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An estimate on the relative Morse index for strongly indefinite functionals *

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Abstract

We extend the Benci and Rabinowitz linking theorem to strongly indefinite functionals satisfying the Palais-Smale condition. More precisely, we show an upper estimate for a relative Morse index of critical points.

1 Introduction

Let E be a real Hilbert space, with scalar product $(\cdot,\cdot)$. On this space. we consider the functional

$$f(u) = \frac{1}{2}(Lu, u) + h(u)\,, \tag{1}$$

where L is an invertible self-adjoint operator and $h \in C^2(E)$ with h' is compact. In particular, we are interested in the strongly indefinite case, that is when both the positive and the negative eigenspaces of L are infinite dimensional. It is well known that critical points of such functionals always have infinite Morse index.

For a closed subspace of $V \subset E$, we denote by P_V the orthogonal projection onto V and by $V^\perp$ the orthogonal complement of V. Following [1] and [2], we say that the closed subspaces V, W of E are commensurable if $P_{V^\perp}P_W$ and $P_{W^\perp}P_V$ are compact operators.

If V and W are commensurable, the *relative dimension of W with respect to V* is defined as

$$\dim_V W = \dim W \cap V^\perp - \dim W^\perp \cap V.$$

Commensurability guarantees that both terms in the above formula are finite. It can be proved that if two self-adjoint Fredholm operators differ by a compact operator, then their negative (resp. positive) eigenspaces are commensurable.

Denote by E^+ and E^- the positive and the negative eigenspaces of L. Since $f''(x) = L + h''(x)$ and $h''(x)$ is compact, the *relative Morse index of a critical*

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point x of functional f with respect to the splitting $E = E^+ \oplus E^-$ can be defined as the integer

$$m_{(E^+,E^-)}(x) = \dim_{E^-}[\,\textit{negative eigenspace of } f''(x)\,].$$

For degenerate critical points, a *relative large Morse index* can be defined as well:

$$m^*_{(E^+,E^-)}(x) = m_{(E^+,E^-)}(x) + \dim\ker f''(x).$$

In [3] it is shown that this definition of the Morse index coincides with a definition given by Chang, Liu and Liu [7] which uses finite dimensional projections. Chang's definition fits very well in the framework of Galerkin reductions. Assuming the (PS^*) condition, Galerkin reductions and the equivalence of the two indices are used in [3] to prove Morse index estimates for critical points coming from Benci and Rabinowitz's generalized Mountain Pass theorem.

Here we give another proof of the upper index estimate without using finite dimensional reductions. The advantage of this approach is that we have only to assume the usual (PS) condition, and not the stronger (PS^*) condition. Here is our result.

Theorem 1.1 *Let $E = W^+ \oplus W^-$ be an orthogonal decomposition of E, where W^+ is commensurable with E^+ and W^- is commensurable with E^-. Let $e \in \partial B_1 \cap W^+$ and set*

$$S = \partial B_\rho \cap W^+, \quad Q = B_s \cap W^- \oplus [0, r]e,$$

where $r > \rho > 0$ and $s > 0$. Denote by ∂Q the boundary of Q in $W^- \oplus \mathbb{R}e$. Assume that there exist numbers $\alpha < \beta$ such that

$$\sup_{\partial Q} f < \alpha < \inf_S f, \quad \sup_Q f < \beta,$$

and that f satisfies the $(PS)_c$ condition for every $c \in [\alpha, \beta]$.

Then f has a critical point x such that $\alpha \le f(x) \le \beta$. Moreover, if $f \in C^2(E)$, the following index estimate hold

$$m_{(E^+,E^-)}(x) \le \dim_{E^-} W^- + 1.$$

This estimate generalizes the usual index estimate of finite dimensional min-max critical points, proved by Lazer and Solimini [9], Solimini [13] and Benci [5].

