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Three solutions for quasilinear equations in $\mathbb{R}^n$ near resonance *

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Abstract

We use minimax methods to prove the existence of at least three solutions for a quasilinear elliptic equation in $\mathbb{R}^n$ near resonance.

1 Introduction

J. Mawhin and K. Smichtt [7], proved the existence of at least three solutions for the two-point boundary value problem

$$\begin{gathered} -u'' - u + \varepsilon u = f(x,u) + h(x) \\ u(0) = u(\pi) = 0 \end{gathered}$$

for $\varepsilon > 0$ small enough, h orthogonal to $\sin x$ and f bounded satisfying the sign condition $uf(x,u) > 0$. In [9], To Fu Ma and L. Sanchez considered the problem

$$-\Delta_p u - \lambda_1 |u|^{p-2}u + \varepsilon |u|^{p-2}u = f(x,u) + h(x) \tag{1.1}$$

in $W_0^{1,p}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ a bounded domain, and λ_1 the first eigenvalue of

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.2}$$

They proved the following result.

Theorem 1.1 *Suppose that $p \geq 2$ and that the following two conditions hold:*

(H1) $f : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and there exist $\theta > \frac{1}{p}$ such that $\theta s f(x,s) - F(x,s) \to -\infty$ as $|s| \to \infty$

(H2) There exists $R > 0$ such that $sf(x,s) > 0$ for all $x \in \Omega$, $|s| \geq R$

Then for every $h \in L^{p'}(\Omega)$ with $\int_\Omega h(x)\varphi_1(x)dx = 0$, where φ_1 is the first eigenfunction of (1.2), the equation (1.1) has at least three solutions for $\varepsilon > 0$ small enough.

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We recall that the assumptions on f imply the growth condition

$$|f(x,s)| \leq c_1 + c_2|s|^\sigma$$

with $\sigma = \frac{1}{\theta} < p$.

These problems have been studied for several authors, see [3, 4, 5, 8].

The functional setting

Our aim is to extend this result to equations in $\mathbb{R}^n$. As $W^{1,p}(\mathbb{R}^n)$ is no longer compactly imbedded into $L^p(\mathbb{R}^n)$, we shall work in the space $D^{1,p}$, the closure of $C_0^1(\mathbb{R}^n)$ with the norm

$$\|u\|_{1,p} = \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p dx \Big)^{1/p}$$

By the Sobolev inequality we have: $D^{1,p} \subset L^{p^*}(\mathbb{R}^n)$ with $p^* = \frac{Np}{N-p}$, this imbedding is not compact, however in proposition 2.1 we prove that the imbedding $D^{1,p} \subset L^p_g(\mathbb{R}^n)$ is compact for $g \in L^{N/p} \cap L^{N/p+\varepsilon}_{loc}$.

Simplicity of the first eigenvalue

We recall the simplicity of the first eigenvalue of the p-laplacian that is proved in [4]. They studied the problem:

$$\begin{gathered} -\Delta_p u = g(x)|u|^{p-2}u \quad x \in \mathbb{R}^n \\ 0 < u \quad \text{in } \mathbb{R}^n, \quad \lim_{|x|\to+\infty} u(x) = 0\,, \end{gathered} \tag{1.3}$$

where $1 < p < n$. They proved the theorem below, assuming the following conditions:

(G) g is a smooth function, at least $C^{0,\gamma}_{loc}(\mathbb{R}^n)$ for some $\gamma \in (0,1)$, such that $g \in L^{N/p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $g(x) > 0$ in Ω^+ with $|\Omega^+| > 0$. Also g satisfies one the following two conditions

(G^+) $g(x) \geq 0$ a.e. in $\mathbb{R}^n$

(G^-) $g(x) < 0$ for $x \in \Omega^-$, with $|\Omega^-| > 0$.

Theorem 1.2 *1. Let g satisfy (G) and (G^+). Then equation (1.3) admits a positive first eigenvalue,*

$$\lambda_1 = \inf_{B(u)=1} \|u\|^p_{D^{1,p}} \tag{1.4}$$

with $B(u) = \int_{\mathbb{R}^n} |u(x)|^p g(x)\, dx$.

