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Infinitely many solutions for an elliptic system with nonlinear boundary conditions *

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Abstract

In this paper we prove the existence of infinitely many nontrivial solutions of the system $\Delta u = u$, $\Delta v = v$, with nonlinear coupling at the smooth boundary of a bounded domain of $\mathbb{R}^N$. The proof, under suitable assumptions on the Hamiltonian, is based on variational arguments and on the Fountain Theorem of the critical point theory.

1 Introduction.

In this paper we study the existence of infinitely many nontrivial solutions of the elliptic system

$$\Delta u = u, \qquad \Delta v = v, \tag{1.1}$$

in Ω with nonlinear coupling at the boundary given by

$$\frac{\partial u}{\partial \eta} = H_v(x,u,v)\,, \quad \frac{\partial v}{\partial \eta} = H_u(x,u,v)\,, \quad x \in \partial\Omega\,. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary (say $C^{2,\alpha}$), $\frac{\partial}{\partial \eta}$ is the outer normal derivative and $H : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth positive function (say C^1) with growth control on H and its first derivatives.

Existence results for nonlinear elliptic systems have received a great deal of interest in recent years, in particular when the nonlinear term appears as a source in the equation, complemented with Dirichlet boundary conditions. For the system (1.1)-(1.2), existence of solutions and of positive solutions, have been proved in [9] under similar assumptions on the Hamiltonian H that we made here. See also [10] for an existence result for (1.1)-(1.2) without any variational assumption on the nonlinearities. For this type of results in the semilinear case see, among others, [1, 2, 4, 5, 7, 12] and the survey [3].

This work is inspired by the articles [1] and [8] where the authors study

$$-\Delta u = H_v(x,u,v)$$
$$-\Delta v = H_u(x,u,v),$$

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with Dirichlet boundary conditions in $\Omega \subset \mathbb{R}^n$, a smooth bounded domain.

The crucial part in the nonlinear boundary conditions case, is to find the proper functional setting for (1.1)-(1.2) that allows us to treat our problem variationally. We accomplish this by defining a selfadjoint operator that takes into account the boundary conditions together with the equations and considering its fractional powers that satisfy a suitable “integration by parts” formula. For the proof of our multiplicity result we use a Fountain-type theorem (see [17]) in a version due to Felmer and Wang [8] (see also [1]).

Let us now state the precise assumptions on the Hamiltonian H.

Hypotheses on H:

$$|H(x,u,v)| \le C\left(|u|^{p+1} + |v|^{q+1} + 1\right), \tag{1.3}$$

and for small positive r, if $|(u,v)| \le r$, then

$$|H(x,u,v)| \le C\left(|u|^{\alpha} + |v|^{\beta}\right), \tag{1.4}$$

where the exponents satisfy $p+1 \ge \alpha > p > 0$ and $q+1 \ge \beta > q > 0$ with

$$1 > \frac{1}{\alpha} + \frac{1}{\beta}, \tag{1.5}$$

$$\max\left\{\frac{p}{\alpha} + \frac{q}{\beta};\ \frac{q}{q+1}\frac{p+1}{\alpha} + \frac{p}{p+1}\frac{q+1}{\beta}\right\} < 1 + \frac{1}{N-1}\,, \tag{1.6}$$

$$\frac{p}{p+1}\frac{q+1}{\beta} < 1 \quad \text{and} \quad \frac{q}{q+1}\frac{p+1}{\alpha} < 1\,. \tag{1.7}$$

If $N \ge 4$, we have to impose the additional hypothesis

$$\max\left\{\frac{p}{\alpha};\ \frac{q}{\beta};\ \frac{q}{q+1}\frac{p+1}{\alpha};\ \frac{p}{p+1}\frac{q+1}{\beta}\right\} < \frac{N+1}{2(N-1)}\,. \tag{1.8}$$

On the derivatives of H we impose the following conditions

$$\begin{aligned} \left|\tfrac{\partial H}{\partial u}(x,u,v)\right| &\le C\left(|u|^{p} + |v|^{p(q+1)/(p+1)} + 1\right), \\ \left|\tfrac{\partial H}{\partial v}(x,u,v)\right| &\le C\left(|u|^{q(p+1)/(q+1)} + |v|^{q} + 1\right). \end{aligned} \tag{1.9}$$

And for R large, if $|(u,v)| \ge R$,

$$\frac{1}{\alpha}\frac{\partial H}{\partial u}(x,u,v)u + \frac{1}{\beta}\frac{\partial H}{\partial v}(x,u,v)v \ge H(x,u,v) > 0, \tag{1.10}$$

We also impose the following symmetry condition

$$H(x,u,v) = H(x,-u,-v). \tag{1.11}$$

Remark 1.1 (1.5)-(1.8), imply that there exist s and t with $s+t=1$, $s,t>1/4$ such that

$$\frac{\alpha-p}{\alpha}>\frac{1}{2}-\frac{2s-1/2}{N-1},\quad \frac{\beta-q}{\beta}>\frac{1}{2}-\frac{2t-1/2}{N-1},$$

$$1-\frac{p(q+1)}{\beta(p+1)}>\frac{1}{2}-\frac{2s-1/2}{N-1},\quad 1-\frac{q(p+1)}{\alpha(q+1)}>\frac{1}{2}-\frac{2t-1/2}{N-1}.$$

Remark 1.2 When $\alpha=p+1$ and $\beta=q+1$, conditions (1.5), (1.6) and (1.8) become

$$1>\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{1}{N-1},\quad p,q\le\frac{N+1}{N-3}\quad \text{if } N\ge 4.$$

Remark 1.3 We observe that from (1.10), it follows that (see [6])

$$|H(x,u,v)|\ge c\left(|u|^{\alpha}+|v|^{\beta}\right)-C.$$

The main result in this paper is the following Theorem.

