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A REMARK ON THE HALF-LINEAR EXTENSION OF THE HARTMAN-WINTNER THEOREM

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Abstract. We establish a Hartman-Wintner type theorem for the half-linear second order differential equation

$$(r(t)\Phi_p(x'))' + c(t)\Phi_p(x) = 0, \quad \Phi_p(x) := |x|^{p-2}x,\ p > 1\,.$$

This equation is viewed as a perturbation of the non-oscillatory equation

$$(r(t)\Phi_p(x'))' + \tilde{c}(t)\Phi_p(x) = 0$$

with $\tilde{c}(t) \neq 0$ eventually.

1. Introduction

The classical Hartman-Wintner theorem concerns the non-oscillatory second order linear equation

$$x'' + c(t)x = 0 \tag{1.1}$$

and states that for any solution x of (1.1) and for $w := \frac{x'}{x}$ we have

$$\int^{\infty} w^2(t)\,dt < \infty \quad \iff \quad \liminf_{t\to\infty} \frac{1}{t}\int_T^t \int_T^s c(\tau)d\tau\,ds > -\infty,$$

where $T \in \mathbb{R}$ is sufficiently large, see [6, Chap. XI]. If r is a positive function such that $\int^\infty r^{-1}(t)\,dt = \infty$, using the change of dependent variable $s = \int^t r^{-1}(\tau)\,d\tau$, one can directly verify that the above statement extends also to the non-oscillatory equation in the Sturm-Liouville form

$$(r(t)x')' + c(t)x = 0, \tag{1.2}$$

namely, if $w := \frac{r(t)x'}{x}$, then

$$\int^\infty \frac{w^2(t)}{r(t)} < \infty \tag{1.3}$$

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if and only if

$$\liminf_{t\to\infty} \frac{1}{\int_T^t r^{-1}(s)\,ds} \int_T^t r^{-1}(s) \int_T^s c(\tau)\,d\tau\,ds > -\infty. \tag{1.4}$$

Note that $w = \frac{r(t)x'}{x}$ solves the Riccati equation associated with (1.2)

$$w' + c(t) + \frac{w^2}{r(t)} = 0. \tag{1.5}$$

A half-linear second order equation is the differential equation of the form

$$(r(t)\Phi_p(x'))' + c(t)\Phi_p(x) = 0, \quad \Phi_p(x) := |x|^{p-2}x, \quad p > 1, \tag{1.6}$$

where the functions r, c are continuous and $r(t) > 0$ in the interval under consideration. If $p = 2$, then (1.6) reduces to linear equation (1.2). The terminology "half-linear equation" comes from the fact that the solution space of (1.6) has just one half of the properties which characterize linearity, namely the homogeneity.

Oscillation theory of (1.6) is very similar to that of (1.2). The Sturmian separation and comparison theory extends directly to (1.6), in particular, all solutions of this equation are either oscillatory or non-oscillatory, see [5,9].

The direct half-linear extension of the Hartman-Wintner theorem can be found in [8] and reads as follows.

Proposition 1.1. *Suppose that* (1.6) *is non-oscillatory,* $\int^\infty r^{1-q}(t)\,dt = \infty$*, where* q *is the conjugate number of* p*, i.e.* $\frac{1}{p} + \frac{1}{q} = 1$*. Further, let* x *be any non-oscillatory solution of* (1.6) *and* $w := \frac{r(t)\Phi_p(x')}{\Phi_p(x)}$*. Then*

$$\int^\infty r^{1-q}(t)|w(t)|^q\,dt < \infty \iff \liminf_{t\to\infty} \frac{1}{\int_T^t r^{1-q}(s)\,ds} \int_T^t r^{1-q}(s) \int_T^s c(\tau)\,d\tau\,ds > -\infty. \tag{1.7}$$

Actually, the half-linear version of the Hartman-Wintner theorem is formulated in [8] for equation (1.6) with $r \equiv 1$ and some weight function appears in (1.7). However, taking a suitable weight function, the result in [8] gives essentially the statement of Proposition 1.1.

