by reversing the inequality in (2.6).

We now establish an iteration that generates a monotone sequence. In order to start the iteration, let us first assume that there exist an upper solution $\overline{\rho}(t)$ that is in Γ and a lower solution $\underline{\rho}(t)$ (not necessarily in Γ) of (2.1) with $0 \leq \underline{\rho}(t) \leq \overline{\rho}(t) \leq q$ for $t \in \mathbb{R}$. We assume $\underline{\rho}$ is a nontrivial lower solution (that is , $\underline{\rho} \not\equiv 0$ on $\mathbb{R}$). It is easy to verify that $y_1 : \mathbb{R} \to \mathbb{R}$ given by

$$y_1(t) = e^{-\frac{\mu t}{c}} \int_{-\infty}^{t} \frac{1}{c} e^{\frac{\mu s}{c}} H_\beta(\overline{\rho})(s)\, ds, \quad t \in \mathbb{R} \tag{2.7}$$

is a well defined C^1- function. Some of the important properties of y_1 are formulated as follows:

Proposition 2.2. *The function y_1 defined by (2.7) satisfies*

(i) $\frac{d}{dt} y_1(t) \geq 0$ *for* $t \in \mathbb{R}$*;*

(ii) $\underline{\rho}(t) \leq y_1(t) \leq \overline{\rho}(t)$ *for* $t \in \mathbb{R}$*;*

(iii) $\lim_{t\to-\infty} y_1(t) = 0$ *and* $\lim_{t\to+\infty} y_1(t) = q$*.*

Proof. Using the monotonicity of $\overline{\rho}$ and (i) of Proposition 2.1, we get

$$\begin{aligned}
\frac{d}{dt} y_1(t) =& -\frac{\mu}{c} e^{-\frac{\mu t}{c}} \int_{-\infty}^{t} e^{\frac{\mu s}{c}} H_\beta(\overline{\rho})(s)\, ds + \frac{1}{c} H_\beta(\overline{\rho})(t) \\
=& -\frac{\mu}{c} e^{-\frac{\mu t}{c}} \int_{-\infty}^{t} e^{\frac{\mu s}{c}} H_\beta(\overline{\rho})(s)\, ds + \frac{\mu}{c} e^{-\frac{\mu t}{c}} \int_{-\infty}^{t} e^{\frac{\mu s}{c}} H_\beta(\overline{\rho})(t)\, ds \\
=& \frac{\mu}{c} e^{-\frac{\mu t}{c}} \int_{-\infty}^{t} e^{\frac{\mu s}{c}} [H_\beta(\overline{\rho})(t) - H_\beta(\overline{\rho})(s)]\, ds \geq 0.
\end{aligned}$$

Applying the L' Hospital's rule, we get

$$\begin{aligned}
\lim_{t\to-\infty} y_1(t) &= \lim_{t\to-\infty} \frac{\frac{1}{c} e^{\frac{\mu t}{c}} H_\beta(\overline{\rho})(t)}{\frac{\mu}{c} e^{\frac{\mu t}{c}}} = \lim_{t\to-\infty} \frac{1}{\mu} H_\beta(\overline{\rho})(t) = 0; \\
\lim_{t\to\infty} y_1(t) &= \lim_{t\to\infty} \frac{\frac{1}{c} e^{\frac{\mu t}{c}} H_\beta(\overline{\rho})(t)}{\frac{\mu}{c} e^{\frac{\mu t}{c}}} = \lim_{t\to-\infty} \frac{1}{\mu} H_\beta(\overline{\rho})(t) = \frac{(\beta + 2n\alpha) q}{\mu} = q.
\end{aligned}$$

The inequality $\underline{\rho}(t) \leq y_1(t) \leq \overline{\rho}(t)$ for $t \in \mathbb{R}$ follows from the definition of y_1, the upper solution and the monotonicity $H_\beta(\overline{\rho})(t) \geq H_\beta(\underline{\rho})(t)$ for $t \in \mathbb{R}$. This completes the proof.