2. Let g satisfy (G) and (G^-). Then problem (1.3) admits two first eigenvalues of opposite sign:

$$\lambda_1^+ = \inf_{B(u)=1} \|u\|^p_{D^{1,p}} \quad \lambda_1^- = -\inf_{B(u)=-1} \|u\|^p_{D^{1,p}}$$

In both cases the associated eigenfunctions φ_1^+, φ_1^- belong to $D^{1,p} \cap L^\infty$.

3. The set of eigenvectors corresponding to λ_1 is a one dimensional subspace.

Remark 1.3 *The first eigenfunction φ_1 does not change its sign in Ω, so we may assume $\varphi_1 \geq 0$.*

Proof. Taking φ^- as a test function in (1.3) with $\lambda = \lambda_1$ we see that

$$\int_{\mathbb{R}^n} |\nabla(\varphi^-)|^p = \lambda_1 \int_{\mathbb{R}^n} |\varphi_1^-|^p g(x)dx$$

It follows that $\varphi^- = 0$ (and $\varphi \geq 0$), or φ_1^- is also a solution of the minimization problem (1.4). In the last case, from the simplicity of the first eigenvalue $\varphi_1^- = c\varphi_1$. It follows that $\varphi^- = -\varphi_1$, so $\varphi_1 \leq 0$. ♢

Existence of multiple solutions

In this paper we study quasilinear elliptic equation

$$-\Delta_p u = (\lambda_1 - \varepsilon)g(x)|u|^{p-2}u + f(x,u) + h(x) \tag{1.5}$$

in $\mathbb{R}^n$. We assume the following:

1. $1 < p < n$ and $\varepsilon > 0$
2. On the weight g we make the assumptions (G) and (G^+) of [4]
3. $h \in L^{p^{*\prime}}$ and $\int_{\mathbb{R}^n} h\varphi_1 dx = 0$
4. We assume that the non linearity $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

   (H0) Growth condition.
   $$|f(x,s)| \leq c_1(x) + c_2(x)|s|^{\sigma-1}$$
   with $\sigma < p$ and $c_1 \in L^{(p^*)'}$, $c_2 \in L^{(p^*/\sigma)'} \cap L^{(p/\sigma)'+\eta}_{loc}$ for some $\eta > 0$.

   (H1) If $F(x,s) = \int_0^s f(x,t)dt$ then $\frac{1}{p}sf(x,s) - F(x,s) \to -\infty$ as $|s| \to \infty$.

   (H2) Sign condition. There exists $R > 0$ such that: $sf(x,s) > 0$ for all $x \in \mathbb{R}^n$, $|s| \geq R$.

For example we may take $f(x,s) = c_2(x)|s|^{\sigma-1}s \cdot \text{sgn s}$ where $c_2(x)$ satisfies the conditions above, $c_2(x) > 0$, and $\sigma < p$.

Note that integrating on condition $(H0)$ we get

$$F(x,s) \leq c_1(x)|s| + c_2(x)\frac{|s|^\sigma}{\sigma}.$$

In the next section we will see that for the functional $C(u) = \int_{\mathbb{R}^n} F(x,u)dx$ to be of class $C^1(D^{1,p}(\mathbb{R}^n))$, condition $(H0)$ is the natural choice.

Our main result is the following theorem:

Theorem 1.4 *Under the assumptions above, problem (1.5) has at least three solutions for $\varepsilon > 0$ small enough.*

2 Technical Lemmas

For the proof of theorem 1.4 we will need the following results:

A compactness result in weighted L^p spaces

If $u \in D^{1,p}$, $1 \le q \le p^*$, $\frac{1}{r} + \frac{q}{p^*} = 1$ and $g \in L^r, g \ge 0$, then from Hölder and Sobolev inequalities, we have that

$$\int_{\mathbb{R}^n} |u|^q g \le C \int_{\mathbb{R}^n} |\nabla u|^p \tag{2.1}$$

and it follows that $D^{1,p} \subset L^q_g$. The following result proves that under appropriate conditions, this imbedding is also compact. (Other previous results can be found in [6]).