In this paper, we are going to present a slightly different extension of the Hartman-Wintner theorem along the following line. Let us consider (1.6) as a "perturbation" of the one-term equation

$$(r(t)\Phi_p(x'))' = 0 \tag{1.8}$$

and suppose that $\int^\infty r^{1-q}(t)\,dt = \infty$. Then any solution of the Riccati equation $w' + (p-1)r^{-1}(t)|w|^q = 0$ associated with (1.8) satisfies

$$\int^\infty r^{1-q}(t)|w(t)|^q\,dt < \infty$$

as can be verified by a direct computation. The Hartman-Wintner theorems states that this property of solutions of the Riccati equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0 \tag{1.9}$$

(which corresponds to the "perturbed" equation (1.6)) also holds if and only if the function c is not "too negative", i.e., $\liminf$ in (1.7) is $> -\infty$. In this paper, we consider (1.6) not as a perturbation of (1.8), but as a perturbation of the general non-oscillatory equation

$$(r(t)\Phi_p(x'))' + \tilde{c}(t)\Phi_p(x) = 0, \tag{1.10}$$

where $\tilde{c}(t) \neq 0$ eventually. This idea has been used in the recent paper [4] when investigating oscillatory properties of (1.6).

We show in this paper that the convergence of a certain improper integral is related to a limit inferior involving the difference $c - \tilde{c}$. The paper is organized as follows. In the next section, we collect some auxiliary material, including the recently established half-linear version of the so-called Picone identity which plays the crucial role in our investigation. In Section 3, we present our modified extension of the Hartman-Wintner theorem for (1.6), when this equation is viewed a perturbation of (1.10). We deal with the case $1 < p \leq 2$ only and it is an open question whether our results extend also to $p > 2$. The last section is devoted to remarks on results of Section 3, in particular, we show that in case $p = 2$ the results of this section reduce essentially to the classical Hartman-Wintner theorem for (1.4).

2. Auxiliary results

As we have already mentioned in the previous section, if x is a solution of (1.6) for which $x(t) \neq 0$ in the interval under consideration, then

$$w := \frac{r(t)\Phi_p(x')}{\Phi_p(x)} \tag{2.1}$$

solves the associated Riccati equation (1.9).

Lemma 2.1. ([7, Lemma 1]) *Suppose that w is a solution of* (1.9) *defined in some interval $I \subset \mathbb{R}$. Then for any continuously differentiable y the following identity holds:*

$$\begin{aligned} &r(t)|y'|^p - c(t)|y|^p \\ &= [w|y|^p]' + p\left[\frac{1}{p}\,r(t)|y'|^p - w\Phi_p(y)y' + \frac{1}{q}\,r^{1-q}(t)|w|^q|y|^p\right] \\ &= [w|y|^p]' + pP(r^{\frac{1}{p}}y', r^{-1/p}w\Phi_p(y)) \\ &= [w|y|^p]' + pr^{1-q}(t)P(r^{q-1}y', \Phi_p(y)w), \end{aligned} \tag{2.2}$$

where

$$P(u,v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0 \tag{2.3}$$

for any u, v, with the equality holding if and only if $v = \Phi_p(u)$.

Observe that in the linear case $P(u,v) = \frac{1}{2}(u-v)^2$. The following lemma presents some estimates for the just introduced function $P(u,v)$.

Lemma 2.2. ([3]) *The function* $P(u,v)$ *defined in* (2.3) *satisfies the following two inequalities*

$$P(u,v)-\frac{1}{2}|u|^{2-p}\left(\Phi_p(u)-v\right)^2 \gtrless 0 \qquad p\lessgtr 2, \quad \Phi_p(u)\neq v, \tag{2.4}$$

$$P(u,v)-\frac{1}{2(p-1)}|u|^{2-p}\left(\Phi_p(u)-v\right)^2 \lessgtr 0 \quad p\lessgtr 2,\ |\Phi_p(u)|>|v|,\ uv>0. \tag{2.5}$$

3. An extension of the Hartman-Wintner theorem

In this section we present our modified extension of the Hartman-Wintner theorem. In contrast to [8], equation (1.6) is viewed as a perturbation of non-oscillatory equation (1.10) where $\tilde c(t)\neq 0$ eventually.