Note that by (ii) of Proposition 2.1, we have

$$\begin{aligned}
& c\frac{d}{dt} y_1(t) \\
&= -\mu y_1(t) + H_\beta(\overline{\rho})(t) \\
&\geq -\mu y_1(t) + H_\beta(y_1)(t) \\
&= f(y_1(t), y_1(t - c\tau)) + \alpha \sum_{k=1}^{n} [y_1(t + \sigma_k) + y_1(t - \sigma_k)] - 2n\alpha y_1(t), \quad t \in \mathbb{R}.
\end{aligned}$$

Therefore, y_1 is also an upper solution of (2.1) and is in Γ. Thus, we can repeat the above process for the pair $(y_1, \underline{\rho})$ to obtain another upper solution

$$y_2(t) = \frac{1}{c} e^{-\frac{\mu t}{c}} \int_{-\infty}^{t} e^{\frac{\mu s}{c}} H_\beta(y_1)(s)\, ds, \quad t \in \mathbb{R}. \tag{2.8}$$

Inductively, we can define

$$y_n(t) = \frac{1}{c} e^{-\frac{\mu t}{c}} \int_{-\infty}^{t} e^{\frac{\mu s}{c}} H_\beta(y_{n-1})(s)\, ds, \quad t \in \mathbb{R}, \ \ n \geq 2 \tag{2.9}$$

and obtain:

Proposition 2.3. *The above sequence is well-defined and satisfies*

(i) $\frac{d}{dt} y_n(t) \geq 0$ *for* $t \in \mathbb{R}$*;*

(ii) $\lim_{t\to-\infty} y_n(t) = 0, \quad \lim_{t\to+\infty} y_n(t) = q$*;*

(iii) $\underline{\rho}(t) \leq y_n(t) \leq y_{n-1}(t) \leq \overline{\rho}(t)$ *for* $t \in \mathbb{R}$ *and* $n \geq 2$*.*

The monotonicity (iii) in the above result ensures the existence of

$$y(t) = \lim_{n\to\infty} y_n(t), \quad t \in \mathbb{R}. \tag{2.10}$$

Clearly, the limit function $y : \mathbb{R} \to \mathbb{R}$ is nondecreasing. Moreover, we claim

Proposition 2.4. $y : \mathbb{R} \to \mathbb{R}$ *obtained from (2.7), (2.8), (2.9) and (2.10) is a solution of the asymptotic boundary value problem (2.1)-(2.2).*

Proof. Applying the Lebesgue's Dominated Convergence Theorem to (2.9), we can establish

$$y(t) = \frac{1}{c} e^{-\frac{\mu t}{c}} \int_{-\infty}^{t} e^{\frac{\mu s}{c}} H_\beta(y)(s)\, ds \tag{2.11}$$

from which it follows that y satisfies (2.1). $\lim_{t\to-\infty} y(t) = 0$ is obvious since $0 \leq \underline{\rho}(t) \leq y(t) \leq \overline{\rho}(t)$ and $\overline{\rho} \in \Gamma$. It remains to show that $\lim_{t\to\infty} y(t) = q$. Note that y is nondecreasing and bounded. So $y^* := \lim_{t\to\infty} y(t) \leq q$ exists. Taking limit as $t \to \infty$ in (2.1), we get $f(y^*, y^*) = 0$. On the other hand, we have $y_n(t) \geq \underline{\rho}(t)$ for $n \geq 1$ and $t \in \mathbb{R}$. Therefore, $y(t) \geq \underline{\rho}(t)$ and hence $y^* \geq \sup_{t\in\mathbb{R}} \underline{\rho}(t) > 0$. Consequently, in view of (A1), we must have $y^* = q$. This completes the proof.

Summarizing the above propositions, we have

Theorem 2.5. *Assume (A1) and (A2) are satisfied. Suppose (2.1) has an upper solution* $\overline{\rho}$ *in* Γ *and a non-trivial lower solution* $\underline{\rho}$ *(not necessarily in* Γ*) satisfying*

(H1) $0 \leq \underline{\rho}(t) \leq \overline{\rho}(t) \leq q, \quad t \in \mathbb{R}.$

Then, (2.1)-(2.2) has a solution in Γ*, that is, (1.8) has a traveling wave front.*

Remark 2.1. In the proof of Theorem 2.5, the assumption $f(r, r,) \neq 0$ for $r \in (0, q)$ in (A1) is used only in proving $\lim_{t\to\infty} y(t) = q$. Therefore, any replacement that ensures $\lim_{t\to\infty} y(t) = q$ will not change the conclusion of Theorem 2.5. So, we have

Theorem 2.5*. *Assume f is continuous and (A2) is satisfied. Suppose (2.1) has an upper solution $\overline{\rho}$ in Γ and a non-trivial lower solution $\underline{\rho}$ (not necessarily in Γ) satisfying (H1) and*

(A1)* *$f(u,u) \neq 0$ for $u \in (m,q)$, where $m = \sup_{t\in\mathbb{R}} \underline{\rho}(t)$.*

Then, (2.1)-(2.2) has a solution in Γ, that is, (1.8) has a traveling wave front.

