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SYSTEMS OF MULTI-DIMENSIONAL LAPLACE TRANSFORMS AND A HEAT EQUATION

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Abstract. The object of this paper is to establish several new theorems involving systems of two-dimensional Laplace transforms containing five to seven equations. These systems can be used to calculate new Laplace transform pairs. In the second part, a boundary value problem is solved by using the double Laplace transformation.

1. Introduction

The two-dimensional Laplace transform of function $f(x, y)$ is defined by Ditkin and Prudnikov [5] as follows:

$$F(p, q) = pq \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y) dx\ dy$$

and symbolically is denoted by $F(p, q) \doteqdot f(x, y)$ where the symbol $\doteqdot$ is called "operational". The correspondence between $f(x, y)$ and $F(p, q)$ may be interpreted as a transformation which transforms the function $f(x, y)$ into the function $F(p, q)$. Thus we call $F(p, q)$ the image of $f(x, y)$ and $f(x, y)$ is the original of $F(p, q)$. In this paper we derive new rules on systems involving double Laplace transformations. We also solve a boundary value problem.

2. Systems of two-dimensional Laplace transforms.

2.1. The Image of $G(\frac{1}{4x}, \frac{1}{4y})$.

The six systems of two dimensional Laplace transforms are obtained in this s ection. Each system contains five to seven equations. These systems can be used to calculate one of the functions, when the others are known, especially to compute the image of $x^{i/2}y^{j/2}G(\frac{1}{4x}, \frac{1}{4y})$ when $i = -3, \pm 5$ and $j = \pm 1, \pm 3$. They are further used to obtain new Laplace transform pairs. These systems are proved in six theorems and some typical examples are given after each theorem. The image of $x^{i/2}y^{j/2}G(\frac{1}{4x}, \frac{1}{4y})$ when $i = 3$ and $j = \pm 1, 3$ is given by Dahiya [3].

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Theorem 2.1. *Let*

(I) $F(p,q) \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} f(x,y)$

(II) $\psi(p,q) = -pq\frac{\partial}{\partial p}\left(\frac{F(p,q)}{pq}\right) \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} xf(x,y)$

(III) $\sigma(p,q) = -pq\frac{\partial}{\partial q}\left(\frac{F(p,q)}{pq}\right) \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} yf(x,y)$

(IV) $\varphi(p,q) = pq\frac{\partial^2}{\partial p\partial q}\left(\frac{F(p,q)}{pq}\right) \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} xyf(x,y)$

(V) $G(p,q) \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} xyf\left(\sqrt{x},\sqrt{y}\right)$.

Then

$$\begin{aligned}&(pq)^{1/2}F\left(\sqrt{p},\sqrt{q}\right) + pq^{1/2}\psi\left(\sqrt{p},\sqrt{q}\right) + p^{1/2}q\sigma\left(\sqrt{p},\sqrt{q}\right) + pq\varphi\left(\sqrt{p},\sqrt{q}\right)\\ &\qquad \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} \frac{1}{4\pi}(xy)^{-3/2}G\left(\frac{1}{4x},\frac{1}{4y}\right)\end{aligned} \tag{2.1.1}$$

Proof. We start from two operational relations

$$p(1+s\sqrt{p})e^{-s\sqrt{p}} \mathrel{\overset{\cdot}{\underset{\cdot}{=}}} \frac{1}{4\sqrt{\pi}}s^3x^{-5/2}e^{-s^2/4x}$$

$$q(1+t\sqrt{q})e^{-t\sqrt{q}} \mathrel{\overset{\cdot}{\underset{\cdot}{=}}} \frac{1}{4\sqrt{\pi}}t^3y^{-5/2}e^{-t^2/4y}$$

We multiply together the above equations to get

$$pq(1+\sqrt{p})(1+t\sqrt{q})e^{-s\sqrt{p}-t\sqrt{q}} \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} \frac{1}{16\pi}(xy)^{-5/2}(st)^3e^{-\frac{s^2}{4x}-\frac{t^2}{4y}}.$$

Now, we multiply both sides by $f(s,t)$ and integrate with respect to s and t over the positive quarter plane.

$$\begin{aligned}&pq\int_0^\infty\int_0^\infty (1+s\sqrt{p})(1+t\sqrt{q})e^{-s\sqrt{p}-t\sqrt{q}}f(s,t)dsdt\\ &\quad \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} \frac{1}{16\pi}(xy)^{-5/2}\int_0^\infty\int_0^\infty e^{-\frac{s^2}{4x}-\frac{t^2}{4y}}(st)^3f(s,t)dsdt.\end{aligned}$$

We make the change of variables $s=\sqrt{u}$ and $t=\sqrt{v}$ on the right hand side to obtain

$$\begin{aligned}&pq\int_0^\infty\int_0^\infty e^{-s\sqrt{p}-t\sqrt{q}}f(s,t)dsdt + p^{3/2}q\int_0^\infty\int_0^\infty e^{-s\sqrt{p}-t\sqrt{q}}sf(s,t)dsdt\\ &+pq^{3/2}\int_0^\infty\int_0^\infty e^{-s\sqrt{p}-t\sqrt{q}}tf(s,t)dsdt + (pq)^{3/2}\int_0^\infty\int_0^\infty e^{-s\sqrt{p}-t\sqrt{q}}(st)f(s,t)dsdt\\ &\qquad \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} \frac{1}{64\pi}(xy)^{-5/2}\int_0^\infty\int_0^\infty e^{-\frac{u}{4x}-\frac{v}{4y}}(uv)f(\sqrt{u},\sqrt{v})dudv.\end{aligned}$$

