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On two reverse inequalities in the Segal-Bargmann space *

Fernando Galaz-Fontes & Stephen Bruce Sontz

Dedicated to Eyvind H. Wichmann on his 70th birthday

Abstract

We review here two reverse inequalities in the Segal-Bargmann space: a reverse hypercontractivity estimate due to Carlen and a reverse log-Sobolev inequality due to the second author.

1 Notation and Definitions

We start with the scale of spaces

$$\mathcal{A}_p := \{\phi : \mathbf{C}^n \to \mathbf{C} : \phi \text{ is holomorphic and } \|\phi\|_p < \infty\},$$

where $0 < p < \infty$ and $\|\phi\|_p := \left(\int_{\mathbf{C}^n} d\mu_n(z)|\phi(z)|^p\right)^{1/p}$. Here

$$d\mu_n(z) = \pi^{-n} e^{-|z|^2} d^n x d^n y$$

is Gaussian measure on $\mathbf{C}^n$, where $d^n x d^n y$ is the Lebesgue measure, and where $|z|$ is the Euclidean norm of $z \in \mathbf{C}^n$. The spaces $\mathcal{A}_p$ for $1 \le p < \infty$ are Banach spaces with the norm $\|\cdot\|_p$, and $\mathcal{A}_2$ is a Hilbert space known as the *Segal-Bargmann space*. (See [1] and [7].) Moreover, $\mathcal{A}_2$ carries an irreducible representation of the Weyl-Heisenberg group (exponentiated canonical commutation relations) and as such is a Hilbert space ready for use in the quantum mechanics of a system with n degrees of freedom. The infinitesimal form of this representation is given by the following definitions of the creation and annihilation operators:

$$a_k^* \phi(z) := z_k \phi(z)$$
$$a_k \phi(z) := \frac{\partial}{\partial z_k}\phi$$

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for $k = 1, \dots, n$, where $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ and $\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)$. Moreover, the basic Hamiltonian operator in this formalism is defined by

$$N := \sum_{k=1}^{n} a_k^* a_k = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}$$

and is known as the *number operator*. It is unitarily equivalent to the operator $H_{harm} - \frac{n}{2}I$, where H_{harm} is the (isotropic, normalized) quantum harmonic oscillator Hamiltonian with n degrees of freedom and I is the identity operator, both acting in $L^2(\mathbf{R}^n, d^n x)$. It turns out that N, when realized as an operator in $\mathcal{A}_2$, is an unbounded self-adjoint operator satisfying $N \geq 0$. Consequently, for $t \geq 0$ by spectral theory we have a semigroup $\left\{e^{-tN}\right\}$ of contractions on $\mathcal{A}_2$. But more is true; this semigroup also acts contractively on each $\mathcal{A}_p$. To be perfectly clear here, we define $e^{-tN}\phi(z) := \phi(e^{-t}z)$ for any function $\phi : \mathbf{C}^n \to \mathbf{C}$ and any $t \in \mathbf{R}$, where $z \in \mathbf{C}^n$. This is valid since this formula is shown in [1] to hold for $f \in \mathcal{A}_2$ and $t \geq 0$ where e^{-tN} is already defined. We actually have the following *hypercontractivity* result, which is so called since it implies contractivity in each $\mathcal{A}_p$. It says that

$$\|e^{-tN}\|_{\mathcal{A}_p \to \mathcal{A}_q} = \begin{cases} 1 & \text{if } 0 < q \leq pe^{2t} \\ \infty & \text{if } q > pe^{2t} \end{cases}$$

for $0 < p \leq q$ and $t \geq 0$. The credits for this result begin with Janson [5] in 1983, continue with Carlen [2] and Zhou [10] in 1991, come back to Janson [6] in 1997 and conclude with Gross [4] in 1998 whose proof is the most direct in the sense that the method is essentially that of Gross' original paper [3] on log-Sobolev inequalities. In fact, we do get a log-Sobolev inequality from the hypercontractivity result by taking $p = 2$ and $q = pe^{2t} = 2e^{2t}$ (the "critical" index) and considering the inequality

$$\|e^{-tN}\phi\|_{2e^{2t}} \leq \|\phi\|_2,$$

which holds for $t \geq 0$. Differentiating this from the right at $t = 0$ (where it becomes an *equality*) gives the log-Sobolev inequality

