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On matched asymptotic analysis for laminar channel flow with a turning point *

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Abstract

This paper presents a formal analysis of the asimptotic behaviour of solutions of type III for the Berman equation

$$\epsilon f^{iv} = ff''' - f'f'', \quad f(0) = f''(0) = f'(1) = f(1) - 1 = 0\,,$$

where f describes a laminar flow in a channel with porous walls. A solution has a nonlinear turning point $(1-\Delta)$, i.e. $f(1-\Delta) = 0$ for some $\Delta(\epsilon)$. It is shown that

$$f(\eta) \sim -\frac{1-\Delta}{\pi\Delta} \sin \frac{\pi\eta}{1-\Delta},$$

as $\epsilon \to 0^+$, for $\eta \in [0, 1-\Delta)$ where Δ satisfies

$$\frac{\Delta}{\epsilon} e^{\Delta/\epsilon} \sim \frac{1}{2e\pi^9\epsilon^8}.$$

1 Introduction

The laminar flow of a viscous fluid in a rectangular channel with porous walls is governed by the Berman equation [1]

$$f^{iv} = R(ff''' - f'f'') \tag{1}$$

with boundary conditions

$$f(0) = f''(0) = 0, \quad f(1) = 1, \quad f'(1) = 0 \tag{2}$$

where $f = f(\eta)$ is the unknown function related to the stream function of the flow, and R is the Reynold number of the flow. The case $R > 0$ corresponds to suction while $R < 0$ to injection. Setting $\epsilon = 1/R$, one gets a singular perturbation problem for small $|\epsilon|$:

$$\epsilon f^{iv} = ff''' - f'f'' \tag{3}$$

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subject to boundary conditions (2). The boundary value problem (3)-(2) has been investigated by many authors. In [2, 8, 9] it is shown that there exists a unique solution for each $\epsilon \leq 0$. When ϵ is positive and sufficiently small, the boundary value problem admits at least three solutions [7] which are classified as follows: Type I if the solution is increasing concave down, type II if the solution is increasing with an inflection point, and type III if the solution is non-monotone with a turning point.

A turning point, denoted by $z_\epsilon = 1 - \Delta_\epsilon$, is defined as the value in $(0,1)$ at which the solution vanishes. Among the most interesting questions is the asymptotic behavior of the solutions as $\epsilon \to 0$, [2, 4, 10]. Hastings et. al. [2] showed that as $\epsilon \to 0^-$, $f(\eta) \to \sin \frac{2}{\pi}\eta$ uniformly on $[0,1]$. As $\epsilon \to 0^+$, however, type I and type II solutions approach the linear function in compact subsets of $[0,1)$ (see [4]), but type III solutions become unbounded on the left side of the turning point [5]. In addition, for type III solutions, the turning point $z_\epsilon \to 1$ or $\Delta_\epsilon \to 0$, as $\epsilon \to 0^+$. Transcendentally small terms in the asymptotic expansion of the solutions of types I and II have been found by Robinson, Terrill and Lu et. al. Their validity has been confirmed by McLeod [6]. As for the type III solutions, the unboundedness and the existence of turning points cause the asymptotic analysis to be more difficult than for types I and II. Robinson [7] and Zaturska et. al. [11] used the method of matched asymptotic expansions and gave estimates on the asymptotic behavior for the type III solutions in the seventies and the eighties respectively. In 1994, MacGillivrary and Lu [5] considered the transcendental terms and formally obtained an asymptotic formula for type III solutions,

$$f \sim -\kappa \sin \frac{\pi\eta}{1-\Delta}$$

for $\eta \in [0, 1-\Delta)$, where $\kappa \sim \frac{1-\Delta}{\pi\Delta}$ and $\Delta = \Delta_\epsilon$ satisfies the asymptotic formula

$$\frac{\Delta}{\epsilon} e^{\frac{\Delta}{\epsilon}} \sim \frac{1}{2\pi^9\epsilon^8}$$