Theorem 1.4 *Assume that $H:\partial\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies (1.3)-(1.11). Then there exists a sequence of nontrivial strong solutions $\{u_n,v_n\}$ to (1.1)-(1.2) such that*

$$\|u_n\|_{W^{1,(q+1)/(q)}(\Omega)}+\|v_n\|_{W^{1,(p+1)/(p)}(\Omega)}\to\infty.$$

The rest of the paper is organized as follows, in §2 we establish the functional setting in which the problem will be posed and prove a regularity result for weak solutions of (1.1)-(1.2). In §3 we prove the main theorem.

2 The functional setting

In this section we describe the functional setting that allows us to treat (1.1)-(1.2) variationally.

Let us consider the space $L^2(\Omega)\times L^2(\partial\Omega)$ which is a Hilbert space with inner product, that we will denote by $\langle\cdot,\cdot\rangle$, given by

$$\langle(u,v),(\phi,\psi)\rangle=\int_\Omega u\phi+\int_{\partial\Omega}v\psi.$$

Now, let $A:D(A)\subset L^2(\Omega)\times L^2(\partial\Omega)\to L^2(\Omega)\times L^2(\partial\Omega)$ be the operator defined by

$$A(u,u\mid_{\partial\Omega})=(-\Delta u+u,\frac{\partial u}{\partial\eta}),$$

where $D(A)=\{(u,u\mid_{\partial\Omega})/u\in H^2(\Omega)\}$. $D(A)$ is dense in $L^2(\Omega)\times L^2(\partial\Omega)$.

We observe that A is invertible with inverse given by

$$A^{-1}(f,g)=(u,u\mid_{\partial\Omega}),$$

where u is the solution of

$$\begin{aligned} -\Delta u + u = f \quad \text{in } \Omega, \\ \tfrac{\partial u}{\partial \eta} = g \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

By standard regularity theory, see [11, p. 214], it follows that A^{-1} is bounded and compact. Therefore, $R(A) = L^2(\Omega) \times L^2(\partial\Omega)$ thus in order to see that A (and hence A^{-1}) is selfadjoint it remains to check that A is symmetric [15, p. 512]. To see this let $u, v \in D(A)$ and by Green's formula we have

$$\langle Au, v\rangle = \int_\Omega (-\Delta u + u)v + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v = \int_\Omega u(-\Delta v + v) + \int_{\partial\Omega} u \frac{\partial v}{\partial \eta} = \langle u, Av\rangle;$$

therefore, A is symmetric. Moreover, A (and hence A^{-1}) is positive. In fact, let $u \in D(A)$ and using again Green's formula,

$$\langle Au, u\rangle = \int_\Omega (-\Delta u + u)u + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} u = \int_\Omega |\nabla u|^2 + u^2 \geq 0.$$

Therefore, there exists a sequence of eigenvalues $(\lambda_n) \subset \mathbb{R}$ with eigenfunctions $(\phi_n, \psi_n) \in L^2(\Omega) \times L^2(\partial\Omega)$ such that $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ... \nearrow +\infty$ and $\phi_n \in H^2(\Omega)$, $\phi_n \mid_{\partial\Omega} = \psi_n$,

$$\begin{aligned} -\Delta \phi_n + \phi_n = \lambda_n \phi_n \quad \text{in } \Omega, \\ \tfrac{\partial \phi_n}{\partial \eta} = \lambda_n \phi_n \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Let us consider the fractional powers of A, namely for $0 < s < 1$,

$$A^s : D(A^s) \to L^2(\Omega) \times L^2(\partial\Omega), \quad \text{with} \quad A^s u = \sum_{n=1}^{\infty} \lambda_n^s a_n (\phi_n, \psi_n),$$

where $u = \sum a_n(\phi_n, \psi_n)$. Let $E^s = D(A^s)$, which is a Hilbert space under the inner product

$$(u, \phi)_{E^s} = \langle A^s u, A^s \phi\rangle.$$

Note that $E^s \subset H^{2s}(\Omega)$. In fact, if we define $A_1 : H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$A_1 u = -\Delta u + u,$$

and $A_2 : H^2(\Omega) \subset D(A_2) \subset L^2(\partial\Omega) \to L^2(\partial\Omega)$ by

$$A_2 u = \frac{\partial u}{\partial \eta},$$

then $\tilde{A} = (A_1, A_2)$ satisfies

$$A = \tilde{A} \mid_{(u,u)} \quad u \in D(A_1) \cap D(A_2),$$

and hence

$$A^s = \tilde{A}^s \mid_{(u,u)} \quad u \in D(A_1^s) \cap D(A_2^s).$$

As $D(A_1) = H^2(\Omega) \subset D(A_2)$ we have, $D(A_1^s) \subset D(A_2^s)$, therefore

$$E^s = D(A^s) = D(A_1^s).$$

Now, by the results of [16, p. 187] (see also [13], [15]), as Ω is smooth, it follows that $E^s = D(A_1^s) \subset H^{2s}(\Omega)$.

