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Bounded solutions of nonlinear parabolic equations with time delay *

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Abstract

The existence and uniqueness of a bounded solution is established for a nonlinear parabolic equation with piecewise continuous time delay. This problem may be considered as a generalization of Fisher's equation which has applications in certain ecological studies.

1 Introduction

In this paper, we are interested in finding sufficient conditions for the existence of a unique bounded solution to a nonlinear parabolic equation with time delay. The initial-value problem under investigation is the following

$$u_t(x,t) = a^2 u_{xx}(x,t) + f(u(x,t), u(x,[t])), \tag{1}$$

$$u(x,0) = \varphi(x), \quad -\infty < x < \infty \tag{2}$$

in the domain $\mathcal{D} = (-\infty,\infty) \times (0,\infty)$, where $[t]$ denotes the greatest-integer function. In this case, the delay function is constant on unit intervals and has jump discontinuities at their endpoints. Within each of these intervals the dynamical system is described by a PDE without delay. Continuity of a solution at the endpoints of two consecutive intervals leads to a difference equation of an integer argument for the values of the solution at the endpoints. Therefore, equations with piecewise constant delay describe hybrid (continuous and discrete) dynamical systems and combine the properties of both differential and difference equations. They include, as particular cases, loaded and impulsive equations of control theory. This note continues our earlier research on bounded solutions of nonlinear hyperbolic equations with time delay [1-4].

Definition. A function $u(x,t)$ is called a solution of the initial-value problem (1)-(2) if it satisfies the following conditions:

(i) $u(x,t)$ is continuous in $\mathcal{D} = (-\infty,\infty) \times (0,\infty)$.

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(ii) u_t and u_{xx} exist and are continuous in $\mathcal{D}$, with the possible exception of the points $(x,[t])$ where one-sided derivatives exist.

(iii) $u(x,t)$ satisfies (1) in $\mathcal{D}$, with the possible exception of the points $(x,[t])$, and initial conditions (2).

2 Existence and uniqueness theorem

The method of proof is based on reducing equations (1), (2) to an integral equation in two variables and the use of successive approximations.

Theorem 1 *Assume the following hypotheses:*

(i) The function $\varphi(x)$ is twice continuously differentiable and bounded on $\mathbb{R}$.

(ii) The function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and bounded in $\mathcal{D}$, and satisfies the Lipschitz condition $|f(u,w) - f(v,w)| \le L\,|u-v|$ uniformly with respect to w, where L is a positive constant and $u, v \in (-\infty,\infty)$.

Then there exists a unique solution to problem (1), (2) which is bounded in $\mathcal{D}$.

Proof. On the interval $0 \le t < 1$, equation (1) becomes

$$u_t(x,t) = a^2 u_{xx}(x,t) + f(u(x,t),\varphi(x))\,. \tag{3}$$

Using integration by parts twice for equation (1) we obtain, after some computations, the following integral equation for the solution of problem (1), (2):

$$\begin{aligned} u(x,t) \;&=\; \frac{1}{\sqrt{4a^2\pi t}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4a^2t}\varphi(\xi)\,d\xi \qquad (4)\\ &+\frac{1}{\sqrt{2\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\Big(\frac{1}{\sqrt{4a^2\pi t}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4a^2t} f(u(\xi,\tau),\varphi(\xi))\,d\xi\Big)\,d\tau\,. \end{aligned}$$

According to the method of successive approximations, put

$$u_0(x,t) = \frac{1}{\sqrt{4a^2\pi t}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4a^2t}\varphi(\xi)\,d\xi\,.$$

Since $|\varphi(x)| \le M$ for $x \in (-\infty,\infty)$, the substitution $(x-\xi)/2a\sqrt{t} = \alpha$ implies that

$$|u_0(x,t)| \le \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\alpha^2} 2a\sqrt{t}\,|\varphi(\xi)|\,d\alpha = |\varphi(x)| \le M\,, \tag{5}$$

and from equation (4) one obtains the estimates

$$\begin{aligned} |u_1 - u_0| \;&\le\; \frac{1}{\sqrt{2\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\Big(\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4a^2(t-\tau)}\,|f(u_0,\varphi(\xi))|\,d\xi\Big)\,d\tau\\ &\le\; \frac{1}{\sqrt{2\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\Big(\int_{-\infty}^{\infty} e^{-\beta^2}\cdot \overline{M}\cdot 2a\sqrt{t-\tau}\,d\beta\Big)\,d\tau\\ &=\; \sqrt{2}a\overline{M}t, \quad 0 \le t < 1\,, \qquad (6) \end{aligned}$$

where $|f(u_0(x,t),\varphi(x))| \leq \overline{M}$ in $\mathcal{D}$ and $(x-\xi)/2a\sqrt{t-\tau} = \beta$, with some constants M and $\overline{M}$. Therefore, from equation (4) and inequality (6), it follows that

$$|u_1| \leq M + \sqrt{2}a\overline{M}t, \quad 0 \leq t < 1\,. \tag{7}$$

Furthermore, we use equation (4) for the second approximation

$$u_2(x,t) = u_0(x,t) + \frac{1}{\sqrt{2\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}(\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4a^2(t-\tau)} f(u_1,\varphi)\,d\xi)\,d\tau$$

and obtain the estimate

$$|u_2 - u_1| \leq \frac{1}{\sqrt{2\pi}}\int_0^t (\int_{-\infty}^{\infty} e^{-\gamma^2} L\,|u_1 - u_0| \cdot 2a\sqrt{t-\tau}\,d\gamma)\,d\tau \leq 2L\overline{M}\frac{a^2t^2}{2!},$$

which implies

$$|u_2| \leq M + 2\frac{\overline{M}}{L}\frac{(Lat)^2}{2!}, \quad 0 \leq t < 1\,. \tag{8}$$