Proposition 2.1 *Let $1 \le q < p^*$, $\frac{1}{r} + \frac{q}{p^*} = 1$, $g \in L^r \cap L^{r+\varepsilon}_{loc}$ for some $\varepsilon > 0$. Then the imbedding*

$$D^{1,p} \subset L^q_g(\mathbb{R}^n)$$

is compact.

Proof. Let $(u_n) \subset D^{1,p}$ be a bounded sequence:

$$\|u_n\|_{1,p} \le C$$

Then, as $D^{1,p}$ is reflexive, we may extract a weakly convergent subsequence (u_{n_k}). For simplicity we assume that $u_n \rightharpoonup u$. We want to prove that in fact $u_n \to u$ strongly. From Hölder and Sobolev inequalities we have:

$$\int_{|x|>R} g|u-u_n|^q \le \Big(\int_{|x|>R} |g|^r\Big)^{1/r} \Big(\int_{|x|>R} |u_n-u|^{p^*}\Big)^{p/p^*} \le C\Big(\int_{|x|>R} |g|^r\Big)^{1/r}$$

Given $\varepsilon > 0$, as $g \in L^r$ we can choose $R > 0$ verifying

$$\int_{|x|>R} g|u-u_n|^q \le \frac{\varepsilon}{2}$$

Now $D^{1,p}(\mathbb{R}^n) \subset W^{1,p}_{loc}(\mathbb{R}^n)$ continously and by the Rellich-Kondrachov theorem

$$u_n \to u \text{ strongly in } L^t(B_R)$$

if $1 \le t < p^*$. We choose $s > 1$ such that $s' = r + \varepsilon$, then $s < \frac{p^*}{q}$, and

$$\int_{|x|\le R} g|u_n - u|^q \le \Big(\int_{|x|\le R} |g|^{s'}\Big)^{1/s'} \Big(\int_{|x|<R} |u - u_n|^{qs}\Big)^{1/s} \le \frac{\varepsilon}{2}$$

if $n \ge n_0(\varepsilon)$. So $u_n \to u$ in $L^p_g(\mathbb{R}^n)$. ♢

Some results about the Associated Functional

Under the same assumptions of theorem 1.4, we have the following results:

Lemma 2.2 *Let* $C : D^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ *given by* $C(u) = \int_{\mathbb{R}^n} F(x,u)dx$. *Then* $C \in C^1(D^{1,p}(\mathbb{R}^n))$ *and* $C'(u)(h) = \int_{\mathbb{R}^n} f(x,u)h$

Proof. From the Hölder inequality we have that

$$|C(u)| \leq \int_{\mathbb{R}^n} c_1(x)|u| + c_2(x)\frac{|u|^\sigma}{\sigma}dx \leq \|c_1\|_{(p^*)'}\|u\|_{p^*} + \frac{1}{\sigma}\|c_2\|_{(p^*/\sigma)'}\|u\|_{p^*}^\sigma$$

From the imbedding $D^{1,p} \subset L^{p^*}$ we conclude that $C(u)$ is well defined. In a similar way,

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} f(x,u)h\Big| &\leq \int_{\mathbb{R}^n} c_1(x)|h| + c_2|u|^{\sigma-1}|h| \\ &\leq \|c_1\|_{(p^*)'}\|h\|_{p^*} + \|c_2\|_{(p^*/\sigma)'}\|u\|_{p^*}^{\sigma-1}\|h\|_{p^*} \end{aligned}$$

and we have that $\int_{\mathbb{R}^n} f(x,u)h$ is also well defined. Using a similar argument as in [8], we conclude the proof. ♢

Lemma 2.3 *Assume that* $f(x,y)$ *is a Caratheodory function, verifying that*

$$|f(x,u)| \leq c_1(x) + c_2(x)|u|^{\sigma-1}$$

where $1 \leq \sigma < p^*$, $c_1 \in L^{s_1}(\mathbb{R}^n)$ *with* $s_1 = p^{*\prime}$, *and* $c_2 \in L^{s_2} \cap L^{s_2+\varepsilon}_{loc}$ *with* $s_2 = \frac{p^*}{p^*-\sigma}$. *Then the Nemitski operator* $N_f : D^{1,p}(\mathbb{R}^n) \to L^{p^{*\prime}}(\mathbb{R}^n)$ *given by* $N_f(u) = f(x,u)$ *is compact.*

Proof. Let (u_n) be a sequence in $D^{1,p}$ such that $u_n \rightharpoonup u$ weakly in $D^{1,p}$. We may assume, passing to a subsequence, that $u_n \to u$ a.e..