Remark 2.2. In (1.8), we just incorporated a single discrete delay. The approach used in Section 2 is also applicable to lattice differential equations with general delay, i.e., equations of the form

$$u'_\eta(t) = \alpha(\Delta_n u)_\eta + f((u_\eta)_t), \quad \eta \in \mathbb{Z}^n \tag{2.12}$$

where $f : C([-\tau,0];\mathbb{R}) \to \mathbb{R}$ and $(u_\eta)_t \in C([-\tau,0];\mathbb{R})$ is defined by $(u_\eta)_t(\theta) = u_\eta(t+\theta)$ for $\theta \in [-\tau,0]$. In such a general case, the quasi-monotonicity condition (A2) should be replaced by

(A2)* There exists a $\beta > 0$ such that for any $\phi, \quad \psi \in C([-\tau,0];\mathbb{R})$ with $0 \leq \phi \leq \psi \leq q$, one has

$$f(\psi) - f(\phi) + \beta[\psi(0) - \phi(0)] \geq 0.$$

Moreover, the monotonicity condition (A2) ((A2)*) can be relaxed to some extent, but as a cost, the requirements on the ordered pair of upper and lower solutions will be more restrictive. For the details of this idea, see [22,23].

3. Applications

In this section, we apply Theorem 2.5 to a particular system. Consider

$$u'_\eta(t) = \alpha(\Delta_n u)_\eta + u_\eta(t-\tau)[1 - u_\eta(t)], \quad t \in \mathbb{R}, \ \eta \in \mathbb{Z}^n. \tag{3.1}$$

This is a spatial discretization of

$$\frac{\partial u}{\partial t} = \alpha \Delta u(t,x) + u(t-\tau)[1 - u(t,x)], \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \tag{3.2}$$

which was derived from branching theory in [14]. The corresponding wave equation of (3.1) is

$$cy'(t) = \alpha \sum_{k=1}^{n} [y(t+\sigma_k) + y(t-\sigma_k) - 2y(t)] + y(t-c\tau)[1-y(t)], \tag{3.3}$$

and the boundary conditions for wave fronts are

$$\lim_{t\to-\infty} y(t) = 0, \quad \lim_{t\to\infty} y(t) = 1. \tag{3.4}$$

The nonlinear function $f(u,v) = v(1-u)$ is obviously continuous and satisfies (A1) with $q = 1$. For (A2), we have

Lemma 3.1. *$f(u,v) = v(1-u)$ satisfies (A2) with $q = 1$ and $\beta = 1$.*

Proof. Let u_1, u_2, and v_2 be such that $0 \le u_1 \le u_2 \le 1$ and $0 \le v_1 \le v_2 \le 1$. Then

$$\begin{aligned} f(u_2, v_2) - f(u_1, v_1) &= v_2(1-u_2) - v_1(1-u_1) \\ &= (1-u_1)(v_2 - v_1) - v_2(u_2 - u_1) \\ &\ge -v_2(u_2 - u_1) \ge -(u_2 - u_1) \end{aligned}$$

which completes the proof of this lemma.

Let $G(s)$ be defined by

$$G(s) = \alpha \sum_{k=1}^{n} \left[e^{s\sigma_k} + e^{-s\sigma_k} - 2 \right] + e^{-c\tau s} - cs, \quad s \in \mathbb{R}.$$

Then,

Lemma 3.2. *There exists a $c^* > 0$ such that*
(i) when $c < c^*$, $G(s) > 0$ for $s \in \mathbb{R}$;
(ii) when $c = c^*$, $G(s) = 0$ has a unique positive solution; and
(iii) when $c > c^*$, there exist $0 < s_1 < s_2$ such that
$G(s_1) = G(s_2) = 0$,
$G(s) < 0$ for $s \in (s_1, s_2)$, and
$G(s) > 0$ for $s \in (-\infty, s_1) \cup (s_2, \infty)$.

Proof. Denote $g(s) = \alpha \sum_{k=1}^{n} [e^{s\sigma_k} + e^{-s\sigma_k} - 2]$ and $h_c(s) = cs - e^{-c\tau s}$. Then, $G(s) = g(s) - h_c(s)$, and elementary analysis of $g(s)$ and $h_c(s)$ (see Figure 1) leads to the conclusion of this lemma.