Finally by using (V) on the right hand side and (I), (II), (III), (IV) on the left hand side we get

$$\begin{aligned}&(pq)^{1/2}F(\sqrt{p},\sqrt{q}) + pq^{1/2}\psi(\sqrt{p},\sqrt{q}) + p^{1/2}q\sigma(\sqrt{p},\sqrt{q}) + pq\varphi(\sqrt{p},\sqrt{q})\\ &\quad \mathrel{\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}}} \frac{1}{4\pi}(xy)^{-3/2}G\left(\frac{1}{4x},\frac{1}{4y}\right). \quad \square\end{aligned}$$

Thus we have the following system of operational relations

$$\begin{aligned} F(p,q) &\doteqdot f(x,y) & \varphi(p,q) &\doteqdot xyf(x,y) \\ \psi(p,q) &\doteqdot xf(x,y) & \sigma(p,q) &\doteqdot yf(x,y) \\ G(p,q) &\doteqdot xyf(\sqrt{x},\sqrt{y}) \end{aligned}$$

$$\begin{aligned} K(p,q) &= (pq)^{1/2}F(\sqrt{p},\sqrt{q}) + pq^{1/2}\psi(\sqrt{p},\sqrt{q}) + p^{1/2}q\sigma(\sqrt{p},\sqrt{q}) + pq\psi(\sqrt{p},\sqrt{q}) \\ &\doteqdot \frac{1}{4\pi}(xy)^{-3/2}G\left(\frac{1}{4x},\frac{1}{4y}\right) \end{aligned}$$

and it is always possible to calculate one of the twelve functions, when the others are known.

Hence by using the above system we can derive twelve rules. For example, we obtain:

1) the original of the function $G(p,q)$ from the original of the function $F(p,q)$ by replacing x and y by $\sqrt{x}$ and $\sqrt{y}$ respectively, and finally multiplying by xy.
2) the original of the function $F(p,q)$ from the original of the function $\varphi(p,q)$ by multiplying by $\frac{1}{xy}$.
3) the image of the function $xyf(\sqrt{x},\sqrt{y})$ from the original of the function $K(p,q)$ by replacing x and y by $\frac{1}{4p}$ and $\frac{1}{4q}$ respectively, and finally multiplying by $\frac{\pi}{16}(pq)^{-3/2}$
4) the image of the function $xf(x,y)$ from the image of the function $\frac{1}{4\pi}(xy)^{-3/2}G(\frac{1}{4x},\frac{1}{4y})$ by replacing p and q by p^2 and q^2 respectively, then subtracting $pqF(p,q)+pq^2\sigma(p,q)+(pq)^2\varphi(p,q)$, and finally multiplying by $\frac{1}{p^2q}$.

Example 2.1. Let $f(x,y) = yJ_2(2\sqrt{xy})$, then

$$\begin{aligned} F(p,q) &= \frac{p}{(pq+1)^2}, & \sigma(p,q) &= \frac{p(3pq+1)}{q(pq+1)^3} \quad [5;\ p.137] \\ \psi(p,q) &= \frac{2pq}{(pq+1)^3}, & \varphi(p,q) &= \frac{6p^2q}{(pq+1)^4} \\ G(p,q) &= \frac{\sqrt{\pi}}{64}\cdot\frac{48pq-16\sqrt{pq}+1}{p^{5/2}q^3}e^{-\frac{1}{2\sqrt{pq}}} \end{aligned}$$

Using (2.1.1) and simplifying a bit, results in

$$\frac{p\sqrt{q}(6pq+4\sqrt{pq}+1)}{(\sqrt{pq}+1)^4} \doteqdot \frac{1}{\sqrt{\pi}}(4xy-16\sqrt{xy}+3)\sqrt{y}e^{-2\sqrt{xy}}. \quad \square$$

Example 2.2. Let $f(x,y)=\frac{1}{\sqrt{x^2+y^2}}$, then

$$F(p,q)=\frac{pq}{\sqrt{p^2+q^2}}\mathrm{Ln}\frac{p+q+\sqrt{p^2+q^2}}{p+q-\sqrt{p^2+q^2}} \qquad [5;\ \text{p.147}]$$

$$\psi(p,q)=\frac{p^2q}{\sqrt{(p^2+q^2)^3}}\mathrm{Ln}\frac{p+q+\sqrt{p^2+q^2}}{p+q-\sqrt{p^2+q^2}}+\frac{q(q-p)}{p^2+q^2}$$

$$\sigma(p,q)=\frac{pq^2}{\sqrt{(p^2+q^2)^3}}\mathrm{Ln}\frac{p+q+\sqrt{p^2+q^2}}{p+q-\sqrt{p^2+q^2}}+\frac{p(p-q)}{p^2+q^2}$$

$$\varphi(p,q)=\frac{2p^2q^2}{\sqrt{(p^2+q^2)^5}}\mathrm{Ln}\frac{p+q+\sqrt{p^2+q^2}}{p+q-\sqrt{p^2+q^2}}+\frac{(p+q)(p^2-3pq+q^2)}{(p^2+q^2)^2}$$

$$G(p,q)=\frac{\sqrt{\pi}}{2}\cdot\frac{(p+q+3\sqrt{pq})}{2\sqrt{pq}(\sqrt{p}+\sqrt{q})^3} \qquad [5;\ \text{p.127}]$$