for sufficiently small $\epsilon > 0$. It was shown in [5], with numerical comparison, that this result is better than those given by Robinson and Zaturska et. al.. In 1997, Lu [3] rigorously proved that for the type III solutions, as $\epsilon \to 0^+$,

$$f(\eta) \sim -\frac{1-\Delta}{\pi\Delta} \sin \frac{\pi\eta}{1-\Delta}$$

uniformly on $[0, 1-\Delta]$, and Δ satisfies

$$\frac{\Delta}{\epsilon} e^{\frac{\Delta}{\epsilon}} \sim \frac{1}{2e\pi^9\epsilon^8}.$$

Interested readers may find that the formal result of [5] and the rigorous proof in [3] are of the same orders, but the coefficients are different. The purpose of this paper is to improve the formal matching method used in [5] to get the correct asymptotic result. It implies that the rigorous analysis is important for not only confirming but also improving the formal matching method, although sometimes it is hard and tedious. Also, the paper shows that the formal matching technique can exactly provide the asymptotic behavior of the solution when it is properly applied. The aim of the paper is to provide a general method of the formal asymptotic analysis to deal with the singular perturbation problems with boundary and internal layers. The paper is organized as follows. The outer solution that behaves like a sine function on the left side of the turning point is studied in §2. The inner solution near the turning point is given in §3 and §4. The Poincaré expansion is studied in §3, and the transcendentally small terms in §4. Finally, in §5, the relation between Δ and ϵ is determined by a formal matching process on the right side of the turning point.

Throughout the paper, the process $\epsilon \to 0$ means $\epsilon \to 0^+$, and the solution of the boundary value problem means the type III solution. Also, the crucial property $f^{iv}(\eta) < 0$ for $\eta \in (0,1]$, whose proof can be found in [2], is applied repeatedly in the paper.

2 Approximation on the left side of the turning point

Setting $\epsilon = 0$ in (3), we obtain the reduced equation

$$ff''' - f'f'' = 0\,. \tag{4}$$

The initial condition $f(0) = f''(0) = 0$ is then imposed. We see that equation (4) may have three possible solutions: $C\eta$, $C\sin\lambda\eta$, and $C\sinh\lambda\eta$ where C and λ are constants. Recall that the turning point is defined by $f(1-\Delta) = 0$. Thus, the outer solution on $[0, 1-\Delta]$ that satisfies $f(1-\Delta) = 0$ must be neither the linear function nor the hyperbolic sine function. Thus, the solution of (4) with $f(0) = f(1-\Delta) = 0$ is $C\sin\lambda\eta$. Since $f(\eta) \leq 0$ on $[0, 1-\Delta]$, we can write

$$f(\eta) \sim -\kappa \sin\frac{\pi\eta}{1-\Delta}, \tag{5}$$

where $\kappa > 0$ is a constant related to ϵ, and assume the convergence is valid uniformly on compact subintervals of $[0, 1-\Delta)$. This is the outer solution on the left side of the turning point. It then remains to determine the values of κ and Δ in terms of ϵ as $\epsilon \to 0$.

3 The Poincaré expansion at the turning point

Noting that the turning point is moving toward the right end-point of the interval [0,1] as $\epsilon \to 0$, we introduce the interior-layer variable $\tau = \frac{\eta-1+\Delta}{\Delta}$. Then, the turning point, which is $\tau = 0$, becomes immobilized as $\epsilon \to 0$, and the interval $[1-\Delta, 1]$ of η becomes the interval $[0,1]$ of τ.

Let $f(\eta) = f(\Delta\tau + 1 - \Delta) = \bar{f}(\tau)$ and denote $\bar{\epsilon} = \frac{\epsilon}{\Delta}$. The original equation (3) then takes the form

$$\bar{\epsilon}\bar{f}^{iv} = f\bar{f}''' - \bar{f}'\bar{f}'', \tag{6}$$

with boundary conditions

$$\bar{f}(0) = 0, \quad \bar{f}(1) = 1, \quad \bar{f}'(1) = 0. \tag{7}$$

As suggested in [5], assume $\bar{\epsilon} \to 0$ as $\epsilon \to 0$, which implies that (6)-(7) is again a singular perturbation problem.

