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THE MEASURABLE DISTINCTION BETWEEN THE SPIN AND ORBITAL ANGULAR MOMENTA OF ELECTROMAGNETIC RADIATION

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ABSTRACT. We show how the angular momentum of electromagnetic radiation may be decomposed into spin and orbital parts, of which the spin part is measurable in terms of Stokes parameters, thereby providing an unambiguous, gauge-invariant, distinction between the two parts.

1. INTRODUCTION

We are concerned with two issues in this paper. First, to what extent are those physical quantities which are transported by electromagnetic radiation measurable in terms of Stokes parameters? These quantities would include energy, momentum and angular momentum. We find that, in general, the angular momentum is not measurable, but only its spin part. Thus, our second issue: how can the spin and orbital angular momentum of electromagnetic radiation be distinguished?

An example will illustrate these issues. Consider a left-circularly-polarized monochromatic plane wave incident on a dielectric sphere. The problem is to determine the force and torque exerted by the incoming wave on the sphere [1, 2]. It would seem reasonable (but it turns out to be unnecessary) to expect that it would be important to know experimentally how much momentum and angular momentum would be transported by the scattered radiation. Detectors could be set up in the radiation zone, with appropriate polarization filters, to measure Stokes parameters at all scattering angles. The question is then, is such a detection scheme adequate to determine the momentum and angular momentum flux of the scattered radiation? The answer is yes for momentum, but no for angular momentum. The detection of the right- and left-circular polarizations of the scattered light permits only a partial measurement – the spin part – of the angular momentum.

Formulas for the force and torque exerted by beams of light on arbitrarily-shaped target objects depend on a precise knowledge of the electric and magnetic multipoles induced in the target [2, 3, 4]. We are not assuming such precise knowledge, but are, in effect, inquiring whether measurements of the Stokes parameters in the radiation zone are sufficient to specify such detailed information of the source.

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Relating the Stokes parameters to the angular momentum of light, particularly to the spin angular momentum, is not a new idea. Jauch and Rohrlich developed a quantum-mechanical version of the Stokes parameters to discuss the spin of photons [5]. Here we are working with the classical fields and the classical Stokes parameters, but we will be able to exploit the close relationship between circularly-polarized light and the helicity states of photons in developing and interpreting our classical results.

In Section 2, we define the Stokes parameters and review the relationships among angular momentum, helicity, and the Stokes parameters. In Section 3, we obtain expressions for the rate of transport of the physical quantities of interest from a finite source through a spherical surface far removed from the source. We express these quantities in terms of Stokes parameters insofar as possible. An understanding of the measurement of the angular momentum consistent with the quantum nature of light is presented in Section 4 and related to the Stokes parameters. The well-known decomposition of the angular momentum of electromagnetic radiation into spin and orbital parts is reviewed in Section 5 and applied to the problem raised in Sections 3 and 4. The problem of the gauge invariance of this decomposition is discussed in Section 6. We give some specific examples of the problem of non-measurability in Section 7 and save some concluding remarks for Section 8. In an appendix, we present formulas for the rate of transport of physical quantities, given the source multipole moments. These formulas are useful for the examples of Section 7.

2. Plane waves and Stokes parameters

Consider a monochromatic electromagnetic plane wave propagating in the z-direction. With real functions, we could express such a wave with transverse components as follows:

$$E_x = E_{0x} \cos(kz - \omega t + \alpha), \tag{1a}$$

$$E_y = E_{0y} \cos(kz - \omega t + \beta), \tag{1b}$$

where the amplitudes E_{0x}, E_{0y} are real and positive. In the physically-realizable case, they must decrease to zero at some finite distance from the z-axis. Such a variation in x and y gives a non-vanishing longitudinal component to the field which, as we shall see, is related to the angular momentum carried by the wave [6].

In taking time-averages, it will be convenient to work with complex fields:

$$E_x = |E_{0x}| e^{i(kz-\omega t+\alpha)}, \tag{2a}$$

$$E_y = |E_{0y}| e^{i(kz-\omega t+\beta)}, \tag{2b}$$

whose real parts are the physical fields.

The Stokes parameters are measures of the energy flux and the polarization of the electromagnetic wave. Using the convention of Bohren and Huffman, we take these to be [7]:

$$I = |E_{0x}|^2 + |E_{0y}|^2, \tag{3a}$$

$$Q = |E_{0x}|^2 - |E_{0y}|^2, \tag{3b}$$

$$U = 2\,\mathrm{Re}\left(E_x E_y^*\right) = 2|E_{0x}||E_{0y}| \cos(\alpha - \beta), \tag{3c}$$

$$V = 2\,\mathrm{Im}\left(E_x E_y^*\right) = 2|E_{0x}||E_{0y}| \sin(\alpha - \beta). \tag{3d}$$

For the monochromatic radiation under consideration, these parameters are not all independent and are related by [7, 8, 9]: $I^2 = Q^2 + U^2 + V^2$.

