Mathematical Physics and Quantum Field Theory,
Electronic Journal of Differential Equations, Conf. 04, 2000, pp. 231–242
http://ejde.math.swt.edu or http://ejde.math.unt.edu
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Quantum electrodynamics and the fundamental constants *

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Abstract

The fundamental constants are evaluated by comparison of the results of critical experiments and the theoretical expressions for these results written in terms of the constants. Many of the theoretical expressions are based on quantum electrodynamics (QED), so the consistency of the comparison provides a critical test of the validity of the theory.

1 Introduction

This paper reviews the basic approach used in the 1998 adjustment of the values of the fundamental constants, which is the subject of a long article by the authors written under the auspices of the Committee on Data for Science and Technology (CODATA) Task Group on Fundamental Constants [1, 2].

The purpose of the adjustment is to determine and recommend values of various fundamental constants such as the

fine-structure constant α

Rydberg constant R_∞

Avogadro constant $N_{\rm A}$

Planck constant h

electron mass $m_{\rm e}$

muon mass m_μ

and many others, which provide the greatest consistency between critical experiments and predictions based on quantum electrodynamics (QED) theory and condensed matter theory.

* *Mathematics Subject Classifications:* 81V10.
Key words: Fundamental physical constants, quantum electrodynamics.

Published August 28, 2000.

2 Background

A pioneering comprehensive determination of constants was done by Raymond T. Birge at the University of California, Berkeley, and published in 1929. Adjustments done since 1950, including those sponsored by the National Research Council/National Academy of Sciences, USA (NRC/NAS) and by CODATA, are the following:

Bearden and Watts (1951) [3, 4, 5]
NRC/NAS: DuMond and Cohen (1951) [6, 7, 8];
Bearden and Thomsen (1957) [9];
NRC/NAS: Cohen and DuMond (1965) [10];
Taylor, Parker, and Langenberg (1969) [11];
CODATA: Cohen and Taylor (1973) [12];
CODATA: Cohen and Taylor (1987) [13].

Prior to the 1998 adjustment of the constants [1, 2], the CODATA recommended values were based on the 1986 adjustment [13]. The current CODATA recommended values for the constants, based on the 1998 adjustment, are available on the World Wide Web at

physics.nist.gov/constants

3 Method of least squares

The method of least squares, as it is applied to the 1998 adjustment, is summarized here. Input data, which are the results of measurements, or in some cases calculations, are denoted by $q_1,\ q_2, \ldots, q_N$. The variables of the least-squares adjustment, which are members of a suitable subset of the fundamental constants termed adjusted constants, are denoted by $z_1,\ z_2, \ldots, z_M$, where $M \leq N$. The input data and the adjusted constants are related by observational equations which are theoretical expressions for the q_i as functions of the z_j and are denoted by

$$q_i \doteq f_i(z) \equiv f_i(z_1, z_2, \ldots, z_M) \ . \tag{1}$$

We employ the symbol $\doteq$ to indicate the unsymmetric relation between the item of input data on the left-hand side and the corresponding theoretical expression for that item as a function of the adjusted constants on the right-hand side. In general, this set of equations is overdetermined, so the left- and right-sides will not be equal, even for optimized values of the constants.

Almost all of the observational equations are non-linear, but they can be linearized by expanding about less-accurately known starting values s for the variables z by writing

$$q_i \doteq f_i(s) + \sum_{j=1}^{M} \frac{\partial f_i(s)}{\partial s_j}(z_j - s_j) + \cdots \tag{2}$$

or

$$y_i \doteq \sum_{j=1}^{M} A_{ij} x_j + \cdots \ , \tag{3}$$

where $y_i = q_i - f_i(s)$, $x_j = z_j - s_j$, and $A_{ij} = \dfrac{\partial f_i(s)}{\partial s_j}$. Equation (3) can be written compactly in terms of matrices as

$$Y \doteq AX \tag{4}$$

with the standard uncertainties and covariances of the observational data expressed as elements of the covariance matrix

$$V = \mathrm{cov}(Y) \ . \tag{5}$$

The least-squares adjustment is based on the solution $\hat{X}$ for X that minimizes