As $\sigma < p^*$, we apply proposition 2.1 with $q = (\sigma-1)s_1 < p^*$, $g = c_2^{s_1}$. We note that $g \in L^r \cap L^{r+\varepsilon'}_{loc}$ with $r = \frac{p^*-1}{p^*-\sigma}$. We get, passing to a subsequence, that $u_n \to u$ in L^q_g.

From theorem IV.9 in [2], we obtain, after passing again to a subsequence, a function $m \in L^q_g(\mathbb{R}^n)$ such that

$$|u_n(x)| \leq m(x)$$

a.e. with respect to the measure $g(x)dx$. Then, from condition $(H0)$ we deduce that

$$\begin{aligned} |f(x,u) - f(x,u_n)|^{s_1} &\leq 2^{s_1}[|f(x,u)|^{s_1} + |f(x,u_n)|^{s_2}] \\ &\leq 2^{s_1+1}[c_1(x)^{s_1} + c_2(x)^{s_1}|m|^{(\sigma-1)s_1}]\,. \end{aligned}$$

Applying the bounded convergence theorem to $\int_{\mathbb{R}^n} |f(x,u) - f(x,u_n)|^{s_1}dx$ we obtain that $f(x,u_n) \to f(x,u)$ in $L^{s_1}(\mathbb{R}^n)$. ♢

Remark 2.4 *The weak solutions of equation (1.5) are the critical points in $D_0^{1,p}$ of the functional*

$$J_\epsilon(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p dx - \frac{\lambda_1 - \varepsilon}{p}\int_{\mathbb{R}^n} |u|^p g(x) dx - \int_{\mathbb{R}^n} (F(x,u) + h(x)u)$$

Under the previous assumptions it is easy to check that $J_\varepsilon \in C^1(D^{1,p})$.

Let

$$W = \left\{ w \in D^{1,p} : \int_{\mathbb{R}^n} g(x)|\varphi_1|^{p-2}\varphi_1 w = 0 \right\}$$

We recall that as a consequence of proposition 2.1 W is a weakly closed linear subspace.

Lemma 2.5 *If $\varepsilon < \lambda_1$, J_ϵ is coercive in $D^{1,p}$, and there exist $m > 0$ such that $\inf_{u\in W} J_\epsilon(u) \geq -m$.*

Proof. We suppose $0 < \varepsilon < \lambda_1$, then

$$J_\varepsilon(u) \geq \frac{1}{p}\left(1 - \frac{\lambda_1 - \epsilon}{\lambda_1}\right)\int_{\mathbb{R}^n} |\nabla u|^p - \int_{\mathbb{R}^n} (F(x,u) + hu)$$

and

$$J_\varepsilon(u) \geq \frac{\epsilon}{p\lambda_1} \|u\|_{1,p}^p - C_1 - C_2 \|u\|_{1,p}^\sigma - \|h\|_{(p^*)'} \|u\|_{p^*}$$

As $\sigma < p$, it follows that J_ε is coercive.

We define

$$\lambda_W = \inf\left\{ \int_{\mathbb{R}^n} |\nabla w|^2 : w \in W, \int_{\mathbb{R}^n} g(x)|w(x)|^p = 1 \right\}$$

We claim that $\lambda_W > \lambda_1$. In fact if $\lambda_1 = \lambda_W$ then we would have $w \in W$ verifying

$$\int_{\mathbb{R}^n} |w|^p = \lambda_1, \int_{\mathbb{R}^n} |w|^p g(x) dx = 1$$

So by the simplicity of the first eigenvalue, $w = c\varphi_1$ but this contradicts the definition of W.