Now we set $\bar{\epsilon} = 0$ to get the reduced equation,

$$\bar{f}\bar{f}''' - \bar{f}'\bar{f}'' = 0, \tag{8}$$

which must have a linear solution in a outer region $[0, 1 - \mu(\bar{\epsilon})]$ of τ where $\mu(\bar{\epsilon}) = o(1)$. This is seen from $\bar{f}(0) = 0, \bar{f}' > 0$ for all $\tau \geq 0$, and $\bar{f}(1) = 1$. In general, equation (8) would have three solutions: a linear function, $\sin \frac{\pi}{2}\tau$ and $\frac{\sinh \lambda\tau}{\sinh \lambda}$ where $\lambda > 0$ is a constant. The other two are excluded because they have positive fourth derivatives, which contradicts the property $f^{iv} < 0$ for $\eta > 0$. Thus, the outer solution, up to order $O(1)$, for $\tau \in [0, 1)$ is a linear function, i.e.,

$$\overline{f}(\tau) = \lambda\tau + \ldots \tag{9}$$

where λ is to be determined.

The inner solution near the right end-point $\tau = 1$ can be formally found by introducing the boundary layer variable x^* defined by

$$\bar{\epsilon}x^* = 1 - \tau. \tag{10}$$

Let $\tilde{f}(x^*) = \bar{f}(\frac{1-\tau}{\bar{\epsilon}})$. Then, the equation (6) becomes

$$\tilde{f}^{iv} = -\tilde{f}\tilde{f}''' + \tilde{f}'\tilde{f}''. \tag{11}$$

At $x^* = 0$, the initial conditions are $\tilde{f}(0) = 1$ and $\tilde{f}'(0) = 0$. Assume that $\tilde{f} \sim 1$ is valid for $x^* \in [0, A]$ where A is any positive constant. Then, we can set

$$\tilde{f}(x^*) = 1 + \bar{\epsilon}f_1(x^*) + \bar{\epsilon}^2 f_2(x^*) + \ldots. \tag{12}$$

Substituting (12) into (11) and extract the powers of $\bar{\epsilon}$, we get the following system of equations:

$$\begin{aligned} f_1^{iv} + f_1''' &= 0, \quad f_1(0) = f_1'(0) = 0, \\ f_2^{iv} + f_2''' &= -f_1 f_1''' + f_1' f_1'', \quad f_2(0) = f_2'(0) = 0, \\ f_k^{iv} + f_k''' &= -\sum_{i=1}^{k-1} f_i f_{k-i}''' + \sum_{i=1}^{k-1} f_i' f_{k-i}'', \ f_k(0) = f_k'(0) = 0, k = 3, 4, \ldots. \end{aligned} \tag{13}$$

The general solution of (13) is

$$f_1(x^*) = a_1(-1 + x^* + e^{-x^*}) + \frac{1}{2}b_1 x^{*2}$$

where a_1 and b_1 are constant to be determined by matching with appropriate terms from the outer solution. So, the inner expansion is

$$\tilde{f}(x^*) = 1 + \bar{\epsilon}[a_1(-1 + x^* + e^{-x^*}) + \frac{1}{2}b_1 x^{*2}] + o(\bar{\epsilon}). \tag{14}$$

Recall that the leading term of outer solution is given in (9). Thus, the Poincaré expansion of the outer solution can be written as

$$\bar{f}(\tau) = \lambda\tau + \sum_{i=1}^{\infty} \bar{\epsilon}^i g_i(\tau). \tag{15}$$

