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Gluing approximate solutions of minimum type on the Nehari manifold *

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Abstract

In the last decade or so, variational gluing methods have been widely used to construct homoclinic and heteroclinic type solutions of nonlinear elliptic equations and Hamiltonian systems. This note is concerned with the procedure of gluing mountain-pass type solutions. The first procedure to glue mountain-pass type solutions was developed through the work of Séré, and Coti Zelati - Rabinowitz. This procedure and its variants have been extensively used in many problems by now for nonlinear equations with superlinear nonlinearities. In this note we provide an alternative device to the by now standard procedure which allows us to glue minimizers on the Nehari manifold together as genuine, multi-bump type, solutions.

1 Introduction

In the last decade or so, variational gluing methods have been widely used to construct homoclinic and heteroclinic type solutions of nonlinear elliptic equations and Hamiltonian systems (see, e.g. Rabinowitz [7] and references therein). The idea is to first construct some basic solutions (or approximate solutions) which are characterized by minimax method and which are used as building blocks for construction of multi-bump type solutions. These multi-bump type solutions then are obtained by some gluing procedures and look roughly like sums of the basic solutions. The general idea is clear by now, though for different types of basic solutions one has to employ different procedures for the concrete problems. Different type of basic solutions have been glued together by various authors, which include minimizers and mountain-pass type solutions. In fact even $cat > 1$ solutions have been glued together, see for example Giannoni and Rabinowitz [4].

This note is concerned with the procedure of gluing mountain-pass type solutions. The first procedure to glue mountain-pass type solutions was developed through the work of Séré ([8] [9]) and Coti Zelati - Rabinowitz ([2] [3]), and this procedure and its variants have been extensively used in many problems by now for nonlinear equations with superlinear nonlinearities (see, e.g. Rabinowitz

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[7] and references therein). In these papers the basic solutions are mountain-pass type solutions. On the other hand, under slightly stronger conditions these mountain-pass solutions can also be characterized as minimizers of a constrained problem, namely, minimizers on the Nehari manifold. In this paper we provide an alternative device to the by now standard procedure which allows us to glue minimizers on the Nehari manifold (or local minimizers, approximate local minimizers) together as genuine (multi-bump type) solutions. Though the new procedure is somewhat parallel to the original one for mountain-pass solutions there are still technical complications needed to be fixed. On the other hand, it seems the new device in gluing minimizers on Nehari manifold is simpler than those for gluing mountain-pass solutions in the full space. For instance, one step involved in [2] and [3] is to do a minimization problem on some annulus regions and to use elliptic estimates to achieve the smallness of certain map. This step has to be done on a case by case basis for ODEs, PDEs with subcritical exponents and PDEs with critical exponents and seems to be somewhat laborsome for PDE problems, especially for those involving critical exponents ([5] [6]). Our device given here will *avoid* this step and treat all problems uniformly.

For simplicity we only present our device for an ODE problem to demonstrate the procedure. Although the results are not new, the procedure we use is different from the known one and may prove to be of advantage in dealing with some other problems with the presence of a Nehari manifold. The same device clearly works for analogous subcritical exponent periodic PDEs

$$-\Delta u + a(x)u = f(x,u), \text{ in } \mathbb{R}^N,$$

with suitable growth condition on f and periodic dependency in x; and presumably should also work for analogous critical exponent periodic PDEs.

2 An ODE problem

Consider

$$-u'' + a(t)u = f(t,u), \quad t \in \mathbb{R} \tag{1}$$

We look for homoclinic solutions of this equation, i.e.,solutions such that $\lim_{|t|\to\infty} u(t) = 0$ and $\lim_{|t|\to\infty} u'(t) = 0$. Assume

(f1) $a(t) \in C(\mathbb{R},\mathbb{R})$ is $T-$periodic and $\min_{\mathbb{R}} a(t) > 0$.

(f2) $f(t,u) \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is $T-$periodic in t.

(f3) $f_u(t,0) = 0$ and $|f_u(t,u)| \le C(1+|u|^p)$ for some $p > 1$.

(f4) There is a $\theta > 1$ such that $f'(t,u)u^2 \ge \theta f(t,u)u$ for all t and u.

There is a variational formulation of the problem. Namely,

$$I(u) = \frac{1}{2}\int_{\mathbb{R}} (|\dot{u}|^2 + au^2)dt - \int_{\mathbb{R}} F(t,u)dt$$

for $u \in X := H^1(\mathbb{R})$. Then critical points of I are solutions of (1). We use $\|\cdot\|$ to denote the norm in X.