$$S(\phi) \leq \langle \phi, N\phi \rangle ,$$

where

$$S(\phi) = \int_{\mathbf{C}^n} d\mu_n(z) |\phi(z)|^2 \log |\phi(z)|^2 - \|\phi\|_2^2 \log \|\phi\|_2^2$$

is the entropy of ϕ and

$$\langle \phi, N\phi \rangle = \sum_{k=1}^{n} \int_{\mathbf{C}^n} d\mu_n(z) \left|\frac{\partial \phi}{\partial z_k}\right|^2$$

is the (expected) energy of ϕ. Here log means the natural logarithm (base e), and we take $0 \log 0 = 0$. Note that both $S(\phi)$ and $\langle \phi, N\phi \rangle$ are defined for *all* $\phi \in \mathcal{A}_2$, and that $S(\phi) \geq 0$ (by Jensen's inequality) and $\langle \phi, N\phi \rangle \geq 0$ hold. It is possible that $S(\phi) = \infty$ and $\langle \phi, N\phi \rangle = \infty$ for some $\phi \in \mathcal{A}_2$. We will have more to say about this later.

2 Two Reverse Inequalities

The reverse hypercontractivity inequality due to Carlen [2] in the Segal-Bargmann space says the following.

Theorem: [2] We have for all $\phi \in \mathcal{A}_p$

$$\|e^{-tN}\phi\|_q \le (A(t,p,q))^n \|\phi\|_p, \tag{1}$$

where $A(t,p,q) = (1 - qe^{-2t}/p)^{-1/q}$, provided that $0 < q < p$ and $\frac{1}{2}\log(q/p) < t < 0$. It seems that this is the only reverse hypercontractivity inequality that is explicitly given in the literature.

In [2] Carlen notes but does not prove that the constant $(A(t,p,q))^n$ is not optimal. However, it was recently shown in [9] that this indeed is so, though the proof was by contradiction (i.e., assume that the constant $(A(t,p,q))^n$ is optimal and derive from that a false statement) and so one gets no further improvement in the constant. But now we have an improvement in the following result.

Proposition: For $0 < q < p$ and $\frac{1}{2}\log(q/p) < t < 0$ we have

$$\|e^{-tN}\phi\|_q \le (B(t,p,q))^n \|\phi\|_p \tag{2}$$

for all $\phi \in \mathcal{A}_p$ where

$$B(t,p,q) = \left(1 - \frac{q}{p}\right)^{1/q} \left(\frac{pe^{2t} - q}{p - q}\right)^{1/p} A(t,p,q).$$

However, $\|e^{-tN}\|_{\mathcal{A}_p \to \mathcal{A}_q} < (B(t,p,q))^n$, that is to say, the constant in (2) is not optimal.

Observation: The hypotheses on p, q and t imply that $B(t,p,q) < A(t,p,q)$, so that this is a better bound than Carlen's, as claimed.

Proof: One simply calculates

$$\begin{aligned}
\|e^{-tN}\phi\|_q^q &= \int_{\mathbf{C}^n} d\mu_n(z) |e^{-tN}\phi(z)|^q \\
&= \int_{\mathbf{C}^n} d^n x \, d^n y \, \pi^{-n} e^{-|z|^2} |\phi(e^{-t}z)|^q \\
&= e^{2nt} \int_{\mathbf{C}^n} d^n u \, d^n v \, \pi^{-n} e^{-|w|^2} e^{(1-e^{2t})|w|^2} |\phi(w)|^q \\
&\le e^{2nt} \| \, |\phi|^q \, \|_r \left\{ \int_{\mathbf{C}^n} d^n u \, d^n v \, \pi^{-n} \, e^{-|w|^2} e^{r'(1-e^{2t})|w|^2} \right\}^{1/r'},
\end{aligned}$$

where we have used $w = u + iv = e^{-t}z$ and Hölder's inequality for $1 < r < \infty$. Here the norm $\| \cdot \|_r$ is with respect to the measure μ_n. Taking $r = p/q$, we obtain

$$\|e^{-tN}\phi\|_q^q \le e^{2nt}\|\phi\|_p^q \left(\frac{pe^{2t}-q}{p-q}\right)^{-n\left(1-\frac{q}{p}\right)}$$

since the Gaussian integral converges for p, q and t satisfying the hypotheses. Now the result follows by simple algebra manipulations.