$$(Y - AX)^\top V^{-1} (Y - AX) \ . \tag{6}$$

The solution is [14]

$$\hat{X} = (A^\top V^{-1} A)^{-1} A^\top V^{-1} Y \tag{7}$$

$$\mathrm{cov}(\hat{X}) = (A^\top V^{-1} A)^{-1} \ , \tag{8}$$

so the constants and their covariance matrix are given by

$$\hat{Z} = S + \hat{X} \tag{9}$$

$$\mathrm{cov}(\hat{Z}) = \mathrm{cov}(\hat{X}) \ , \tag{10}$$

and, as a corollary, the best estimates of the measured quantities are given by

$$\hat{Y} = A\hat{X} \tag{11}$$

$$\hat{q}_i = f_i(s) + \hat{y}_i \ . \tag{12}$$

It should be remarked that the initial linear correction described above is usually not sufficient. In most trials and variants of the 1998 least-squares adjustment, several iterations were needed in which the resulting values of the constants of one iteration become the starting values in the next.

Motivation for using the solution $\hat{X}$ obtained by minimization of Eq. (6) is provided by the following considerations. In the case where the observations Y are uncorrelated, V is diagonal, and Eq. (6) reduces to the simple weighted sum of differences given by

$$(Y - AX)^\top V^{-1} (Y - AX) = \sum_{i=1}^{N} \frac{(Y - AX)_i^2}{(\delta Y)_i^2} \ , \tag{13}$$

where $(\delta Y)_i^2 = V_{ii}$ is the square of the standard uncertainty associated with the input datum q_i. In this case, the solution $\hat{X}$ provides the best match of the theoretical expressions to the input data as depicted in Eq. (13). However, in general V is not diagonal, and we seek to optimize the constants taking covariances of the input data into account. In this case, there is an alternative criterion that may be applied. From Eqs. (7) and (11) we have

$$\hat{Y} = C\,Y \ , \tag{14}$$

where

$$C = A(A^\top V^{-1} A)^{-1} A^\top V^{-1} \ . \tag{15}$$

The elements of $\hat{Y}$ so obtained are the best estimates for the quantities represented by Y in the following sense: If we consider an estimate of the quantities represented by Y of the form $Y' = DY$ such that the sum of the squares of the uncertainties of Y' as given by the trace of the covariance matrix $\mathrm{cov}(Y') = DVD^\top$ is a minimum, subject to the condition that the matrix D reproduces any set of data of the form AX (that is, $DAX = AX$ for any X), then $D = C$, where C is just the matrix in Eq. (15), and hence $Y' = \hat{Y}$ [14]. Thus, the solution $\hat{X}$ obtained by minimizing Eq. (6) provides the set of constants for which the observational equations make the most accurate theoretical predictions.

As mentioned above, a subset of the constants is adjusted by minimizing Eq. (6); the rest are calculated from this subset. The choice of the subset is arbitrary, provided the constants in it are independent. The results of the adjustment will be the same for any such choice, as can be seen from the form of the solution. If an alternate subset W were selected, then since both X and W are independent sets, we can write $W = BX$, where B has an inverse, and $Y \doteq AB^{-1}W$. The solution would then be

$$\hat{W} = ((AB^{-1})^\top V^{-1} AB^{-1})^{-1} (AB^{-1})^\top V^{-1} Y = B\hat{X} \tag{16}$$

$$\mathrm{cov}(\hat{W}) = ((AB^{-1})^\top V^{-1} AB^{-1})^{-1} = B\mathrm{cov}(\hat{X})B^\top \ , \tag{17}$$

where the second line is just the standard formula for the propagation of uncertainty. Hence, we reproduce the same optimized constants $\hat{X}$ by finding the least-squares solution for any independent set W, with its corresponding matrix B, and forming

$$\hat{X} = B^{-1}\hat{W} \tag{18}$$

$$\mathrm{cov}(\hat{X}) = B^{-1}\mathrm{cov}(\hat{W})(B^{-1})^\top \ . \tag{19}$$

4 Adjusted Constants

The subset of constants used in the 1998 adjustment is given in Table 1. In the table, the symbol $A_\mathrm{r}(\mathrm{x})$ denotes the relative atomic mass of particle x,

Table 1: Variables of the 1998 least-squares adjustment.