There is alternative approach to the above, namely the Nehari manifold. Define

$$\gamma(u) := \int_{\mathbb{R}} f(t,u)u dt - \int_{\mathbb{R}} (|\dot{u}|^2 + au^2) dt,$$

and let

$$V = \{u \in X \setminus \{0\} \mid \gamma(u) = 0\}.$$

Then it is well known that under conditions (f_1 - f_4), V is a C^1 manifold and critical points of I on V are also critical points of I in X and therefore solutions of (1). We use the usual notations. $I^c = \{u \in V \mid I(u) \le c\}$, $I_c = \{u \in V \mid I(u) \ge c\}$, $I_a^b = I^b \cap I_a$, $K = \{u \in V \mid I'(u) = 0\}$, $K^c = K \cap I^c$. For an integer j, $\tau_j u = u(t-j)$ the translation of u. Then for any j, $\tau_j w \in K^c$.

Let

$$c := \inf_V I(u),$$

the ground state energy of I. Using the following compactness results for (PS) sequences of I one easily gets that c is always achieved at some u which is a ground state solution of (1).

Proposition 2.1 *Let $(u_n) \subset V$ be such that $I(u_n) \to b$ and $(I_{|V})'(u_n) \to 0$. Then there is an $l \in N$ (depending on b), $v_1, ..., v_l \in K \setminus \{0\}$, a subsequence of u_n and corresponding $(j_{i,n})_{i=1}^l \subset Z^l$ such that*

$$\|u_n - \sum_{i=1}^l \tau_{j_{i,n}} v_i\| \to 0, \quad \sum_{i=1}^l I(v_i) = b,$$

and for $i \neq \ell$, $|j_{i,n} - j_{\ell,n}| \to \infty$.

This is just a reformulation of Prop. 2.31 in [3], since V is a natural constraint of I in the sense that $(I_{|V})'(u) = 0$ iff $I'(u) = 0$.

Due to the translation invariance of the problem, there may be many solutions on the energy level c. We shall assume

$$K_c^c \text{ has an isolated point, say, } w. \tag{$*$}$$

For an integer $k \ge 2$, let $\vec{j} = (j_1, \cdots, j_k)$, a k-tuples of integers. We shall show that there are real solutions of (1) which roughly look like $\sum_{i=1}^k \tau_{j_i} w$. More precisely, let

$$2r_0 = \min\{\nu, \mu\} > 0,$$

where $\nu = \inf\{\|u\| \mid u \in K \setminus \{0\}\}$ and $\mu = \inf\{\|u - w\| \mid u \in K\}$.

Theorem 2.2 *Assume (f1 - f4) and K_c^c has an isolated point. For $0 < \alpha < \frac{c}{2}$ and $0 < r < r_0$ there is $j_0 > 0$ such that for all k-tuples of integers $\vec{j}$ satisfying $\min_{i \neq \ell} |j_i - j_\ell| > j_0$*

$$K_{kc-\alpha}^{kc+\alpha} \cap N_r\Big(\sum_{i=1}^k \tau_{j_i} w\Big) \neq \emptyset.$$

Here, $N_r(\cdot)$ denotes the r-neighborhood in X.

The proof of Theorem 2.2 is based on an indirect argument with the basic idea going back to [8] [2] [3]. Our procedure below is somewhat different from the one used in the original argument ([8] [2] [3]), and in a way simpler.

Step 1. First, for $R > 0$ we define a cut-off operator

$$T_R(u) = \rho(2R^{-1}|x|)u(x)$$

where $\rho(t) = 1$ for $0 \le t \le 1$ and $\rho(t) = 0$ for $t \ge 2$. With $\vec{j} = (j_1, \cdots, j_k)$ satisfying $\inf_{i\neq\ell} |j_i - j_\ell| > 2R$, for $y = (y_1, ..., y_k)$ with $y_i \ge 0$, $i = 1, ..., k$ and $\sum_{i=1}^k y_i = 1$, we define

$$G_0(y) = b(y) \sum_{i=1}^k y_i \tau_{j_i} T_R(w)$$

where $b(y) > 0$ is such that $G_0(y) \in V$. We fix a $\delta_0 \in (0, 1/k)$ so that $\max_y \gamma(\delta_0 b(y)w) < 0$ (which can be done due to (f_3)) and define

$$\Delta_k = \left\{ y = (y_1, ..., y_k) \in \mathbb{R}^k | \sum_{i=1}^k y_i = 1, y_i \ge \delta_0 \right\},$$

a $(k-1)$-dimensional simplex. Then $G_0 \in C(\Delta_k, V)$. By the explicit form of G_0 we have, as $R \to \infty$,

$$I(G_0(y)) = \sum_{i=1}^k I(b(y)y_i\tau_{j_i}T_R(w)) = \sum_{i=1}^k I(b(y)y_i w) + o(1) \le kc + o(1).$$

