2004-Fez conference on Differential Equations and Mechanics
Electronic Journal of Differential Equations, Conference 11, 2004, pp. 135–142.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ON A NONLINEAR PROBLEM MODELLING STATES OF THERMAL EQUILIBRIUM OF SUPERCONDUCTORS

MOHAMMED EL KHOMSSI

Abstract. Thermal equilibrium states of superconductors are governed by the nonlinear problem
$$\sum_{i=1}^{i=N} \frac{\partial}{\partial x_i}\big(k(u)\frac{\partial u}{\partial x_i}\big) = \lambda F(u) \quad \text{in } \Omega\,,$$
with boundary condition $u = 0$. Here the domain Ω is an open subset of $\mathbb{R}^N$ with smooth boundary. The field u represents the thermal state, which we assume is in $H_0^1(\Omega)$. The state $u = 0$ models the superconductor's state which is the unique physically meaningful solution. In previous works, the superconductor domain is unidirectional while in this paper we consider a domain with arbitrary geometry. We obtain the following results: A set of criteria that leads to uniqueness of a superconductor state, a study of the existence of normal states and the number of them, and optimal criteria when the geometric dimension is 1.

1. Statement of the problem

General model. In the framework of superconductivity, the energy conservation in a physical volume Ω_S, having as boundary the closed surface $\partial\Omega_S$ can be written as
$$\frac{\partial}{\partial \widetilde{t}}\iiint_{\Omega_S} E\,dv = -\iiint_{\Omega_S} \operatorname{div}(\overrightarrow{q})\,dv + \iiint_{\Omega_S} W\,dv + \iiint_{\Omega_S} P\;dv. \tag{1.1}$$
Where the left hand side of the equation is made of the inner quantity of accumulated energy inside Ω_S during $d\widetilde{t}$. The first member of the second hand side is the heat flux going by conduction in the closed $\partial\Omega_S$, and P is a parasite volume supply of heat of nature, responsible partly, of the thermal perturbation of the environment. The Fourier hypothesis relates the flux density, $\widetilde{q}$ at temperature T by
$$\widetilde{q} = -K(X,T)\cdot \operatorname{grad} T\,, \tag{1.2}$$
where K is the tensor of thermal conductivity. Note at this stage that, it is well known that the application of first Principle of Thermodynamics Theory to a continuous environment is reduced, without matter transfer to the heat equation. This

2000 *Mathematics Subject Classification.* 35J60, 34L30, 35Q99.
Key words and phrases. Equilibrium states; nonlinear; thermal equilibrium; superconductors.

Published October 15,2004.

later is simplified and established by assuming that the previous equation must be valid for any volume Ω_S that one considers and is written as

$$C(X,T)\frac{\partial T}{\partial \widetilde{t}} = \operatorname{div}(K(X,T)\cdot \operatorname{grad} T) + W(X,T) + P(X,\widetilde{t}), \tag{1.3}$$

where C is the heat's capacity of the solid. Note that, the characteristics of problem (1.3), is

$$W = G - Q\,. \tag{1.4}$$

Note that W represents the competition between G and Q, which depends, à priori, of the thermal field, representing respectively the power consumed by volume unit of the conductor and the power exchange between the conductor and the external environment. One should note that equation (1.3) does not make any physical meaning except for conditions well defined and applied to a domain space-time well defined also. These conditions are the reason that specifies the evolution of thermal field. Hence, it is necessary to know the initial distribution in any point of the environment as well as the volume of the field on the boundary of the domain; this, actually, states initial conditions, at $\widetilde{t} = 0$ and limit conditions as well. In practice, T is given at any point of $\partial\Omega_S$; hence limit conditions will be conditions of Dirichlet's type:

$$T(X,\widetilde{t}) = T_b \quad \text{on } \partial\Omega_S \times \mathbb{R}^+ \tag{1.5}$$

and the initial condition is

$$T(X,0) = T_0(X) \quad \text{on } \Omega_S. \tag{1.6}$$

The thermal field T_b is the cryogenic temperature. A dimensional analysis based on the use of floating parameters, numerical characteristics of the environment see for example [1, 2], as well as the isotropy and homogeneity of the environment [7] allow us to rewrite problem (1.3)-(1.4)-(1.5) under a reduced form: Find u a field modelling T defined on $\Omega \times [0,+\infty[$ so that

$$c(u)\frac{\partial u}{\partial t} - \sum_{i=1}^{i=N} \frac{\partial}{\partial x_i}(k(u)\frac{\partial u}{\partial x_i}) = \lambda F(u) + ap(x,t) \quad \text{in}\Omega \times \mathbb{R}^+ \tag{1.7}$$

$$u(x,t) = 0 \quad \text{in } \partial\Omega \times \mathbb{R}^+ \tag{1.8}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega \times \{0\}\,. \tag{1.9}$$