Using (2.1.1) and simplifying a bit, we obtain

$$\frac{pq(2p^2+2q^2+7pq)}{\sqrt{(p+q)^5}}\ \mathrm{Ln}\frac{\sqrt{p}+\sqrt{q}+\sqrt{p+q}}{\sqrt{p}+\sqrt{q}-\sqrt{p+q}}+\frac{pq(\sqrt{p}+\sqrt{q})(p-3\sqrt{pq}+q)}{(p+q)^2}$$
$$\mathrel{\substack{\cdot\cdot\\=\\\cdot\cdot}}\frac{1}{2\sqrt{\pi}}\cdot\frac{x+y+3\sqrt{xy}}{\sqrt{xy}(\sqrt{x}+\sqrt{y})^3}. \quad \square$$

Example 2.3. Let $f(x,y)=\mathrm{Ln}\ (x^2+y^2)$, then

$$F(p,q)=2\Big(\Gamma'(1)+\frac{\frac{\pi}{2}pq-p^2\mathrm{Ln}\ q-q^2\mathrm{Ln}\ p}{p^2+q^2}\Big) \qquad [5;\ \text{p. 147}]$$

$$\psi(p,q)=2\Big(\frac{\Gamma'(1)}{p}+\frac{p(q^2-p^2)\mathrm{Ln}\ q-q^2(p^{-1}q^2+3p)\ \mathrm{Ln}\ p+pq^2+p^{-1}q^4+\pi p^2q}{(p^2+q^2)^2}\Big)$$

$$\sigma(p,q)=2\Big(\frac{\Gamma'(1)}{q}+\frac{q(p^2-q^2)\mathrm{Ln}\ p-p^2(p^2q^{-1}+3q)\mathrm{Ln}\ p+pq^2+p^4q^{-1}+\pi pq^2}{(p^2+q^2)^2}\Big)$$

$$\psi(p,q)=2\Big(\frac{\Gamma'(1)}{pq}+\frac{pq(3p^2-6q^2-p^{-2}q^4)\ \mathrm{Ln}\ p+pq(3q^2-6p^2-p^4q^{-2})\ \mathrm{Ln}\ q}{(p^2+q^2)^3}$$
$$+\frac{-pq(p^2+q^2)+p^5q^{-1}+p^{-1}q^5+4\pi(pq)^2}{(p^2+q^2)^3}\Big)$$

$$G(p,q)=\frac{q(3-p^{-1}q))\mathrm{Ln}\ p-p(3-pq^{-1})\mathrm{Ln}\ q+p-q+p^{-1}q^2-p^2q^{-1}}{(q-p)^3}+\frac{\Gamma(1)}{pq}.$$

Using (2.1.1) and manipulating a bit, we get

$$\frac{2\sqrt{pq}(2p^3+2q^3+3p^2q+3pq^2-pq)+\pi pq(3p^2+3q^2+14pq)}{(p+q)^3}$$
$$-\frac{4q^2\sqrt{pq}(3p+q)\mathrm{Ln}\ p+4p^2\sqrt{pq}(3p+q)\mathrm{Ln}\ q}{(p+q)^3}+8\Gamma'(1)\sqrt{pq}$$
$$\mathrel{\substack{\cdot\cdot\\=\\\cdot\cdot}}\frac{4}{\pi}\left(\frac{x^2(x-3y)\mathrm{Ln}\ (4x)+y^2(3x-y)\mathrm{Ln}\ (4x)+(x-y)(x^2+y^2)}{\sqrt{xy}(x-y)^3}+\frac{\Gamma'(1)}{\sqrt{xy}}\right). \quad \square$$

Theorem 2.2. *Let*

(I) $F(p,q) \mathrel{\overset{..}{\underset{..}{=}}} f(x,y)$ *(II)* $\varphi(p,q) \mathrel{\overset{..}{\underset{..}{=}}} xyf(x,y)$ *(III)* $\psi(p,q) \mathrel{\overset{..}{\underset{..}{=}}} xf(x,y)$

(IV) $\sigma(p,q) \mathrel{\overset{..}{\underset{..}{=}}} yf(x,y)$ *(V)* $G(p,q) \mathrel{\overset{..}{\underset{..}{=}}} xy^{-1/2}f(\sqrt{x},\sqrt{y})$

Then

$$\begin{aligned} K(p,q) =& p^{1/2}q^{-1}F(\sqrt{p},\sqrt{q}) + pq^{-1}\psi(\sqrt{p},\sqrt{q}) \\ &+ p^{1/2}q^{-1/2}\sigma(\sqrt{p},\sqrt{q}) + pq^{-1/2}\varphi(\sqrt{p},\sqrt{q}) \\ \mathrel{\overset{..}{\underset{..}{=}}}& \frac{2}{\pi}x^{-3/2}y^{3/2}G\left(\frac{1}{4x},\frac{1}{4y}\right). \end{aligned} \tag{2.1.2}$$

Proof. We know that

$$p(1+s\sqrt{p})e^{-s\sqrt{p}} \doteqdot \frac{1}{4\pi}s^3x^{-5/2}e^{-\frac{s^2}{4x}}, q^{-1/2}(1+t\sqrt{q})e^{-t\sqrt{q}} \doteqdot \frac{2}{\sqrt{\pi}}y^{1/2}e^{-\frac{t^2}{4y}}$$