The energy, momentum and angular momentum densities of electromagnetic fields are given by the following well-known formulas (in Gaussian units) [8]:

$$e = \frac{1}{8\pi}\left(E^2 + B^2\right), \tag{4a}$$

$$p_z = \frac{1}{4\pi c}\left(\mathbf{E}\times\mathbf{B}\right)_z, \tag{4b}$$

$$j_z = \frac{1}{4\pi c}\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\right]_z. \tag{4c}$$

For the case of plane waves, we can obtain expressions for the time-averaged flux of energy, momentum and angular momentum by considering the amount of each in a slab of thickness $c\delta t$ perpendicular to the z-axis. Then:

$$\langle\dot{\varepsilon}\rangle = \frac{c}{4\pi}\int dx\, dy\, \langle E^2 + B^2\rangle = \frac{c}{8\pi}\int dx\, dy\, (\mathbf{E}^*\cdot\mathbf{E}), \tag{5a}$$

$$\langle\dot{P}_z\rangle = \frac{1}{4\pi}\int dx\, dy\, \langle\mathbf{E}\times\mathbf{B}\rangle_z = \frac{1}{8\pi}\int dx\, dy\, (\mathbf{E}^*\cdot\mathbf{E}), \tag{5b}$$

$$\langle\dot{J}_z\rangle = \frac{1}{8\pi}\int dx\, dy\, \left[\mathbf{r}\times(\mathbf{E}^*\times\mathbf{B})\right]_z. \tag{5c}$$

In these formulas, we are using the standard procedure for time-averaging sinusoidally-varying waves, where the fields are now given by the complex quantities of Eqs. (2a) and (2b). The brackets $\langle\ldots\rangle$ denote the time-average of the quantity inside. Thus:

$$\langle\dot{\varepsilon}\rangle = \frac{c}{8\pi}\int dx\, dy\, \left(|E_{0x}|^2 + |E_{0y}|^2\right) = \frac{c}{8\pi}\int dx\, dy\, I, \tag{6a}$$

$$\langle\dot{P}_z\rangle = \frac{1}{8\pi}\int dx\, dy\, I. \tag{6b}$$

We see in Eqs. (6a) and (6b) that the energy and momentum fluxes can be expressed in terms of Stokes parameters integrated over the surface normal to the propagation direction. The corresponding expression for angular momentum flux requires some careful treatment. It is useful to go to a cylindrical coordinate system. In the source-free region, the divergence of $\mathbf{E}$ vanishes, giving:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\rho}\frac{\partial(\rho E_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z} = 0.$$

Thus, for a z-dependence e^{ikz}, we have:

$$-ikE_z = \frac{1}{\rho}\frac{\partial(\rho E_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi}.$$

Likewise, for a time-dependence $e^{-i\omega t}$, Faraday's law gives:

$$\frac{1}{\rho}\frac{\partial(E_z)}{\partial\phi} - ikE_\phi = ikB_\rho,$$

$$ikE_\rho - \frac{\partial E_z}{\partial\rho} = ikB_\phi,$$

$$\frac{1}{\rho}\frac{\partial(\rho E_\phi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial(E_\rho)}{\partial\phi} = ikB_z.$$

Substituting these results in Eq. (5c) (neglecting small quantities) gives:

$$\begin{aligned}\langle \dot{J}_z\rangle &= \frac{1}{8\pi}\int \rho\, d\rho\, d\phi\, \rho(E_z^* B_\rho - E_\rho^* B_z)\\ &= \frac{1}{8\pi i k}\int \rho\, d\rho\, d\phi \left(E_\rho^* \frac{\partial E_\rho}{\partial\phi} + E_\phi^* \frac{\partial E_\phi}{\partial \phi}\right),\end{aligned}$$

neglecting "small" terms and integrating by parts, as necessary. Returning to Cartesian coordinates, using:

$$\begin{aligned}E_\rho &= E_x\cos\phi + E_y \sin\phi,\\ E_\phi &= -E_x \sin\phi + E_y\cos\phi,\end{aligned}$$

we have finally:

$$\begin{aligned}\langle \dot{J}_z\rangle &= \frac{1}{8\pi i k}\int dx\, dy \left[E_x^* \frac{\partial E_x}{\partial\phi} + E_y^*\frac{\partial E_y}{\partial\phi} - 2i\,\mathrm{Im}\,(E_x E_y^*)\right]\\ &= \frac{1}{8\pi k}\int dx\, dy \left[\frac{1}{2} I \frac{\partial}{\partial\phi}(\alpha+\beta) + \frac{1}{2}\frac{\partial}{\partial\phi}(\alpha-\beta) - V\right].\end{aligned} \tag{6c}$$

We note here that the angular momentum flux cannot be expressed completely in terms of Stokes parameters as explained below Eq. (10). This fact is the central issue of this paper and which will be examined for outgoing spherical waves in more detail in later sections.