2 The relative dimension

Let E be a real Hilbert space, with scalar product $(\cdot, \cdot)$. We deal with functionals of the form

$$f(u) = \frac{1}{2}(Lu, u) + h(u),$$

where L is an invertible self-adjoint operator, $h \in C^2(E)$ and h' is compact. Moreover, denote by E^+ and E^- the positive and the negative eigenspaces of L. For a closed subspace $V \subset E$, denote by P_V the orthogonal projection onto V and by $V^\perp$ the orthogonal complement of V.

Definition 2.1 *Two closed subspaces V and W of E are called commensurable if the operator $P_V - P_W$ is compact. In this case the relative dimension of W with respect to V is the integer*

$$\dim_V W := \dim W \cap V^\perp - \dim W^\perp \cap V.$$

The identities

$$P_V - P_W = P_V P_{W^\perp} + P_V P_W - P_W = (P_{W^\perp} P_V)^* - P_{V^\perp} P_W,$$
$$P_{V^\perp} P_W = (P_W - P_V) P_W, \quad P_{W^\perp} P_V = (P_V - P_W) P_V,$$

show that V and W are commensurable if and only if both $P_{W^\perp} P_V$ and $P_{V^\perp} P_W$ are compact. Hence the spaces $W \cap V^\perp$ and $W^\perp \cap V$ in the above definition are finite dimensional because they are the spaces of fixed points of the compact operators $P_{V^\perp} P_W$ and $P_{W^\perp} P_V$, respectively.

Easy examples of pairs of commensurable subspaces are built by adding or removing a finite number of dimensions: $V \oplus W_r$ is commensurable with V if W_r has dimension r, and $\dim_V (V \oplus W_r) = r$. If W^s is a s-codimensional subspace of V, then it is commensurable with V and $\dim_V W^s = -s$. Clearly the definition of commensurability is more general than this: for example the intersection of two commensurable spaces could be $\{0\}$. Here are some useful facts which follow directly from the definitions. For V, W, Z commensurable subspaces,

(i) $\dim_W V = -\dim_V W$;

(ii) $\dim_Z V = \dim_W V + \dim_Z W$;

(iii) $V^\perp$ and $W^\perp$ are commensurable and $\dim_{V^\perp} W^\perp = -\dim_V W$.

Formula (ii) with $Z = \{0\}$ implies that $\dim_W V = \dim V - \dim W$ when V and W are finite dimensional.

Remark 2.1 *Commensurability is an equivalence relation. The notion of commensurability is stable with respect to operators of the form identity + compact*

The proof of the following result it is found in [3] [Theorem 2.6]

Proposition 2.1 *Let T be a self-adjoint Fredholm operator. If K is a self-adjoint compact operator, the positive (negative) eigenspaces of T and $T + K$ are commensurable.*

Vice versa, if T is an invertible self-adjoint operator with positive (negative) eigenspace E^+ (E^-) and $E = W^+ \oplus W^-$, where W^+ (W^-) is commensurable with E^+ (E^-), then there exists an invertible self-adjoint operator M whose positive (negative) eigenspace is W^+ (W^-) and such that $M - T$ is compact.

From the above proposition we get the following

Remark 2.2 *Let X, Y two closed commensurables subspaces of the Hilbert space H. If $\dim_Y X > 0$ then there exists $T : H \to H$ with $T = I + K$ invertible such that $T(X) = Y \oplus Y_r$ with $Y_r \neq 0$ and $\dim Y_r < \infty$.*

Definition 2.2 *Let x be a critical point of the functional f. Assume that f is twice differentiable in x. Let $E = V^+ \oplus V^- \oplus V^0$ be the decomposition of E into the positive, the negative and the null eigenspaces of $f''(x) = L + K$, where K is a compact self-adjoint operator. The relative Morse index of x with respect to the splitting $E^+ \oplus E^-$ of H is the integer*

$$m_{(E^+,E^-)}(x) = m_{(E^+,E^-)}(x; f) = \dim_{E^-} V^-.$$

The following results are from [3], Theorem 1.5 and Proposition 1.3 respectively.