If we match (14) with (15) to order $O(1)$, then $\lambda = 1$. Substituting (15) with $\lambda = 1$ into (6) and comparing the powers of $\bar{\epsilon}$, we obtain

$$\tau g_1''' - g_1'' = 0, \tag{16}$$

$$\tau g_i''' - g_i'' = g_{i-1}^{iv} - \sum_{j=1}^{i-1} g_j g_{i-j}''' + \sum_{j=1}^{i-1} g_j' g_{i-j}'', \tag{17}$$

for $i = 2, 3, \ldots$. Since $f(0) = 0$, the boundary conditions

$$g_i(0) = 0, i = 1, 2, \ldots \tag{18}$$

are imposed. We may also assume that all the functions g_i, $i = 1, 2, \ldots$, are analytic at $\tau = 0$. Thus, the function g_1 can be solved from (16) and (18), which is

$$g_1 = \frac{1}{6} A_1 \tau^3 + B_1 \tau$$

for some constant A_1 and B_1. Writing the outer expansion $\overline{f}(\tau) = \tau + \bar{\epsilon} g_1(\tau) + \ldots$ in terms of the variable x^*, one obtains

$$\begin{aligned} \bar{f}(\tau) &= (1 - \bar{\epsilon}x^*) + \bar{\epsilon}[\frac{1}{6} A_1 (1 - \bar{\epsilon}x^*)^3 + B_1(1 - \bar{\epsilon}x^*)] + o(\bar{\epsilon}) \\ &= 1 + \bar{\epsilon}[-x^* + \frac{1}{6} A_1 + B_1] + o(\bar{\epsilon}). \end{aligned} \tag{19}$$

Now, matching the outer and inner expansions (14) and (19) to order $O(\bar{\epsilon})$, we find that

$$a_1 = -1, b_1 = 0, \quad \frac{1}{6} A_1 + B_1 = 1.$$

Thus, at this stage, we have the following outer and inner approximations up to $O(\bar{\epsilon})$:

$$\text{outer:} \quad \bar{f} = \tau + \bar{\epsilon}[\frac{1}{6} A_1 \tau^3 + B_1 \tau] + o(\bar{\epsilon}), \tag{20}$$

and

$$\text{inner:} \quad \tilde{f} = 1 + \bar{\epsilon}[1 - x^* - e^{-x^*}] + o\left(\bar{\epsilon}\right). \tag{21}$$

Next, we want to argue $A_1 = 0$, which implies that $B_1 = 1$, hence, g_1 is linear. To see this, we set $i = 2$ in (17) to obtain

$$\tau g_2''' - g_2'' = \frac{1}{3} A_1^2 \tau^3. \tag{22}$$

Differentiating the both sides of (22) produces $\tau g_2^{iv} = A_1^2\tau^2$. If $A_1 \neq 0$, the fourth derivative formally satisfies

$$\bar{f}^{iv}(\tau) = \bar{\epsilon}^2 A_1^2\tau + o(\bar{\epsilon}^2),$$

hence, $\bar{f}^{iv} > 0$ for $\tau \geq \frac{1}{2}$, which contradicts the property $f^{iv} < 0$ for $\eta > 0$. Moreover, it can be argued that every function g_i, $i = 1, 2, 3, ...$, is linear. To see this, we use an argument by contradiction. Suppose not. Then, there would be the smallest integer greater than two, say k, such that g_k is the first nonlinear function, and $g_1, g_2, ..., g_{k-1}$ are linear. Substitution of $g_1, g_2, ..., g_{k-1}$ into (17) for $i = k$ yields

$$\tau g_k''' - g_k'' = 0,$$

which implies $g_k = \frac{1}{6}A_k\tau^3 + B_k\tau$. If $A_k \neq 0$, from (17), the function g_{k+1} satisfies

$$\tau g_{k+1}''' - g_{k+1}'' = \frac{1}{3}A_k^2\tau^3,$$

which leads to $\tau g_{k+1}^{iv} = A_k^2\tau^2$. Similar to the case $k = 1$ above, we would have $f^{iv} > 0$ for $\eta > 0$, again a contradiction. Thus, the Poincaré expansion is a linear function $\gamma(\bar{\epsilon})\tau$ where

$$\gamma(\bar{\epsilon}) = 1 + \bar{\epsilon} + B_2\bar{\epsilon} + B_3\bar{\epsilon}^3 + \ldots. \tag{23}$$