To illustrate the use of these expressions, consider a standard case, a left-circularly-polarized (LCP) plane wave [9]:

$$\begin{aligned}E_x &= \frac{E_0}{\sqrt{2}}\cos(kz-\omega t) = \mathrm{Re}\left[\frac{E_0}{\sqrt{2}} e^{i(kz-\omega t)}\right],\\ E_y &= -\frac{E_0}{\sqrt{2}}\sin(kz-\omega t) = \mathrm{Re}\left[\frac{E_0}{\sqrt{2}} e^{i(kz-\omega t+\pi/2)}\right].\end{aligned}$$

We obtain:

$$\begin{aligned}\langle\dot{\varepsilon}\rangle_{LCP} &= \frac{c}{8\pi}\int dx\, dy\, |E_0|^2,\\ \langle \dot{P}_z\rangle_{LCP} &= \frac{1}{8\pi}\int dx\, dy\, |E_0|^2,\\ \langle \dot{J}_z\rangle_{LCP} &= \frac{1}{8\pi k}\int dx\, dy\, |E_0|^2.\end{aligned}$$

Thus: $\langle\dot{\varepsilon}\rangle_{LCP} = c\langle\dot{P}_z\rangle_{LCP} = \omega\langle\dot{J}_z\rangle_{LCP}$, with obvious quantum-mechanical interpretation that $\hbar$ of angular momentum corresponds to $\hbar\omega$ of energy for this wave. In the language of particle physics, such a wave would be said to have positive helicity: the angular momentum component in the direction of motion is $+\hbar$. Likewise, the right-circularly-polarized (RCP) states have negative helicity. For such waves:

$$\langle\dot{J}_z\rangle_{RCP} = -\frac{\langle\dot{P}_z\rangle_{RCP}}{k} = -\frac{\langle\dot{\varepsilon}\rangle_{RCP}}{\omega}.$$

It will be useful later to have the transverse components of the electric field decomposed into LCP and RCP components. This is done as follows:

$$\mathbf{e}_L \equiv -\frac{(\mathbf{i}+i\mathbf{j})}{\sqrt{2}}, \quad \mathbf{e}_R \equiv \frac{(\mathbf{i}-i\mathbf{j})}{\sqrt{2}}.$$

Then:

$$\mathbf{E} = E_x\mathbf{i} + E_y\mathbf{j} = E_L\mathbf{e}_L + E_R\mathbf{e}_R,$$

where:

$$E_L = -\frac{E_x - iE_y}{\sqrt{2}}, \quad E_R = \frac{E_x + iE_y}{\sqrt{2}}.$$

(For the example given above: $E_L = -E_0 e^{i(kz-\omega t)}$, $E_R = 0$.)

Expressing the Stokes parameters in terms of these helicity amplitudes yields:

$$I = |E_L|^2 + |E_R|^2, \tag{7a}$$

$$Q = -2\,\mathrm{Re}\,(E_L E_R^*), \tag{7b}$$

$$U = -2\,\mathrm{Im}\,(E_L E_R^*), \tag{7c}$$

$$V = -\left(|E_L|^2 - |E_R|^2\right). \tag{7d}$$

3. Transport of physical quantities by outgoing spherical waves

In this section, we develop expressions for the fluxes of energy, momentum and angular momentum carried by outing spherical waves in the radiation zone. Far from the source of the radiation, the transverse components of the electric field, given appropriately in a spherical polar coordinate system, have a $1/r$ behavior:

$$E_\theta, E_\phi \sim \frac{1}{r} e^{i(kr-\omega t)}$$

as do the transverse magnetic field components, which, through Faraday's law, are related to the transverse electric field components:

$$B_\theta = -E_\phi, \quad B_\phi = E_\theta,$$

the error in these equations being of order $1/r^2$. There is a radial component for each field, of order $1/r^2$, obtained from the divergence conditions. Thus:

$$-ikE_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\theta) + \frac{\partial E_\phi}{\partial\phi}\right] + O\left(1/r^3\right),$$

$$ikB_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\phi) - \frac{\partial E_\theta}{\partial\phi}\right] + O\left(1/r^3\right).$$

In terms of the transverse $\mathbf{E}$ components, the time-averaged fluxes of energy, momentum and angular momentum through a spherical surface of radius r are given by:

$$\langle\dot{\varepsilon}\rangle = \frac{c}{8\pi}\int r^2\,d\Omega\,\left(|E_\theta|^2 + |E_\phi|^2\right), \tag{8a}$$

$$\langle\dot{P}_z\rangle = \frac{1}{8\pi}\int r^2\,d\Omega\,\left(|E_\theta|^2 + |E_\phi|^2\right)\cos\theta, \tag{8b}$$

$$\langle\dot{J}_z\rangle = \frac{1}{8\pi k}\int r^2\,d\Omega\,\left[E_\theta^*\left(-i\frac{\partial E_\theta}{\partial\phi}\right) + E_\phi^*\left(-i\frac{\partial E_\phi}{\partial\phi}\right)\right], \tag{8c}$$

where the integration is over the solid angle $d\Omega = \sin\theta\,d\theta\,d\phi$. Some integration by parts was required to bring the expression for $\langle\dot{J}_z\rangle$ into its final form.

We again follow Bohren and Huffman in defining Stokes parameters for the outgoing radiation at angles θ and ϕ:

$$I = |E_\theta|^2 + |E_\phi|^2, \tag{9a}$$

$$Q = |E_\theta|^2 - |E_\phi|^2, \tag{9b}$$

$$U = -2\,\mathrm{Re}\,(E_\theta E_\phi^*), \tag{9c}$$

$$V = +2\,\mathrm{Im}\,(E_\theta E_\phi^*). \tag{9d}$$

The expressions for the fluxes in terms of these Stokes parameters are then given by:

$$\langle\dot{\varepsilon}\rangle = \frac{c}{8\pi}\int r^2\,d\Omega\,I, \tag{10a}$$

$$\langle\dot{P}_z\rangle = \frac{1}{8\pi}\int r^2\,d\Omega\,I\cos\theta, \tag{10b}$$

$$\langle\dot{J}_z\rangle = \frac{1}{16\pi k}\int r^2\,d\Omega\,\left[I\frac{\partial}{\partial\phi}(\alpha+\beta) + Q\frac{\partial}{\partial\phi}(\alpha-\beta)\right]. \tag{10c}$$

with $E_\theta = |E_\theta|e^{i\alpha}$ and $E_\phi = |E_\phi|e^{i\beta}$. For outgoing spherical waves, as for plane waves, the angular momentum flux cannot be expressed completely in terms of Stokes parameters, hence is not measurable. The unmeasurable quantity is the sum of the phases, $\alpha+\beta$, the difference being given by the inverse tangent of $-V/U$.