Proposition 2.2 *Let (K_n) be a sequence of self-adjoint compact operators which converges to K in the operators' norm. Moreover, let (f_n) be a C^2 sequence in x and suppose that $f_n''(x) = L + K_n$ and $f'' = L + K$. Then*

$$m_{(E^+,E^-)}(x; f) \leq \liminf_{n\to\infty} m_{(E^+,E^-)}(x; f_n). \tag{2}$$

3 The non degenerate case

In this section we will see that in order to prove Theorem 1.1, is suffices to show only the case of non degenerate critical points.

Theorem 3.1 *Under the hypothesis of Theorem 1.1 and assuming that all critical points at level c are non degenerate there exits a critical point u_0 such that $f(u_0) = c$ and $m_{(W^+,W^-)}(u_0) \leq 1$.*

Definition 3.1 *Let $U(x)$ be an open neighborhood of $x \in E$. We say that $f \in C^2(U(x))$ is a Morse function on $U(x)$, if the following condition holds.*

(i) $f(x)$ satisfies (PS) on $U(x)$.

(ii) f has only non degenerate critical points on $U(x)$.

Remark 3.1 *Let x be a degenerate critical point of $f \in C^2(U(x))$. Assume that, f satisfies the (PS) condition on $U(x)$ and $f''(x) = T + K$ where T is an invertible self-adjoint operator and K is self-adjoint compact operator.*

Then from the celebrated Marino-Prodi's Theorem, there exists a Morse sequence (f_n) on $U(x)$ such that

$$f_n = f \text{ on } X - B(x, \tfrac{1}{n}) \text{ for all } n$$
$$f_n \to f \text{ in } C^2(U(x)).$$

Remark 3.2 *Observe that assuming Theorem 3.1, the prove of the Theorem 1.1 is easy.*

In fact, let u_0 be a degenerate critical point of f. So, there exists a sequence $\{u_n\}$ such that $u_n \to u_0$ and $f'(u_n) = 0$. On the other hand, it is easy to see that f satisfies all assumptions of Remark 3.1. So, let (f_n) be a strong Morse sequence such that

$$f_n = f \text{ on } X - B(u_0, \tfrac{1}{n^2}) \text{ for all } n$$
$$f_n \to f \text{ in } C^2(U(u_0)).$$

and hence we can assume $f'_n(u_n) = 0$. Moreover note that for all n big, f_n satisfies the hypothesis of Theorem 1.1, and therefore we can assume that u_n is a min-max for f_n with $f_n(u_n) =: c_n$. Observe that the u_n are non degenerate critical points of f_n, so from Theorem 3.1 we obtain that

$$m_{(W^+,W^-)}(u_n, f_n) \leq 1$$

for all n big. Now, by Proposition 2.2 it is follows that

$$m_{(W^+,W^-)}(u_0, f_n) \leq 1$$

and hence, finally, we obtain that $m_{(W^+,W^-)}(u_0, f) \leq 1$.

4 Proof of the Theorem 1.1

From Proposition 2.1 changing L by another invertible self-adjoint operator, and hence changing h by another map with compact gradient, we can assume in Theorem 1.1 that $W^+ = E^+$ and $W^- = E^-$.

Let $\Gamma = \{g \in C([0,1] \times E, E) : g \quad satisfies \quad \Gamma_1, \Gamma_2\}$ where

(Γ_1) $g(u) = u$, for $u \in \partial Q$;

(Γ_2) $\gamma(u) = s(u)$
$exp\{\theta(u)L\}u + K(u)$ where $\theta : E \to \mathbb{R}$ is a Lipschitz function, $K : E \to E$ is a compact and Lipschitz function and $s : E \to]0,1]$ is a Lipschitz function with $s(E) > s_0 > 0$;

As in Theorem 5.3 of [11], is easy to prove that

$$c = \inf_{g \in \Gamma} \sup_{u \in Q} f(g(u))$$

is a critical value of f.

Because of the term of $s(u)$ in (Γ_2) we have to prove that $c \geq \alpha$. To do that, is suffices to show that for all $g \in \Gamma$, $g(Q) \cap S_\rho \neq \emptyset$.