Multiplying together the above equations, results in

$$pq^{-1/2}(1+s\sqrt{p})(1+t\sqrt{q})e^{-s\sqrt{p}-t\sqrt{q}} \mathrel{\overset{..}{\underset{..}{=}}} \frac{1}{2\pi}x^{-5/2}y^{1/2}s^3e^{-\frac{s^2}{4x}-\frac{t^2}{4y}}.$$

If we multiply both sides by $f(s,t)$ and integrate with respect to s and t over the positive quarter plane, we obtain

$$\begin{aligned} &pq^{-1/2}\int_0^\infty\int_0^\infty (1+s\sqrt{p})(1+t\sqrt{q})e^{-s\sqrt{p}-t\sqrt{q}}f(s,t)dsdt \\ &\mathrel{\overset{..}{\underset{..}{=}}} \frac{1}{2\pi}x^{-5/2}y^{1/2}\int_0^\infty\int_0^\infty e^{-\frac{s^2}{4x}-\frac{t^2}{4y}}s^3f(s,t)dsdt \end{aligned}$$

which is

$$\begin{aligned} &pq^{-1/2}\int_0^\infty\int_0^\infty e^{-s\sqrt{p}-t\sqrt{q}}f(s,t)dsdt + p^{3/2}q^{-1/2}\int_0^\infty\int_0^\infty e^{-s\sqrt{p}-t\sqrt{q}}sf(s,t)dsdt \\ &+ p\int_0^\infty\int_0^\infty e^{-s\sqrt{p}-t\sqrt{q}}tf(s,t)dsdt + p^{3/2}\int_0^\infty\int_0^\infty e^{-s\sqrt{p}-t\sqrt{q}}(st)f(s,t)dsdt \\ &\mathrel{\overset{..}{\underset{..}{=}}} \frac{1}{8\pi}x^{-5/2}y^{1/2}\int_0^\infty\int_0^\infty e^{-\frac{u}{4x}-\frac{v}{4y}}uv^{-1/2}f(\sqrt{u},\sqrt{v})dudv. \end{aligned}$$

By using (V) on the right hand side and (I), (II), (III), (IV) on the left hand side, we get the desired result. □

Theorem 2.2 gives a system of six equations (I)–(V) and (2.1.2). For this system we will formulate twelve rules analogous to those mentioned for the system obtained from Theorem 2.1. For example, we can derive:

1) The original of the function $K(p,q)$ from the image of the function $xy^{-1/2}f(\sqrt{x},\sqrt{y})$ by replacing p and q by $\frac{1}{4x}$ and $\frac{1}{4y}$ respectively, and finally multiplying by $\frac{2}{\pi}x^{-3/2}y^{3/2}$.
2) The image of the function $\frac{2}{\pi}x^{-3/2}y^{3/2}G(\frac{1}{4x},\frac{1}{4y})$ from the images of the functions $f(x,y), xf(x,y), yf(x,y)$ and $xyf(x,y)$ by multiplying by pq^{-2}, p^2q^{-2}, pq^{-1} and p^2q^{-1} respectively, then adding them, and finally replacing p and q by $\sqrt{p}$ and $\sqrt{q}$ respectively.

Example 2.4. Let $f(x,y) = \frac{(xy)^a}{\Gamma(2a+1)} J_{2a}(2\sqrt{xy})$ where $\mathrm{Re}(2a+1) > 0$, then

$$F(p,q) = \frac{pq}{(pq+1)^{2a+1}}, \qquad \sigma(p,q) = \frac{(2a+1)p^2q}{(pq+1)^{2a+2}} \qquad [5;\ \text{p.137}]$$

$$\psi(p,q) = \frac{(2a+1)pq^2}{(pq+1)^{2a+2}}, \qquad \psi(p,q) = (2a+1)\frac{((2a+1)pq-1)pq}{(pq+1)^{2a+3}}$$

$$G(p,q) = \frac{\sqrt{\pi}}{4^{a+1}} \cdot \frac{4(a+1)\sqrt{pq}-1}{\Gamma(2a+1)p^{a+\frac{3}{2}}q^a} e^{-\frac{1}{2\sqrt{pq}}} \qquad [5;\ \text{p.144}]$$

Using (2.1.2) and simplifying a bit, we obtain

$$\frac{p((a+1)^2pq + (2a+3)\sqrt{pq}+1)}{\sqrt{q}(\sqrt{pq}+1)^{2a+3}} \overset{\cdot\cdot}{\underset{\cdot\cdot}{=}} \frac{4^{a+1}}{\Gamma(2a+1)\sqrt{\pi}}(a+1-\sqrt{xy})x^{a-\frac{1}{2}}y^{a+1}e^{-2\sqrt{xy}}.$$

Theorem 2.3. *Let*
(I) $F(p,q) \overset{\cdot\cdot}{\underset{\cdot\cdot}{=}} f(x,y)$ *(II)* $\psi(p,q) \overset{\cdot\cdot}{\underset{\cdot\cdot}{=}} xf(x,y)$ *(III)* $G(p,q) \overset{\cdot\cdot}{\underset{\cdot\cdot}{=}} xf(\sqrt{x},\sqrt{y})$
Then

$$(pq)^{1/2}F(\sqrt{p},\sqrt{q}) + pq^{1/2}\psi(\sqrt{p},\sqrt{q}) \overset{\cdot\cdot}{\underset{\cdot\cdot}{=}} \frac{1}{2\pi}x^{-3/2}y^{-1/2}G\left(\frac{1}{4x},\frac{1}{4y}\right) \tag{a}$$