4. A tentative measurement of the angular momentum of outgoing spherical radiation

We have seen that the theoretical expression for the angular momentum flux derived in the previous section does not give much hope for measuring angular momentum in the radiation zone. But certainly we can measure something. If we think of the scattered light as consisting of photons, each of which has a probability for being found to have positive or negative helicity upon measurement, then we can count the contributions of all the photons in carrying a certain component of angular momentum.

Here is the measurement that we propose: set a detector with an appropriate set of polarization filters at given angles θ and ϕ. The detector can be used to measure the irradiance of left-circularly-polarized light (energy carried by photons found to have positive helicity) and that of right-circularly-polarized light. The resulting contribution to angular momentum transport over the total solid angle is (again, the z-component for convenience) we denote by $\langle\dot{J}_z\rangle_M$:

$$\langle\dot{J}_z\rangle_M = \int r^2\,d\Omega\,\left[\frac{I_L}{h\nu}(+\hbar) + \frac{I_R}{h\nu}(-\hbar)\right]\cos\theta. \tag{11}$$

In this expression, the irradiances, I_L and I_R, or left- and right-circularly-polarized light, respectively, are divided by $\hbar\omega$ to give the number of photons per unit area per unit time. An angular momentum $+\hbar$ is attributed to the LCP states and $-\hbar$ to the RCP states. The irradiances called for are obtained by decomposing the electric field into LCP and RCP components as in Section 2 with

$$\mathbf{e}_L = -\frac{(\mathbf{e}_\theta + i\mathbf{e}_\phi)}{\sqrt{2}}, \quad \mathbf{e}_R = \frac{(\mathbf{e}_\theta - i\mathbf{e}_\phi)}{\sqrt{2}}. \tag{12}$$

Here the LCP and RCP field components are:

$$E_L = -\frac{(E_\theta - iE_\phi)}{\sqrt{2}}, \quad E_R = \frac{(E_\theta + iE_\phi)}{\sqrt{2}}. \tag{13}$$

The irradiances are then given by $I_{L,R} = (c/8\pi)|E_{L,R}|^2$, so that

$$\begin{aligned} \langle \dot{J}_z \rangle_M &= \frac{c}{8\pi\omega} \int r^2 \, d\Omega \, \left(|E_L|^2 - |E_R|^2 \right) \cos\theta \\ &= -\frac{1}{8\pi k} \int r^2 \, d\Omega \, V \cos\theta. \end{aligned} \tag{14}$$

Thus, the angular momentum flux that we could measure is expressible in terms of a Stokes parameter, but nevertheless there must be some part of the angular momentum that we have missed, since $\langle \dot{J}_z \rangle_M$ is clearly not equal to $\langle \dot{J}_z \rangle$ as given by Eq. (10c).

5. The Humblet decomposition: orbital and spin angular momenta of light

It is a common feature of quantum field theories that expressions for total field angular momentum can be naturally decomposed into orbital and spin parts [5, 10]. Humblet applied this idea to the classical electromagnetic field [11]. For the time-averaged angular momentum density (i-th Cartesian component),

$$j_i = \frac{1}{8\pi c} \left[\mathbf{r} \times (\mathbf{E}^* \times \mathbf{B}) \right]_i , \tag{15}$$

a simple rearrangement of terms gives the desired result. We start with Faraday's law (for monochromatic fields with time-dependence $e^{-i\omega t}$):

$$\boldsymbol{\nabla} \times \mathbf{E} = ik\mathbf{B}.$$

Then:

$$j_i = \frac{1}{8\pi i\omega} \left\{ \mathbf{r} \times [\mathbf{E}^* \times (\boldsymbol{\nabla} \times \mathbf{E})] \right\}_i = l_i + s_i + j_i^{surf}; \tag{16}$$

with

$$l_i = \frac{1}{8\pi\omega} E_j^* \left[-i(\mathbf{r} \times \boldsymbol{\nabla}) \right]_i E_j, \tag{17a}$$

$$s_i = \frac{1}{8\pi\omega} E_j^* (-i\varepsilon_{ijk}) E_k, \tag{17b}$$

$$j_i^{surf} = -\frac{1}{8\pi\omega} \nabla_j (E_j^* (\mathbf{r} \times \mathbf{E})_i). \tag{17c}$$

(In these expressions, summation over repeated indices is assumed and ε_{ijk} is the Levi–Civita symbol, which is $+1$ (-1) when the indices are an even (odd) permutation of 123 and zero otherwise.)