For $\lambda \in \mathbb{R}$ and $u \in E^-$, we define:

$$\Psi(u) = \Psi(\lambda e + u^-) = \|P_{E^+}\gamma(u)\|e + \frac{1}{s(u)} e^{-\theta(u)L} P_{E^-}\gamma(u).$$

From the fact that $e^{\theta L}$ commutes with P_{E^-}, we get that

$$\Psi(u) = \|P_{E^+}\gamma(\lambda e + u^-)\|e + u^- + \frac{1}{s(\lambda e + u^-)} e^{-\theta(\lambda e + u^-)L} P_{E^-} K(\lambda e + u^-).$$

On the other hand, using the fact that $\frac{1}{|s|} \leq \frac{1}{s_0} < +\infty$, θ is bounded and K is a compact map, we obtain that Ψ is a compact perturbation of the identity.

Moreover, if $u \in \partial Q$, by (Γ_1) and (Γ_2) we get that $\Psi(u) = u$. Hence, $\Psi(u) \neq \rho e$ if $u \in \partial Q$ and

$$deg(\Psi, Q, \rho e) = deg(id, Q, \rho e) = 1.$$

Therefore there exists $\bar{u} \in Q$ such that $\Psi(\bar{u}) = \rho e$, that is, $\gamma(\bar{u}) \in S_\rho$. The remaining part of the proof it follows from Theorem 3.1 and Remark 3.2.

5 Results used for proving Theorem 3.1

Let u_0 be a non degenerate critical point of f with $f(u_0) = c$ and assume that $E = V^- \oplus V^+$ where V^- (respectively V^+) is the negative eigenspace (positive eigenspace) of $f''(u_0)$. Also we write P_- (respectively P_+) the projection of E on V^- (V^+). Moreover, if $u \in E$, by u_- we means $P_-(u)$ (respectively for u_+).

Let $B_- = B(0, r_1) \cap V_-$ and $B_+ = B(0, r_2) \cap V^+$. For $r_1 > 0$ and $r_2 > 0$ small enough, by Morse Lemma, there exists a local C^1-isomorphism $\Psi : 2B_- \oplus B_+ \to U$ where $\Psi(0) = u_0$ and $U = \Psi(2B_- \oplus B_+)$, such that

$$f(\Psi(z_+ + z_-)) = c + ||z_+||^2 - ||z_-||^2 \tag{3}$$

Now for r_1 and r_2 such that $r_2^2 - 4r_1^2 > 0$, we define $\Phi : E \to E$ as follows

$$\Phi(u) = \begin{cases} u & \text{if } u \notin U \\ \Psi(\varphi(\frac{||\Psi^{-1}(u)_-||}{r_1} - 1)\Psi^{-1}(u)_+ + \Psi^{-1}(u)_-) & \text{if } u \in U \end{cases}$$

where $\varphi : \mathbb{R} \to [0, 1]$ is a Lipschitz function defined by

$$\varphi(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ 1 & \text{if } s \geq 1 \end{cases}$$

Note that Φ is continuous on $H - \Psi(2B_- \oplus \partial B_+)$; and Φ is Lipschitz on $f^{-1}(]-\infty, c + \alpha_1])$ with α_1 as in Lemma 5.1 below.

The following Lemma is similar to Lemma 3.1 in [9], except for (iv).

Lemma 5.1 *Let $g \in \Gamma$, $0 < \alpha_1 < r_2^2 - 4r_1^2$, $c = f(u_0)$ and $U_1 = \Psi(B_- \oplus B_+)$*

(i) *If $u \in U - \text{int}(U_1)$ then $f(\Phi(u)) \leq f(u)$;*

(ii) *If $u \in \partial U_1$ and $f(u) \leq c + \alpha_1$ then $\Phi(u) = \Psi(\{\Psi^{-1}(u)\}_-)$ and $\Phi(u) \in \Psi(\partial B_-)$;*

(iii) $\Phi(H - \text{int}(U_1)) \subset H - \text{int}(U_1)$;

(iv) *Suppose that $f(g(Q)) < c + \alpha_1$, and $f(\partial Q) \leq \alpha < c$. Moreover assume that $A \neq \emptyset$ where $A = g^{-1}(\text{int}(U_1)) \cap Q$.*

Then $\Phi(g(\partial A)) \in \Psi(\partial B_-(0, r_1))$.