Moreover if we let
(IV) $H(p,q) \overset{\cdot\cdot}{\underset{\cdot\cdot}{=}} xy^{-1/2}f(\sqrt{x},\sqrt{y})$
then

$$p^{1/2}F(\sqrt{p},\sqrt{q}) = p\psi(\sqrt{p},\sqrt{q}) \overset{\cdot\cdot}{\underset{\cdot\cdot}{=}} \frac{1}{\pi}x^{-3/2}y^{1/2}H\left(\frac{1}{4x},\frac{1}{4y}\right). \tag{b}$$

Proof. We have the following operational relations

$$p(1+s\sqrt{p})e^{-s\sqrt{p}} \doteqdot \frac{1}{4\sqrt{\pi}}s^3x^{-5/2}e^{-\frac{s^2}{4x}} \tag{2.1.3}$$

$$qe^{-t\sqrt{q}} \doteqdot \frac{1}{2\sqrt{\pi}}ty^{-3/2}e^{-\frac{t^2}{4y}} \tag{2.1.4}$$

$$q^{1/2}e^{-t\sqrt{q}} \doteqdot \frac{1}{\sqrt{\pi}}y^{-1/2}e^{-\frac{t^2}{4y}} \tag{2.1.5}$$

To get (a), we start with (2.1.3) and (2.1.4) and to get (b) we start with (2.1.3) and (2.1.5). The rest of the proof is similar to the proof of Theorem 2.2. □

We can derive two systems of operational relations in a similar way we did for the preceding theorem. For each system we can formulate eight rules analogous to those mentioned for the preceding systems.

Example 2.5. Let $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$, then similar to what we performed in Example 2.2, from (a) we obtain

$$\frac{pq(2p+q)}{(p+q)^{3/2}}\mathrm{Ln}\frac{\sqrt{p}+\sqrt{q}+\sqrt{p+q}}{\sqrt{p}+\sqrt{q}-\sqrt{p+q}} + \frac{pq(\sqrt{q}-\sqrt{p})}{p+q}$$
$$\overset{\cdot\cdot}{\underset{\cdot\cdot}{=}} \frac{1}{2\sqrt{\pi}} \cdot \frac{\sqrt{x}+2\sqrt{y}}{\sqrt{xy}(\sqrt{x}+\sqrt{y})^2}. \quad \square$$

Example 2.6. Let $f(x,y) = \mathrm{Ln}(x^2+y^2)$, then as we did in Example 2.3, (a) results in

$$4\Gamma'(1)\sqrt{pq} + \frac{2pq\sqrt{pq}+2q^2\sqrt{pq}-2p^2\sqrt{pq}\ \mathrm{Ln}q - 2q\sqrt{pq}(2p+q)\mathrm{Ln}p + \pi pq(3p+q)}{(p+q)^2}$$
$$\doteqdot \frac{2}{\pi}\left(\frac{x(x-2y)\mathrm{Ln}(4x)+y^2/\ln(4y)+x(x-y)-\Gamma'(1)(x-y)^2}{\sqrt{xy}(x-y)^2}\right) \quad \square$$

Example 2.7. Let $f(x,y) = yJ_2(2\sqrt{xy})$, then similar to what we did in Example 2.1, from (a) we get

$$\frac{p\sqrt{q}(3\sqrt{pq}+1)}{(\sqrt{pq}+1)^3} \doteqdot \frac{2}{\sqrt{\pi}}(3-2\sqrt{xy})\sqrt{y}e^{-2\sqrt{xy}} \quad \square$$

Example 2.8. Let $f(x,y) = \frac{(xy)^a}{\Gamma(2a+1)}J_{2a}(2\sqrt{xy})$ where $\mathrm{Re}(2a+1)>0$, then similar to what we did in Example 2.4, (b) results in

$$\frac{p\sqrt{q}+2(a+1)p^{3/2}q}{(\sqrt{pq}+1)^{2a+2}} \doteqdot \frac{2^{2a+1}}{\sqrt{\pi}\Gamma(2a+1)}(a+1-\sqrt{xy})x^{a-\frac{1}{2}}y^a e^{-2\sqrt{xy}}. \quad \square$$

We now state four more thoerems. The proofs of these theorems are omitted because they are similar to the proofs of the preceding theorems. Some examples are given after each theorem.

Theorem 2.4. *Let*
(I) $F(p,q) \doteqdot f(x,y)$ *(II)* $\eta(p,q) \doteqdot x^2 f(x,y)$
(III) $\psi(p,q) \doteqdot xf(x,y)$ *(IV)* $G(p,q) \doteqdot x^{-1/2}f(\sqrt{x},\sqrt{y})$.
Then

$$3p^{-2}q^{1/2}F(\sqrt{p},\sqrt{q}) + 3p^{-\frac{3}{2}}q^{1/2}\psi(\sqrt{p},\sqrt{q}) + p^{-1}q^{1/2}\eta(\sqrt{p},\sqrt{q}) \doteqdot \frac{8}{\pi}x^{5/2}y^{-1/2}G\left(\frac{1}{4x},\frac{1}{4y}\right). \tag{a}$$

Moreover if we assume
(V) $H(p,q) \doteqdot (xy)^{-1/2}f(\sqrt{x},\sqrt{y})$.
Then

$$3p^{-2}F(\sqrt{p},\sqrt{q}) + 3p^{-3/2}\psi(\sqrt{p},\sqrt{q}) + p^{-1}\eta(\sqrt{p},\sqrt{q}) \doteqdot \frac{16}{\pi}x^{5/2}y^{1/2}H\left(\frac{1}{4x},\frac{1}{4y}\right) \quad \square \tag{b}$$