The "orbital" angular momentum density, l_i , has a striking similarity to a quantum-mechanical density with the orbital angular-momentum operator (except for a factor of $h/2\pi$) sandwiched between a wave function and its complex conjugate. (Elsewhere, it is explained why the electric field components do not make good wave functions for photons [12].)

The "spin" angular momentum density, s_i, is appropriately named in that there is no moment arm. Its relation to spin is even more strongly suggested when we form matrices Σ_i according to the following rule:

$$(\Sigma_i)_{jk} = -i\varepsilon_{ijk}. \tag{18}$$

The set of three 3×3 matrices $\{\Sigma_i\}$ satisfies the angular momentum commutation relations and further:

$$\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = 2I_3, \tag{19}$$

as would be appropriate for spin-one matrices. The "surface" term, j_i^{surf} is a three-divergence: integration of this term throughout a volume is thus equivalent to integration of $E_n^*(\mathbf{r} \times \mathbf{E})_i$ over a surface (E_n being the component of $\mathbf{E}$ normal to that surface) which could be indefinitely far removed from the source. Such an integral can be safely taken to vanish, although this should be checked whenever the Humblet decomposition is used. Humblet showed that, in calculating the time-averaged flux of angular momentum through a spherical surface, the surface term makes no contribution.

Thus, for the angular momentum flux we have:

$$\langle \dot{J_z} \rangle = \langle \dot{L_z} \rangle + \langle \dot{S_z} \rangle, \tag{20a}$$

$$\langle \dot{L_z} \rangle = \frac{1}{8\pi k} \int r^2 \, d\Omega \, E_j^*(-i) \frac{\partial E_j}{\partial \phi}, \tag{20b}$$

$$\langle \dot{S_z} \rangle = \frac{1}{8\pi k} \int r^2 \, d\Omega \, \left(|E_L|^2 - |E_R|^2 \right) \cos\theta, \tag{20c}$$

where in Eq. (20c), we have made use of Eq. (13). We see that $\langle \dot{S_z} \rangle$ is exactly what we suggested could be measured in the hypothetical experiment of Sec. 4, and the orbital part is that which cannot be measured.

Our result shows that there is a measurable distinction between the spin and orbital parts of the angular momentum of electromagnetic radiation. Yet, it is often stated that there is no unique, gauge-invariant, separation of these two kinds of angular momentum [5, 11, 12]. To this problem we turn next.

6. Spin and gauge invariance

The standard decomposition of the angular momentum density uses $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ in the development of Eq. (15). There results the following:

$$j_i = l_i + s_i + j_i^{surf}; \tag{21}$$

with

$$l_i = \frac{1}{8\pi} E_j^* \left[(\mathbf{r} \times \boldsymbol{\nabla}) \right]_i A_j, \tag{22a}$$

$$s_i = \frac{1}{8\pi} E_j^* \varepsilon_{ijk} A_k \tag{22b}$$

$$j_i^{surf} = -\frac{i}{8\pi} \nabla_j \left[E_j^* (\mathbf{r} \times \mathbf{A})_i \right]. \tag{22c}$$

Under a gauge transformation,

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi, \tag{23}$$

the spin angular momentum density transforms as:

$$s_i \to s_i' = s_i + \frac{1}{8\pi}(\mathbf{E}^* \times \boldsymbol{\nabla}\chi)_i. \tag{24}$$

Because no measurement can distinguish between $\mathbf{A}$ and $\mathbf{A}'$, it would seem that the spin density is not unique. We suggest below a number of ways out of this difficulty.

First of all, if we restrict the physical situation to one of monochromatic radiation, the decomposition performed in Sec. 5 is valid, and the decomposition there is expressed in terms of gauge-invariant quantities, i.e., the transverse electric-field components in the radiation zone.

Second, we point out that the change in s_i, to leading order in powers of $1/r$, is:

$$\delta\mathbf{s} = \frac{1}{8\pi}(\mathbf{E}^* \times \boldsymbol{\nabla}\chi) = -\frac{ik}{8\pi r}(\mathbf{r} \times \mathbf{E}^*\chi), \tag{25}$$

i.e., it has a moment arm. It appears then that to the spin density has been added an orbital part. But what is true of the orbital angular momentum density is true here also: the change in the spin density is transverse to the propagation direction and cannot be measured. Our gauge-invariant decomposition gives gauge-invariant quantities, one of which is measurable.

Finally, we note that the electromagnetic fields can be uniquely decomposed into longitudinal ($\|$) and transverse ($\perp$) parts [12]:

$$\mathbf{E} = \mathbf{E}_\| + \mathbf{E}_\perp, \quad \mathbf{B} = \mathbf{B}_\| + \mathbf{B}_\perp, \tag{26}$$

with

$$\boldsymbol{\nabla} \times \mathbf{E}_\| = \boldsymbol{\nabla} \times \mathbf{B}_\| = 0; \quad \boldsymbol{\nabla} \cdot \mathbf{E}_\perp = \boldsymbol{\nabla} \cdot \mathbf{B}_\perp = 0. \tag{27}$$