The function F, containing all information on energy assessment in the domain Ω, which is an open and bounded subset of $\mathbb{R}^N$ containing zero. The term p is null in the stationary case (no perturbation at initial instant). The thermal equilibrium states of the superconductor are solutions in $H_0^1(\Omega)$ of

$$-\sum_{i=1}^{i=N} \frac{\partial}{\partial x_i}(k(u)\frac{\partial u}{\partial x_i}) = \lambda F(u) \quad \text{in } \Omega \text{ with } u \in H_0^1(\Omega) \tag{1.10}$$

Hypotheses. The term F depends explicitly on the cooling process of the environment. This helps in defining conditions that are satisfied by the classes of admissible functions. This is stated as the hypothesis

(HG) (1) $F \in C^2(\mathbb{R}^+)$, $F(u) = 0$ for $u \le 0$ and $(\frac{dF}{du})(0^+) \le 0$
(2) u_1 is so that $F(u_1) = 0$ and $0 < u_1 < 1$.

Additional hypotheses are stated as follows:

(H1) $F(u) \le 1$ for all $u \ge u_1$

(H2) There is $u_2 > 1$ satisfying $F(u_2) = 0$ and $(\frac{dF}{du})(u) \leq 0$ for all $u \geq u_2$.

(H3) There is $\gamma_0 > 0$ so that $\lim_{u\to+\infty}(\frac{F(u)}{u}) \leq \gamma_0$ and $\lim_{u\to+\infty} F(u) = +\infty$.

Denote by U^1_{ad} the set of functions F satisfying (HG) and (H1); U^2_{ad} the set of functions satisfying (HG) and (H2); and U^3_{ad} the set of functions satisfying (HG) and (H3). Also set $U^G_{ad} = \bigcup_{m=1}^{m=3} U^m_{ad}$.

Canonical transform and consequences. A large part of the thermal stability analysis is based on the nature and the number of the possible stationary solutions. It seems interesting to transform the differentiable operator of (1.10) so as $k(u)$ does not appear. This is possible due to the following Kirchhoff's transform, related to a function k, and defined by

$$y = Y(u) = \int_0^u k(\omega)d\omega\,. \tag{1.11}$$

The function k is continuous and strictly positive, this implies Y is strictly increasing sequence of positive numbers; so it is invertible. It is possible to clear out $k(u)$ with the introduction of the following transformation

We remark that $k(u)\partial_i u = \partial_i y$ and the equation of problem (1.10) has a new form:

$$-\Delta y = \lambda\widetilde{F}(y) = \lambda F \circ Y^{-1}(y) \qquad \forall x \in \Omega \text{ and } y = 0 \text{ on } \Gamma\,. \tag{1.12}$$

The stationary problem (1.10) is transformed in a more simple one (1.12). It not difficult to check that

Proposition 1.1 ([2]). *Let $F \in U^G_{ad}$. Then $F \in U^m_{ad}$ if and only if $\widetilde{F} \in U^m_{ad}$*

Remark 1.2. The case, $F(u) \leq 0$, is an optimal physical case since it shows the domination of Joule's effect by the cryogenic system ($G(u) \leq Q(u)$). Hence, the conductor stays always in the superconductor state. This situation is known as Stekly criterion; see [3].

Classes U^m_{ad} model physical reality; noticing that U^1_{ad} represents a cooling system based an Helium II.

Furthermore, Proposition 1.1 shows that y and u have the same property (due to the properties of Y).

Definition. A fundamental state or a superconductor state is a state in which $y \equiv 0$.

Definition. We say that a state is normal if every state y which is not null is a solution in $H^1_0(\Omega)$ of the problem (1.12).