Variable name	Symbol
electron relative atomic mass	$A_{\rm r}({\rm e})$
proton relative atomic mass	$A_{\rm r}({\rm p})$
neutron relative atomic mass	$A_{\rm r}({\rm n})$
deuteron relative atomic mass	$A_{\rm r}({\rm d})$
helion relative atomic mass	$A_{\rm r}({\rm h})$
alpha particle relative atomic mass	$A_{\rm r}(\alpha)$
electron-muon mass ratio	$m_{\rm e}/m_{\mu}$
fine-structure constant	α
Planck constant	h
Rydberg constant	R_{∞}
proton rms charge radius	$R_{\rm p}$
deuteron rms charge radius	$R_{\rm d}$
molar gas constant	R
Newtonian constant of gravitation	G
electron-proton magnetic moment ratio	$\mu_{\rm e}/\mu_{\rm p}$
deuteron-electron magnetic moment ratio	$\mu_{\rm d}/\mu_{\rm e}$
electron to shielded proton magnetic moment ratio	$\mu_{\rm e}/\mu'_{\rm p}$
shielded helion to shielded proton magnetic moment ratio	$\mu'_{\rm h}/\mu'_{\rm p}$
neutron to shielded proton magnetic moment ratio	$\mu_{\rm n}/\mu'_{\rm p}$
{220} lattice spacing of Si crystal X	$d_{220}({\rm X})$
{220} lattice spacing of an ideal Si crystal	d_{220}
Cu x unit	$\mathrm{xu}(\mathrm{Cu\,K}\alpha_1)$
Mo x unit	$\mathrm{xu}(\mathrm{Mo\,K}\alpha_1)$
ångstrom star	Å*
corrections to hydrogen level theory	$\delta_{\rm H}(n{\rm L}_j)$
corrections to deuterium level theory	$\delta_{\rm D}(n{\rm L}_j)$
correction to $a_{\rm e}$(th)	$\delta_{\rm e}$
correction to a_{μ}(th)	δ_{μ}
correction to $\Delta\nu_{\rm Mu}$(th)	$\delta_{\rm Mu}$

i.e., the ratio $m_{\rm x}/m_{\rm u}$, where the atomic mass constant $m_{\rm u}$ is 1/12 times the mass of a ^{12}C atom. The ^{3}He nucleus (helion) is denoted by h. Also in that table, d_{220}(X) refers to the lattice spacing of any of 7 different Si crystals, with names represented by X, used in various experiments that provide information on the constants. The last five entries termed "corrections" are correction terms that are added to the theoretical expressions for the corresponding quantities (hydrogen and deuterium energy levels, anomalous magnetic moment of the electron and muon, and muonium hyperfine splitting) to take into account the uncertainty of the theory. There are a total of 25 correction terms for the hydrogen and deuterium energy levels.

Other quantities can be derived from the adjusted constants. For example, based on familiar formulas, we can express the elementary charge in terms of constants contained in the adjusted subset:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \quad \Rightarrow \quad e = \sqrt{2\epsilon_0 h c \alpha} \ . \tag{20}$$

Similarly, for the electron mass, we have

$$R_\infty = \frac{\alpha^2 m_{\rm e} c}{2h} \quad \Rightarrow \quad m_{\rm e} = \frac{2hR_\infty}{\alpha^2 c} \ . \tag{21}$$

It should be noted that it is not necessary to select the most accurately known constants for the adjusted subset. Since we calculate covariances and use them to calculate values of the derived constants, it is possible to obtain values for the derived constants with relative uncertainties that are smaller than the relative uncertainties of the values of the constants from which they are derived. The final result of the adjustment is a set of values for over 300 constants that are either in the adjusted subset or calculated as above.