Now let us show that the constant $(B(t,p,q))^n$ is not optimal. As shown in [9], the inequality

$$\|e^{-tN}\phi\|_q \le M(t,p,q,n)\|\phi\|_p,$$

where $M(t,p,q,n)$ is the optimal constant for this inequality (i.e., the operator norm $\|e^{-tN}\|_{\mathcal{A}_p\to\mathcal{A}_q}$), does have an optimizer $\phi_0 \in \mathcal{A}_p$. This means that $\phi_0 \ne 0$ and

$$\|e^{-tN}\phi_0\|_q = M(t,p,q,n)\|\phi_0\|_p.$$

This is shown using a standard compactness argument. If $(B(t,p,q))^n$ were the optimal constant, then we would have equality at the step in the above argument where we applied the Hölder inequality. This would imply that

$$|\phi_0(w)|^q = ce^{(r'-1)\left(1-e^{2t}\right)|w|^2}$$

for some constant $c > 0$ and all $w \in \mathbf{C}^n$. But it is well known that a nonzero holomorphic function can not satisfy such an equality. To see this, simply note that we would have for all $w \in \mathbf{C}^n$ that

$$\phi_0(w) = c_1 e^{i\theta(w)} e^{c_2|w|^2}$$

for some phase $\theta(w) \in \mathbf{R}$, some $c_1 > 0$ and some $c_2 \ne 0$. Then taking the real part of (say, the principal branch of) the logarithm of $\phi_0(w)$ gives a harmonic function (on some open subset of $\mathbf{C}^n$). But on the other hand, this gives us $\log c_1 + c_2|w|^2$, which is harmonic if and only if the constant c_2 is zero. QED

The fact that $(B(t,p,q))^n$ is not optimal has to do with the fact that the inequality (2) actually holds for all ϕ in $L^p(\mathbf{C}^n,\mu_n)$, since the proof of the inequality (2) given above never uses the holomorphicity of ϕ. Of course, the constant $(B(t,p,q))^n$ is optimal for (2) if ϕ is allowed to run over all of $L^p(\mathbf{C}^n,\mu_n)$.

Another observation about reverse hypercontractivity is that for the "critical" index we do not have a bounded operator. This is unlike the hypercontractivity result where we do have boundedness at the "critical" index, though not compactness. Specifically, we have the next result.

Proposition: For $0 < p < \infty$ and $t < 0$, we have

$$\|e^{-tN}\|_{\mathcal{A}_p\to\mathcal{A}_q} = \infty$$

for $q = pe^{2t}$.

Proof: Let $\phi_\alpha(z) = e^{\alpha z^2}$ for real α and $z \in \mathbf{C}^n$. Then

$$\|\phi_\alpha\|_p = \left(1 - \alpha^2 p^2\right)^{-n/(2p)}$$

provided that $-1/p < \alpha < 1/p$. Since $e^{-tN}\phi_\alpha(z) = \phi_\alpha(e^{-t}z) = e^{\alpha e^{-2t} z^2} = \phi_{\alpha e^{-2t}}(z)$, we have that $\|e^{-tN}\phi_\alpha\|_q = \|\phi_{\alpha e^{-2t}}\|_q = \left(1 - \alpha^2 e^{-4t} q^2\right)^{-n/(2q)}$ provided that $-1/q < \alpha e^{-2t} < 1/q$, which is equivalent to $-1/p < \alpha < 1/p$ since $q = pe^{2t}$. Next, note that

$$\frac{\|e^{-tN}\phi_\alpha\|_q}{\|\phi_\alpha\|_p} = \left(1 - \alpha^2 p^2\right)^{n\left(e^{2t}-1\right)/(2pe^{2t})}.$$

But $t < 0$ implies that $e^{2t} - 1 < 0$, and so taking the limit as $|\alpha| \to 1/p$ gives us $+\infty$. QED

The other reverse inequality in the Segal-Bargmann space is a reverse log-Sobolev inequality.

Theorem: For every $c > 1$ there exists a constant $R(c)$ independent of the dimension n such that

$$\langle \phi, N\phi \rangle \leq cS(\phi) + nR(c)\|\phi\|_2^2 \tag{3}$$

for all $\phi \in \mathcal{A}_2$. One can take $R(c) = -1 + c\log\left(\frac{c}{c-1}\right)$.