Following a similar procedure, one can determine the values of $B_2, B_3, \ldots$. However, for the purpose of this paper, the result for $B_1 (= 1)$ is enough. In addition, we can assume that B_n, $n = 2, 3, \ldots$, are all of order $O(1)$ to guarantee the convergence $\gamma(\bar{\epsilon}) \to 1$ as $\bar{\epsilon} \to 0$. From the inner expansion (21) we see that the exponentially small terms should be studied. This suggests that to have a precise outer expansion $\bar{f}(\tau)$ for $\tau \in [0, 1)$, we may use a general form including both power terms of $\bar{\epsilon}$ and transcendentally small terms,

$$\sum_{i=0}^{\infty} \bar{\epsilon}^i g_i(\tau) + \sum_{i=1}^{\infty} \delta_i h_i(\tau) \tag{24}$$

where the first series is the Poincaré expansion and the second are transcendental terms. It turns out, from the discussion above, that

$$\bar{f}(\tau) = \gamma(\bar{\epsilon})\tau + \sum_{i=1}^{\infty} \delta_i h_i(\tau) \tag{25}$$

where $\delta_1 = o(\bar{\epsilon}^n)$ for all positive integers n, $\delta_i = \delta_i(\bar{\epsilon})$, and $\delta_{i+1} \ll \delta_i$ for all $i = 1, 2, 3, ...$, are transcendentally small. The outer expansion (25) is valid in a neighborhood of the turning point, including $[0, 1-\nu(\bar{\epsilon})]$ of τ, where $\nu(\bar{\epsilon}) = o(1)$. It is interesting to point out here that the results in this section also show that $\bar{f}''(0), \bar{f}'''(0)$, and $\bar{f}^{iv}(0)$ are all transcendentally small, which agrees to the rigorous proof in [3]. But there is no more information on those quantities that can be provided from the right side of the turning point at this stage. We must return to the left side of the turning point to determine the leading transcendental terms in the next section.

Remark The power expansion should be fully studied as above, but it was not completely given in [5].

4 The transcendental terms at the turning point

Note that the outer approximation on the right side of the turning point is the interior layer approximation which, from the previous section, is assumed as

$$\bar{f} = \gamma\tau + \delta_1 h_1(\tau) + \delta_2 h_2(\tau) + \dots \tag{26}$$

where $\gamma = \gamma(\bar{\epsilon})$ is given in (23), and $\delta_1 = o(1)$, $\delta_2 = o(\delta_1)$, and h_i, $i = 1, 2$, are to be determined. Here $o(1) = o(\bar{\epsilon}^n)$, for all positive integers n, is transcendentally small in terms of $\bar{\epsilon}$ (not the original ϵ). Since, from here on, the reasoning is similar to that in [5], only key steps of the formal analysis will be given in the remainder of the paper for completeness. The interior layer approximation (26) must be valid in the small neighborhood of $\eta = 1 - \Delta$ which should also include a small part of left side of the turning point, in terms of the original variable η. Thus, it may be assumed, for the interior variable τ, that the expression (26) is valid on any compact intervals of $[-M, 1)$, where $M > 0$ is any constant, for sufficiently small $\bar{\epsilon} > 0$. Substitution of (26) into (6) leads to

$$\gamma^{-1}\bar{\epsilon}h_1^{iv} = \tau h_1''' - h_1'' \tag{27}$$

which has a solution (see [5] for details)

$$h_1 = \frac{\tau^3}{6} + r_1\tau, \tag{28}$$

where r_1 is a constant. Note that this h_1 is valid for not only $\tau < 0$ but also for $\tau \in [0, 1 - \nu(\bar{\epsilon})]$. Replace h_1 using (28), and substitute (26) into (6) again. We then balance the terms to get $\delta_2 = \delta_1^2$ and