5 Observational Equations

Some of the observational equations, represented by Eq. (1), employed in the 1998 least-squares adjustment are given in Table 2. In each observational equation, the symbol on the left-hand side represents one of the 93 measured, or in some cases calculated, input data of the adjustment. The right-hand side expresses that quantity as a function of the constants that are taken as variables of the adjustment. (This distinction explains the appearance of equations such as $\delta_{\rm e} \doteq \delta_{\rm e}$, which expresses the relation between an input datum and the corresponding adjusted constant, even though in general their values will not be equal.)

For example, in the first equation in Table 2, the quantity $A_{\rm r}(^1{\rm H})$ represents the measured value of the relative atomic mass of the hydrogen atom. The right-hand side is the sum of the relative atomic masses of the proton and electron, both of which are variables of the adjustment, minus the binding energy equivalent mass in the same units, which is relatively small and sufficiently well known to be taken as exact.

Table 2: Some observational equations used in the 1998 adjustment.

$$
\begin{aligned}
A_{\mathrm{r}}(^1\mathrm{H}) &\doteq A_{\mathrm{r}}(\mathrm{p}) + A_{\mathrm{r}}(\mathrm{e}) - E_{\mathrm{b}}(^1\mathrm{H})/m_{\mathrm{u}}c^2 \\
\frac{\lambda_{\mathrm{meas}}}{d_{220}(\mathrm{ILL})} &\doteq \frac{\alpha^2 A_{\mathrm{r}}(\mathrm{e})}{R_\infty d_{220}(\mathrm{ILL})} \frac{A_{\mathrm{r}}(\mathrm{n}) + A_{\mathrm{r}}(\mathrm{p})}{[A_{\mathrm{r}}(\mathrm{n}) + A_{\mathrm{r}}(\mathrm{p})]^2 - A_{\mathrm{r}}^2(\mathrm{d})} \\
\frac{m_{\mathrm{e}}}{m(^{12}\mathrm{C}^{6+})} &\doteq \frac{6A_{\mathrm{r}}(\mathrm{e})}{12 - 6A_{\mathrm{r}}(\mathrm{e}) + E_{\mathrm{b}}(^{12}\mathrm{C})/m_{\mathrm{u}}c^2} \\
a_{\mathrm{e}} &\doteq a_{\mathrm{e}}(\alpha, \delta_e) \\
\delta_{\mathrm{e}} &\doteq \delta_{\mathrm{e}} \\
\frac{\mu_{\mathrm{e}^-}(\mathrm{H})}{\mu_{\mathrm{p}}(\mathrm{H})} &\doteq \frac{g_{\mathrm{e}^-}(\mathrm{H})}{g_{\mathrm{e}^-}} \left(\frac{g_{\mathrm{p}}(\mathrm{H})}{g_{\mathrm{p}}} \right)^{-1} \frac{\mu_{\mathrm{e}^-}}{\mu_{\mathrm{p}}} \\
\nu(f_{\mathrm{p}}) &\doteq \nu\Big(f_{\mathrm{p}}; R_\infty, \alpha, \tfrac{m_{\mathrm{e}}}{m_\mu}, \tfrac{\mu_{\mathrm{e}^-}}{\mu_{\mathrm{p}}}, \delta_{\mathrm{e}}, \delta_\mu, \delta_{\mathrm{Mu}}\Big) \\
\frac{\mu_{\mu^+}}{\mu_{\mathrm{p}}} &\doteq -\frac{1 + a_\mu(\alpha, \delta_\mu)}{1 + a_{\mathrm{e}}(\alpha, \delta_{\mathrm{e}})} \frac{m_{\mathrm{e}}}{m_\mu} \frac{\mu_{\mathrm{e}^-}}{\mu_{\mathrm{p}}} \\
\Delta\nu_{\mathrm{Mu}} &\doteq \Delta\nu_{\mathrm{Mu}}\Big(R_\infty, \alpha, \tfrac{m_{\mathrm{e}}}{m_\mu}, \delta_\mu, \delta_{\mathrm{Mu}}\Big) \\
\delta_{\mathrm{Mu}} &\doteq \delta_{\mathrm{Mu}} \\
K_{\mathrm{J}} &\doteq \left(\frac{8\alpha}{\mu_0 c h}\right)^{1/2} \\
R_{\mathrm{K}} &\doteq \frac{\mu_0 c}{2\alpha} \\
K_{\mathrm{J}}^2 R_{\mathrm{K}} &\doteq \frac{4}{h} \\
\frac{h}{m_{\mathrm{n}} d_{220}(\mathrm{W04})} &\doteq \frac{c A_{\mathrm{r}}(\mathrm{e}) \alpha^2}{2 R_\infty A_{\mathrm{r}}(\mathrm{n}) d_{220}(\mathrm{W04})} \\
\frac{d_{220}(\mathrm{X}) - d_{220}(\mathrm{Y})}{d_{220}(\mathrm{Y})} &\doteq \frac{d_{220}(\mathrm{X}) - d_{220}(\mathrm{Y})}{d_{220}(\mathrm{Y})} \\
d_{220}(\mathrm{X}) &\doteq d_{220}(\mathrm{X}) \\
\frac{\lambda(\mathrm{W\,K}\alpha_1)}{d_{220}(\mathrm{N})} &\doteq \frac{0.209\,010\,0\ \text{Å}^*}{d_{220}(\mathrm{N})} \\
\nu_{\mathrm{H}}(n_1\mathrm{L}_{1j_1} - n_2\mathrm{L}_{2j_2}) &\doteq \Big[E_{\mathrm{H}}\Big(n_2\mathrm{L}_{2j_2}; R_\infty, \alpha, A_{\mathrm{r}}(\mathrm{e}), A_{\mathrm{r}}(\mathrm{p}), R_{\mathrm{p}}, \delta_{\mathrm{H}}(n_2\mathrm{L}_{2j_2})\Big) \\
&\qquad - E_{\mathrm{H}}\Big(n_1\mathrm{L}_{1j_1}; R_\infty, \alpha, A_{\mathrm{r}}(\mathrm{e}), A_{\mathrm{r}}(\mathrm{p}), R_{\mathrm{p}}, \delta_{\mathrm{H}}(n_1\mathrm{L}_{1j_1})\Big)\Big] \Big/ h \\
\delta_{\mathrm{H}}(n\mathrm{L}_j) &\doteq \delta_{\mathrm{H}}(n\mathrm{L}_j) \\
R_{\mathrm{p}} &\doteq R_{\mathrm{p}}
\end{aligned}
$$

