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On the L^∞-regularity of solutions of nonlinear elliptic equations in Orlicz spaces *

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Abstract

Our main result is a maximum principle bounding the absolute values of the solution in terms of the supremum of the absolute values of the boundary data.

1 Introduction

Let Ω be a bounded Lipshitz domain in $\mathbb{R}^n$ $(n \geq 1)$, let $M(t)$ be an N-function i.e. continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$, and $m(t)$ be its right derivatives. Consider the nonlinear boundary-value problem

$$\begin{gathered} Au = -\operatorname{div} a(\nabla u) = f \quad \text{in } \Omega \\ u = \theta \quad \text{in } \partial\Omega \end{gathered} \tag{1.1}$$

with prescribed boundary datum θ, where $a = \{a_i, 1 \leq i \leq n\}$ is a vector of Carathéodory functions defined on $\mathbb{R}^n$ satisfying the hypotheses

(H1) $|a(\xi)| \leq \lambda_1 m(|\xi|)$ for all $\xi \in \mathbb{R}^n$ and some positive constant λ_1.

(H2) $a(\xi).\xi \geq \lambda_2 |\xi| m(|\xi|)$ for all $\xi \in \mathbb{R}^n$ and some positive constant λ_2.

Definition 1.1 Let $\theta \in W^1 L_M(\Omega)$ and $f \in L^1(\Omega)$. A function $u \in W^1 L_M(\Omega)$ is called a weak solution of the boundary-value problem (1.1) if $u - \theta \in W_0^1 L_M(\Omega)$ and

$$\int_\Omega a(\nabla u).\nabla\varphi = \int_\Omega f.\varphi \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\Omega).$$

Here $W_0^1 L_M(\Omega)$ and $W^1 L_M(\Omega)$ denote the Orlicz-Sobolev Spaces associated to the N-function M (see section 2).

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Recently, Fuchs and Gongbao proved in [6, Theorem 1.1] that if u is a weak solution of (1.1) with the second member f lies in $L^\infty(\Omega)$, and $\sup_{\partial\Omega}\theta(x)<\infty$, then u is bounded from above; i.e.,

$$\sup_{\Omega} u(x) \leq \mathrm{const}(\sup_{\partial\Omega}\theta(x), \|u\|_{L^1(\Omega)}, n, |\Omega|, M, \|f\|_{L^\infty\Omega)}, \lambda_1, \lambda_2) < \infty.$$

For this, the authors have supposed additionally to (H1)-(H2) that the N-function M satisfies the Δ_2-condition near infinity.

The aim of this paper is to prove (Theorem 3.2) the previous statement for the general operators,

$$Au = -\operatorname{div}(a(x,u,\nabla u)) \tag{1.2}$$

without assuming the Δ_2-condition. To do this, we replace the hypothesis (H1) by the more general growth condition,

$$|a(x,s,\xi)| \leq c(x) + k_1\overline{P}^{-1}M(k_2|s|) + k_3\overline{M}^{-1}M(k_4|\xi|) \tag{1.3}$$

and the hypothesis (H2) by

$$a(x,s,\xi)\xi \geq \alpha M(\frac{|\xi|}{\beta}). \tag{1.4}$$

(see section 3). To generalize theorem of [6], in our case, we need to prove the following approximating result (see theorem 3.2)

$$W_0^{1,1}(\Omega)\cap W^1L_M(\Omega) = W_0^1L_M(\Omega).$$

which guaranties that $\varphi = \max(u-k,0)$ can be taken as a test function (for details see theorem 3.2).

When $M(t)=|t|^p$ $(p>1)$ (i.e., a satisfies the polynomial growth condition), the regularity result of the solution of (1.1) are investigated in [14] and [11]. Non-standard examples of $M(t)$ which occur in the mechanics of solids and fluids are $M(t) = t\ln(1+t)$, $M(t) = \int_0^t s^{1-\alpha}(\operatorname{arsinh} s)^\alpha ds$ $(0\leq\alpha\leq 1)$ and $M(t) = t\ln(1+\ln(1+t))$ (see [8, 9, 10, 6]) for more details). When $Au=-\Delta u$ (corresponding to the Poisson equation), the reader is referred to [3], where the regularity of u is studied in Orlicz Spaces with respect to the second member f (in particular where f is a measure).