Theorem 3.1. *Suppose that* $p\in(1,2]$, $\tilde c(t)\neq 0$ *eventually, equations* (1.6) and (1.10) *are non-oscillatory and* (1.10) *possesses a solution* h *such that*

$$\int^\infty G^{-1}(t)\,dt=\infty, \quad G(t):=r(t)h^2(t)|h'(t)|^{p-2}. \tag{3.1}$$

If

$$\liminf_{t\to\infty}\frac{1}{\int_T^t G^{-1}(s)\,ds}\int_T^t G^{-1}(s)\left[\int_T^s (c(\tau)-\tilde c(\tau))h^p(\tau)\,d\tau\right]ds>-\infty, \tag{3.2}$$

then for any solution x *of* (1.6) *and* w *given by* (2.1) *we have*

$$\int^\infty \frac{v^2(t)}{G(t)}\,dt<\infty, \quad v(t):=h^p(t)(w_h(t)-w(t)), \tag{3.3}$$

with $w_h=\frac{r(t)\Phi_p(h')}{\Phi_p(h)}$.

Proof. Let x be any (non-oscillatory) solution of (1.6) and w be given by (2.1). Then by the Picone identity

$$\begin{aligned}\int_T^t [r|h'|^p-ch^p]\,ds &= wh^p\,\big|_T^t + p\int_T^t r^{1-q}P(r^{q-1}h',\Phi_p(h)w)\,ds\\ &= wh^p\,\big|_T^t + p\int_T^t r^{1-q}h^pP(\Phi_q(w_h),w)\,ds,\end{aligned}$$

where $\Phi_q(s):=|s|^{q-2}s$. Simultaneously, integration by parts gives

$$\begin{aligned}\int_T^t [r|h'|^p-ch^p]\,ds &= \int_T^t [r|h'|^p-\tilde ch^p]ds-\int_T^t (c-\tilde c)h^p\,ds\\ &= rh\Phi_p(h')\,\big|_T^t-\int_T^t h[(r\Phi_p(h'))'+\tilde c\Phi_p(h)]\,ds-\int_T^t (c-\tilde c)h^p\,ds\\ &= h^pw_h\,\big|_T^t-\int_T^t (c-\tilde c)h^p\,ds.\end{aligned}$$

Denote $v := h^p(w_h - w)$; then we have

$$v(t) - v(T) = \int_T^t (c - \tilde{c})h^p\, ds + p\int_T^t r^{1-q}h^p P(\Phi_q(w_h), w)\, ds.$$

Hence, multiplying this equality by $G^{-1}(t)$ and integrating it from T to t, we have

$$\begin{aligned}\int_T^t G^{-1}(s)v(s) &= \int_T^t G^{-1}(s)\, ds \left[v(T) + \frac{\int_T^t G^{-1}(s)\left(\int_T^s (c-\tilde{c})h^p d\tau\right) ds}{\int_T^t G^{-1}(s)\, ds}\right] \\ &\quad + p\int_T^t G^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi_q(w_h), w)d\tau\right) ds.\end{aligned}$$

By Lemma 2.2

$$\begin{aligned} r^{1-q}h^p P(\Phi_q(w_h), w) &\geq \frac{1}{2} r^{1-q}h^p \left|\Phi_q(w_h)\right|^{2-p}(w_h - w)^2 \\ &= \frac{1}{2} r^{1-q+(q-1)(2-p)}h^p \left|\frac{h'}{h}\right|^{2-p} \left[h^p(w_h - w)\right]^2 \\ &= \frac{1}{2}\, \frac{\left[h^p(w_h - w)\right]^2}{rh^2|h'|^{p-2}} = \frac{v^2}{2G}, \end{aligned}$$