So we have the following inclusions

$$E^s \hookrightarrow H^{2s}(\Omega) \hookrightarrow H^{2s-1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega).$$

More precisely, we have the following immersion Theorem,

Theorem 2.1 *Given $s > 1/4$ and $p \geq 1$ so that $\frac{1}{p} \geq \frac{1}{2} - \frac{2s-1/2}{N-1}$ the inclusion map $i : E^s \to L^p(\partial\Omega)$ is well defined and bounded. Moreover, if above we have strict inequality, then the inclusion is compact.*

Let us now set $E = E^s \times E^t$ where $s + t = 1$, s, t given by Remark 1.1 and define $B : E \times E \to \mathbb{R}$ by

$$B((u, v), (\phi, \psi)) = \langle A^s u, A^t \psi\rangle + \langle A^s \phi, A^t v\rangle.$$

E is a Hilbert space with the usual product structure, and hence B is a bounded, bilinear, symmetric form. Therefore, there exists a unique bounded, selfadjoint, linear operator $L : E \to E$, such that

$$B(z, \gamma) = (Lz, \gamma)_E.$$

Now we define

$$\mathcal{Q}(z) = \frac{1}{2}B(z, z) = \frac{1}{2}(Lz, z)_E = \langle A^s u, A^t v\rangle.$$

The following Lemma gives us a characterization of L,

Lemma 2.2 *The operator L defined above can be written as*

$$L(u, v) = (A^{-s}A^t v, A^{-t}A^s u).$$

Proof. Let $z = (u, v)$, $\eta = (\phi, \psi)$ and $Lz = (w, y)$. Then we have

$$(Lz, \eta)_E = ((w, y), (\phi, \psi))_E = (w, \phi)_{E^s} + (y, \psi)_{E^t} = \langle A^s w, A^s \phi\rangle + \langle A^t y, A^y \psi\rangle.$$

On the other hand

$$(Lz, \eta)_E = B(z, \eta) = \langle A^s u, A^t \psi\rangle + \langle A^s \phi, A^t v\rangle.$$

Now if we take $\psi = 0$ we obtain,

$$\langle A^s w, A^s \phi\rangle = \langle A^t v, A^s \phi\rangle,$$

then

$$\langle A^s w - A^t v, A^s \phi\rangle = 0.$$

As A^s is invertible, it follows that $A^s w = A^t v$ and hence $w = A^{-s}A^t v$. Analogously, $y = A^{-t}A^s u$. $\square$

Next, we consider the eigenvalue problem $Lz = \lambda z$. Using Lemma 2.2 we can rewrite this as

$$A^{-s}A^t v = \lambda u, \qquad A^{-t}A^s u = \lambda v,$$

where $z = (u, v)$. As A^s and A^t are isomorphisms, it follows that $\lambda = 1$ or $\lambda = -1$. The associated eigenvectors are

$$\begin{aligned} \text{for } \lambda = 1, &\quad (u, A^{-t}A^s u)\ \forall u \in E^s, \\ \text{for } \lambda = -1, &\quad (u, -A^{-t}A^s u)\ \forall u \in E^s. \end{aligned}$$

We can define the eigenspaces

$$\begin{aligned} E^+ &= \{(u, A^{-t}A^s u) : u \in E^s\}, \\ E^- &= \{(u, -A^{-t}A^s u) : u \in E^s\}, \end{aligned} \tag{2.3}$$

which gives the natural splitting

$$E = E^+ \oplus E^-. \tag{2.4}$$

By (1.3), Remark 1.1 and Theorem 2.1 we can define the functional, $\mathcal{H} : E \to \mathbb{R}$ as

$$\mathcal{H}(u, v) = \int_{\partial\Omega} H(x, u, v).$$

Proposition 2.3 *The functional $\mathcal{H}$ defined above is of class C^1 and its derivative is given by*

$$\mathcal{H}'(u, v)(\phi, \psi) = \int_{\partial\Omega} H_u(x, u, v)\phi + \int_{\partial\Omega} H_v(x, u, v)\psi.$$

Moreover, $\mathcal{H}'$ is compact.

Proof. From (1.9) we have

$$\int_{\partial\Omega} \left| \frac{\partial H}{\partial u}(x, u, v)\phi \right| \le C \int_{\partial\Omega} \left(|u|^p + |v|^{p(q+1)/(p+1)} + 1 \right) |\phi|.$$

By Hölder inequality and Theorem 2.1 we have

$$\int_{\partial\Omega} \left| \frac{\partial H}{\partial u}(x, u, v)\phi \right| \le C \left(\|u\|_{E^s}^p + \|v\|_{E^t}^{p(q+1)/(p+1)} + 1 \right) \|\phi\|_{E^s}.$$

In a similar way we obtain the analogous inequality for H_v.