Finally. note that some problems of the calculus of variations (see [6, remark 1.2]) can be lead to the equation (3.1) as in section 3, contributions in this sense include the works [5, 2].

2 Preliminaries

2-1 Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t)>0$ for $t>0$, $M(t)/t\to 0$ as $t\to 0$ and $M(t)/t\to\infty$ as $t\to\infty$.

The N-function conjugate to M is defined as $\overline{M}(t)=\sup\{st-M(t), s\geq 0\}$. We will extend these N-functions into even functions on all $\mathbb{R}$.

The N-function M is said to satisfy the Δ_2-condition if, for some k

$$M(2t) \leq kM(t) \quad \forall t \geq 0. \tag{2.1}$$

When this inequality holds only for $t \geq$ some $t_0 > 0$, M is said to satisfy the Δ_2-condition near infinity. Moreover, we have the following Young's inequality

$$\forall s, t \geq 0, \quad st \leq M(s) + \overline{M}(t) \tag{2.2}$$

Let P and M be two N-functions. $P << M$ means that P grows essentially less rapidly than M, i.e. for each $\varepsilon > 0$, $\frac{P(t)}{M(\varepsilon t)} \to 0$ as $t \to \infty$ This is the case if and only if

$$\lim_{t\to\infty} \frac{M^{-1}(t)}{P^{-1}(t)} = 0$$

2-2 Let Ω be an open subset of $\mathbb{R}^n$. The Orlicz class $K_M(\Omega)$ [resp. The Orlicz space $L_M(\Omega)$] is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that,

$$\int_\Omega M(u(x))dx < +\infty \quad (\text{resp. } \int_\Omega M(\frac{u(x)}{\lambda})dx \text{ for some } \lambda > 0).$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_M = \inf\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})dx \leq 1\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.
2-3 We now turn to the Orlicz-Sobolev spaces, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1.M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. Now, we recall the following concept.

Definition 2.1 A domain Ω has the segment property if for every $x \in \partial\Omega$ there exists an open set G_x and a nonzero vector y_x such that $x \in G_x$ and if $z \in \overline{\Omega} \cap G_x$, then $z + ty_x \in \Omega$ for all $0 < t < 1$.

Recall that if Ω is a bounded Lipshitz domain it satisfies the segment property (see [1]).

3 Main results

Let Ω be a bounded Lipshitz domain in $\mathbb{R}^n$. Our first aim of this section is to prove the following result which play an important role in the proof of the regularity result (Theorem 3.2). Note that some ideas of the proof of Theorem 3.1 are inspired from the analogous of Theorem 1.3 of [12].

Theorem 3.1 *Let M be an N-function, Ω be a bounded open domain of $\mathbb{R}^N$ satisfying the segment property. Then*

$$W_0^{1,1}(\Omega) \cap W^1L_M(\Omega) = W_0^1L_M(\Omega).$$

Proof. We use the notation

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^N \backslash \Omega \end{cases}$$

Step 1 We show that, $W_0^{1,1}(\Omega) \cap W^1L_M(\Omega) \subset W_0^1L_M(\Omega)$.
Let $u \in W_0^{1,1}(\Omega) \cap W^1L_M(\Omega)$, set $K =: \overline{\{x \in \Omega, u(x) \neq 0\}}^{\mathbb{R}^N}$ (closure in $\mathbb{R}^N$). Then K is a compact in $\mathbb{R}^N$, and $K \subset \overline{\Omega}$.

If $K \subset \Omega$. Let j_ε a mollifier function, the convolution $j_\varepsilon * u$ belongs in $C_0^\infty(\Omega)$, for all $0 < \varepsilon < dist(K, \partial\Omega)$. By [12, Lemma 6] we get $j_\varepsilon * u \to u$ in $W^1L_M(\Omega)$ for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$, as $\varepsilon \to 0^+$, this proves that $u \in W_0^1L_M(\Omega)$.