Consequently,

$$\begin{aligned}\int_T^t G^{-1}(s)v(s)\, ds &\geq \int_T^t G^{-1}(s)\, ds \left[v(T) + \frac{\int_T^t G^{-1}(s)\left(\int_T^s (c-\tilde{c})h^p d\tau\right) ds}{\int_T^t G^{-1}(s)\, ds}\right] \\ &\quad + \frac{p}{2}\int_T^t G^{-1}(s)\left(\int_T^s \frac{v^2(\tau)}{G(\tau)} d\tau\right) ds.\end{aligned}$$

The Cauchy-Schwarz inequality yields

$$\int_T^t G^{-1}(s)v(s)\, ds \leq \left(\int_T^t G^{-1}(s) ds\right)^{1/2} \left(\int_T^t \frac{v^2(s)}{G(s)} ds\right)^{1/2},$$

thus, taking into account (3.2), there exists a constant $K \in \mathbb{R}$ such that

$$\begin{aligned} &\left(\int_T^t G^{-1}(s) ds\right)^{1/2} \left(\int_T^t \frac{v^2(s)}{G(s)} ds\right)^{1/2} \\ &\geq \int_T^t G^{-1}(s) ds \left[K + \frac{p\int_T^t G^{-1}(s)\left(\int_T^s \frac{v^2(\tau)}{G(\tau)} d\tau\right) ds}{2\int_T^t G^{-1}(s)\, ds}\right]. \end{aligned} \tag{3.4}$$

Suppose, by contradiction, that $\int^\infty \frac{v^2(t)}{G(t)}\, dt = \infty$. Since (3.1) holds, by L'Hospital's rule

$$\lim_{t\to\infty} \frac{\int_T^t G^{-1}(s)\left(\int_T^s \frac{v^2(\tau)}{G(\tau)} d\tau\right) ds}{\int_T^t G^{-1}(s) ds} = \infty$$

as well, i.e.,

$$K+\frac{p\int_T^t G^{-1}(s)\left(\int_T^s \frac{v^2(\tau)}{2\,G(\tau)}d\tau\right)ds}{2\int_T^t G^{-1}(s)ds}\ge \frac{p}{4}\,\frac{\int_T^t G^{-1}(s)\left(\int_T^s \frac{v^2(\tau)}{G(\tau)}d\tau\right)ds}{\int_T^t G^{-1}(s)ds} \tag{3.5}$$

for t sufficiently large. Let

$$S(t):=\int_T^t G^{-1}(s)\left(\int_T^s \frac{v^2(\tau)}{G(\tau)}d\tau\right)ds.$$

Then by (3.4) and (3.5),

$$\left(\int_T^t G^{-1}(s)\,ds\right)^{1/2}\left(\int_T^t \frac{v^2(s)}{G(s)}ds\right)^{1/2}\ge \frac{p}{4}S(t)$$

which means

$$\left(\int_T^t G^{-1}(s)\,ds\right)^{1/2}\left(S'(t)G(t)\right)^{1/2}\ge \frac{p}{4}\,S(t),$$

and hence

$$\frac{S'(t)}{S^2(t)}\ge \frac{p^2}{16\,G(t)\left(\int_T^t G^{-1}(s)ds\right)}. \tag{3.6}$$

Integrating (3.6) from t_1 $(>T)$ to t, we have

$$\frac{1}{S(t_1)}>\frac{1}{S(t_1)}-\frac{1}{S(t)}\ge \frac{p^2}{16}\ln\left(\int_{t_1}^t G^{-1}(s)\,ds\right)\to\infty \quad \text{as} \quad t\to\infty,$$

which is a contradiction and completes the proof.