Proof.

(i) It is follows easily from (3).

(ii) There result that $\Psi^{-1}(u) \in (\partial B_- \oplus B_+) \cup (B_- \oplus \partial B_+)$. But, $\Psi^{-1}(u) \in B_- \oplus \partial B_+$ it is not possible otherwise

$$f(u) = c + ||\Psi^{-1}(u)_+||^2 - ||\Psi^{-1}(u)_-||^2 > c + \alpha$$

Thus, $\Psi^{-1}(u) \in \partial B_- \oplus B_+$ and therefore $\varphi(\frac{||\Psi^{-1}(u)_-||}{r_1} - 1) = 0$ and $\Phi(u) = \Psi[\Psi^{-1}(u)_-] \in \Psi(\partial B_-)$.

(iii) It is follows from definition.

(iv) It is easy to see that $g(u) \in \partial U_1$ when $u \in \partial A$, and

$$f(g(\partial A)) \leq c + \alpha_1$$

On the other hand, we can assume that $g^{-1}(\mathrm{int}(U_1)) \cap \partial Q = \emptyset$.

Otherwise, if z is such that $\Psi(z) \in \partial Q \cap (\mathrm{int}(U_1))$, then $f(\Psi(z)) \leq \alpha_1$. But the last is not possible, if we take r_1 in the definition of Ψ, such that $c - \alpha_1 > r_1^2$.

The two following Lemmas will be important in the next section; the first one is known as Mac Shane Lemma and its proof can be found in [8]. The proof of the second one can be found in [10].

Lemma 5.2 *Let X be a metric space, $Y \subset X$ and $\alpha > 0$. If $f : Y \to \mathbb{R}$ is a Lipschitz function with constant α, then f can be extended to X as a Lipschitz function with the same constant α.*

Lemma 5.3 *Let H be a Hilbert space and let $f : H \to \mathbb{R}$ be a uniformly continuous function. Then f can be approximate uniformly by C^1 functions.*

We also need the following result

Lemma 5.4 *Let E and F two separable Banach spaces. Moreover let h be a C^1 Fredholm map of index 0 of E into F, such that $w \notin h(E)$ for some $w \in F$. Then h is not locally surjective.*

Proof. Let us assume by contradiction that h is locally surjective. Then for $y_1 = h(x_1)$ we can take y near to y_1 such that there exists $x \in h^{-1}(y)$. Hence, from Sard-Smale Theorem, $h'(x)$ is surjective and we get that h is locally invertible in x. But this would imply that h is globally invertible, which contradicts the fact that $w \notin h(E)$.

6 Proof of Theorem 3.1

We observe that there only finitely many critical points in K_c since f satisfies (PS) and all critical points are non degenerate.

Remember that, cf [11], given $\epsilon > 0$ and W any neighborhood of the set of critical points at level c, there exist $\epsilon_1 \in]0, \epsilon[$ and $\eta : [0,1] \times E \to E$ continuous such that

(*i*) $\eta(0, u) = u$ for all u.

(*ii*) $\eta(t, u) = u$ for all $t \in [0,1]$ if $f(u) \notin [c - \epsilon_1, c + \epsilon_1])$.

(*iii*) $\eta(1, f^{c+\epsilon_1} - W) \subset f^{c-\epsilon_1}$

We first treat the case when K_c is the singleton $\{u_0\}$. The general case is obtained by induction.