Thus $\mathcal{H}'$ is well defined and bounded in E. Next, a standard argument gives that $\mathcal{H}$ is Fréchet differentiable with $\mathcal{H}'$ continuous. The fact that $\mathcal{H}'$ is compact comes from Theorem 2.1 (see [14] for the details). $\square$

Now we can define the functional $\mathcal{F} : E \to \mathbb{R}$ as

$$\mathcal{F}(z) = \mathcal{Q}(z) - \mathcal{H}(z). \tag{2.5}$$

$\mathcal{F}$ is of class C^1 and in the next section we prove that it has the structure needed in order to apply the minimax techniques.

Let us now give the definition of weak solution of (1.1)-(1.2).

Definition 2.4 *We say that $z = (u, v) \in E = E^s \times E^t$ is an $(s,t)-$weak solution of (1.1)-(1.2) if z is a critical point of $\mathcal{F}$. In other words, for every $(\phi, \psi) \in E$ we have*

$$\langle A^s u, A^t \psi \rangle + \langle A^s \phi, A^t v \rangle - \int_{\partial\Omega} H_u(x,u,v)\phi - \int_{\partial\Omega} H_v(x,u,v)\psi = 0. \tag{2.6}$$

Now, we prove a Theorem that gives us the regularity of (s,t)-weak solutions. In [9] Theorem 2.2, it is claimed that (s,t)-weak solutions of (1.1)-(1.2) are strong solutions. However, the proof given there only shows that they are weak solutions in the following sense

$$\int_\Omega \nabla u \nabla \phi + u\phi - \int_{\partial\Omega} H_v(x,u,v)\phi = 0,$$

$$\int_\Omega \nabla v \nabla \varphi + v\varphi - \int_{\partial\Omega} H_u(x,u,v)\varphi = 0,$$

for all smooth $\phi,\ \varphi$.

Theorem 2.5 *If $(u, v) \in E^s \times E^t$ is an (s,t)-weak solution of (1.1)-(1.2) then $u \in W^{1,(q+1)/q}(\Omega)$, $v \in W^{1,(p+1)/p}(\Omega)$ and (u, v) is in fact a weak solution of (1.1)-(1.2).*

Proof. Let us first consider $\psi = 0$ in (2.6), then

$$\langle A^s \phi, A^t v \rangle - \int_{\partial\Omega} H_u(x,u,v)\phi = 0, \tag{2.7}$$

for all $\phi \in E^s$. If we take $\phi \in H^2(\Omega)$, we have

$$\langle A^s \phi, A^t v \rangle = \langle A\phi, v \rangle = \int_\Omega (-\Delta\phi + \phi)v + \int_{\partial\Omega} \frac{\partial\phi}{\partial\eta} v. \tag{2.8}$$

On the other hand, using (1.9) we find

$$H_u(x, u(x), v(x)) \in L^{(p+1)/p}(\partial\Omega).$$

Then from basic elliptic theory (see [11]), there exists a function $w \in W^{1,\frac{p+1}{p}}(\Omega)$ such that

$$\begin{aligned} \Delta w = w &\quad \text{in } \Omega, \\ \frac{\partial w}{\partial \eta} = H_u(x, u(x), v(x)) &\quad \text{on } \partial\Omega\,. \end{aligned}$$

Now, integration by parts gives us

$$0 = \int_\Omega (-\Delta w + w)\phi = \int_\Omega w(-\Delta\phi + \phi) + \int_{\partial\Omega} w\frac{\partial\phi}{\partial\eta} - \int_{\partial\Omega} H_u(x,u,v)\phi. \quad (2.9)$$

Combining (2.7),(2.8) and (2.9), we obtain

$$\langle v - w, A\phi\rangle = \int_\Omega (v-w)(-\Delta\phi + \phi) + \int_{\partial\Omega}(v-w)\frac{\partial\phi}{\partial\eta} = 0,$$

from where it follows that $v = w$. We argue similarly for u. □

3 Proof of Theorem 1.4

In this section we present an abstract theorem from critical point theory from [8] (see also [1]), that provides us with infinitely many critical points. Next, we prove that it can be applied to our functional setting stated in the previous section.

Let E be a Hilbert space with inner product $(\cdot,\cdot)_E$. Assume that E has a splitting $E = X \oplus Y$ where X and Y are both infinite dimensional subspaces. Assume there exists a sequence of finite dimensional subspaces $X_n \subset X, Y_n \subset Y$, $E_n = X_n \oplus Y_n$ such that $\overline{\cup_{n=1}^\infty E_n} = E$. Let $T : E \to E$ be a linear bounded invertible operator.

Let $\mathcal{F} \in C^1(E,\mathbb{R})$. Instead of the usual Palais-Smale condition we will require that the functional $\mathcal{F}$ satisfies the so-called $(PS)^*$ conditions with respect to E_n, i.e. any sequence $z_k \in E_{n_k}$ with $n_k \to \infty$ as $k \to \infty$, satisfying $\mathcal{F}|'_{E_{n_k}}(z_k) \to 0$ and $\mathcal{F}(z_k) \to c$ has a subsequence that converges in E.