Then, for $u \in W$ we have

$$J_\varepsilon(u) \geq \frac{\lambda_W - \lambda_1}{p\lambda_W} \|u\|_{1,p}^p - C_1 - C_2 \|u\|_{1,p}^\sigma - \|h\|_{(p^*)'} \|u\|_{p^*}$$

Then J_ε is uniformly coercive in W respect to ε, and in particular is uniformly bounded from below. ♢

For stating the next result we need the two open sets:

$$O^+ = \left\{ w \in D^{1,p} : \int_{\mathbb{R}^n} g(x)|\varphi_1|^{p-2}\varphi_1 w > 0 \right\},$$

$$O^- = \left\{ w \in D^{1,p} : \int_{\mathbb{R}^n} g(x)|\varphi_1|^{p-2}\varphi_1 w < 0 \right\}$$

The next condition is a variant of the Palais-Smale condition (PS).

We will say that a functional $\phi : D^{1,p} \to \mathbb{R}$ verifies the $(PS)_{O^{\pm},c}$ condition if any sequence (u_n) in O^+ (respectively in O^-) with $\phi(u_n) \to c$, $\phi'(u_n) \to 0$, has a subsequence $(u_{n_k}) \to u \in O^+$.

Proposition 2.6 *The operator $-\Delta_p : D^{1,p} \to (D^{1,p})^*$ satisfies the (S_+) condition: if $u_n \rightharpoonup u$ (weakly in $D^{1,p}(\mathbb{R}^n)$) and $\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u\rangle \leq 0$, then $u_n \to u$ (strongly in $D^{1,p}$)*

Proof. This follows from the uniform convexity of $D^{1,p}(\mathbb{R}^n)$ (see [3])

Lemma 2.7 *J_ϵ satisfies the (PS) condition, and it verifies $(PS)_{O^{\pm},c}$ if $c < -m$.*

Proof. Let $(u_n) \subset D^{1,p}$ be a (PS) sequence such that

$$J_\varepsilon(u_n) \to c, J'_\varepsilon(u_n) \to 0$$

Since J_ε is coercive, it follows that (u_n) is bounded in $D^{1,p}$, which is reflexive, so (after passing to a subsequence) we may assume that $u_n \to u$ weakly. We want to show that in fact, $u_n \to u$ strongly. We have that

$$\begin{aligned} J'_\varepsilon(u_n)(u_n - u) &= \int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u) \\ &\quad -(\lambda_1 - \varepsilon)\int |u_n|^{p-2}u_n(u_n - u)g(x)dx \\ &\quad -\int h(u_n - u) - \int f(x, u_n)(u_n - u) \end{aligned}$$

Clearly $\int h(u_n - u) \to 0$ since $u_n \rightharpoonup u$ weakly. Then $u_n \to u$ strongly in $L^p_g(\mathbb{R}^n)$ since the imbedding $D^{1,p} \subset L^p_g$ is compact. It follows that: $\int |u_n|^{p-2}u_n(u_n - u)g(x)dx \to 0$

From proposition 2.3 and the Hölder inequality

$$\int f(x, u_n)(u_n - u)dx = \int [f(x, u) - f(x, u_n)](u_n - u)dx + \int f(x, u)(u_n - u) \to 0\,.$$

Since $J'_\varepsilon(u_n)(u_n - u) \to 0$, it follows that

$$\int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u)dx \to 0$$

or equivalently, $\langle -\Delta_p u_n, u_n - u\rangle \to 0$. By the S_+ condition, this implies that $u_n \to u$ strongly in $D^{1,p}$.

To prove that J_ϵ satisfies $(PS)_{O^{\pm},c}$ for $c < -m$, consider $(u_n) \subset O^{\pm}$ be a $(PS)_c$ sequence. There exists a convergent subsequence: $u_{n_k} \to u$, and it is enough to prove that $u \in O^{\pm}$, but if $u \in \partial O^{\pm} = W$, then $c = J(u) \geq -m$, a contradiction. ♢

Lemma 2.8 *If $\varepsilon > 0$ is small enough, there exists two numbers, $t^- < 0 < t^+$, such that $J_\varepsilon(t^{\pm}\varphi_1) < -m$.*

Proof. From $\int h(x)\varphi_1(x)dx = 0$, we have that

$$J_\varepsilon(t\varphi_1) = \frac{1}{p}\int_{\mathbb{R}^n} 94\varepsilon t^p \varphi_1^p g(x) - \int_{\mathbb{R}^n} F(x, t\varphi_1(x))dx\,.$$

Since $\varphi_1 \in L^\infty$, we can assume that $0 \leq \varphi_1(x) \leq 1$ for all $x \in \mathbb{R}^n$.