If $K \cap \partial\Omega \neq \emptyset$. For all $x \in \partial\Omega$, let G_x and y_x be respectively the open set and the nonzero vector given by Definition 2.1. Set $F = K \cap (\overline{\Omega} \backslash \cup_{x\in\partial\Omega} G_x)$, then F is a subset compact of Ω. Hence there exist an open G_0 such that $F \subset G_0 \subset\subset \Omega$. Since K is compact, we can found finitely sets G_x; (let us rename them $G_1, \dots, G_k$) such that $K \subset G_0 \cup \dots \cup G_k$. Moreover, as in the proof of [1, Theorem 3.18], we can construct some open sets $\tilde{G}_0, \tilde{G}_1, \dots, \tilde{G}_k$ which cover K, such that $\overline{\tilde{G}_j} \subset G_j$ for every j. Now, let $\Theta = \{\theta_j, 0 \leq j \leq k\}$ be a partition of unity subordinate to $\{\tilde{G}_j, 0 \leq j \leq k\}$ and put $u_j = \theta_j u$, for every $j = 0, \dots, k$. We have $u = \sum_{j=0}^k u_j$ and $\operatorname{supp} u_j \subset \tilde{G}_j$, for every $j = 0, \dots, k$. So, it suffices to show that any u_j belongs in $W_0^1L_M(\Omega \cap G_j)$. Since $\overline{\tilde{G}_0} \subset \Omega$, as in the case $K \subset \Omega$ above we prove that $u_0 \in W_0^1L_M(\Omega \cap G_j)$.

Since $u_j \in W^1L_M(\Omega \cap G_j)$ for all $j \geq 1$ we claim that $\tilde{u}_j \in W^1L_M(\mathbb{R}^N)$ (indeed: the function $\tilde{u}_j$ lies in $W_0^{1,1}(\mathbb{R}^N)$ due to $u_j \in W_0^{1,1}(\Omega \cap G_j)$. Moreover, $\nabla\tilde{u}_j = \tilde{\nabla} u_j$ in the distributional sense and a.e. on $\mathbb{R}^N$. On the other hand, $\tilde{u}_j \in L_M(\mathbb{R}^N)$ and $\nabla\tilde{u}_j \in (L_M(\mathbb{R}^N))^N$, because $u_j \in W^1L_M(\Omega \cap G_j)$). Let $K_j = \operatorname{supp} u_j$ and $u_{j,t} = \tilde{u}_j(x - ty_j)$, with $0 < t < \min\{1, |y_j|^{-1} \operatorname{dist}(\tilde{G}_j, G_j^c)\}$, where y_j be the nonzero vector associated to the set G_j (see Definition 2.1). We claim that $\operatorname{supp} u_{j,t} \subset \Omega \cap G_j$, for all t satisfying $0 < t < \min\{1, |y_j|^{-1} \operatorname{dist}(\tilde{G}_j, G_j^c)\}$. In fact, by the segment property, we have

$$\operatorname{supp} u_{j,t} = K_j + ty_j \subset (G_j \cap \overline{\Omega}) + ty_j \subset \Omega.$$

On the other hand, let $x \in \operatorname{supp} u_{j,t}$. Then $\operatorname{dist}(x, \tilde{G}_j) \leq \operatorname{dist}(x, x - ty_j) + \operatorname{dist}(x - ty_j, K_j) + \operatorname{dist}(K_j, \tilde{G}_j) = \operatorname{dist}(x, x - ty_j)$,
which implies that,

$$\operatorname{dist}(x, \tilde{G}_j) \leq \operatorname{dist}(x, x - ty_j) = |ty_j|.$$

Then, $\operatorname{dist}(x, \tilde{G}_j) < \operatorname{dist}(\tilde{G}_j, G_j^c)$, hence $x \in G_j$.
Since $u_{j,t} \in W_0^1 L_M(\mathbb{R}^N)$ and $\operatorname{supp} u_{j,t} \subset \Omega \cap G_j$, in virtue of lemma 1.5 of [12], we see that $u_{j,t} \to 0$ in $W^1 L_M(\Omega \cap G_j)$ for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ as $t \to 0$. Moreover, by using lemma 1.6 of [12] we can approximate $u_{j,t}$ by a sequences of elements of $\mathcal{D}(\Omega \cap G_j))$, $W^1 L_M(\Omega \cap G_j)$ for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$, hence gives the result.