Then we define the basic sets over which the linking process will take place. For $\rho > 0$ we define

$$S = S_\rho = \{y \in Y |\ \|y\|_E = \rho\}$$

and for some fixed $y_1 \in Y$ with $\|y_1\|_E = 1$ and subspaces X_1 and X_2, we consider

$$X \oplus \text{span}\{y_1\} = X_1 \oplus X_2.$$

Without loss of generality we may assume that $y_1 \in X_2$. Next, we define for $M, \sigma > 0$

$$D = D_{M,\sigma} = \{x_1 + x_2 \in X_1 \oplus X_2 |\ \|x_1\|_E \le M,\ \|x_2\|_E \le \sigma\}.$$

Now we can state our abstract critical point result whose proof can be found in [8]:

Theorem 3.1 *Let $\mathcal{F} \in C^1(E,\mathbb{R})$ be an even functional satisfying the $(PS)^*$ condition with respect to E_n. Assume that $T : E_n \to E_n$, for n large. Let $\rho > 0$ and $\sigma > 0$ be such that $\sigma\|Ty_1\|_E > \rho$. Assume that there are constants $\alpha \le \beta$ such that*

$$\inf_{S\cap E_n} \mathcal{F} \ge \alpha, \qquad \sup_{T(\partial D\cap E_n)} \mathcal{F} < \alpha, \qquad \sup_{T(D\cap E_n)} \mathcal{F} \le \beta$$

for all n large. Then $\mathcal{F}$ has a critical value $c \in [\alpha,\beta]$.

Next, we show how the functional setting introduced in §2 can be applied in Theorem 3.1. Let ϕ_n be the eigenfunctions defined in (2.2). We define

$$E_n = \text{span}\{\phi_1, ..., \phi_n\} \times \text{span}\{\phi_1, ..., \phi_n\}$$

and it is easy to see that $\overline{\cup_{n=1}^{\infty} E_n} = E$.

Next, we prove that $\mathcal{F}$ satisfies the $(PS)^*$ condition with respect to the family E_n.

Lemma 3.2 *The functional $\mathcal{F}$ satisfies the $(PS)^*$ condition with respect to E_n*

Proof. Let $(z_k)_{k\geq 1} \subset E_{n_k}$ be a sequence such that

$$\mathcal{F}(z_k) \to c \quad \text{and} \quad \mathcal{F}' |_{E_{n_k}} (z_k) \to 0. \tag{3.1}$$

Let us first prove that (3.1) implies that (z_k) is bounded. From (3.1) it follows that there exists a sequence $\varepsilon_k \to 0$ such that

$$|\mathcal{F}'(z_k)w| \leq \varepsilon_k \|w\|_E, \ \forall w \in E_{n_k}. \tag{3.2}$$

Let us take

$$w_k = ((w_k)_1, (w_k)_2) = \frac{\alpha\beta}{\alpha+\beta}(\frac{1}{\alpha}u_k, \frac{1}{\beta}v_k), \quad \text{where } z_k = (u_k, v_k).$$

Now, using (3.1) and (3.2), for k large,

$$\begin{aligned}
c &+ 1 + \varepsilon_k \|w_k\|_E \\
&\geq \mathcal{F}(z_k) - \mathcal{F}'(z_k) w_k \\
&= \langle A^s u_k, A^t v_k \rangle - \int_{\partial\Omega} H(x, u_k, v_k) - \langle A^s u_k, A^t (w_k)_2 \rangle \\
&\quad - \langle A^s (w_k)_1, A^t v_k \rangle + \int_{\partial\Omega} H_u(x, u_k, v_k)(w_k)_1 + \int_{\partial\Omega} H_v(x, u_k, v_k)(w_k)_2 \\
&= \frac{\alpha\beta}{\alpha+\beta} \int_{\partial\Omega} \frac{1}{\alpha} H_u(x, u_k, v_k) u_k + \frac{1}{\beta} H_v(x, u_k, v_k) v_k - H(x, u_k, v_k) \\
&\quad + \left(\frac{\alpha\beta}{\alpha+\beta} - 1 \right) \int_{\partial\Omega} H(x, u_k, v_k).
\end{aligned}$$

Now, by (1.10) and (1.5) we obtain $C(1 + \|z_k\|_E) \geq \int_{\partial\Omega} H(x, u_k, v_k)$, and then, by Remark 1.3,

$$\int_{\partial\Omega} |u_k|^\alpha + |v_k|^\beta \leq C(1 + \|u_k\|_{E^s} + \|v_k\|_{E^t}). \tag{3.3}$$

Next we consider $w = (\phi, 0)$, $\phi \in E^s_{n_k}$. From (3.2) we have

$$\langle A^s \phi, A^t v_k \rangle \leq \int_{\partial\Omega} |H_u(x, u_k, v_k)\phi| + \varepsilon_k \|\phi\|_{E^s}.$$

Now, by (1.9)

$$\int_{\partial\Omega} |H_u(x,u_k,v_k)\phi| \le C\left(\int_{\partial\Omega} |u_k|^p|\phi| + |v_k|^{p\frac{q+1}{p+1}}|\phi| + |\phi|\right).$$

