Nonlinear Differential Equations,
Electron. J. Diff. Eqns., Conf. 05, 2000, pp. 237–??
http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu or ejde.math.unt.edu (login: ftp)

# On critical points for noncoercive functionals and subharmonic solutions of some Hamiltonian systems *

Klaus Schmitt & Zhi-Qiang Wang

**Abstract**

The paper is concerned with existence results about subharmonic solutions of some Hamiltonian systems. The existence of such solutions is established using a variational approach and results about minima of noncoercive functionals.

## 1 Introduction

In this paper we use some critical point theorems for noncoercive functionals (see [4], [5], [6]) (and also [2]) to deduce the existence of periodic solutions of some second order Hamiltonian systems and similar problems for semilinear elliptic partial differential equations. Extensions of these results to quasilinear differential equations are also indicated. The results will be used to obtain existence results for subharmonic solutions of such problems. The nonlinear terms involved have superquadratic growth. Thus we obtain existence results for subharmonic solutions complementing those in [1] and [3], though the perturbation terms considered here are different. For results concerning subharmonic solutions of equations with subquadratic perturbation terms, we refer to [8].

### The setting

Let $E$ be a reflexive Banach space with norm norm $\|\cdot\|$, and pairing $\langle\cdot,\cdot\rangle$. Let

$$A : E \to E^*$$

be a mapping such that the functional $\varphi : E \to \mathbb{R}$ given by

$$u \mapsto \langle Au, u\rangle$$

is weakly lower semicontinuous. Let us denote by

$$\psi : E \to \mathbb{R}$$

---

**Mathematics Subject Classifications:* 35J15, 37J45, 34C15.

*Key words:* Noncoercive functionals, Hamiltonian systems, subharmonic solutions.



Published October 25, 2000.

a weakly continuous mapping whose level surfaces will be denoted by

$$S = \{u \in E : \psi(u) = \gamma \in \mathbb{R}\}.$$

Suppose that $\varphi$ is nonnegative and positive homogeneous of degree $p > 1$. Further assume that

$$\ker \varphi = \{u : \varphi(u) = 0\}$$

is a finite dimensional subspace of $E$ such that

$$\varphi(u + v) = \varphi(v),\ \forall v \in E,\ \forall u \in \ker \varphi.$$

We also assume that $E = \ker \varphi \oplus X$, where $X$ is a closed subspace of $E$ and $\varphi|_X$ is coercive in the sense that there exists $c > 0$ such that

$$\varphi(v) \geq c\|v\|^p,\ \forall v \in X\,.$$

Concerning the functional $\psi$ we also require that it be positive homogeneous of degree $\alpha > 1$.

We then have the following theorem. The theorem is established in [4], [5]. The first part follows from properties of noncoercive functionals which are coercive on some subsets, satisfying suitable properties, and the second part from Liusternik's theorem on Lagrange multipliers coupled with scaling arguments.

**Theorem 1** *(a) Assume the above and also that*

$$\begin{aligned} \varphi(v-u) &\leq \varphi(v) \\ \psi(v-u) &= \psi(v),\ \forall v \in E,\ \forall u \in \ker\varphi \cap \ker\psi. \end{aligned}$$

*Then the minimization problem*

$$\varphi(u) = \min_{v \in S} \varphi(v)$$

*has a solution $u \in S$.*

*(b) If $\varphi,\ \psi$ belong to class $C^1$, $\alpha \neq p$, and if*

*(i) $\psi(u) < 0$, for some $u \in E$*

*(ii) $\psi(u) \geq 0, \quad \forall u \in \ker \varphi$*

*(iii) if $u \in \ker \varphi$ is such that $\psi(u) = 0$, then*

$$\begin{aligned} \varphi(v-u) &\leq \varphi(v) \\ \psi(v-u) &= \psi(v),\ \forall v \in E\,. \end{aligned}$$

*Then the functional $f = \varphi + \psi$ has a nontrivial critical point.*

## 2 Periodic solutions of Hamiltonian systems

Let us consider the following system of second order ordinary differential equations defined by a function

$$G : (0, T) \times \mathbb{R}^N \to \mathbb{R}^N$$

$$-u''(t) + \nabla_u G(t, u) = 0, \ 0 < t < T, \tag{1}$$

subject to the periodic boundary conditions

$$u(0) = u(T), \ u'(0) = u'(T). \tag{2}$$

With appropriate conditions imposed on $G$ solutions of (1), (2) are critical points of the functional

$$f(u) = \frac{1}{2}\int_0^T |u'|^2 \, dt + \int_0^T G(t, u) dt \tag{3}$$

on the space

$$\begin{array}{rcl} E & = & W_T^{1,2}\left((0,T), \mathbb{R}^N\right) \\ & = & \{u|_{[0,T]} : u \in W_{loc}^{1,2}\left(\mathbb{R}, \mathbb{R}^N\right), \ u(t+T) = u(t), \ t \in \mathbb{R}\}, \end{array} \tag{4}$$

and conversely. (See e.g. [7].)