The opposite implication holds under a slightly stronger assumption than (3.3). We use the notation introduced in the previous theorem.

Theorem 3.2. *Suppose that* $p\in(1,2]$, $\tilde c(t)\ne 0$ *eventually, equations* (1.6) and (1.10) *are non-oscillatory, and* (1.10) *possesses a solution* h *satisfying* (3.1). *If*

$$\int^\infty r^{1-q}(t)h^p(t)P(\Phi_q(w_h),w)\,dt<\infty, \quad w_h:=\frac{r(t)\Phi_p(h')}{\Phi_p(h)} \tag{3.7}$$

for any solution x *of* (1.6) *and* w *given by* (2.1), *then*

$$\lim_{t\to\infty}\frac{1}{\int_T^t G^{-1}(s)\,ds}\int_T^t G^{-1}(s)\left[\int_T^s (c(\tau)-\tilde c(\tau))h^p(\tau)\,d\tau\right]ds$$

exists and is finite.

Proof. Suppose that (3.7) holds for any solution x of (1.6) and associated solution w of (1.9). Similar to the previous proof, we have

$$\int_T^t G^{-1}(s)v(s)ds=\int_T^t G^{-1}(s)\,ds\left[v(T)+\frac{\int_T^t G^{-1}(s)\left(\int_T^s (c-\tilde c)h^p d\tau\right)ds}{\int_T^t G^{-1}(s)\,ds}\right]$$
$$+\int_T^t G^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi_q(w_h),w)d\tau\right)ds.$$

Divergence of the integral in (3.1) implies (by L'Hospital's rule) that

$$\begin{aligned}\lim_{t\to\infty} &\frac{\int_T^t G^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi_q(w_h),w)\,d\tau\right)ds}{\int_T^t G^{-1}(s)\,ds}\\ &= \int_T^\infty r^{1-q}(s)h^p(s)P(\Phi_q(w_h),w)\,ds\end{aligned}$$

exists and is finite, and by the Cauchy-Schwarz inequality,

$$0 \le \frac{\left|\int_T^t G^{-1}(s)v(s)\,ds\right|}{\int_T^t G^{-1}(s)\,ds} \le \frac{\left(\int_T^t G^{-1}(s)\,ds\right)^{1/2}\left(\int_T^t \frac{v^2(s)}{G(s)}\,ds\right)^{1/2}}{\int_T^t G^{-1}(s)\,ds}.$$

Now, by Lemma 2.2,

$$\infty > \int^\infty r^{1-q}(s)h^p(s)P(\Phi_q(w_h),w)\,ds \ge \int^\infty \frac{v^2(s)}{G(s)}\,ds$$

and this implies

$$\frac{\int_T^t G^{-1}(s)v(s)\,ds}{\int_T^t G^{-1}(s)\,ds} \to 0 \quad \text{as } t\to\infty.$$

Consequently,

$$\begin{aligned}\lim_{t\to\infty} &\frac{\int_T^t G^{-1}(s)\left(\int_T^s (c(\tau)-\tilde c(\tau))h^p(\tau)d\tau\right)ds}{\int_T^t G^{-1}(s)\,ds}\\ &= -v(T) - \int_T^\infty r^{1-q}(s)h^p(s)P(\Phi_q(w_h),w)\,ds.\end{aligned}$$

4. Remarks and comments

(i) If $p=2$, then Theorems 3.1 and 3.2 state the *equivalence*

$$\begin{gathered}\int^\infty \frac{v^2(t)}{G(t)}dt < \infty \qquad \Longleftrightarrow\\ \liminf_{t\to\infty} \frac{\int_T^t G^{-1}(s)\left(\int_T^s (c(\tau)-\tilde c(\tau))h^p(\tau)\right)ds}{\int_T^t G^{-1}(s)ds} > -\infty\end{gathered} \tag{4.1}$$

and this statement agrees with the classical Hartman-Wintner theorem. Indeed, any non-oscillatory equation (1.10) possesses a so-called principal solution which is characterized by