Using Hölder inequality the last term is bounded by

$$\|u_k\|^p_{L^\alpha(\partial\Omega)}\|\phi\|_{L^{\frac{\alpha}{\alpha-p}}(\partial\Omega)} + \|v_k\|^{p\frac{q+1}{p+1}}_{L^\beta(\partial\Omega)}\|\phi\|_{L^{\frac{\beta(p+1)}{\beta(p+1)-p(q+1)}}(\partial\Omega)} + \|\phi\|_{L^1(\partial\Omega)}.$$

Now, by Theorem 2.1 and Remark 1.1, we get that the last equation is bounded by

$$\|u_k\|^p_{L^\alpha(\partial\Omega)}\|\phi\|_{E^s} + \|v_k\|^{p\frac{q+1}{p+1}}_{L^\beta(\partial\Omega)}\|\phi\|_{E^s} + \|\phi\|_{E^s}.$$

Thus,

$$|\langle A^s\phi, A^t v_k\rangle| \le C\|\phi\|_{E^s}\left(\|u_k\|^p_{L^\alpha(\partial\Omega)} + \|v_k\|^{p\frac{q+1}{p+1}}_{L^\beta(\partial\Omega)} + 1\right).$$

By duality (A^s in invertible over E^s) we get

$$\|v_k\|_{E^t} \le C\left(\|u_k\|^p_{L^\alpha(\partial\Omega)} + \|v_k\|^{p\frac{q+1}{p+1}}_{L^\beta(\partial\Omega)} + 1\right). \tag{3.4}$$

Analogously, we obtain

$$\|u_k\|_{E^s} \le C\left(\|v_k\|^q_{L^\beta\partial\Omega)} + \|u_k\|^{q\frac{p+1}{q+1}}_{L^\alpha(\partial\Omega)} + 1\right). \tag{3.5}$$

Now combining (3.3), (3.4) and (3.5), we obtain

$$\|u_k\|_{E^s} + \|v_k\|_{E^t} \le c\left(\|u_k\|^{p/\alpha}_{E^s} + \|v_k\|^{p\frac{q+1}{\beta(p+1)}}_{E^t} + \|v_k\|^{q/\beta}_{E^t} + \|u_k\|^{q\frac{p+1}{\alpha(q+1)}}_{E^s} + 1\right),$$

and as all the exponents are less than one, we get that z_k in bounded.

Now, by the compactness of $\mathcal{H}'$ and the invertibility of L we can extract a subsequence of z_k that converges in E. In fact, we can take a subsequence z_{k_j} that converges weakly in E, as $\mathcal{H}'$ is compact, it follows that $\mathcal{H}'(z_{k_j})$ converges strongly in E. Hence, using the fact that $\mathcal{F}'(z_{k_j}) \to 0$ strongly and the invertibility of L, the result follows. □

Now we define the splitting of E_n. Fix $k \in \mathbb{N}$ and for $n \ge k$ let

$$X_n = \left(E_1^- \oplus \cdots \oplus E_n^-\right) \oplus \left(E_1^+ \oplus \cdots \oplus E_{k-1}^+\right) \quad \text{and} \quad Y_n = \left(E_k^+ \oplus \cdots \oplus E_n^+\right), \tag{3.6}$$

where $E_j^+ = \operatorname{span}\{(\phi_j, A^{-t}A^s\phi_j)\}$ and $E_j^- = \operatorname{span}\{(\phi_j, -A^{-t}A^s\phi_j)\}$. By (2.4) we have $E_n = X_n \oplus Y_n$.

Lemma 3.3 *There exist $\alpha_k > 0$ and $\rho_k > 0$ independent of n such that for all $n \ge k$*

$$\inf_{z\in S_{\rho_k}\cap Y_n} \mathcal{F}(z) \ge \alpha_k$$

where $S_{\rho_k} = \{y \in E^+ \mid \|y\| = \rho_k\}$. Moreover, $\alpha_k \to \infty$ as $k \to \infty$.

Proof. We first recall that by Theorem 2.1, E^s is embedded in $L^\gamma(\partial\Omega)$ for any $\gamma \in [1, \frac{2N-2}{N-4s}]$, hence there exists $a = a(\gamma)$ such that

$$\|u\|_{L^\gamma(\partial\Omega)} \le a\|u\|_{E^s} \qquad \text{for all } u \in E^s.$$

Also for $z \in E_k^+ \oplus \cdots \oplus E_j^+ \oplus \cdots$ we have

$$\|z\|_E \ge \lambda_k^{\min\{s,t\}} \|z\|_{L^2(\partial\Omega)}$$

with $\lambda_k \to \infty$ as $k \to \infty$.

Now consider $z = (u, v) \in Y_n$. For a constant a independent of n, we observe that there exists $\kappa > 0$ such that

$$\|u\|^{p+1}_{L^{p+1}(\partial\Omega)} \le \|u\|^{2/\kappa}_{L^2(\partial\Omega)} \|u\|^{(2N-2)/[(N-4s)\kappa']}_{L^{(2N-2)/(N-4s)}(\partial\Omega)} \le \frac{a}{\lambda_k^{\min\{s,t\}(2/\kappa)}} \|u\|^{p+1}_{E^s}$$

Analogously, we obtain

$$\|v\|^{q+1}_{L^{q+1}(\partial\Omega)} \le \frac{a}{\lambda_k^{\min\{s,t\}(2/\theta)}} \|v\|^{q+1}_{E^t}$$

for some $\theta > 0$.