This was originally proved in [8], but Gross has given a more elegant proof, reproduced in [9]. His proof gives the constant $R(c)$ quoted here, which is better than the constant found in [8]. However, the optimal value of the coefficient of the norm term remains unknown. This seems to be the only reverse log-Sobolev inequality in the literature.

Two applications of this reverse log-Sobolev inequality to the analysis of the Segal-Bargmann transform can be found in [8]. Also, one surprising result of the reverse log-Sobolev inequality together with the (regular) log-Sobolev inequality is that the entropy $S(\phi)$ is finite if *and only if* the energy $\langle \phi, N\phi \rangle$ is finite. However, the reverse log-Sobolev inequality may also be considered as a "generalized" Heisenberg uncertainty principle, which may be its more fundamental role. The idea here is that, by a theorem of Stone and von Neumann, all irreducible representations of the Weyl-Heisenberg group are unitarily equivalent, and moreover by a unitary (and intertwining) operator that is unique up to multiplication by a complex number of modulus one. So all properties expressible in terms of the inner product and the creation and annihilation operators alone are invariant under such Stone-von Neumann operators. However, certain quantities are representation dependent and may not be preserved by the Stone-von Neumann unitary operator. The usual example given in an introductory physics course in quantum mechanics is the *variance* $\mathrm{Var}(\phi)$ of a state ϕ in $L^2(\mathbf{R}^n, d^n x)$, the position space. By going to the momentum space representation (which is in fact another representation of the Weyl-Heisenberg group as is the position space), one finds that the Stone-von Neumann unitary operator in this case is the Fourier transform $\mathcal{F}$, and it is well known that the Fourier transform does *not* preserve variance. In fact, we have the Heisenberg inequality $\mathrm{Var}(\phi)\mathrm{Var}(\mathcal{F}\phi) \geq c > 0$ for all states ϕ (i.e., $\|\phi\|_2 = 1$) where c is a constant independent of ϕ that depends only on how one normalizes Planck's constant. Other quantities (actually nonlinear functionals on the Hilbert space) not necessarily preserved by the Stone-von Neumann unitary operator are the L^p norm of a state and the entropy of a state, provided that the Hilbert space carries enough extra structure to allow us to define these quantities, namely, that it is the L^2 space of a measure space. But the energy $\langle \phi, N\phi \rangle$ in the Segal-Bargmann space (which also is a nonlinear functional in $\phi \in \mathcal{A}_2$) *is* preserved by the Stone-von Neumann unitary operators, since it has a definition in terms of creation and annihilation operators. So inequalities (such as the regular or reverse log-Sobolev inequalities) express relations that hold in one particular representation, but not in all representations, and thereby serve to distinguish that representation from the others. In this sense we can think of such inequalities as "generalized" Heisenberg uncertainty principles. Returning to the position and momentum representations, we see for example that the Hausdorff-Young inequality (of the Fourier transform $\mathcal{F}$ on Euclidean space) is a "generalized" Heisenberg inequality, which is related to the fact that $\mathcal{F}$ does not preserve L^p norms for $p \neq 2$.

3 Differentiating Hypercontractivity

Much as the log-Sobolev inequality arises from the hypercontractivity inequality by differentiation, we would like to be able to derive the reverse log-Sobolev inequality by differentiating the reverse hypercontractivity inequality. Unfortunately, this does not work out as anticipated. Let us write

$$M(t, p, q) := \|e^{-tN}\|_{\mathcal{A}_p \to \mathcal{A}_q}$$

even though this notation omits the (possible) dependence on the dimension n. (See [9] for a discussion of this dependence on n.) So we have

$$\|e^{-tN}\phi\|_q \leq M(t, p, q)\|\phi\|_p \tag{4}$$

for all $\phi \in \mathcal{A}_p$. Moreover, for $\frac{1}{2}\log(q/p) < t < 0$ and $0 < q < p$, we have that $M(t, p, q)$ is finite. However, it remains an open problem to find an explicit formula for $M(t, p, q)$. Proceeding with (4) above, we can not take $q = pe^{2t}$ (the "critical" index) since we have already shown that $M(t, p, q) = \infty$ in that case. Instead, we choose some function $s : (-\epsilon, 0] \to (1, \infty)$ for some $\epsilon > 0$ such that: (a) $s(t) < pe^{2t}$ for $-\epsilon < t < 0$; (b) $s(0) = p$; and (c) the derivative of s from the left at $t = 0$, denoted $s'(0^-)$, exists. (Here we are only considering the case $p > 1$.) Using condition (a), we have