$$\gamma^{-1}\bar{\epsilon}h_2^{iv} = \tau h_2''' - h_2'' - \gamma^{-1}\frac{\tau^3}{3}, \tag{29}$$

from which one obtains

$$h_2^{iv} = -\frac{1}{\bar{\epsilon}}e^{\frac{\gamma}{2\bar{\epsilon}}\tau^2}\int_0^\tau s^2 e^{-\frac{\gamma}{2\bar{\epsilon}}s^2}ds - D_2 e^{\frac{\gamma}{2\bar{\epsilon}}\tau^2}, \tag{30}$$

where D_2 is a constant. To determine D_2, we choose $\tau < 0$ bounded away from $\tau = 0$ and apply integration by parts to evaluate the integral of (30). The result is

$$\begin{aligned} h_2^{iv} \quad \sim \quad & -\frac{1}{\bar{\epsilon}}e^{\frac{\gamma}{2\bar{\epsilon}}\tau^2}\{-(\frac{2\bar{\epsilon}}{\gamma})^{3/2}[\frac{\sqrt{\pi}}{4} - (\frac{\sqrt{\frac{\gamma}{2\bar{\epsilon}}}\,|\tau|}{2}e^{-\frac{\gamma}{2\bar{\epsilon}}\tau^2} + \frac{1}{4|\tau|\sqrt{\gamma/(2\bar{\epsilon})}}e^{-\frac{\gamma}{2\bar{\epsilon}}\tau^2})]\} \\ & -D_2 e^{\frac{\gamma}{2\bar{\epsilon}}\tau^2}. \end{aligned} \tag{31}$$

To have $\delta_2 h_2(\tau) \ll \bar{\epsilon}^n$ for all positive integers n, the first and last terms in the right hand side of (31) must be balanced each other, for otherwise, $h_2(\tau)$ would be exponentially large if τ is bounded away from zero. This may cause a contradiction. Thus, balancing the two terms, one obtains

$$D_2 = \frac{\sqrt{2\bar{\epsilon}\pi}}{2\gamma\sqrt{\gamma}}.$$

Then, from (31), for $\tau < 0$ bounded away from 0,

$$h_2^{iv} \sim \gamma^{-1}\tau$$

which gives that h_2 is asymptotically equal to a polynomial of fifth degree, $p(x) = \gamma^{-1}(\frac{\tau^5}{5!} + \ldots)$. Then, the inner expansion on the left side of the turning point is

$$\bar{f} = \gamma\tau + \delta_1(\frac{\tau^3}{6} + r_1\tau) + \delta_1^2\gamma^{-1}(\frac{\tau^5}{5!} + \ldots) + \ldots \tag{32}$$

for $\tau < 0$. Now, we match the expression (32) with the outer solution (5). Let $\eta - (1-\Delta) = t$. Then, $t = \tau\Delta$. Expand the sine function at the turning point:

$$-\kappa \sin\frac{\pi\eta}{1-\Delta} = \kappa\frac{\pi\tau\Delta}{1-\Delta} - \frac{\kappa}{3!}\left(\frac{\pi\tau\Delta}{1-\Delta}\right)^3 + \frac{\kappa}{5!}\left(\frac{\pi\tau\Delta}{1-\Delta}\right)^5 + \ldots\,. \tag{33}$$

Compare the corresponding terms in (32) and (33). We then see from the linear term that $\kappa\frac{\pi\Delta}{1-\Delta} \sim \gamma$. Since $\gamma \sim 1$, we have

$$\kappa \sim \frac{1-\Delta}{\pi\Delta}. \tag{34}$$

Then, from the cubic term,

$$\delta_1 \sim -\left(\frac{1-\Delta}{\pi\Delta}\right)^2.$$

In closing this section, we write the interior layer approximation which is the out expansion of the solution on the right side of the turning point as follows:

$$\bar{f}(\tau) = \gamma\tau - \left(\frac{1-\Delta}{\pi\Delta}\right)^2\frac{\tau^3}{6} + \left(\frac{1-\Delta}{\pi\Delta}\right)^4 h_2(\tau) + \ldots, \tag{35}$$

where h_2 satisfies (30).