The vector potential $\mathbf{A}$ can be similarly decomposed. A gauge transformation is then represented as:

$$\mathbf{A}_\| \to \mathbf{A}_\|' = \mathbf{A}_\| + \boldsymbol{\nabla}\chi, \tag{28a}$$

$$\mathbf{A}_\perp \to \mathbf{A}_\perp' = \mathbf{A}_\perp. \tag{28b}$$

The angular momentum density can also be expressed in terms of longitudinal and transverse parts:

$$\mathbf{j}_\| = \frac{1}{4\pi}\mathbf{r} \times (\mathbf{E}_\| \times \mathbf{B}), \tag{29a}$$

$$\mathbf{j}_\perp = \frac{1}{4\pi}\mathbf{r} \times (\mathbf{E}_\perp \times \mathbf{B}). \tag{29b}$$

Making the substitution $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ as before, it can be shown that the longitudinal part can be distributed to the moment of (canonical) momentum of the charged particles constituting the source of the fields. The longitudinal part is not gauge-invariant. However, the transverse part, which can be expressed as:

$$\mathbf{j}_\perp = \frac{1}{4\pi}\left[E_{\perp i}(\mathbf{r} \times \boldsymbol{\nabla})A_{\perp i} + \mathbf{E}_\perp \times \mathbf{A}_\perp\right], \tag{30}$$

is obviously gauge-invariant. It is this transverse part with which we are dealing in the radiation zone.

Our conclusion, then, is that the spin flux $\langle \dot{S}_z \rangle$ is measurable and certainly satisfies gauge invariance as much as any other measurable quantity.

7. Examples of measurable spin and unmeasurable orbital angular momentum

We now present some examples to illustrate this measurability problem with the angular momentum of outgoing spherical electromagnetic waves.

First, we consider the field of an electric dipole rotating in the x-y plane with angular frequency ω:

$$\mathbf{D} = D(\mathbf{i}\cos\omega t + \mathbf{j}\sin\omega t), \tag{31a}$$

or, in terms of complex quantities:

$$\mathbf{D} = D(\mathbf{i} + i\mathbf{j})e^{-i\omega t}. \tag{31b}$$

The electric-field components in the radiation zone are [13]:

$$E_\theta = \frac{k^2 D}{r}\cos\theta e^{i\phi} e^{i(kr-\omega t)}, \tag{32a}$$

$$E_\phi = i\frac{k^2 D}{r} e^{i\phi} e^{i(kr-\omega t)}. \tag{32b}$$

Using Eqs. (8a) – (8c), we obtain the fluxes of interest:

$$\langle\dot{\varepsilon}\rangle = \frac{2}{3}k^3 D^2\omega, \tag{33a}$$

$$\langle\dot{P}_z\rangle = 0, \tag{33b}$$

$$\langle\dot{J}_z\rangle = \frac{2}{3}k^3 D^2, \tag{33c}$$

and, from Eqs. (20b) and (20c):

$$\langle\dot{L}_z\rangle = \langle\dot{S}_z\rangle = \frac{1}{3}k^3 D^2. \tag{33d}$$

In this example, in which we know what the radiated angular momentum will be because we know the source of the fields, we see that half of the total angular momentum, in the orbital form, will escape unmeasured: it cannot in principle be measured from Stokes parameters.

Consider next the example given in the introduction: the scattering of LCP light from a dielectric sphere. The measurement of spin angular momentum is predicted to be:

$$\langle\dot{S}_z\rangle = \frac{1}{2k^3}\sum_n \left[\frac{2n+1}{n(n+1)}\left(|a_n|^2 + |b_n|^2\right) + \frac{2n(n+2)}{n+1}\,\mathrm{Re}\,\left(a_n b_{n+1}^* + b_n a_{n+1}^*\right)\right] \tag{34}$$

where the a_n, b_n are the Mie scattering coefficients [14]. (This expression is derived in the Appendix.) This is another case where we know, theoretically, the source of the fields, so the measurement of the spin would be a test of the theory. The total angular momentum, however, is given theoretically by a much simpler expression. Matching boundary conditions of incoming and scattered waves on the surface of the sphere gives to the spherical polar components of the scattered field a simple ϕ-dependence: $e^{i\phi}$. The expression for the total angular momentum is then:

$$\langle\dot{J}_z\rangle = \frac{1}{8\pi k}\int r^2\, d\Omega\, I = \frac{\langle\dot{\varepsilon}\rangle}{\omega}. \tag{35}$$

We must emphasize that this simple result, whose expression in terms of Mie coefficients is given in the Appendix, is a theoretical one which cannot be verified by measurements in the radiation zone.

As a final example, we use multipole sources which are nonvanishing only for $m = 0$ components (in the notation of the Appendix):

$$a_{E,M}(n,m) = a_{E,M}(n,0)\delta_{m0}, \tag{36}$$

Then, setting $a_E(n,0) = c_n$ and $a_M(n,0) = d_n$, we have:

$$\langle\dot{\varepsilon}\rangle = \frac{c}{\pi k^2}\sum_n \left(|c_n|^2 + |d_n|^2\right), \tag{37a}$$

$$\langle\dot{P}_z\rangle = -\frac{1}{4\pi k^2}\sum_n \left[\frac{n(n+2)}{(2n+1)(2n+3)}\right]^{1/2} \mathrm{Im}\,\left(c_n c_{n+1}^* + d_n d_{n+1}^*\right), \tag{37b}$$

$$\langle\dot{J}_z\rangle = 0, \tag{37c}$$

$$\langle\dot{S}_z\rangle = \frac{1}{4\pi k^3}\sum_n \left[\frac{n(n+2)}{(2n+1)(2n+3)}\right]^{1/2} \mathrm{Re}\,\left(d_n c_{n+1}^* - c_n d_{n+1}^*\right). \tag{37d}$$

Here, without further specification of the multipole sources, we have an example in which the unmeasurable orbital angular momentum just cancels the measurable spin to give zero (z-component) total angular momentum. As in the previous cases, measurement of the spin provides no useful information concerning the total angular momentum.