$$\|e^{-tN}\phi\|_{s(t)} \leq M(t, p, s(t))\|\phi\|_p \tag{5}$$

for $-\epsilon < t < 0$. Moreover, at $t = 0$ both the sides of this inequality become $\|\phi\|_p$, using condition (b) and $M(0, p, s(0)) = 1$. So we can take the left-sided derivative at $t = 0$ on both sides of (5) to get a new inequality. (Warning: The sense of the inequality in (5) reverses on taking this derivative since $t = 0$ is the *right* end point.) Using the chain rule together with condition (c), we get formally

$$Re\,\langle \phi_p, N\phi \rangle \leq \frac{1}{p^2} s'(0^-) S_p(\phi) - \kappa_p \|\phi\|_p^p, \tag{6}$$

where

$$\kappa_p := \frac{d}{dt}\bigg|_{t=0^-} M(t, p, s(t)).$$

Moreover, $\phi_p = \mathrm{sgn}(\phi)|\phi|^{p-1}$, a usual notation in L^p analysis, and $S_p(\phi) := S(|\phi|^{p/2})$ is called the *index-p entropy* of ϕ.

One would now like to take $p = 2$ so that $Re\,\langle \phi_p, N\phi \rangle = \langle \phi, N\phi \rangle$ and $S_2(\phi) = S(\phi)$, and then (6) reduces to what seems to be a perfectly well-behaved reverse log-Sobolev inequality. But actually what is happening here is that $\kappa_p = -\infty$ for $p \geq 2$, that is, the derivative used to define κ_p does not exist as a finite number. (The proof is given in [8].) But there may be an escape hatch here. First, it may be that the argument above for (6) is nontrivial for $1 < p < 2$, namely, κ_p is finite. But to verify this it would behoove us to find an explicit formula for $M(t, p, q)$. Then one would hope that one could take the limit of (6) as p increases to 2, and that the resulting limit would be the reverse log-Sobolev inequality. The upshot of this paragraph is a conjectural approach to relating reverse hypercontractivity to reverse log-Sobolev, but with some sturdy open problems. The following proposition, which we have already referred to, may be useful in addressing these problems.

Proposition: If $1 \leq q < p$ and $\frac{1}{2}\log(q/p) < t < 0$, then $e^{-tN} : \mathcal{A}_p \to \mathcal{A}_q$ is a compact operator and has a maximizer $\phi_0 \neq 0$ in $\mathcal{A}_p$.

The proof is given in [9]. If one could identify one of these maximizers ϕ_0, then one could read off the value of $M(t, p, q)$ as $\|e^{-tN}\phi_0\|_q / \|\phi_0\|_p$. But this is also an open problem which does not appear to be trivial.

We conclude with a list of open problems.

1. Is the derivation of (6) valid for $1 < p < 2$?
2. Find an explicit formula for $M(t, p, q)$.
3. Does an inequality of the form (6) hold for $p \neq 2$?
4. For each $c > 1$, what is the optimal constant for the coefficient of the norm term in (3)? Does this optimal constant depend on n? Is there some value of $c > 1$ such that this optimal constant is zero?
5. Find the optimal constant in the reverse hypercontractivity inequality (1).

6. What are the maximizers in the last proposition? In particular, it would be helpful to know if it is true that every n-dimensional maximizer can be expressed as a product of one dimensional maximizers. Specifically, this means that for any $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ we can write any maximizer ϕ_0 as in the above proposition in the form $\phi_0(z) = \psi_1(z_1) \cdots \psi_n(z_n)$, where each ψ_j is a maximizer for dimension $n = 1$. This is what Zhou proves in his article [10] for the case of (regular) hypercontractivity in the Segal-Bargmann space.

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Fernando Galaz-Fontes
Centro de Investigación en Matemáticas
Guanajuato, Gto., Mexico
email: galaz@fractal.cimat.mx

Stephen Bruce Sontz
Universidad Autónoma Metropolitana-Iztapalapa
México, D.F. 09340, Mexico
email: sontz@xanum.uam.mx