5 On Δ and ϵ

The asymptotic relation between κ and Δ has been determined by (34). To complete the formal asymptotic analysis, our final task is to determine the asymptotic value of Δ in terms of ϵ. This can be done by matching (35) with the inner expansion (21). For $\tau \sim 1$, from §4,

$$h_2^{iv} = -\frac{1}{\bar{\epsilon}} e^{\frac{\gamma}{2\bar{\epsilon}}\tau^2} \int_0^\tau s^2 e^{-\frac{\gamma}{2\bar{\epsilon}}s^2} ds - \frac{\sqrt{2\bar{\epsilon}\pi}}{2\gamma\sqrt{\gamma}} e^{\frac{\gamma}{2\bar{\epsilon}}\tau^2} = -\frac{\sqrt{2\bar{\epsilon}\pi}}{\gamma\sqrt{\gamma}} e^{\frac{\gamma}{2\bar{\epsilon}}\tau^2} + \tau + \dots. \tag{36}$$

Differentiating (35) four times and applying (36), we find

$$\frac{d^4 \bar{f}}{d\tau^4} = -\pi^{9/2} \frac{\Delta^{7/2}}{(1-\Delta)^4} \sqrt{2\bar{\epsilon}} e^{\frac{\gamma}{2\bar{\epsilon}}\tau^2} + \left(\frac{\pi\Delta}{1-\Delta}\right)^4 \tau + \dots \tag{37}$$

Writing this equation in terms of the boundary-layer variables x^* by substituting (10) into (37), we obtain

$$\frac{1}{\bar{\epsilon}^4} \frac{d^4 \tilde{f}}{dx^{*4}} = -\pi^{9/2} \frac{\Delta^{7/2}}{(1-\Delta)^4} \sqrt{2\bar{\epsilon}} e^{\frac{\gamma}{2\bar{\epsilon}}} e^{\frac{\bar{\epsilon}\gamma}{2}x^{*2}} e^{-\gamma x^*} + \left(\frac{\pi\Delta}{1-\Delta}\right)^4 (1-\bar{\epsilon}x^*) + \dots. \tag{38}$$

Then, from the inner expansion (21),

$$\frac{1}{\bar{\epsilon}^4} \frac{d^4 \tilde{f}}{dx^{*4}} = -\frac{1}{\bar{\epsilon}^3} e^{-x^*} + o\left(\frac{1}{\bar{\epsilon}^2}\right) \tag{39}$$

Comparison of (38) and (39) shows the overlap domain must be such that

$$\frac{\bar{\epsilon}\gamma}{2} x^{*2} \ll 1 \text{and} \quad x^* \gg 1,$$

and hence

$$-\pi^{9/2} \frac{\Delta^{7/2}}{(1-\Delta)^4} \sqrt{2\bar{\epsilon}} e^{\frac{\gamma}{2\bar{\epsilon}}} \sim -\frac{1}{\bar{\epsilon}^3}. \tag{40}$$

Using the expression (23) and $\bar{\epsilon} = \epsilon/\Delta$, and comparing the leading terms in (40), we finally obtain

$$\frac{\Delta}{\epsilon} e^{\Delta/\epsilon} = \frac{1}{2e\pi^9\epsilon^8}.$$

This result is the same as the rigorous result in [3].

6 Conclusion

The paper shows that when formal matching method is applied on the singular perturbation problem with a nonlinear turning point, like the Berman problem, an application of a full expansion of the asymptotic approximation is very important (see §2). This means that both the power terms and transcendental terms should involve. To determine the coefficients of their leading terms, the formal matching, in usual, must travel around the turning point several times. In other words, we have to match the inner and outer solutions repeatedly. Of course, it makes the computation tedious, but is necessary and important.

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