2. Analysis of the equilibrium problem

Uniqueness criterion of an equilibrium state. Recall that the only interesting physical state is when $y \equiv 0$. Hence, we will be looking for possibilities to avoid any equilibrium solution that is not zero.

Theorem 2.1. *Let $\widetilde{F}$ be an element of U_{ad} such that*

$$\widetilde{F}(y) \leq \frac{\lambda_1}{\lambda} y \quad \forall y \geq 0. \tag{2.1}$$

Then, problem (1.12) has $y \equiv 0$ as a unique solution in $H^1_0(\Omega)$.

Proof. Let the domain Ω be a bounded open subset in $\mathbb{R}^N$, $F \in U_{ad}^G$ satisfy (HG) and only one of the hypothesis (H1), (H2), (H3). Hence, by Proposition 1.1, $F \in U_{ad}^G$, problem (1.10) and problem (1.12) are equivalent. The operator $Ay = -\Delta y$ is an elliptic self-adjoint operator.

Next, we recall that the first eigenvalue of $Ay = -\Delta y$ is characterized by

$$\lambda_1 = \inf \Big\{ \int_\Omega |\nabla v|^2 \, dx : v \in H_0^2(\Omega) \text{ and } \|v\|_{L(\Omega)}^2 = 1 \Big\} \tag{2.2}$$

Next consider the energy function associated with equation (1.12),

$$\Phi(y) = \int_\Omega |\nabla y|^2 \, dx - \lambda \int_\Omega \int_0^y \widetilde{F}(s) ds \, dx \,. \tag{2.3}$$

It is known that solutions of problem (1.10) are the critical points of Φ. Hence, problem (1.10) has a critical solution obtained as a minimal point of Φ. So, by hypothesis (1.8), we have

$$\Phi(y) \geq \frac{1}{2}(\|\nabla y\|_{L(\Omega)}^2 - \lambda_1 \|y\|_{L(\Omega)}^2). \tag{2.4}$$

Since Ω is bounded and $y \in H_1^2(\Omega)$

Now, the Poincaré inequality applied to (2.4) allows us to derive $\Phi(y) \geq 0$ for all $y \in H_0^1(\Omega)$. Hence, $y \equiv 0$ is the only critical point of Φ and since $y \equiv 0$ is a trivial solution, the proof is complete. □

Corollary 2.2. *Let $F \in U_{ad}^G$ and assume one of the following conditions holds:*

1. $\widetilde{F} \in U_{ad}^2$ *and* $\int_0^{y_2} \widetilde{F}(\omega) d\omega \leq 0$ *with* $y_2 = \int_0^{u_2} k(\omega) d\omega$
2. $\widetilde{F} \in U_{ad}^1$ *and* $\lambda \leq \frac{\lambda_1}{\sup(1, \widetilde{F}_m)}$ *with* $\widetilde{F}_m = \sup_{y \geq 0} \widetilde{F}(y)$
3. $\widetilde{F} \in U_{ad}^3$ *and* $\lambda \leq \lambda_1 \sup(\frac{1}{\gamma_0}, \frac{1}{\gamma_3})$ *with* $\gamma_3 = \sup_{y_3 \geq y \geq y_1} \widetilde{F}(y)$

Where y_3 is the value so that $\widetilde{F}(y) \leq \gamma_0 y$ for all $y \geq y_3$. Then, problem (1.10) *has only one solution which is the fundamental state.*

Proof. If (1) is satisfied, it would be enough to look at two possibilities: $\max_{x \in \Omega} y(x) \leq y_2$ or $\max_{x \in \Omega} y(x) \geq y_2$. For these two cases,

$$\Phi_{\widetilde{F}}(y) = \lambda \int_\Omega \int_0^y \widetilde{F}(s) ds \, dx \leq \Phi_{\widetilde{F}}(y_1) + \lambda \int_\Omega \int_{y_1}^y \widetilde{F}(s) ds \, dx \,.$$

Hence, by the previous theorem the proof is complete in this case.