Example 2.9. Let $f(x,y) = \frac{(xy)^a}{\Gamma(2a+1)}J_{2a}(2\sqrt{xy})$ where $\mathrm{Re}(2a+1)>0$, then (a) gives

$$\frac{q(4(a+1)(a+2)pq+3(2a+3)\sqrt{pq}+3)}{p^{3/2}(\sqrt{pq}+1)^{2a+3}} \doteqdot \frac{4^{a+1}}{\sqrt{\pi}\Gamma(2a+1)}x^{a+2}y^{a-\frac{1}{2}}e^{-2\sqrt{xy}}. \quad \square$$

Example 2.10. Let

$$f(x,y) = x^{2c_1} y^{2c_2} G^{i,j}_{u,v}\left(c(xy)^2 \middle| \begin{matrix} a_1, \ a_2, \ \dots, \ a_u \\ b_1, \ b_2, \ \dots, \ b_v \end{matrix}\right)$$

where $(u+v) < 2(i+j), |argc| < \frac{\pi}{2}(i+j-\frac{u}{2}-\frac{v}{2})$ and $\mathrm{Re}(2c_k+b_\ell+1) > 0, k = 1,2;\ \ell = 1,2,\dots,i$. Then

$$\begin{aligned}
F(p,q) =& \frac{2^{2c_1+2c_2}}{\pi p^{2c_1} q^{2c_2}} G^{i,j}_{u+4,v+4}\left(\frac{16c}{(pq)^2} \middle| \begin{matrix} -c_1, \ -c_1+\frac{1}{2}, \ -c_2, \ -c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right) \\
\psi(p,q) =& \frac{2^{2c_1+2c_2+1}}{\pi p^{2c_1+1} q^{2c_2}} G^{i,j}_{u+4,v+4}\left(\frac{16c}{(pq)^2} \middle| \begin{matrix} -c_1, \ -c_1-\frac{1}{2}, \ -c_2, \ -c_2+\frac{1}{2}, (a_u) \\ (b_v) \end{matrix}\right) \\
\eta(p,q) =& \frac{2^{2c_1+2c_2+2}}{\pi p^{2c_1+2} q^{2c_2}} G^{i,j}_{u+4,v+4}\left(\frac{16c}{(pq)^2} \middle| \begin{matrix} -c_1-\frac{1}{2}, \ -c_1-1, \ -c_2, \ -c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right) \\
H(p,q) =& p^{-2c_1+\frac{1}{2}} q^{-2c_2+\frac{1}{2}} G^{i,j}_{u+2,v+2}\left(\frac{c}{pq} \middle| \begin{matrix} -2c_1+\frac{1}{2}, \ -2c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right)
\end{aligned}$$

where $(a_u) = a_1, a_2, \dots, a_u$ and (b_v) is defined similarly. Hence (b) gives

$$\begin{aligned}
& \frac{(3)2^{2c_1+2c_2}}{\pi p^{c_1+2} q^{c_2}} G^{i,j}_{u+4,v+4}\left(\frac{16c}{pq} \middle| \begin{matrix} -c_1, \ -c_1+\frac{1}{2}, \ -c_2, \ -c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right) \\
& + \frac{(3)2^{2c_1+2c_2+1}}{\pi p^{c_1+2} q^{c_2}} G^{i,j}_{u+4,v+4}\left(\frac{16c}{pq} \middle| \begin{matrix} -c_1, \ -c_1-\frac{1}{2}, \ -c_2, \ -c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right) \\
& + \frac{2^{2c_1+2c_2+2}}{\pi p^{c_1+2} q^{c_2}} G^{i,j}_{u+4,v+4}\left(\frac{16c}{pq} \middle| \begin{matrix} -c_1-\frac{1}{2}, \ -c_1-1, \ -c_2, \ -c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right) \\
& \quad \doteqdot \frac{2^{4c_1+4c_2+2}}{\pi} x^{2c_1+2} y^{2c_2} G^{i,j}_{u+2,v+2}\left(16c(xy) \middle| \begin{matrix} -2c_1+\frac{1}{2}, \ -2c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right).
\end{aligned}$$

Recall

$$F(ap,bq) \doteqdot f\left(\frac{x}{a}, \frac{y}{b}\right) \tag{2.1.6}$$

where $F(p,q)$ is the Laplace Carson transform of $f(x,y)$. If we use (2.1.6) with $a = b = 4$, we get

$$\begin{aligned}
& p^{-c_1-2} q^{-c_2} \left[3G^{i,j}_{u+4,v+4}\left(\frac{c}{pq} \middle| \begin{matrix} -c_1, \ -c_1+\frac{1}{2}, \ -c_2, \ -c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right) \right. \\
& + 6G^{i,j}_{u+4,v+4}\left(\frac{c}{pq} \middle| \begin{matrix} -c_1, \ -c_1-\frac{1}{2}, \ -c_2, \ -c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right) \\
& \left. +4G^{i,j}_{u+4,v+4}\left(\frac{c}{pq} \middle| \begin{matrix} -c_1-\frac{1}{2}, \ -c_1-1, \ -c_2, \ -c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right) \right] \\
& \quad \doteqdot 4x^{2c_1+2} y^{2c_2} G^{i,j}_{u+2,v+2}\left(c(xy) \middle| \begin{matrix} -2c_1+\frac{1}{2}, \ -2c_2+\frac{1}{2}, \ (a_u) \\ (b_v) \end{matrix}\right). \quad \square
\end{aligned}$$

Theorem 2.5. *Let*

(I) $F(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} f(x,y)$ *(II)* $\eta(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} x^2 f(x,y)$

(III) $\psi(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} xf(x,y)$ *(IV)* $G(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} x^2 f(\sqrt{x},\sqrt{y})$.