So we get

$$\lim_{R\to\infty} \max_{\Delta_k} I(G_0(y)) \le kc. \tag{2}$$

Note that $I(G_0(y_c)) \ge kc$, where $y_c = (\frac{1}{k}, ..., \frac{1}{k})$ the center of Δ_k.

Define

$$\Gamma = \left\{ G \in C(\Delta_k, V) \mid G_{|\partial\Delta_k} = G_0 \right\}.$$

For $\vec{j} = (j_1, ..., j_k)$ satisfying $\inf_{i\neq\ell} |j_i - j_\ell| > 2R$, used for G_0, we define for any $u \in V$

$$u^{(i)}(x) = \rho(R^{-1}|x - j_i|)u(x), \;\; i = 1, ..., k.$$

Lemma 2.3 *Given G_0 as above with $\vec{j} = (j_1, ..., j_k)$ and R fixed, for any $G \in \Gamma$ there exists $y_0 \in \Delta_k$ such that*

$$\gamma(G(y_0)^{(i)}) = 0, \;\; i = 1, ..., k.$$

Proof. Regarding Δ_k as a part of an affine $(k-1)$-plane which we denote by A^{k-1}, we see $A^{k-1} - (\frac{1}{k}, ..., \frac{1}{k})$ is a $(k-1)-$plane passing through the origin in $\mathbb{R}^k$ which we denote by $\tilde{A}^{k-1}$. For any $G \in \Gamma$ we introduce a map from $\tilde{\Delta}_k = \Delta_k - (\frac{1}{k}, ..., \frac{1}{k})$ into $\tilde{A}^{k-1}$ (with $\tilde{y} = y - (\frac{1}{k}, ..., \frac{1}{k})$), by

$$h(\tilde{y}) = (h_1, ..., h_k) := (\gamma(G(y)^{(1)}), ..., \gamma(G(y)^{(k)})).$$

Then the claim is proved if we can show that

$$\deg(h, \tilde{\Delta}_k, 0) \neq 0.$$

Note that 0 is the center of $\tilde{\Delta}_k$. But this degree only depends on G_0 because all G agree with G_0 on the boundary of $\tilde{\Delta}_k$. Finally, we compute the above degree for G_0 and we claim it is 1. We prove this by an induction in k. For $k = 2$ it is easy to check it by hand: when $y_1 = \delta_0$ and $y_2 = 1 - \delta_0$ we have $\tilde{y}_1 = \delta_0 - \frac{1}{2} < 0$ and $\tilde{y}_2 = 1 - \delta_0 - \frac{1}{2} > 0$; and $h_1 = \gamma(b\delta_0\tau_{j_1}T_R(w)) < 0$ and therefore $h_2 = \gamma(b(1-\delta_0)\tau_{j_2}T_R(w)) > 0$ for $h_1 + h_2 = 0$. At the other end point of $\tilde{\Delta}_2$ we have similar computations, which together shows that h is homotopy to the identity map.

Now for $k \geq 3$, $\tilde{\Delta}_k$ has k faces (opposite to each vertex and denoted by F_i). On the ith-face F_i, if $y = (y_1, ..., y_k) \in F_i$ then $y_i = \delta_0 - \frac{1}{k}$. And we get $h_i = \gamma(b\delta_0\tau_{j_i}T_R(w)) < 0$. Using this fact, we may first project (by radial scalings on $\tilde{A}^{k-1}$) the image of h to $\epsilon\tilde{\Delta}_k$ for some $\epsilon >$ small. Then using an expansion scaling we may have the image of $\partial\tilde{\Delta}_k$ into itself. We denote this operation by P, i.e., Ph is a map from $\tilde{\Delta}_k \to \tilde{A}^{k-1}$ such that $Ph(\partial\tilde{\Delta}_k) \subset \partial\tilde{\Delta}_k$. By the homotopy property, $\deg(h, \tilde{\Delta}_k, 0) = \deg(Ph, \tilde{\Delta}_k, 0)$. Note that taking $y = 0$ we see $\gamma(g_1(y)) = \cdots = \gamma(g_k(y)) = 0$. By some standard properties of the degree (see, e.g. [1]),

$$\deg(Ph, \tilde{\Delta}_k, 0) = \deg(Ph, \partial\tilde{\Delta}_k, \partial\tilde{\Delta}_k).$$