In the same fashion,

$$\begin{aligned}
|u_3 - u_2| \quad &\leq \quad \frac{1}{\sqrt{2\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}(\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4a^2t}\Big| f(u_2(\xi,\tau),\varphi(\xi)) \\
&\qquad\qquad -f(u_1(\xi,\tau),\varphi(\xi))\Big|\,d\xi)\,d\tau \\
&\leq \quad \frac{1}{\sqrt{2\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}(\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4a^2t} L\,|u_2 - u_1|\;d\xi)\,d\tau \\
&\leq \quad \frac{1}{\sqrt{2}} \cdot 2a \cdot 2L^2\overline{M}a^2\frac{t^3}{3!}\,,
\end{aligned}$$

or

$$|u_3 - u_2| \leq 2^{3/2}\overline{M}L^2\frac{(at)^3}{3!}\,. \tag{9}$$

Similarly,

$$|u_4 - u_3| \leq 2^2\overline{M}L^3\frac{(at)^4}{4!}\,.$$

In general, this procedure leads to the estimate

$$|u_n(x,t) - u_{n-1}(x,t)| \leq 2^{n/2}\frac{\overline{M}}{L}\frac{(Lat)^n}{n!}, \quad n = 1,2,\ldots \tag{10}$$

and since

$$u(x,t) = u_0(x,t) + \sum_{i=1}^{\infty}(u_{i+1}(x,t) - u_i(x,t)),$$

it follows

$$|u(x,t)| \leq M + \sum_{i=1}^{\infty}|u_{i+1} - u_i|\,.$$

With the account of inequality (10),

$$|u(x,t)| \le M + \frac{\overline{M}}{L} e^{\sqrt{2}aLt}, \quad 0 \le t < 1\,,$$

which proves the existence of a bounded solution for (1), (2) in the domain $0 \le t \le 1$, $-\infty < x < \infty$. Now, for the solution of (1), (2) on the interval $1 \le t < 2$, we note that $0 \le t-1 < 1$ and replacing t with $t-1$, arrive at the estimates

$$\begin{aligned}
|u_0| &\le \frac{1}{\sqrt{4a^2\pi(t-1)}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4a^2(t-1)}\,|\varphi_1(\xi)|\,d\xi \le M_1,\\
|u_1 - u_0| &\le \frac{1}{\sqrt{2\pi}} \int_0^{t-1} \frac{1}{\sqrt{t-1-\tau}} \Big(\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4a^2(t-1)}\,|f(u_0,\varphi_1)|\,d\xi\Big)\,d\tau\\
&= \sqrt{2}a\overline{M_1}(t-1),\\
|u_1| &\le M_1 + \sqrt{2}a\overline{M_1}(t-1), \quad 1 \le t < 2\,,
\end{aligned}$$

with some constants M_1 and $\overline{M_1}$, where

$$\varphi_1(x) = u(x,1), \quad |\varphi_1(x)| \le M_1, \quad \text{and} \quad |f(u_0(x,t),\varphi_1(x))| \le \overline{M_1}\,.$$

On the interval $1 \le t < 2$, we have the inequalities

$$|u_2 - u_1| \le 2L\overline{M_1}\frac{a^2(t-1)^2}{2!}\,, \tag{11}$$

and

$$|u_2| \le M_1 + 2\frac{\overline{M_1}}{L}\frac{L^2a^2(t-1)^2}{2!}, \quad 1 \le t < 2\,.$$

Continuation of the above procedure yields the general estimates

$$|u_n(x,t) - u_{n-1}(x,t)| \le 2^{n/2}L^{n-1}\overline{M_1}\frac{(a(t-1))^n}{n!}, \quad 1 \le t < 2\,. \tag{12}$$

These inequalities imply that

$$|u(x,t)| \le M_1 + \frac{\overline{M_1}}{L} e^{\sqrt{2}aL(t-1)}, \quad 1 \le t < 2\,,$$

which proves the existence of a bounded solution to problem (1), (2) in the domain $1 \le t < 2$, $-\infty < x < \infty$. In the next step, we obtain

$$|u(x,t)| \le M_2 + \frac{\overline{M_2}}{L} e^{\sqrt{2}aL(t-2)}, \quad 2 \le t < 3\,,$$

and, in general,

$$|u(x,t)| \le M_n + \frac{\overline{M_n}}{L} e^{\sqrt{2}aL(t-n)}, \quad n \le t < n+1\,,$$

where the variants M_n and $\overline{M_n}$ are bounded. Therefore, the function $u(x,t)$ constructed is a solution of problem (1), (2) which is bounded in $\mathcal{D}$. Uniqueness of this solution is a simple consequence of the Lipschitz condition.

Remark 1. The method of successive approximations also enables one to prove, under certain assumptions, the existence of a unique bounded solution for the differential equation with constant delay

$$u_t(x,t) = a^2 u_{xx}(x,t) + f(u(x,t), u(x,t-h)),$$

satisfying the initial condition

$$u(x,t) = \varphi(x,t), \qquad (-h \le t \le 0, -\infty < x < \infty).$$

We may also generalize problem (1), (2) to include equations of the form

$$u_t(x,t) = a^2 u_{xx}(x,t) + f(u(x,t), u(x,[t]), u_x(x,[t])).$$

Remark 2. Equation (1) may be considered as a generalization of Fisher's equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + f(u),$$

the discrete and continuous versions of which have been used as models for population dynamics and the propagation of a favored gene in a population. The discrete model might be more appropriate in certain situations, for example, population dispersal in a patchy environment. Equation (1) is also a semi-discretization of a continuous nonlinear diffusion equation.

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