Then

$$\begin{aligned} &3(pq)^{1/2}F(\sqrt{p},\sqrt{q}) + 3pq^{1/2}\psi(\sqrt{p},\sqrt{q}) + p^{3/2}q^{1/2}\eta(\sqrt{p},\sqrt{q}) \\ &\mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} \frac{1}{4\pi}x^{-5/2}y^{-1/2}G\left(\frac{1}{4x},\frac{1}{4y}\right). \end{aligned} \tag{a}$$

Moreover if we assume

(V) $H(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} x^2 y^{-1/2} f(\sqrt{x},\sqrt{y})$.

Then

$$\begin{aligned} &3p^{1/2}F(\sqrt{p},\sqrt{q}) + 3p\psi(\sqrt{p},\sqrt{q}) + p^{3/2}\eta(\sqrt{p},\sqrt{q}) \\ &\mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} \frac{1}{2\pi}x^{-5/2}y^{1/2}H\left(\frac{1}{4x},\frac{1}{4y}\right) \quad \square \end{aligned} \tag{b}$$

Example 2.11. Let $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$, then from (a), we get

$$\begin{aligned} &\frac{8p^2+8pq+3q^2)}{(p+q)^{5/2}}\mathrm{Ln}\frac{\sqrt{p}+\sqrt{q}+\sqrt{p+q}}{\sqrt{p}+\sqrt{q}-\sqrt{p+q}} + \frac{pq(5p\sqrt{q}-q\sqrt{p}-4p^{3/2}+2q^{3/2}}{(p+q)^2} \\ &\mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} 2\sqrt{\pi}\ \frac{x^{5/2}(6\sqrt{x}+16\sqrt{y}+9)}{(\sqrt{x}+\sqrt{y})^3}. \quad \square \end{aligned}$$

Example 2.12. Let $f(x,y) = yJ_2(2\sqrt{xy})$, then (a) gives

$$\frac{3p\sqrt{q}(5pq+4\sqrt{pq}+1)}{(\sqrt{pq}+1)^4} \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} \frac{2}{\sqrt{\pi}}(4xy-18\sqrt{xy}+15)\sqrt{y}\ e^{-2\sqrt{xy}}. \quad \square$$

Example 2.13. Let $f(x,y) = \frac{(xy)^a}{\Gamma(2a+1)}J_{2a}(2\sqrt{xy})$ where $\mathrm{Re}(2a+1) > 0$, then (b) gives

$$\begin{aligned} &\frac{p\sqrt{q}(4(a+1)(a+2)pq+3(2a+3)\sqrt{pq}+3)}{(\sqrt{pq}+1)^{2a+3}} \\ &\mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} \frac{4^{a+\frac{1}{2}}}{\sqrt{\pi}}(2xy-(4a+7)\sqrt{xy}+2(a+1)(a+2))x^{a-\frac{1}{2}}y^a e^{-2\sqrt{xy}}. \quad \square \end{aligned}$$

Theorem 2.6. *Let*

(I) $F(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} f(x,y)$ *(II)* $\eta(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} x^2 f(x,y)$

(III) $\psi(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} xf(x,y)$ *(IV)* $\xi(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} x^2 y f(x,y)$

(V) $\sigma(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} yf(x,y)$ *(VI)* $\varphi(p,q) \mathrel{\underset{\cdot\cdot}{\overset{\cdot\cdot}{=}}} xyf(x,y)$

(VII) $G(p,q) \doteqdot x^{-1/2} y f(\sqrt{x}, \sqrt{y})$.

Then

$$3p^{-2}q^{1/2}F(\sqrt{p},\sqrt{q}) + 2p^{-3/2}q^{1/2}\psi(\sqrt{p},\sqrt{q}) + p^{-1}q^{1/2}\eta(\sqrt{p},\sqrt{q}) \qquad \text{(a)}$$
$$+ 3p^{-2}q\sigma(\sqrt{p},\sqrt{q}) + 3p^{-3/2}q\psi(\sqrt{p},\sqrt{q}) + p^{-1}q\xi(\sqrt{p},\sqrt{q})$$
$$\doteqdot \frac{4}{\pi} x^{5/2} y^{-3/2} G\left(\frac{1}{4x}, \frac{1}{4y}\right).$$

Moreover if we define
(VII) $H(p,q) \doteqdot (x,y)^{-1/2} f(\sqrt{x}, \sqrt{y})$
then

$$3p^{-2}q^{-1}F(\sqrt{p},\sqrt{q}) + 3p^{-3/2}q^{-1}\psi(\sqrt{p},\sqrt{q}) + p^{-1}q^{-1}\eta(\sqrt{p},\sqrt{q}) \qquad \text{(b)}$$
$$+ 3p^{-2}q^{-1/2}\sigma(\sqrt{p},\sqrt{q}) + 3p^{-3/2}q^{-1/2}\varphi(\sqrt{p},\sqrt{q}) + p^{-1}q^{-1/2}\xi(\sqrt{p},\sqrt{q})$$
$$\doteqdot \frac{32}{\pi} x^{5/2} y^{3/2} H\left(\frac{1}{4x}, \frac{1}{4y}\right). \quad \square$$