8. Conclusion

It should be noted that restricting our development to the z-components of momentum, angular momentum and spin does not limit our conclusions. It is simply an artifact of the spherical polar coordinate system that the corresponding expressions for x- and y-components are somewhat more complicated. Nevertheless, $\langle\dot{P}_x\rangle$, $\langle\dot{P}_y\rangle$, $\langle\dot{S}_x\rangle$ and $\langle\dot{S}_y\rangle$ can be expressed completely in terms of Stokes parameters but $\langle\dot{J}_x\rangle$ and $\langle\dot{J}_y\rangle$ cannot.

We have shown above, on the basis of classical electromagnetic theory, that it is possible to measure energy, momentum and spin angular momentum transported by electromagnetic radiation from a finite source region *and* that it is *not* possible to measure total angular momentum using Stokes parameters.

Such a finding is surprising on the one hand because of the lack of gauge invariance when decomposing the total angular momentum into spin and orbital parts. On the other hand, it is not surprising when we consider these measurements from the standpoint of quantum theory. The detectors used to measure the Stokes parameters determine photon momentum, all three components. Physical quantities whose corresponding operators do not commute with the momentum operator are left undetermined, unmeasurable. The orbital angular momentum is one such quantity. The helicity, however, is the projection of angular momentum along the momentum direction and the corresponding operator commutes with the momentum operator. Hence, helicity is measurable along with momentum. We have shown then that the classical theory is consonant with the quantum theory in this regard.

Appendix A. Rate of transport of energy, momentum, angular momentum and spin from a given source of electromagnetic radiation

To obtain the formulas for energy, momentum, angular momentum and spin transport in terms of sources, we need the transverse components of the electric field in the radiation zone. The most general solution of Maxwell's equations corresponding to outgoing spherical waves in the radiation zone is, according to Jackson:

$$\mathbf{E} = \frac{1}{kr} e^{i(kr-\omega t)} \sum_{n,m} (-i)^{n+1} \left[a_E(n,m)\mathbf{X}_{nm} + a_M(n,m)\mathbf{n} \times \mathbf{X}_{nm} \right] \times \mathbf{n}, \tag{38}$$

where the $\mathbf{X}_{nm}$ are vector spherical harmonics,

$$\mathbf{X}_{nm} = \left[n(n+1) \right]^{-1/2} \mathbf{n} \times \boldsymbol{\nabla} Y_{nm},$$

$\mathbf{n}$ is a unit vector radially outward, and the $a_E(n,m)$, $a_M(n,m)$ are the electric and magnetic multipole moments of the source. Helicity components E_L, E_R are found to be:

$$E_{L,R} = \pm \frac{1}{kr} e^{i(kr-\omega t)} \sum_{n,m} (-i)^{n+1} (-1)^m \left[\frac{2n+1}{8\pi} \right]^{1/2} d^n_{\pm 1,m}(\theta) e^{im\phi} \times \left[a_M(n,m) \pm i a_E(n,m) \right]. \tag{39}$$

In Eq. (39) the $d^n_{mm'}$ are the reduced Wigner rotation matrices. Various properties of the vector spherical harmonics, d-matrices and Clebsch–Gordan coefficients needed in these derivations can be found in the monograph of Varshalovich, Moskalev and Khersonkii [15].

The time-averaged energy flux at distance r from the source in terms of helicity components is given by:

$$\langle \dot{\varepsilon} \rangle = \frac{c}{8\pi} \int r^2 \, d\Omega \, \left(|E_L|^2 + |E_R|^2 \right). \tag{40a}$$

Substitution of E_L and E_R from Eq. (39) and integration yield:

$$\langle \dot{\varepsilon} \rangle = \frac{c}{8\pi k^2} \sum_{n,m} \left(|a_E(n,m)|^2 + |a_M(n,m)|^2 \right). \tag{40b}$$

The time-averaged angular momentum flux (z-component for convenience) is:

$$\begin{aligned} \langle \dot{J}_z \rangle &= \frac{1}{8\pi k} \int r^2 \, d\Omega \, \left[E_L^*(-i)\frac{\partial}{\partial \phi} E_L + E_R^*(-i)\frac{\partial}{\partial \phi} E_R \right] \\ &= \frac{1}{8\pi k^3} \sum_{n,m} m \left(|a_E(n,m)|^2 + |a_M(n,m)|^2 \right). \end{aligned} \tag{41}$$

The time-averaged momentum flux (z-component) is given by:

$$\langle \dot{P}_z \rangle = \frac{1}{8\pi} \int r^2 \, d\Omega \, \left(|E_L|^2 + |E_R|^2 \right) \cos\theta, \tag{42}$$

while the time-averaged spin angular momentum flux (z-component) is given by:

$$\langle \dot{S}_z \rangle = \frac{1}{8\pi k} \int r^2 \, d\Omega \, \left(|E_L|^2 - |E_R|^2 \right) \cos\theta. \tag{43}$$