In order to apply the above critical point theorem, we let the functionals $\varphi$ and $\psi$ be defined by

$$\varphi(u) = \frac{1}{2}\int_0^T |u'|^2 \, dt$$

and

$$\psi(u) = \int_0^T G(t, u) dt.$$

We also impose the following requirement upon $G$ :

$$\begin{gathered} G(t, \lambda u) = \lambda^\alpha G(t, u), \ \lambda \geq 0, \ \alpha > 2, \ u \in \mathbb{R}^N \\ G(t, \gamma) > 0, \ \forall \gamma \in \mathbb{R}^N, \ \gamma \neq 0 \\ \exists u \in E \text{ such that } \psi(u) < 0\,. \end{gathered}$$

These requirements allow us to apply the critical point theorem of the previous section and we conclude the existence of a nontrivial solution $u$ of (1), (2). In particular one minimizes the functional $\varphi$ on sets $S$ given by

$$S = \{u \in E : \int_0^T G(t, u) dt = c, \}$$

where $c$ is a negative constant.

## Subharmonic solutions

We note that whenever the function $G : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is such that

$$G(t+T, u) = G(t, u),\ t \in \mathbb{R},\ u \in \mathbb{R}^N$$

then along with the problem (1), (2) we may also consider, for any natural number $k \geq 1$, the equations

$$-u''(t) + \nabla_u G(t, u) = 0, \tag{5}$$

subject to the periodic boundary conditions

$$u(0) = u(kT),\ u'(0) = u'(kT) \tag{6}$$

whose solutions will be critical points of the functional

$$f_k(u) = \frac{1}{2}\int_0^{kT} |u'|^2\, dt + \int_0^{kT} G(t, u) dt \tag{7}$$

on the space

$$E_k = W^{1,2}_{kT}\left((0, kT), \mathbb{R}^N\right). \tag{8}$$

We recall how the existence of a critical point for the functional $f_k$ was obtained. We write

$$f_k = \varphi_k + \psi_k,$$

where the functionals $\varphi_k$ and $\psi_k$ are defined by

$$\varphi_k(u) = \frac{1}{2}\int_0^{kT} |u'|^2\, dt$$

and

$$\psi_k(u) = \int_0^{kT} G(t, u) dt$$

and have that the set

$$S_k = \{u : \psi_k(u) = -1\} \neq \emptyset$$

with the Fréchet derivative $\psi_k' \neq 0$ on $S_k$. (Note that the manifold $S_k$ could equally well have been chosen as

$$S_k = \{u : \psi_k(u) = -c\} \neq \emptyset,$$

where $c$ is any positive number. Since this manifold is weakly closed and the functionals $\varphi_k$ and $\psi_k$ satisfy the conditions of Theorem 1 we have that the minimization problem

$$\min_{v \in S_k} \varphi_k(v) = \varphi_k(u_k) := m_k \tag{9}$$

has a solution $u_k$, and hence there exists a Lagrange multiplier $\mu_k$ such that

$$\varphi_k'(u_k) + \mu_k \psi_k'(u_k) = 0, \tag{10}$$

which by the homogeneity of the functionals implies that

$$2\varphi_k(u_k) - \alpha\mu_k = 0,$$

i.e.

$$\mu_k = \frac{2}{\alpha} m_k := n_k,$$

which implies that $\mu_k$ is positive and hence we may let

$$v_k = n_k^{1/(\alpha-2)} u_k,$$

and obtain that $v_k$ is a nontrivial critical point of $f_k$.

The question now arises whether, for $k > 1$, it is possible that $v_k$ is a critical point of $f_p$, $p < k$, or phrased differently: whether $v_k$, which is a $kT$ periodic solution of (5), can already be a $pT$ periodic solution of the same system. Note that, of course, the periods of $v_k$ and $u_k$ are the same.

We shall give an answer to this question in a more specific case; namely we shall assume that

$$G(t, u) = \frac{1}{\alpha} g(t)|u|^\alpha,$$

where $g : \mathbb{R} \to \mathbb{R}$ is a $T$ periodic function which is essentially bounded and has the property that

$$\text{meas}\{t : g(t) < 0\} \neq 0,$$

and

$$\int_0^T g > 0,$$

where meas$\{A\}$ denotes the Lebesgue measure of a set $A$.