Then for $z = (u, v)$ we have

$$\mathcal{F}(z) \ge \|z\|_E^2 - C \left(\frac{a}{\lambda_k^{\min\{s,t\}\min\{2/\kappa,\ 2/\theta\}}} \max\{\|z\|_E^{p+1},\ \|z\|_E^{q+1}\} + 1 \right)$$

Then we choose $\rho_k^{\max\{p+1,q+1\}} = \lambda_k^{\min\{s,t\}\min\{2/\kappa,\ 2/\theta\}}$ and observe that $\rho_k \to \infty$ as $k \to \infty$.

Therefore, for $z \in S_{\rho_k} \cap Y_n$ we find that

$$\mathcal{F}(z) \ge \rho_k^2 - C. \tag{3.7}$$

Defining α_k as the right hand side of (3.7) and noting that both ρ_k and α_k are independent of $n \ge k$ we complete the proof of the Lemma. □

Next we define, for $z = (u, v) \in E$

$$T_\sigma(z) = (\sigma^{\mu-1}u, \sigma^{\nu-1}v) \tag{3.8}$$

where μ and ν are such that

$$\frac{1}{\alpha} < \frac{\mu}{\mu+1}, \qquad \frac{1}{\beta} < \frac{\nu}{\nu+1},$$

α and β are given by (1.4).

Lemma 3.4 *There exist $\beta_k > 0$, σ_k and $M_k > 0$ independent of n such that for all $n \ge k$ they satisfy $\sigma_k > \rho_k$,*

$$\sup_{T_{\sigma_k}(\partial D \cap E_n)} \mathcal{F} \le 0 \qquad \textit{and} \qquad \sup_{T_{\sigma_k}(D \cap E_n)} \mathcal{F} \le \beta_k$$

where

$$D = \{z \in E^- \oplus E_1^+ \oplus \cdots \oplus E_k^+ \mid \|z^-\| \le M_k,\ \|z^+\| \le \sigma_k\}.$$

Proof. Let us consider $z = T_\sigma(u,v)$ with $(u,v) \in D$. Then we can write $z = (\sigma^{\mu-1}u^+, \sigma^{\nu-1}v^+) + (\sigma^{\mu-1}u^-, \sigma^{\nu-1}v^-)$. Using the definition of $\mathcal{Q}$ and the spaces E^+ and E^- we have

$$\mathcal{Q}(z) = \sigma^{\mu+\nu-2}(\|z^+\|^2 - \|z^-\|^2).$$

On the other hand we have that

$$\int_{\partial\Omega} H(z,x)\,dS \geq C\left(\int_{\partial\Omega} \sigma^{\alpha(\mu-1)}|u^+ + u^-|^\alpha + \sigma^{\beta(\nu-1)}|v^+ + v^-|^\beta\,dS - |\partial\Omega|\right).$$

The functions u^+ and u^- can be written as

$$u^+ = \sum_{i=1}^k \theta_i\phi_i \qquad \text{and} \qquad u^- = \sum_{i=1}^k \gamma_i\phi_i + \tilde{u}^-,$$

where $\tilde{u}^-$ is orthogonal to ϕ_i, $i = 1, ..., k$ in $L^2(\Omega) \times L^2(\partial\Omega)$. Using Hölder inequality we get

$$\begin{aligned} \sum_{i=1}^k \lambda_i^{s-t}(\theta_i^2 + \theta_i\gamma_i) &= \langle u^+ + u^-, A^{s-t}u^+\rangle \\ &\leq \|u^+ + u^-\|_{L^\alpha(\Omega)\times L^\alpha(\partial\Omega)}\|A^{s-t}u^+\|_{L^{\alpha'}(\Omega)\times L^{\alpha'}(\partial\Omega)}. \end{aligned}$$

Then there exists a constant C_k such that

$$\sum_{i=1}^k \lambda_i^{s-t}(\theta_i^2 + \theta_i\gamma_i) \leq C_k\|u^+ + u^-\|_{L^\alpha(\Omega)\times L^\alpha(\partial\Omega)}\|u^+\|_{L^2(\Omega)\times L^2(\partial\Omega)}. \tag{3.9}$$

In a similar way, using that $v^+ = A^{s-t}u^+$ and $v^- = -A^{s-t}u^-$ (see (2.3)) we have that there exists a constant C_k such that

$$\sum_{i=1}^k \lambda_i^{s-t}(\theta_i^2 - \theta_i\gamma_i) \leq C_k\|v^+ + v^-\|_{L^\beta(\Omega)\times L^\beta(\partial\Omega)}\|u^+\|_{L^2(\Omega)\times L^2(\partial\Omega)}. \tag{3.10}$$