Let us assume that $\dim_{W_1} V^- > 0$ (here $W_1 = W^- \oplus \{\lambda e : \lambda \in \mathbb{R}\}$). It is enough to prove the existence of an open set $\tilde{U}$ containing u_0 so that for all $\epsilon > 0$ small enough there exists $\gamma \in \Gamma$ (see Section 4 for definition of Γ) such that

$$(F) \qquad \sup_{u \in Q} f(\gamma(u)) < c + \epsilon \quad \text{and} \quad \gamma(Q) \cap \tilde{U} = \emptyset$$

This together with a deformation argument lead to a contradiction:

In fact, because $c \geq \alpha$ and $\sup_{\partial Q} f < \alpha$, there exists ϵ_1, with $0 < \epsilon_1 < \epsilon$ such that

$$\max_{u \in \partial Q} f(u) < c - 2\epsilon_1$$

Let $g \in \Gamma$ such that g satisfies (F). If we define $g_1(u) = \eta(1, g(u))$, we have that for $u \in \partial Q$, $g_1(u) = \eta(1, g(u)) = g(u) = u$, so $g_1 \in \Gamma$ (see Theorem 5.29 of [11]).

Now, from (F) we get that $g_1(Q) \subset f^{-1}(] - \infty, c - \epsilon])$ and hence $\sup_{u \in Q} f(g_1(u)) < c - \epsilon < c$ which is a contradiction.

Construction of γ: Let $g \in \Gamma$ such that $\sup_{u \in Q} f(g(u)) < c + \epsilon_1$ where $\epsilon_1 < \alpha_1$ with α_1 as in Lemma 5.1. If we write $A = g^{-1}(\mathrm{int}(U_1)) \cap Q$, we can assume that $A \neq \emptyset$, since otherwise $g(Q) \cap U_1 = \emptyset$ and we get the condition (F). Moreover, as in the proof of Lemma 5.1 (*iv*), we can assume that $g^{-1}(\mathrm{int}(U_1)) \cap \partial Q = \emptyset$.

From Lemma 5.1, $\Phi(g(\partial A)) \in \Psi(\partial B_-(0, r_1))$. Actually, for $u \in \partial A$ there result that $\Phi(g(u)) = \Psi[\Psi^{-1}(g(u))_-]$.

On the other hand, it is easy to see that (cf. [12])

$$\Psi = (I - K)|f''(u_0)|^{-\frac{1}{2}} \qquad \Psi^{-1} = |f''(u_0)|^{\frac{1}{2}}(I + K_1)$$

where K and K_1 are compact.

From now on K_i always will denote a compact map. We claim that $(\Psi^{-1} \circ g)|_{\partial A}$ has a smooth extension on $\mathrm{cl}(A)$. In fact for $u \in \partial A$

$$\begin{aligned} \Psi^{-1}(g(u)) &= |f''(u_0)|^{\frac{1}{2}}(I + K_1)(g(u)) && (4) \\ &= |f''(u_0)|^{\frac{1}{2}} s(u) \exp\{\Theta(u)L\}u + K(u). && (5) \end{aligned}$$

For all $\epsilon > 0$ small enough there exists a compact map K_ϵ with finite rank defined on ∂A such that $||K_\epsilon(u) - K(u)|| < \epsilon$. By using Lemma 5.2 and 5.3 there exists an smooth extension K_4 of K_ϵ. In the same way, let θ_1 be an extension of class $C^1(\mathrm{cl}(A))$ of $\theta_{|\partial A}$. Now, the extension to $\mathrm{cl}(A)$ of $(\Psi^{-1} \circ g)|_{\partial A}$ is $\Psi_1(u) = |f''(u_0)|^{\frac{1}{2}} s(u) \exp\{\Theta_1(u)L\}u + K_4(u)$.

Because,

$$\exp\{\Theta_1(u)L\} - \exp\{\Theta_1(u)[L + h''(u_0)]\}eqno(C)$$

is compact, we get that $\Psi_1(u) = |f''(u_0)|^{\frac{1}{2}} s(u) \exp\{\Theta(u)[L + h''(u_0)]\}u + K_5$

On the other hand, from $\dim_{W_1} V^- > 0$ and using Remark 2.2, we get for $u \in \mathrm{cl}(A)$, that $P_-(u) = u - K_A(u)$ where K_A is compact. Hence,

$$P_-(\Psi_1)(u) = s(u)|f''(u_0)|^{\frac{1}{2}} \exp\{\Theta_1(u)[L + h''(u_0)]\}u + K_6(u).$$

Moreover, using again (C), we get that $\Psi_2 = P_-(\Psi_1)$ is Fredholm map of index 0 ($\Psi_2|_{\mathrm{cl}(A)} : \mathrm{cl}(A) \to V^-$).