We hence have the problems

$$-u''(t) + g(t)|u|^{\alpha-2}u = 0, \tag{11}$$

subject to the same periodic boundary conditions.

Let us choose a nonzero function $\eta \in C([0,T], \mathbb{R}^N)$ such that $\eta(0) = 0 = \eta(T)$ whose support supp$\eta \subset A$ with $A = \{t : g(t) < 0\}$. It then follows that $\eta \in E_k$, for any $k \geq 1$ and $\psi(\eta) < 0$. Hence

$$\tilde{\eta} = \frac{\eta}{\left(-\frac{1}{\alpha}\int_0^{kT} g(t)|\eta|^\alpha\right)^{\frac{1}{\alpha}}} \in S_k$$

and therefore

$$m_k \leq \frac{1}{2}\int_0^T |\tilde{\eta}'|^2 := c.$$

We therefore have the lemma.

**Lemma 1** *There exists a constant $c > 0$, independent of $k$, such that*

$$m_k \leq c,$$

*where $m_k$ is defined by (9).*

We recall that

$$\frac{1}{\alpha}\int_0^{kT} g(t)|u_k|^\alpha = -1. \tag{12}$$

Hence, since $u_k$ is not constant, if the minimal period of $u_k$ is not $kT$ it must equal $pT$, where $\theta = \frac{k}{p}$ is a positive integer. It therefore follows that

$$\frac{1}{\alpha}\theta\int_0^{pT} g(t)|u_k|^\alpha = -1 \tag{13}$$

and

$$\frac{1}{\alpha}\int_0^{pT} g(t)|\theta^{\frac{1}{\alpha}}u_k|^\alpha = -1, \tag{14}$$

i.e. $w_p := \theta^{\frac{1}{\alpha}}u_k \in S_p$. We therefore obtain the following:

$$\begin{aligned} m_p \leq \varphi_p(w_p) &= \frac{1}{2}\int_0^{pT} |w_p'|^2 \\ &= \frac{1}{2}\theta^{2/\alpha}\int_0^{pT} |u_k'|^2 \\ &= \frac{1}{2}\theta^{2/\alpha}\theta^{-1}\int_0^{kT} |u_k'|^2 \\ &= \theta^{(2-\alpha)/\alpha}m_k\,, \end{aligned} \tag{15}$$

or

$$\left(\frac{k}{p}\right)^{\alpha/(\alpha-2)} m_p \leq m_k \leq c\,.$$

The above formula lets us deduce the following theorem. The proof is a simple indirect argument using this formula.

**Theorem 2** *Let $\{k_i\}$ be an unbounded increasing sequence of positive integers. Then the minimal periods of $\{u_{k_i}\}$ tend to infinity. In particular for all primes $k$, sufficiently large, $u_k$ has minimal period $kT$.*

# 3 Periodic solutions of elliptic problems

In this section we briefly point out how the results on Hamiltonian systems may be extended to elliptic problems. We shall discuss this in the case of dimension 2, it will be clear that similar results may be obtained (with suitable restrictions on the power $\alpha$) for systems in more independent variables.

## Periodic and subharmonic solutions

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that there exists $X = (X_1, X_2),\ X_i > 0,\ i = 1, 2$ such that

$$g(x + X) = g(x),\ x \in \mathbb{R}^2. \tag{16}$$

(We say $g$ is periodic with period $X$.) Further assume that $g$ is essentially bounded and $X_1, X_2$ are the least positive numbers such that (16) holds. We let

$$\mathbb{P} = \{x = (x_1, x_2) : 0 \le x_1 \le X_1, 0 \le x_2 \le X_2\}$$

and assume $g$ has the property that

$$\text{meas}\{x \in \mathbb{P} : g(x) < 0\} \neq 0\,,$$

and $\int_{\mathbb{P}} g > 0$.

We now consider the problem (again $\alpha > 2$)

$$-\Delta u + g(x)|u|^{\alpha-2}u = 0,\ x \in \mathbb{R}^2 \tag{17}$$

subject to the periodic boundary condition

$$u(x + X) = u(x),\ x \in \mathbb{R}^2. \tag{18}$$

We set

$$X_{1,1} = X,\ X_{k,l} = (kX_1, lX_2),$$

where $k, l$ are positive integers. We, of course, then have

$$g(x + X_{k,l}) = g(x),\ x \in \mathbb{R}^2$$

and hence, in analogy with the previous section, we also consider equation (17) subject to the (subharmonic) constraints