Example 2.14. Let $f(x,y)\frac{(xy)^a}{\Gamma(2a+1)} J_{2a}(2\sqrt{xy})$ where $\text{Re}(2a+1) > 0$, then from (a) we obtain

$$\frac{q(8(a+1)^2(a+2)(pq)^{3/2} + (8a^2+20a+17)pq + 6(a+2)\sqrt{pq} + 3)}{p^{3/2}(\sqrt{pq}+1)^{2a+4}}$$
$$\doteqdot \frac{4^{a+\frac{3}{2}}}{\sqrt{\pi}\Gamma(2a+1)} (a+1-\sqrt{xy}) x^{a+2} y^{a-\frac{1}{2}} e^{-2\sqrt{xy}}. \quad \square$$

3. Second Order Parabolic Equation.

Consider

$$\begin{gathered} u_{xx} - u_y = f(x,y) \quad 0 < x < \infty,\ 0 < y < \infty \\ u(x,0) = \sin x \\ u(0,y) = y^a \quad \text{Re}(a) > -1 \\ \lim_{x\to\infty} u(x,y) \text{ bounded.} \end{gathered} \qquad (3.1)$$

In order to apply the operational calculus and find the image of u_{xx}, we need the Laplace transform of $u_x(0,y)$. Let us assume

$$U(p,q) \doteqdot u(x,y), \quad F(p,q) \doteqdot f(x,y), \quad H(q) \doteq u_x(0,y).$$

Then the transformed equation (3.1) takes the form

$$p^2\left[U(p,q) - \frac{\Gamma(a+1)}{q^a}\right] - pH(q) - q\left[U(p,q) - \frac{p}{p^2+1}\right] = F(p,q)$$

from which we derive

$$U(p,q) = \frac{F(p,q)}{p^2-q} + \frac{pH(q)}{p^2-q} + \frac{\Gamma(a+1)p^2}{q^a(p^2-q)} - \frac{pq}{(p^2+1)(p^2-q)}.$$

First we assume $f(x,y)=0$. Then we have

$$U(p,q)=\frac{pH(q)}{p^2-q}+\frac{\Gamma(a+1)p^2}{q^a(p^2-q)}-\frac{pq}{(p^2+1)(p^2-q)}.$$

To find $u(x,y)$, we take the inverse of each term with respect to p only, i.e.,

$$\frac{p}{p^2-q}\doteqdot\frac{\sin h\sqrt{q}\ x}{\sqrt{q}},\qquad \frac{p^2}{p^2-q}\doteqdot\cosh\sqrt{q}\ x$$

$$\frac{p}{(p^2+1)(p^2-q)}\doteqdot\frac{\sin h\sqrt{q}\ x-\sqrt{q}\ \sin x}{\sqrt{q}(q+1)}\qquad [6;\ \text{p.197}].$$

Then we obtain

$$U(p,q)\doteqdot\frac{H(q)}{\sqrt{q}}\sin h\sqrt{q}\ x+\frac{\Gamma(a+1)}{q^a}\cosh(\sqrt{q}\ x)-\frac{\sqrt{q}}{q+1}(\sin h\sqrt{q}\ x-\sqrt{q}\sin x)$$

$$U(p,q)\doteqdot\frac{H(q)}{2\sqrt{q}}(e^{\sqrt{q}\ x}-e^{-\sqrt{q}\ x})+\frac{\Gamma(a+1)}{2q^a}(e^{\sqrt{q}\ x}+e^{-\sqrt{q}\ x})$$
$$-\frac{\sqrt{q}}{2(q+1)}(e^{\sqrt{q}\ x}-e^{-\sqrt{q}\ x})+\frac{q}{q+1}\sin x$$

$$u(x,q)=\frac{q^{a-\frac{1}{2}}(q+1)H(q)+\Gamma(a+1)(q+1)-q^{a+1}}{2q^a(q+1)}\ e^{\sqrt{q}\ x} \tag{3.3}$$
$$+\frac{-q^{a-\frac{1}{2}}(q+1)H(q)+\Gamma(a+1)(q+1)+q^{a+1}}{2q^a(q+1)}e^{-\sqrt{q}\ x}+\frac{q}{q+1}\sin x.$$

Since the limit of $u(x,q)$ is bounded as x approaches infinity, we must have

$$q^{a-\frac{1}{2}}(q+1)H(q)+\Gamma(a+1)(q+1)-q^{a+1}=0$$

which implies

$$H(q)=\frac{q^{a+1}-\Gamma(a+1)(q+1)}{q^{a-\frac{1}{2}}(q+1)}.$$

Replacing $H(q)$ in (3.3) results in

$$u(x,q)=\frac{\Gamma(a+1)}{q^a}e^{-\sqrt{q}\ x}+\frac{q}{q+1}\sin x.$$

Inverting the above function with respect to q and using

$$q^{-a}e^{-\sqrt{q}\ x}\doteqdot\frac{2^{a+\frac{1}{2}}}{\sqrt{\pi}}\ y^a e^{-\frac{x^2}{8y}}D_{-2a-1}\left(\frac{x}{\sqrt{2y}}\right),\qquad [6;\ \text{p. }246]$$

we obtain

$$u(x,y)=\frac{2^{a+\frac{1}{2}}\Gamma(a+1)}{\sqrt{\pi}}\ y^a e^{-\frac{x^2}{8y}}D_{-2a-1}\left(\frac{x}{\sqrt{2y}}\right)+e^{-y}\sin x. \tag{3.4}$$

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