6 Electron magnetic moment anomaly

It is well known that the magnetic moment of the electron is not equal to the value predicted by the Dirac equation $g_{\rm e}(\text{Dirac}) = -2$, and the deviation from that value is given in terms of the electron magnetic moment anomaly $a_{\rm e}$ by

$$g_{\rm e} = -2(1 + a_{\rm e}) \ .$$

The anomaly has been measured for the electron and positron at the University of Washington with the results [15]

$$\begin{aligned} a_{\rm e^-}(\text{exp}) &= 1\,159\,652\,188.4(4.3) \times 10^{-12} \\ a_{\rm e^+}(\text{exp}) &= 1\,159\,652\,187.9(4.3) \times 10^{-12} \ . \end{aligned} \tag{22}$$

The average of these values (assuming as we do that the value of the anomaly is CPT invariant) yields the input datum corresponding to $a_{\rm e}$ on the left-hand side of the fourth equation in Table 2. The theoretical expression on the right-hand side of that equation is

$$\begin{aligned} a_{\rm e}(\alpha, \delta_{\rm e}) &= C_{\rm e}^{(2)} \left(\frac{\alpha}{\pi}\right) + C_{\rm e}^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + C_{\rm e}^{(6)} \left(\frac{\alpha}{\pi}\right)^3 \\ &\quad + C_{\rm e}^{(8)} \left(\frac{\alpha}{\pi}\right)^4 + a_{\rm e}(\text{had}) + a_{\rm e}(\text{weak}) \\ &\quad + \delta_{\rm e} \ , \end{aligned} \tag{23}$$