Now on F_1 the center c_1 has coordinates $y_1 = \delta_0 - \frac{1}{k}$ and for $i = 2, ..., k$, $y_i = \frac{1-\delta_0}{k-1} - \frac{1}{k}$. Using this it is easy to see that c_1 is not covered by $Ph(\cup_{i=2}^k F_i)$, for if not $c_1 = Ph(y)$ for some $y \in F_i$ with $i \geq 2$, then we have $h_i(y) < 0$ and therefore $(Ph)_i(y) < 0$, this is a contradiction with $y_i > 0$ for c_1. By the excision property

$$\deg(Ph, \partial\tilde{\Delta}_k, \partial\tilde{\Delta}_k) = \deg(Ph, F_1, c_1).$$

However, this is what we would get from the $(k-1)$-map. The induction is complete. ♢

We need another technical result.

Lemma 2.4 *Let $u \in V$ be such that $u^{(i)} \in V$ for all $i = 1, ..., k$ (obtained by using $\vec{j} = (j_1, ..., j_k)$ satisfying* $\inf_{i\neq\ell} |j_i - j_\ell| > 2R$*). Then $I(u) \geq kc$.*

Proof. First, we write $W_R = \bigcup_{i=1}^k B_R(j_i)$. Then

$$I(u) \quad = \quad \frac{1}{2}\int |\nabla u|^2 + a|u|^2 - \int F(x, u)$$

$$
\begin{aligned}
= \quad & \sum_{i=1}^{k} \Big(\frac{1}{2} \int_{B_{2R}(j_i)} |\nabla u^{(i)}|^2 + a|u^{(i)}|^2 - \int_{B_{2R}(j_i)} F(x, u^{(i)}) \Big) \\
& + \frac{1}{2} \int_{\mathbb{R} \setminus W_{2R}} |\nabla u|^2 + a|u|^2 - \int_{\mathbb{R} \setminus W_{2R}} F(x, u) \\
& + \sum_{i=1}^{k} \frac{1}{2} \int_{B_{2R}(j_i) \setminus B_R(j_i)} \Big(|\nabla((1-\rho)u)|^2 + (1-\rho)^2 a u^2 \\
& + 2\nabla(\rho u) \nabla((1-\rho)u) + 2\rho(1-\rho) a u^2 \Big) \\
& - \sum_{i=1}^{k} \int_{B_{2R}(j_i) \setminus B_R(j_i)} (F(x, u) - F(x, u^{(i)}))
\end{aligned}
$$

Using $u \in V$ and $u^{(i)} \in V$ for all $i = 1, ..., k$, we get

$$
\begin{aligned}
& \int_{\mathbb{R} \setminus W_{2R}} |\nabla u|^2 + a|u|^2 - \int_{\mathbb{R} \setminus W_{2R}} f(x, u) u \\
& + \sum_{i=1}^{k} \int_{B_{2R}(j_i) \setminus B_R(j_i)} \Big(|\nabla((1-\rho)u)|^2 + (1-\rho)^2 a u^2 \\
& \qquad + 2\nabla(\rho u) \nabla((1-\rho)u) + 2\rho(1-\rho) a u^2 \Big) \\
& - \sum_{i=1}^{k} \int_{B_{2R}(j_i) \setminus B_R(j_i)} (f(x, u) u - f(x, u^{(i)}) u^{(i)}) \quad = \quad 0\,.
\end{aligned}
$$