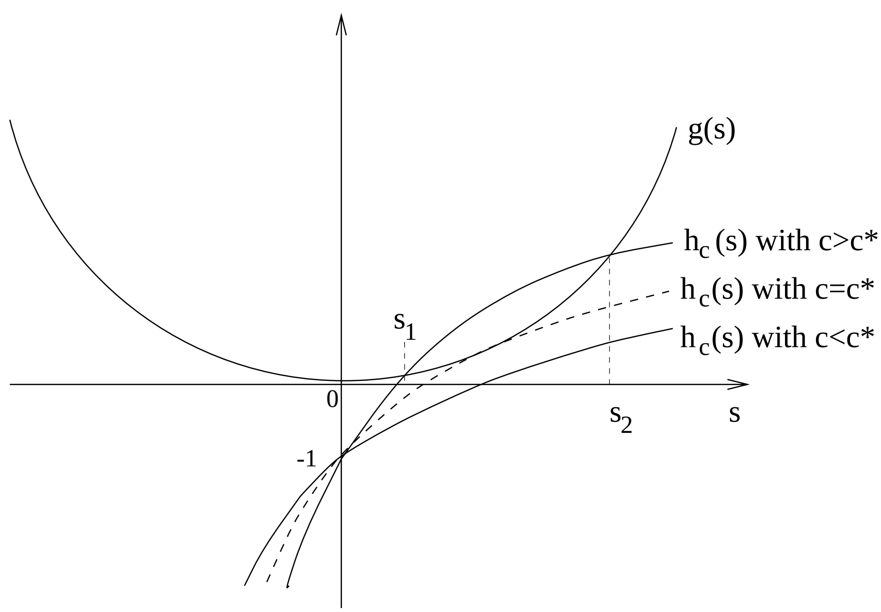

Figure 1

Note that c^* depends on the direction $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$, dimension n as well as the diffusion coefficient α and the delay τ. Using c^*, s_1 and s_2 in Lemma 3.2, we can construct the required ordered pair of upper and lower solutions.

Lemma 3.3. *Assume $c > c^*$ and s_1 be as in Lemma 3.2. Then, $\overline{\rho}(t) = \min\{e^{s_1 t}, 1\}$ is in Γ with $q = 1$ and is an upper solution of (3.3).*

Proof. $\overline{\rho} \in \Gamma$ is obvious. For $t > 0$, $\overline{\rho}(t) = 1$, and

$$\begin{aligned}
&\alpha \sum_{k=1}^{n} [\overline{\rho}(t+\sigma_k) + \overline{\rho}(t-\sigma_k) - 2\overline{\rho}(t)] + \overline{\rho}(t-c\tau)[1-\overline{\rho}(t)] \\
&= \alpha \sum_{k=1}^{n} [\overline{\rho}(t+\sigma_k) + \overline{\rho}(t-\sigma_k) - 2] \\
&\leq \alpha \sum_{k=1}^{n} [1+1-2] = 0 = c\overline{\rho}'(t).
\end{aligned}$$

For $t < 0$, $\overline{\rho}(t) = e^{s_1 t}$, and

$$\begin{aligned}
&\alpha \sum_{k=1}^{n} [\overline{\rho}(t+\sigma_k) + \overline{\rho}(t-\sigma_k) - 2\overline{\rho}(t)]\overline{\rho}(t-c\tau)[1-\overline{\rho}(t)] \\
&= \alpha \sum_{k=1}^{n} \left[\overline{\rho}(t+\sigma_k) + \overline{\rho}(t-\sigma_k) - 2e^{s_1 t}\right] + e^{s_1(t-c\tau)}\left(1-e^{s_1 t}\right) \\
&\leq \alpha \sum_{k=1}^{n} \left[e^{s_1(t+\sigma_k)} + e^{s_1(t-\sigma_k)} - 2e^{s_1 t}\right] + e^{s_1(t-c\tau)}\left(1-e^{s_1 t}\right) \\
&= e^{s_1 t}\left[\alpha \sum_{k=1}^{n} \left(e^{s_1\sigma_k} + e^{-s_1\sigma_k} - 2]\right) + e^{-s_1 c\tau}\left(1-e^{s_1 t}\right)\right] \\
&\leq e^{s_1 t}\left[\alpha \sum_{k=1}^{n} \left(e^{s_1\sigma_k} + e^{-s_1\sigma_k} - 2]\right) + e^{-s_1 c\tau}\right] \\
&= e^{s_1 t}(cs_1) = c\overline{\rho}'(t).
\end{aligned}$$

This completes the proof.