In fact, $d\Psi_2(u)(v) = s(u)|f''(u_0)|^{\frac{1}{2}} \exp\{\Theta_1(u)[L + h''(u_0)]\}\{I + K\}(v)$.

Now let T be as in Remark 2.2. It follows from Lemma 5.4 that $T \circ \Psi_2$ is not locally surjective and hence Ψ_2 it is not locally surjective either. That is, there exists $\delta \in]\frac{1}{2}r_1, \frac{3}{4}r_1[$ and $z \in B(0, \delta) \cap V^-$ such that $z \notin Im(\Psi_2)$. Now, we can project $Im(\Psi_2) \cap B(0, \delta)$ on $\partial B(0, \delta)$ from z. Thus, we can think that Ψ_2 maps $\mathrm{cl}(A)$ into $V^- - B(0, \delta)$. Moreover, we note that Ψ_2 is bounded on bounded sets.

Now, we define $\Psi_3(u) = \frac{\delta}{||\Psi_2(u)||}\Psi_2(u)$, that is,

$$\Psi_3(u) = \frac{\delta}{||\Psi_2(u)||}|f''(u_0)|^{\frac{1}{2}} s(u) \exp\{\Theta_1(u)L\}u + K_6(u)\,.$$

Note that actually $\Psi_3(\mathrm{cl}(A)) \subset \partial B(0, \delta) \cap V^-$.

Thus, we get that $\Psi \circ \Psi_3(u) = s_1(u) \exp\{\Theta_1(u)L\}u + K_7(u)$ where $s_1(u) = \frac{\delta}{||\Psi_2(u)||}s(u) \in]s_0, 1[$, with $s_0 > 0$, and $\Psi \circ \Psi_3(\mathrm{cl}(A)) \subset U - \Psi(B(0, \delta))$.

Finally, we define

$$\gamma(u) = \begin{cases} \Phi(g(u)) & \text{if } u \notin A \\ \Psi \circ \Psi_3(u) & \text{if } u \in \mathrm{cl}(A) \end{cases}$$

Observe that actually our extension $\Psi \circ \Psi_3$ may differ from $\Phi \circ g$ on some points of the boundary ∂A, but since $\Psi \circ \Psi_3(\partial A)$ is outside $\Psi(B(0, \delta))$, we can think of $\Phi \circ g$ and $\Psi \circ \Psi_3$ as being the same on ∂A.

Note that the condition (F) holds for γ and $B(u_0, \delta)$. Suppose that $u \notin A$, so we can assume that $g(u) \notin \mathrm{int}(U_1)$ (and $u \in Q$). From Lemma 5.1, $\Phi(g(u)) \in H - \mathrm{int}(U_1)$. Therefore, if $u \notin A$, $\gamma(u) \notin \Psi(B(0, \delta))$. Assume now that $u \in \mathrm{cl}(A)$. By definition $\gamma(u) = \Psi \circ \Psi_3(u) \in U - \Psi(B(0, \delta))$ and therefore $\gamma(Q) \cap \Psi(B(0, \delta)) = \emptyset$.

Now, we claim that $\sup_{u \in Q} f(\gamma(u)) < c + \epsilon$. If $u \notin A$, as before, we can assume that $g(u) \notin \mathrm{int}(U_1)$ (and $u \in Q$); so we have two possibilities: $g(u) \in$

$U - U_1$ or $g(u) \notin U$. In the first case, from Lemma 5.1 $f(\gamma(u)) \leq f(g(u)) < c + \epsilon$; in the second case, by definition of Φ, $\gamma(u) = g(u)$, and therefore we have the claim. Finally, if $u \in A$, then $f(\gamma(u)) = c - ||P_-(\Psi_3(u))]||^2 < c + \epsilon$.

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