Bringing this into the earlier formula we have

$$
\begin{aligned}
I(u) \quad \geq \quad & kc + \int_{\mathbb{R} \setminus W_{2R}} (\frac{1}{2} f(x, u) u - F(x, u)) \\
& + \sum_{i=1}^{k} \int_{B_{2R}(j_i) \setminus B_R(j_i)} \Big(\frac{1}{2} f(x, u) u - \frac{1}{2} f(x, u^{(i)}) u^{(i)} \\
& - F(x, u) + F(x, u^{(i)}) \Big)
\end{aligned}
$$

which implies $I(u) \geq kc$ since the last two terms on the right hand side are both non-negative. Indeed, by (f_4) we have

$$
\frac{1}{2} f(x, u) u - F(x, u) \geq 0,
$$

and, writing $g(t) = \frac{1}{2} f(x, tu) tu - F(x, tu)$ by the mean value theorem, we have for some $\xi \in (0, 1)$,

$$
\frac{1}{2} f(x, u) u - \frac{1}{2} f(x, u^{(i)}) u^{(i)} - F(x, u) + F(x, u^{(i)})
$$

$$
\begin{aligned}
&= \frac{1}{2}f(x,u)u - \frac{1}{2}f(x,\rho(R^{-1}|x-j_i|)u)\rho(R^{-1}|x-j_i|)u - F(x,u) \\
&\quad +F(x,\rho(R^{-1}|x-j_i|)u) \\
&= g(1) - g(\rho) \\
&= g'(\rho+\xi(1-\rho))(1-\rho) \\
&= \frac{1}{2}\left\{f'(x,[\xi+\rho(1-\xi)]u)[\xi+\rho(1-\xi)]u^2 - f(x,[\xi+\rho(1-\xi)]u)u\right\}(1-\rho) \\
&\geq 0\,.
\end{aligned}
$$

which completes the present proof $\diamondsuit$

Let $z_R = b_R \sum_{i=1}^k \tau_{j_i} T_R(w)$ with $\inf_{i\neq\ell} |j_\ell - j_i| > 2R$, where $b_R > 0$ is such that $z_R \in V$. Note $b_R \to 1$ as $R \to \infty$.

For any $\epsilon > 0$, by choosing $R > 0$ large we may get, by (2), $\max_{\Delta_k} I(G_0(y)) < kc + \epsilon$. Also we remark that when $r_0 > r > 0$ is fixed, for all small ϵ and large R it holds that $I(G_0(y)) \geq kc - \epsilon$ implies $G_0(y) \in N_{\frac{r}{8}}(z_R)$. We fix $r > 0$ now such that for all $R \geq 1$ if $G \in \Gamma$ satisfying $\|G(y) - G_0(y)\| \leq r$ then $G(y)^{(i)} \neq 0$ for all y and $i = 1, ..., k$.

Step 2. If we assume the conclusion of Theorem 2.2 is not true, using a deformation argument from a pseudo-negative gradient flow we deform G_0 to a map $G_1 : \Delta_k \to V$ such that $\max_{\Delta_k} I(G_1(y)) \leq kc - \epsilon$, $\|G_1(y) - G_0(y)\| \leq r$ and $G_1|_{\partial\Delta_k} = G_0$. Then using Lemmas 2.3 and 2.4 we will have a contradiction. We need the following lemma.

Lemma 2.5 *There exist $\delta_r > 0$ and $R_r > 0$ such that for all $R \geq R_r$ and for all $u \in N_r(z_R) \setminus N_{\frac{r}{8}}(z_R)$*

$$\|I'(u)\| \geq \delta_r.$$

Proof. If the conclusion is not true, we would have a sequence $R_n \to \infty$ and $u_n \in N_r(z_R) \setminus N_{\frac{r}{8}}(z_R)$ such that $I'(u_n) \to 0$. Then (u_n) is a $(\mathrm{PS})_b$ sequence for I with some b. By Proposition 2.1

$$\|u_n - \sum_{i=1}^{l} \tau_{j_{i,n}} v_i\| \to 0$$

for some integer l and $v_1, .., v_l \in K$ and $|j_{i,n} - j_{\ell,n}| \to \infty$ for $i \neq \ell$. Since as $R_n \to \infty$, $\|z_{R_n} - \sum_{i=1}^k \tau_{j_{i,R_n}} w\| \to 0$, we get

$$\|\sum_{i=1}^{l} \tau_{j_{i,n}} v_i - \sum_{i=1}^{k} \tau_{j_{i,R_n}} w\| \to 0.$$

From this it is easy to argue by using (*) that $l = k$, $v_i = w$ for all i and for n large $j_{i,n} = j_{i,R_n}$. This is a contradiction to $\|u_n - z_{R_n}\| \geq \frac{r}{8}$. $\diamondsuit$

Now we can finish the proof of our main theorem.