Step 2 We shall prove that, $W_0^1 L_M(\Omega) \subset W_0^{1,1}(\Omega) \cap W^1 L_M(\Omega)$.
Let $u \in W_0^1 L_M(\Omega)$, by theorem 1.4 of [13] there exists $u_n \in \mathcal{D}(\Omega)$ and $\lambda > 0$ such that

$$\int_\Omega M(\frac{D^\alpha u_n - D^\alpha u}{\lambda})\, dx \to\ 0 \ \text{ as } \ n \to\ \infty\ \ \forall\ \ |\alpha| \leq 1.$$

By using Jensen's inequality, we have

$$M(\frac{1}{\operatorname{meas}(\Omega)} \int_\Omega (\frac{D^\alpha u_n - D^\alpha u}{\lambda})\, dx) \leq \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega M(\frac{D^\alpha u_n - D^\alpha u}{\lambda})\, dx$$

for n large enough. Then,

$$M(\frac{1}{\operatorname{meas}(\Omega)} \int_\Omega (\frac{D^\alpha u_n - D^\alpha u}{\lambda})\, dx) \to 0, \ \text{ as } \ n \to\ \infty\ \ \forall\ \ |\alpha| \leq 1,$$

which gives, since M^{-1} is right continuous in $\mathbb{R}^+$,

$$\int_\Omega |D^\alpha u_n - D^\alpha u|\, dx \to 0 \quad \text{as } n \to \infty\, \forall\, |\alpha| \leq 1.$$

This completes the proof. $\Box$

Let M and P be two N-functions such that $P << M$. Consider the Leray Lions operator A defined from $D(A) \subset W_0^1 L_M(\Omega) \to W^{-1} L_{\overline{M}}(\Omega)$ by

$$Au = -\operatorname{div}(a(x, u, \nabla u)),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \neq \xi^* \in \mathbb{R}^n$:

(H1') $|a(x, s, \xi)| \leq c(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\xi|)$

(H2') $a(x, s, \xi)\xi \ \geq\ \alpha M(\frac{|\xi|}{\beta})$ for some positive constants $k_1, \ldots, k_4, \alpha$ and β, where $c(x)$ belongs to $E_{\overline{M}}(\Omega)$.

Consider the nonlinear boundary-value problem

$$\begin{aligned} -\operatorname{div}(a(x,u,\nabla u)) = f \quad \text{in } \Omega \\ u \equiv \theta \quad \text{in } \partial\Omega. \end{aligned} \tag{3.1}$$

Our objective is the following.

Theorem 3.2 *Let $u \in W^1L_M(\Omega)$ denotes a weak solution of (3.1) and assume that f is in $L^\infty(\Omega)$. Then $\sup_{\partial\Omega}\theta < \infty$ implies that u is bounded from above i.e.*

$$\sup_{\Omega} u \leq \text{const}(\sup_{\partial\Omega}\theta, \|u\|_{L^1(\Omega)}, n, |\Omega|, M, \|f\|_{L^\infty(\Omega)}, \alpha, \beta) < \infty.$$

Remark 3.1 Note that the statement of Theorem 3.2 holds for any N-function M. In particular for the following critical cases:

$$M(t) = t\log(1+t) \quad \text{and} \quad M(t) = e^t - t - 1.$$

We state the following lemmas which are needed below.

Lemma 3.1 ([5]) *Let $k_0 > 0, \gamma > 0, \varepsilon > 0$ and $\alpha \in [0, 1+\varepsilon]$ denote constants and suppose that $u \in L^1(\Omega)$ satisfies the estimate*

$$\int_{A_k}(u-k)dx \leq \gamma k^\alpha |A_k|^{1+\varepsilon},$$

for all $k \geq k_0$, where A_k denotes the set of points $x \in \Omega$ for which $u(x) > k$. Then $\sup_\Omega u$ is bounded by a finite constant depending on $\gamma, \varepsilon, \alpha, k_0$ and $\|u\|_{L^1(A_{k_0})}$.