First, since $\varphi_1^p g \in L^1$, we can choose ρ big enough, such that:

$$\frac{1}{p}\int_{|x|>\rho} \varphi_1^p g dx < \frac{m}{2}$$

and we split the integral J_ε in two parts: $J_\varepsilon = J_\varepsilon^1 + J_\varepsilon^2$, where J_ε^1 is the integral over $|x| \leq \rho$, and J_ε^2 is the integral over $|x| > \rho$.

We define

$$A(t) = \{x : |x| \leq \rho : \varphi_1(x) > R/t\}$$
$$B(t) = \{x : |x| \leq \rho : \varphi_1(x) \leq R/t\}$$

Then

$$\int_{B(t)} [\frac{\varepsilon}{p} t^p \varphi_1^p - F(x, t\varphi_1(x))]dx$$

is uniformly bounded in ε and t for $\varepsilon \leq \varepsilon_0$. Let

$$M_\varepsilon(t) = \int_{A(t)} \left(\frac{1}{p} t\varphi_1(x) f(x, t\varphi_1(x)) - F(x, t\varphi_1(x))\right) + \int_{B(t)} \left[\frac{\varepsilon}{p} t^p \varphi_1^p - F(x, t\varphi_1(x))\right] dx$$

Then, from $(H1)$ and Fatou lemma, $M_\varepsilon(t) < -2m$ for t big enough, and $\varepsilon \leq \varepsilon_0$.

By $(H2)$ there exists $0 < \varepsilon_t \leq \varepsilon_0$ such that

$$\varepsilon_t u^{p-1} g(x) < f(x, u) \text{ in } \overline{B_\rho} \times [R, t]$$

Then if $\varphi_1(x) > R/t$ and $|x| \leq \rho$ we have:

$$\varepsilon_t t^{p-1} \varphi_1(x)^{p-1} g(x) < f(x, t\varphi_1)$$

and

$$J_\varepsilon^1(t\varphi_1) \leq M_\varepsilon(t) < -2m\,.$$

From $(H2)$, since $F(x, t\varphi_1) \geq 0$, if we choose ε_t satisfying $\varepsilon_t < \frac{1}{t^p}$ then,

$$J_{\varepsilon_t}^2(t\varphi_1) \leq \frac{1}{p}\int_{|x|>\rho} \varepsilon_t t^p \varphi_1^p dx < \frac{m}{2}$$

and we conclude that $J_{\varepsilon_t}(t\varphi_1) < -m$ for any $\varepsilon_t \leq \varepsilon_0$. In a similar way, choosing first t big enough, and then ε_t small, we can prove that $J_{\varepsilon_t}(-t\varphi_1) < -m$ ♢

Proof of theorem 1.4

For $\varepsilon > 0$ small enough, from lemmas 2.7 and 2.8 we have that

$$-\infty < \inf_{O^{\pm}} J_\varepsilon < -m$$

and since $(PS)_{c,O^{\pm}}$ holds for all $c < -m$, it follows from the deformation lemma that the above infima are attained, say at $u^- \in O^-$ and $u^+ \in O^+$. Since $O^{\pm}$ are both open in $D^{1,p}$ we have found two critical points of J_ε. Let

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\varepsilon(\gamma(t))$$

with

$$\Gamma = \{\gamma \in C([0,1], D^{1,p}(\mathbb{R}^n) : \gamma(0) = u^-, \gamma(1) = u^+\}$$

We observe that $\gamma([0,1]) \cap W \neq 0$ for any $\gamma \in \Gamma$, so we conclude that

$$c = \inf_{W} J_\varepsilon \geq -m$$

J_ε verifies (PS), and from Ambrossetti-Rabinowitz's Mountain Pass Theorem [1] we conclude that c is a third critical value of J_ε, and since $J_\varepsilon(u^{\pm}) < -m$, the corresponding critical point is different from u^+, u^-.

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