$$\int^\infty \frac{1}{r(t)h^2(t)}dt = \infty,$$

i.e., the integrand in this integral is just the function G, and hence (3.1) is automatically satisfied in the linear case. The transformation $x = h(t)y$ transforms (1.2) into the equation

$$(r(t)h^2(t)y')' + (c(t)-\tilde c(t))h^2(t)y = 0 \tag{4.2}$$

since the identity

$$h(t)\left[(r(t)x')' + \tilde{c}(t)x\right] = (r(t)h^2(t)y')' + h(t)\left[(r(t)h'(t))' + \tilde{c}(t)h(t)\right]y \tag{4.3}$$

holds, see e.g. [1, Chap. I]. Now, applying the "linear" Hartman-Wintner theorem to (4.2) and taking into account that $P(u,v) = \frac{1}{2}(u-v)^2$ if $p=2$, we see that (4.1) really holds.

(ii) Theorems 3.1 and 3.2 concern only the case $p \in (1,2]$, due to the fact that for these p inequalities (2.4) "go in a favorable direction," as can be seen by a closer examination of the proof of these theorems. For $p > 2$, these favorable inequalities are contained in (2.5), but we proved them in [3] only under some restrictions on u, v; it is also the reason why we succeeded in proving Theorems 3.2 and 3.2 only for $p \in (1,2]$. Nevertheless, based on explicit computations for particular functions $r, c, \tilde{c}$ we believe that a Hartman-Wintner type statement extends also to the case $p > 2$, and this leads us to the following conjecture.

Conjecture 4.1. *Suppose that $\tilde{c}(t) \neq 0$ eventually, equations* (1.6) and (1.10) *are non-oscillatory and* (1.10) *possesses a solution h satisfying* (3.1). *Then for any solution x of* (1.6) *and w given by* (2.1)

$$\int^{\infty} r^{1-q}(t)h^p(t)P(\Phi_q(w_h), w)\,dt < \infty, \quad w_h := \frac{r(t)\Phi_p(h')}{\Phi_p(h)}$$

if and only if (3.2) *holds.*

(iii) The assumption "$c(t) \neq 0$ eventually" in Theorems 3.1 and 3.2 is actually slightly stronger than necessary and it may be replaced by a weaker assumption that (1.10) possesses a solution h such that $h'(t) \neq 0$ eventually. Indeed, if $\tilde{c}(t) \neq 0$ eventually, then any non-oscillatory solution has the derivative eventually of one sign by the Rolle mean value theorem of differential calculus. We preferred here the stronger assumption $\tilde{c}(t) \neq 0$ since it is easier to verify in particular cases.

(iv) The classical Hartman-Wintner theorem is closely related to the oscillation criterion for (1.2) with $\int^{\infty} r^{-1}(t)\,dt = \infty$ which states that this equation is oscillatory provided (1.4) holds and the limit of the expression in this liminf does not exist, i.e., "$\limsup > \liminf$". The direct half-linear extension of this statement can be found e.g. in [2]. In our setting, when (1.6) is viewed as a perturbation of (1.10) with $\tilde{c}(t) \neq 0$ eventually, we have not succeeded in proving a similar statement yet, but we believe that such a statement holds, and this leads us to the following conjecture which is closely related to Conjecture 4.1.

Conjecture 4.2. *Suppose that $\tilde{c}(t) \neq 0$ eventually, equation* (1.10) *is non-oscillatory and possesses a solution h satisfying* (3.1). *If*

$$\liminf_{t\to\infty} \frac{1}{\int_T^t G^{-1}(s)\,ds} \int_T^t G^{-1}(s)\left(\int_T^s (c(\tau) - \tilde{c}(\tau)h^p(\tau)\,d\tau\right)ds > -\infty \tag{4.4}$$

and $\limsup$ *of the above expression is greater than the* $\liminf$*, then* (1.6) *is oscillatory.*

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