Proof of Theorem 3.2 Let $k_0 = \sup_{\partial\Omega}\theta < \infty$ and let u be a weak solution of (3.1). Let us remark that by the lemma 3.1, it suffices to show that

$$\int_{A_k}(u-k)dx \leq \text{const}(n, |\Omega|, M, \|f\|_{L^\infty(\Omega)}, \alpha, \beta)|A_k|^{1+\frac{1}{n}} \quad \forall k \geq k_0. \tag{3.2}$$

Observe that

$$\begin{aligned} \int_{A_k} f(u-k)dx \leq & |A_k|^{1/n}\Big(\int_{A_k}|u-k|^{\frac{n}{n-1}}dx\Big)^{\frac{n-1}{n}} \\ \leq & c(n)|A_k|^{1/n}\int_{A_k}|\nabla u|dx. \end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned} \int_{A_k}|\nabla u|dx \leq & \int_{A_k}\beta\frac{|\nabla u|}{\beta}dx \leq \int_{A_k}(\overline{M}(\beta) + M(\frac{|\nabla u|}{\beta}))dx \\ \leq & \overline{M}(\beta)|A_k| + \int_{A_k} M(\frac{|\nabla u|}{\beta})dx. \end{aligned} \tag{3.4}$$

Moreover by theorem 3.1, the function $\varphi = \max(u-k,0)$ lies in the space $W_0^1 L_M(\Omega)$. Hence φ is admissible in (3.1) and we obtain,

$$\int_{A_k} M(\frac{|\nabla u|}{\beta})dx \leq \frac{1}{\alpha}\int_{A_k} f(u-k)dx. \tag{3.5}$$

The right hand side of this inequality can be estimated with Hölder's inequality and the imbedding $W_0^{1,1}(\Omega) \hookrightarrow L^{1^*}(\Omega)$ as follows:

$$\begin{aligned}\frac{1}{\alpha}\int_{A_k} f(u-k)dx \leq & \frac{1}{\alpha}\|f\|_\infty |A_k|^{1/n}(\int_{A_k} |u-k|^{\frac{n}{n-1}}dx)^{\frac{n-1}{n}} \\ \leq & \frac{2\beta}{\alpha}c(n)\|f\|_\infty |A_k|^{1/n}\int_{A_k}\frac{|\nabla u|}{2\beta}dx.\end{aligned}$$

Using Young's inequality we can write

$$\begin{aligned}\frac{1}{\alpha}\int_{A_k} f(u-k)dx \leq & \overline{M}(\frac{2\beta}{\alpha}c(n)\|f\|_\infty |A_k|^{1/n})|A_k| + \int_{A_k} M(\frac{|\nabla u|}{2\beta})dx \\ \leq & \overline{M}(\frac{2\beta}{\alpha}c(n)\|f\|_\infty |A_k|^{1/n})|A_k| + \frac{1}{2}\int_{A_k} M(\frac{|\nabla u|}{\beta})dx.\end{aligned} \tag{3.6}$$

Then the conclusion follow immediately from (3.3)–(3.6).

Remark 3.2 The method used in the proof of the above theorem gives also $\inf_\Omega u > -\infty$ provided that θ is bounded from below. In particular, boundedness of θ implies $u \in L^\infty(\Omega)$ (compare with remark 1.1 [6]).

Example Let $M(t) = e^{|t|} - |t| - 1$, we set $a(x,s,\xi) = (a_i(x,s,\xi))_{1\leq i\leq n}$ such that

$$a_i(x,s,\xi) = \begin{cases} \frac{e^{|\xi_i|}-|\xi_i|-1}{|\xi_i|}\,\mathrm{sign}\,\xi_i & \text{if } |\xi_i| \neq 0 \\ 0 & \text{if } |\xi_i| = 0\,. \end{cases}$$

Then $M(t)$ and $a(x,s,\xi)$ satisfy the conditions (1.3) and (1.4). Note that the N-function $M(t)$ does not satisfy the Δ_2-condition.

Remark 3.3 Let $m(t)$ the right derivative of $M(t)$. Then $m(t)$ and $a(x,s,\xi)$ don't satisfy the condition (H_2). Indeed, take $\xi = (0,\ldots,0,n,0,\ldots,0), n \in \mathbb{N}$. Then we have $a(x,s,\xi)\xi = e^n - n - 1$, and $|\xi|m(|\xi|) = n(e^n-1)$. But for all constant $C > 0$ there exists n large enough, such that $\frac{e^n-n-1}{n(e^n-1)} < C$.

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