which gives the anomaly as a function of the variables α and $\delta_{\rm e}$. The coefficients $C_{\rm e}^{(n)}$, the strong interaction correction $a_{\rm e}(\text{had})$, and the weak interaction correction $a_{\rm e}(\text{weak})$ are calculated from theory. The largest uncertainty in the theory arises from numerical integration uncertainty in the massive calculation of $C_{\rm e}^{(8)}$ [16]. The total uncertainty of the theory is estimated to be 1.1×10^{-12}, so the observational equation for $\delta_{\rm e}$ is

$$0.0(1.1) \times 10^{-12} \doteq \delta_{\rm e} \ . \tag{24}$$

The electron anomalous magnetic moment data provide the most influential information on the value of the fine-structure constant α. When considered alone they yield

$$\alpha^{-1} = 137.035\,999\,58(52) \ . \tag{25}$$

Of course, in the final adjustment, all sources of information on α contribute to the 1998 recommended value:

$$\alpha^{-1} = 137.035\,999\,76(50) \ . \tag{26}$$

This value has an uncertainty that is about 1/12 times the uncertainty of the 1986 recommended value, and differs from the earlier value by about 1.7 times the uncertainty of the earlier value. The adjusted value of $\delta_{\rm e}$ is $0.1(1.1) \times 10^{-12}$, very nearly its input value in Eq. (24).

7 Planck constant from the watt balance

The Planck constant can be measured by comparing a watt of mechanical power expressed in terms of the meter, kilogram, and second to a watt of electrical power expressed in terms of the Josephson constant $K_{\rm J} = 2e/h$ and von Klitzing constant $R_{\rm K} = h/e^2$ in the combination

$$K_{\rm J}^2 \, R_{\rm K} = \frac{4}{h} \; . \tag{27}$$

The apparatus that makes the comparison is called a watt balance [17, 18]. A remarkable aspect of the experiment is that it provides a value of the Planck constant through measurements that involve classical mechanics and classical electromagnetism in a two-story-high apparatus. The Planck constant enters through the current and voltage calibration.

The basic principle of the watt balance is illustrated by one of its implementations [18]. A vertical solenoid is wound in such a way as to produce a radial magnetic flux density in a region outside the solenoid. A horizontal coil of wire enclosing the solenoid is suspended in this region from bands that extend over a balance wheel to a counterweight on the other side. In one phase of the experiment, a precisely known mass is added to the side of the balance with the horizontal coil, and a measurement is made of the current in the coil needed to produce an upward force that balances the weight of the mass. The force is given by the equation

$$F_z = \int {\rm d}^3x \; (\vec{J} \times \vec{B}) \cdot \hat{z} = I \int ({\rm d}\vec{\ell} \times \vec{B}) \cdot \hat{z} \; , \tag{28}$$

where $\vec{J}$ is the current density in the coil, I is the current in the coil, and ${\rm d}\vec{\ell}$ is an element of length of the wire in the coil. In the other phase of the experiment, the coil is slowly moved through the flux density and the voltage induced in the coil is measured. This induced voltage is given by

$$U_{\rm v} = \int {\rm d}\vec{\ell} \cdot (\vec{v} \times \vec{B}) = - \, v \int ({\rm d}\vec{\ell} \times \vec{B}) \cdot \hat{z} \; , \tag{29}$$

where $\vec{v}$ is the (vertical) velocity of the coil measured by laser interferometry and, as a first approximation, it is assumed that the vertical force on the coil in the first phase is in exactly the same direction as the vertical motion in the second phase. Equations (28) and (29) can be combined to obtain

$$F_z \;\; = \;\; -mg = -\frac{I}{v} \, U_{\rm v} \; , \tag{30}$$

where m is the precise mass used in the experiment, and g is the local acceleration of free fall, which is accurately measured with a gravimeter. The key to all versions of the experiment is the fact that the flux density and geometry of the coil drop out of this relation. Since the voltage $U_{\rm v}$ and the voltage and

resistance that determine I are calibrated in terms of the Josephson and von Klitzing constants, we have

$$mgv = I\,U_{\rm v} = \frac{Af_1f_2}{K_{\rm J}^2 R_{\rm K}} = Af_1f_2\,\frac{h}{4}\ , \tag{31}$$

where A is an exactly known constant and f_1 and f_2 are the precisely known frequencies applied to the Josephson junctions in the two phases of the experiment. Equation (31) gives h in terms of simple quantities measured in the experiment.