Thus, we must calculate the integral

$$I_{\pm} = \int r^2\, d\Omega\, \left(|E_L^2 \pm |E_R|^2\right) \cos\theta.$$

Exploiting the fact that $d^1_{00}(\theta) = \cos\theta$, and using various symmetry properties of the d matrices, we obtain:

$$\begin{aligned} I_{\pm} = \frac{1}{6k^2} \sum_{n,m} \sum_{n'} &\left[(2n+1)(2n'+1)\right]^{1/2} (-i)^{n-n'} (-1)^{m-1} \\ &\times \{[a_M(n,m) + ia_E(n,m)]\,[a_M^*(n',m) - ia_E^*(n',m)] \\ &\times (n1n'-1|nn'10)(nmn'-m|nn'10) \\ &\pm [a_M(n,m) - ia_E(n,m)]\,[a_M^*(n',m) + ia_E^*(n',m)] \\ &\times (n1n'1|nn'10)(nmn'-m|nn'10)\}\,, \end{aligned}$$

where the $(j_1 m_1 j_2 m_2 | j_1 j_2 J M)$ are the Clebsch–Gordan coefficients in a standard notation [16]. Explicit evaluation of the C–G coefficients leads to:

$$\begin{aligned} I_{\pm} = \frac{1}{2k^2} \sum_{n,m} &\left\{ \frac{m}{n+1} \left[|a_M(n,m) + ia_E(n,m)|^2 \mp |a_M(n,m) - ia_E(n,m)|^2\right] \right. \\ &- \frac{2}{n+1} \left[\frac{n(n+2)(n-m+1)(n+m+1)}{(2n+1)(2n+3)}\right]^{1/2} \\ &\times \{\operatorname{Im}\left[(a_M(n,m) + ia_E(n,m))\,(a_M^*(n+1,m) - ia_E^*(n+1,m))\right] \\ &\left. \pm \operatorname{Im}\left[(a_M(n,m) - ia_E(n,m))\,(a_M^*(n+1,m) + ia_E^*(n+1,m))\right]\}\right\}. \end{aligned} \tag{44}$$

Taking the upper signs, we have:

$$\begin{aligned} \langle \dot{P}_z \rangle = -\frac{1}{4\pi k^2} \sum_{n,m} &\left\{ \frac{m}{n(n+1)} \operatorname{Im}\left[a_E(n,m) a_M^*(n,m)\right] \right. \\ &+ \frac{1}{(n+1)} \left[\frac{n(n+2)(n-m+1)(n+m+1)}{(2n+1)(2n+3)}\right]^{1/2} \\ &\left. \times \operatorname{Im}\left[a_M(n,m) a_M^*(n+1,m) + a_E(n,m) a_E^*(n+1,m)\right]\right\}. \end{aligned} \tag{45}$$

The lower signs give:

$$\begin{aligned} \langle \dot{S}_z \rangle = \frac{1}{8\pi k^3} \sum_{n,m} &\left\{ \frac{m}{n(n+1)} \left[|a_E(n,m)|^2 + |a_M(n,m)|^2\right] \right. \\ &+ \frac{2}{(n+1)} \left[\frac{n(n+2)(n-m+1)(n+m+1)}{(2n+1)(2n+3)}\right]^{1/2} \\ &\left. \times \operatorname{Re}\left[a_M(n,m) a_E^*(n+1,m) - a_E(n,m) a_M^*(n,m)\right]\right\}. \end{aligned} \tag{46}$$

For the case of a left-circularly-polarized plane wave scattering from a dielectric sphere, we have, in the standard notation:

$$a_E(n,m) = (i)^{n+1} \left[4\pi(2n+1)\right]^{1/2} a_n \delta_{m1},$$
$$a_M(n,m) = (i)^{n+2} \left[4\pi(2n+1)\right]^{1/2} b_n \delta_{m1}.$$

With this restriction, Eqs. (40), (41), (45) and (46) reduce to [17]:

$$\langle \dot{\varepsilon} \rangle_{MS} = \frac{c}{2k^2} \sum_n (2n+1) \left(|a_n|^2 + |b_n|^2 \right), \tag{47}$$

$$\langle \dot{J}_z \rangle_{MS} = \frac{1}{2k^3} \sum_n (2n+1) \left(|a_n|^2 + |b_n|^2 \right), \tag{48}$$

$$\langle \dot{P}_z \rangle_{MS} = \frac{1}{k^2} \sum_n \left[\frac{2n+1}{n(n+1)} \operatorname{Re}\left(a_n b_n^*\right) + \frac{n(n+2)}{n+1} \operatorname{Re}\left(a_n a_{n+1}^* + b_n b_{n+1}^* \right) \right] \tag{49}$$

$$\langle \dot{S}_z \rangle_{MS} = \frac{1}{2k^3} \sum_n \left[\frac{2n+1}{n(n+1)} \left(|a_n|^2 + |b_n|^2 \right) + \frac{2n(n+2)}{n+1} \operatorname{Re}\left(a_n b_{n+1}^* + b_n a_{n+1}^* \right) \right], \tag{50}$$

with subscripts "MS" to denote Mie scattering.

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