We take $0 < \epsilon < \frac{3r\delta_r}{8}$ and $R \geq R_r$ so that $\max_{\Delta_k} I(G_0(y)) < kc + \epsilon$ and that $I(G_0(y)) \geq kc - \epsilon$ implies $G_0(y) \in N_{\frac{r}{8}}(z_R)$.

Next, choose $\epsilon < \epsilon_1 < c$. Let

$$\phi(u) = \frac{\|u - X \setminus N_r(z_R)\|}{\|u - N_{\frac{r}{2}}(z_R)\| + \|u - X \setminus N_r(z_R)\|}.$$

and let U be a locally Lipschitz pseudo-gradient vector field for I on $V \setminus K$ such that

(i) $\|U(u)\| \leq \frac{4\epsilon_1}{\|I'(u)\|}$,

(ii) $I'(u)U(u) \geq 2\epsilon_1$.

Let η be the flow given by the solution of

$$\frac{d\eta}{dt} = -\phi(\eta)U(\eta), \eta(0,u) = u.$$

Let $u = G_0(y)$ be such that $I(u) \geq kc - \epsilon$ so that $u \in N_{\frac{r}{8}}(z_R)$. Using Proposition 2.1 we can show either (i) $\eta(t,u)$ reaches $\partial B_r(z_R)$ for some $t \leq 1$ or (ii) $\eta(t,u)$ remains in $B_r(z_R)$ for $t \in [0,1]$. If (i) occurs, in some time interval $[t_1, t_2]$, $\eta(t,u)$ reaches from $\partial B_{\frac{r}{8}}(z_R)$ to $\partial B_r(z_R)$. Then it must reach $I^{kc-\epsilon}$ already in the time interval. Otherwise,

$$\frac{7r}{8} = \|\eta(t_2,u) - \eta(t_1,u)\| \leq \int_{t_1}^{t_2} \phi(\eta)\|U(\eta(t,u)\|dt \leq \frac{4\epsilon_1}{\delta_r}\int_{t_1}^{t_2} \phi(\eta)dt,$$

and

$$2\epsilon \geq I(\eta(t_2,u)) - I(\eta(t_1,u)) = \int_{t_1}^{t_2} \frac{dI}{dt}(\eta(t,u))dt \geq 2\epsilon_1 \int_{t_1}^{t_2} \phi(\eta)dt.$$

This implies $\epsilon \geq \frac{7r\delta_r}{8}$, a contradiction. Thus if (i) occurs there is a unique $\sigma(u) \leq 1$ such that $I(\eta(\sigma(u),u)) = kc - \epsilon$. If (ii) occurs we may have either $\eta(t,u)$ has to go from $B_{\frac{r}{8}}(z_R)$ to the boundary of $B_{\frac{r}{2}}(z_R)$ and similar argument shows that there is a unique $\sigma(u) \leq 1$ such that $I(\eta(\sigma(u),u)) = kc - \epsilon$, or $\eta(t,u)$ stays in $B_{\frac{r}{2}}(z_R)$ for $t \in [0,1]$. In the latter case if $\eta(t,u)$ does not reach $I^{kc-\epsilon}$ we would have ϕ equal to 1 along $\eta(t,u)$ and we have $2\epsilon \geq I(\eta(0,u)) - I(\eta(1,u)) \geq 2\epsilon_1$, a contradiction. In both cases, we have $\|\eta(\sigma(u),u) - u\| \leq r$. We get $G_1(y) = \eta(\sigma(G_0(y)), G_0(y))$ which is a continuous map from Δ_k into V and agrees with $G_0(y)$ on $\partial\Delta_k$. Moreover,

$$\|G_1(y) - G_0(y)\| \leq r. \tag{3}$$

To finish the proof of Theorem 2.2, let us produce a contradiction as follows. Applying Lemma 2.3 to $G_1(y)$ we conclude that there exists $y \in \Delta_k$ such that

$$\gamma(G_1(y)^{(i)}) = 0, \quad i = 1, ..., k.$$

Due to (3), we obtain $G_1(y)^{(i)} \neq 0$ for $i = 1, ..., k$, i.e., $G_1(y)^{(i)} \in V$ for $i = 1, ..., k$. Applying Lemma 2.4, we get a contradiction with $\max I(G_1(y)) \leq kc - \epsilon$.

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