Lemma 3.4. *Assume $c > c^*$ and let s_1 and s_2 be as in Lemma 3.2. Let $r > 0$ be such that $r < s_1$ and $s_1 + r < s_2$. Then, $\underline{\rho}(t) = \max\{0, (1 - Me^{rt})e^{s_1 t}\}$ is a non-trivial solution of (3.3), provided $M > 0$ is sufficiently large.*

Proof. Let $t_0 < 0$ be such that $Me^{rt_0} = 1$. For $t > t_0$, $\underline{\rho}(t) = 0$, and

$$\begin{aligned}
&\alpha \sum_{k=1}^{n} [\underline{\rho}(t+\sigma_k) + \underline{\rho}(t-\sigma_k) - 2\underline{\rho}(t)] + \underline{\rho}(t-c\tau)[1-\underline{\rho}(t)] \\
&= \alpha \sum_{k=1}^{n} [\underline{\rho}(t+\sigma_k) + \underline{\rho}(t-\sigma_k)] + \underline{\rho}(t-c\tau) \\
&\geq 0 = c\underline{\rho}'(t).
\end{aligned}$$

For $t < t_0$, $\underline{\rho}(t) = (1 - Me^{rt})e^{s_1 t}$ and $\underline{\rho}'(t) = (s_1 - (s_1 + r)Me^{rt})e^{s_1 t}$. Using

Lemma 3.2, we get

$$
\begin{aligned}
&\alpha \sum_{k=1}^{n}[\underline{\rho}(t+\sigma_k)+\underline{\rho}(t-\sigma_k)-2\underline{\rho}(t)]+\underline{\rho}(t-c\tau)[1-\underline{\rho}(t)] \\
&\geq \alpha \sum_{k=1}^{n}\left[\left(1-Me^{r(t+\sigma_k)}\right)e^{s_1(t+\sigma_k)}+\left(1-Me^{r(t-\sigma_k)}\right)e^{s_1(t-\sigma_k)}-2\left(1-Me^{rt}\right)e^{s_1 t}\right] \\
&\qquad\qquad \left(1-Me^{r(t-c\tau)}\right)e^{s_1(t-c\tau)}\left[1-\left(1-Me^{rt}\right)e^{s_1 t}\right] \\
&= e^{s_1 t}\left[\alpha \sum_{k=1}^{n}\left(e^{s_1\sigma_k}+e^{-s_1\sigma_k}-2\right)-\alpha M e^{rt}\sum_{k=1}^{n}\left(e^{(s_1+r)\sigma_k}+e^{-(s_1+r)\sigma_k}-2\right)\right. \\
&\qquad\qquad \left.\left(1-Me^{r(t-c\tau)}\right)e^{-s_1 c\tau}-\left(1-Me^{r(t-c\tau)}\right)\left(1-Me^{rt}\right)e^{s_1 t}e^{-s_1 c\tau}\right] \\
&= e^{s_1 t}\left[\alpha \sum_{k=1}^{n}\left(e^{s_1\sigma_k}+e^{-s_1\sigma_k}-2\right)+e^{-s_1 c\tau}-\alpha M e^{rt}\sum_{k=1}^{n}\left(e^{(s_1+r)\sigma_k}+e^{-(s_1+r)\sigma_k}-2\right)\right. \\
&\qquad\qquad \left.-Me^{rt}e^{-(s_1+r)c\tau}-e^{s_1 t}e^{-s_1 c\tau}\left(1-Me^{r(t-c\tau)}\right)\left(1-Me^{rt}\right)\right] \\
&= e^{s_1 t}\left[cs_1-Me^{rt}G(s_1+r)-c(s_1+r)Me^{rt}\right. \\
&\qquad\qquad \left.-e^{s_1 t}e^{-s_1 c\tau}\left(1-Me^{r(t-c\tau)}\right)\left(1-Me^{rt}\right)\right] \\
&> e^{s_1 t}\left[cs_1-c(s_1+r)Me^{rt}-Me^{rt}G(s_1+r)-e^{rt}e^{-s_1 c\tau}\right] \\
&= c\underline{\rho}'(t)+e^{(s_1+r)t}\left[-MG(s_1+r)-e^{-s_1 c\tau}\right] \\
&\geq c\underline{\rho}'(t),
\end{aligned}
$$

provided $M \geq \frac{e^{-s_1 c\tau}}{-G(s_1+r)}$. This completes the proof.

Combining the above lemmas with Theorem 2.5, we obtain

Theorem 3.5. *For each $c > c^*$ where c^* is as in Lemma 3.2, (3.1) has a traveling wave front with velocity c.*

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