8 Kilogram definition

This experiment suggests a possible new definition of the kilogram [19]:

> *The kilogram is the mass of a body at rest whose equivalent energy equals the energy of a collection of photons whose frequencies sum to* $135\,639\,274\times 10^{42}$ *hertz.*

This definition has the consequence that the Planck constant is exactly defined, because the relations $E = mc^2$ and $E = h\nu_1 + h\nu_2 + \ldots$ imply that

$$h = \frac{mc^2}{\nu_1+\nu_2+\ldots} = \frac{(1\ {\rm kg})(299\,792\,458\ {\rm m\ s^{-1}})^2}{135\,639\,274\times 10^{42}\ {\rm Hz}} \tag{32}$$

$$\tag{33}$$

$$= 6.626\,068\,9\ldots\times 10^{-34}\ {\rm J\ s}\ , \tag{34}$$

exactly.

Since the value of the Planck constant would be exactly defined by this definition, the watt balance apparatus could be viewed as a precise scale. In the present-day mode where h is measured, a precise mass is employed to determine the mechanical energy expenditure. However, if h were exact, than it would be the mass that is being measured instead. Of course, the utility of such a definition depends on the accuracy to which the watt balance apparatus can be developed. If it approaches the current long-term stability of the international prototype of the kilogram, then it becomes an attractive means of defining the kilogram and measuring mass.

9 Rydberg Constant

The Rydberg constant is determined primarily by comparison of theory and experiment for energy levels in hydrogen and deuterium. For example, the observational equation corresponding to the the $1{\rm S}-2{\rm S}$ transition frequency of hydrogen is given approximately by the expression

$$\nu_{\rm H}(1{\rm S}_{1/2}-2{\rm S}_{1/2}) \tag{35}$$

$$\doteq \frac{3}{4}R_\infty c\left[1-\frac{m_{\rm e}}{m_{\rm p}}+\frac{11}{48}\,\alpha^2-\frac{28}{9}\frac{\alpha^3}{\pi}\ln\alpha^{-2}-\frac{14}{9}\left(\frac{\alpha R_{\rm p}}{\lambda\!\!\!^{-}_{\rm C}}\right)^2+\cdots\right],$$

where the input datum for this transition is [20]

$$\nu_{\rm H}(1{\rm S}_{1/2} - 2{\rm S}_{1/2}) = 2\,466\,061\,413\,187.34(84)\ {\rm kHz}\ , \tag{36}$$

which has a relative uncertainty of 3.4×10^{-13}. The expression on the right-hand side of Eq. (35) is approximate and only indicates the leading term of each of several contributions. In particular, the four terms beyond the "1" correspond to contributions from reduced mass, relativistic, radiative, and finite nuclear size effects, respectively. However, it is evident that this expression gives information on the value of the Rydberg constant R_∞.

In the 1998 adjustment, 23 transition frequencies or frequency differences in hydrogen or deuterium were included. The theory was carefully reviewed and the expressions for the energy levels employed in the adjustment were based on many more terms and precise numerical evaluations than are indicated in Eq. (35). The result for the 1998 recommended value for the Rydberg constant is

$$R_\infty = 10\,973\,731.568\,549(83)\ {\rm m}^{-1}\ , \tag{37}$$

where the relative uncertainty is 7.6×10^{-12}. This uncertainty is about 1/160 times the uncertainty of the 1986 recommended value. In addition, the 1998 recommended value for R_∞ differs from the 1986 recommended value by about 2.7 times the uncertainty of the latter value, due to an incorrect experimental result that strongly influenced the 1986 value.

10 Future Adjustments

The availability of the World Wide Web as a resource for making results immediately available to the public raises the possibility of issuing recommended values of the constants more frequently than the 13 year periods that preceded each of the last two adjustments. The present thinking is to recommend new values of the constants every four years, and possibly after two years if developments in theory or experiment warrant it. With this in mind, the computation of the 1998 constants has been automated, so that new information can be incorporated and new values produced with virtually no delay.

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