$$u(x + X_{k,l}) = u(x),\ x \in \mathbb{R}^2. \tag{19}$$

The appropriate Sobolev spaces and functionals will be

$$E_{k,l} = \{u \in W^{1,2}_{loc}\left(\mathbb{R}^2, \mathbb{R}^N\right) : u(x + X_{k,l}) = u(x),\ x \in \mathbb{R}^2\}, \tag{20}$$

$$f_{k,l} = \varphi_{k,l} + \psi_{k,l},$$

where the functionals $\varphi_{k,l}$ and $\psi_{k,l}$ are defined by

$$\varphi_{k,l}(u) = \frac{1}{2}\int_{\mathbb{P}_{k,l}} |\nabla u|^2\,dx$$

and

$$\psi_{k,l}(u) = \int_{\mathbb{P}_{k,l}} g(x)|u|^{\alpha}\,dx$$

and have that the set

$$S_{k,l} = \{u : \psi_{k,l}(u) = -1\} \neq \emptyset,$$

with

$$\mathbb{P}_{k,l} = \{x : 0 \leq x \leq X_{k,l}\},$$

employing the usual partial ordering of $\mathbb{R}^2$. We proceed as in the previous section and find that the functional $\varphi_{k,l}$ will assume its minimum on the manifold $S_{k,l}$ and then one finds a Lagrange multiplier and after rescaling a critical point for the functional $f_{k,l}$. Such critical points, on the other hand will be solutions $u_{k,l}$ of (17) subject to the constraint (19). Following the calculations of the previous section one may now prove the following theorem.

**Theorem 3** *Let $\{(k,l)\}$ be an unbounded increasing sequence of tuples of positive integers. Then the minimal periods of $\{u_{k,l}\}$ tend to infinity. In particular for all tuples of primes $k,l$ with $k+l$, sufficiently large, $u_{k,l}$ has minimal period $(kX_1, lX_2)$.*

# 4 Quasilinear problems

Results similar to the above may be obtained for the quasilinear problem (now $\alpha > p > 1$)

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + g(x)|u|^{\alpha-2}u = 0,\ x \in \mathbb{R}^n,\ n \geq 1, \tag{21}$$

subject to the periodic boundary condition

$$u(x+X) = u(x),\ x \in \mathbb{R}^n,\ X \in \mathbb{R}^n.$$

The appropriate Sobolev spaces and functionals will be

$$E = \{u \in W^{1,p}_{loc}\left(\mathbb{R}^n, \mathbb{R}^N\right) : u(x+X) = u(x),\ x \in \mathbb{R}^n\},$$
$$f = \varphi + \psi\,,$$

where the functionals $\varphi$ are defined by

$$\varphi(u) = \frac{1}{p}\int_{\mathbb{P}} |\nabla u|^p \, dx\,,$$

and $\psi$ as above.

# References

[1] V. Benci and D. Fortunato, *A Birkhoff-Lewis type result for non autonomous differential equations.* pp. 85–96, in Partial Differential Equations, Cardoso, Defigueiredo, Iório, Lopes (editors), Springer Lecture Notes in Mathh., vol. 1324, Berlin, New York, 1988.

[2] A. BenNaoum, C. Troestler, and M. Willem, *Existence and multiplicity results for homogeneous second order differential equations.* J. Differential Equations, **112**(1994), 239–249.

[3] P. Felmer and E. de B. e Silva, *Subharmonics near an equilibrium for some second order Hamiltonian systems.* Proc. Roy. Soc. Edinburgh, **123A**(1993), 819–834.

[4] V. Le and K. Schmitt, *Minimization problems for noncoercive functionals subject to constraints.* Trans. Amer. Math. Soc., **347**(1995), 4485–4513.

[5] V. Le and K. Schmitt, *Minimization problems for noncoercive functionals subject to constraints II.* Advances Differential Equations, **1**(1996), 453–498.

[6] V. Le and K. Schmitt, *On minimizing noncoercive functionals on weakly closed sets.* Topology in Nonlinear Analysis, Banach Center Publications, **35**(1996), 51–72.

[7] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems.* Springer Verlag, New York, 1989.

[8] Q. Wang, Z. Wang, and J. Shi. *Subharmonic oscillations with prescribed minimal period for a class of Hamiltonian systems.* Nonl. Anal., TMA, **28**(1996), 1273–1282.

Klaus Schmitt
Department of Mathematics, University of Utah
155 South 1400 East
Salt Lake City, UT 84112, USA
e-mail: schmitt@math.utah.edu

Zhi-Qiang Wang
Department of Mathematics and Statistics
Utah State University
Logan, UT 84322, USA
e-mail: wang@math.usu.edu