Depending on the sign of $\sum_{i=1}^k \lambda_i^{s-t}\alpha_i\gamma_i$ we use (3.9) or (3.10) to conclude that

$$\|u^+\|_{L^2(\Omega)\times L^2(\partial\Omega)} \leq C_k\|u^+ + u^-\|_{L^\alpha(\Omega)\times L^\alpha(\partial\Omega)}$$

or

$$\|u^+\|_{L^2(\Omega)\times L^2(\partial\Omega)} \leq C_k\|v^+ + v^-\|_{L^\beta(\Omega)\times L^\beta(\partial\Omega)}$$

and hence

$$\mathcal{F}(z) \leq \sigma^{\mu+\nu-2}(\|z^+\|_E^2 - C_k\sigma^{\alpha(\mu-1)}\|u^+\|^\alpha_{L^2(\Omega)\times L^2(\partial\Omega)} + C|\partial\Omega|$$

or

$$\mathcal{F}(z) \leq \sigma^{\mu+\nu-2}(\|z^+\|_E^2 - C_k\sigma^{\beta(\nu-1)}\|u^+\|^\beta_{L^2(\Omega)\times L^2(\partial\Omega)} + C|\partial\Omega|.$$

Thus we may choose $\|z^+\|_E = \sigma_k$ large enough in order to obtain $\sigma_k > \rho_k$ and $\mathcal{F}(z) \leq 0$. Then taking $\|z^+\| \leq \sigma_k$ and $\|z^-\| = M_k$, we get

$$\mathcal{F}(z) \leq \sigma_k^{\mu+\nu-2}\left(\sigma_k^2 - M_k^2\right) + C|\partial\Omega|$$

and then choosing M_k large enough we find that $\mathcal{F} \leq 0$. In this way we have finished with the proof of the first part of Lemma 3.4. Then we choose β_k so that the second inequality holds. $\square$

Proof of Theorem 1.4: For a given $k \geq 1$, Lemmas 3.3 and 3.4 allows us to use Theorem 3.1. As a consequence the functional $\mathcal{F}$ has a critical value $c_k \in [\alpha_k, \beta_k]$. Since $\alpha_k \to \infty$ we get infinitely many critical values of $\mathcal{F}$. Therefore we have infinitely many solutions of (1.1)-(1.2).

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References

[1] T.Bartsch and D.G. de Figueiredo, *Infinitely many solutions of Nonlinear Elliptic Systems*, Progr. Nonlinear Differential Equations Appl., Vol. 35 (1999), 51–67, Birkhäuser Verlag Basel/ Switzerland.

[2] Ph. Clément, D.G. de Figueiredo and E. Mitidieri, *Positive solutions of semilinear elliptic systems*, Comm. Partial Differential Equations, 17 (1992), 923–940.

[3] D.G. de Figueiredo, *Semilinear elliptic systems: a survey of superlinear problems*, Resenhas IME-USP, 2 (1996), 373–391.

[4] D.G. de Figueiredo and P.L. Felmer, *On superquadratic elliptic systems*, Trans. Amer. Math. Soc., 343 (1994), 99–116.

[5] D.G. de Figueiredo and C.A. Magalhães, *On nonquadratic hamiltonian elliptic systems*, Adv. Differential Equations, 1(5) (1996), 881–898.

[6] P. Felmer, *Periodic solutions of 'superquadratic' Hamiltonian systems*, J. Differential Equations, 102 (1993), 188–207.

[7] P. Felmer, R.F. Manásevich and F. de Thélin, *Existence and uniqueness of positive solutions for certain quasilinear elliptic systems*, Comm. Partial Differential Equations, 17 (1992), 2013–2029.

[8] P. Felmer and Z-Q. Wang, *Multiplicity for symmetric indefinite functionals: application to hamiltonian and elliptic systems*, Topol. Methods Nonlinear Anal. 12, No.2, 207–226 (1998).

[9] J. Fernández Bonder, J.P. Pinasco and J.D. Rossi, *Existence results for a Hamiltonian elliptic system with nonlinear boundary conditions*, Electron. J. Differential Equations, Vol 1999(1999), No. 40, pp. 1–15.

[10] J. Fernández Bonder and J.D. Rossi, *Existence for an elliptic system with nonlinear boundary conditions via fixed point methods*. Adv. Differential Equations, Vol. 6, No. 1, January 2001, pp. 1–20.

[11] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, NY (1983).

[12] Hulshof and R.C.A.M. van der Vorst, *Differential systems with strongly indefinite variational structure*, J. Funct. Anal. 114, (1993), 32–58.

[13] J.L. Lions and E. Magenes, Problèmes aux limites non homogènes et applications, Vol. I, Dunod, Paris, (1968).

[14] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., no. 65, Amer. Math. Soc., Providence, R.I. (1986).

[15] M.E. Taylor, Partial Differential Equations. Vol. 1 Basic Theory, TAM 23, Springer-Verlag, New York. 1996.

[16] J. Thayer, Operadores Auto-adjuntos e Equacões Diferenciais Parciais, Projecto Euclides, IMPA. 1987.

[17] M. Willem, Minimax Theorems, Progr. Nonlinear Differential Equations Appl., Vol. 24 (1996), Birkhäuser Verlag Basel/ Switzerland.

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