If condition (2) (or 3) holds, we note that this a restatement of the hypothesis $\widetilde{F}(y) \leq \frac{\lambda_1}{\lambda} y$ for all $y \geq 0$ for U_{ad}^1 (respectively for U_{ad}^3). Hence, the proof of the corollary is complete. □

The existence of a condition of normal states. Let $r_{\max}$ be the maximal radius of a ball of center 0 included in Ω, let $\Psi(r)$ the function defined on $D =]0, \sqrt[N]{2}[$ by

$$\Psi(r) = \Big(\frac{1+r}{r}\Big)^2 \Big(\frac{(1+r)^N - 1}{2 - (1+r)^N}\Big) \tag{2.5}$$

Lemma 2.3. *There is a unique $r_0 \in D$, satisfying $\Psi(r_0) = \min_{r \in D} \Psi(r)$ and* $\lim_{r \to 0} \Psi(r) = \lim_{r \to \sqrt[N]{2} - 1} \Psi(r) = +\infty$.

To justify this lemma, it is enough to study the sign of $\frac{d\Psi}{dr}$ and of $\frac{d^2\Psi}{dr^2}$.

Lemma 2.4. *Let $\widetilde{F} \in U_{ad}$ be such that there exists $\omega_0 > 0$ satisfying*

$$\int_0^{\omega_0} \widetilde{F}(s)ds \geq \omega_0 (r_{\max}\sqrt{2})^{-1}\Psi(r_0)\,. \tag{2.6}$$

Then, there is $\widetilde{y} \in H_0^1(\Omega)$ so that $\Phi(\widetilde{y}) < 0$.

Proof. Set

$$y_a(x) = \sum_{l=1}^{l=3} \alpha_l \Sigma_l(x)\,, \tag{2.7}$$

so that $\alpha_1 = \omega_0$, $\alpha_2 = \omega_0[1 - \frac{1+a}{ar_{\max}}(\|x\| - \frac{r_{\max}}{1+a})]$ ($\|x\|$ is a norm $\mathbb{R}^N$) and $\alpha_3 = 0$.

On the other hand Σ_l are defined as follows:

$$\Sigma_1(x) = \begin{cases} 1 & \text{if } \|x\| \leq \frac{r_{\max}}{1+a} \\ 0 & \text{if } \|x\| > \frac{r_{\max}}{1+a}, \end{cases}$$

$$\Sigma_2(x) = \begin{cases} 1 & \text{if } \frac{r_{\max}}{1+a} < \|x\| \leq r_{\max} \\ 0 & \text{otherwise}, \end{cases}$$

$$\Sigma_3(x) = \begin{cases} 1 & \text{if } x \in \Omega - B(0, r_{\max}) \\ 0 & \text{if } x \in B(0, r_{\max}). \end{cases}$$

By construction, the function y_a is in $H_0^1(\Omega)$ and satisfies $y_a(x) \leq \omega_0$. Next, condition (2.6) implies

$$\Phi(y_a) \leq \frac{1}{2}\|\nabla y_a\|^2_{L^2(\Omega)} - (\frac{\omega_0}{ar_{\max}})^2\Psi(r_0)\,. \tag{2.8}$$

Also, calculating the norm of ∇y_a in $\mathbb{R}^N$ allows us to write

$$\|\nabla y_a\| = (\frac{\omega_0(1+a)}{a\, r_{\max}})^2\|x\|^2\Sigma_4(x) \tag{2.9}$$

with $\Sigma_4(x) = 1$ if $r_1 = \frac{r_{\max}}{1+a} \leq x \leq r_2 = r_{\max}$ and $\Sigma_4(x) = 0$ To finish the proof, it suffices to compute the primitive of $(\frac{\omega_0(1+a)}{a\ r_{\max}})^2\|x\|^2\Sigma_4(x)$ and to choose the constant $a = a_0$ so that

$$\|\nabla y_{a_0}\|^2_{L^2(\Omega)} < (\frac{\omega_0}{a_0 r_{\max}})^2\Psi(r_0)\,.$$

□

Theorem 2.5. *Assume the following two hypothesis*

(1) *There exist $\omega_0 > 0$ such that*

$$\int_0^{\omega_0} \widetilde{F}(s)ds \geq \omega_0(r_{\max}\sqrt{2})^{-1}\Psi(r_0) \tag{2.10}$$

(2) *There exists $\omega_1 > 0$ such that*

$$\widetilde{F}(s) \leq \gamma_0 y \quad \forall y \geq \omega_1\,. \tag{2.11}$$

Then the problem (1.12) *has two equilibrium states y and $\widetilde{y}$.*

Proof. The proof of this theorem uses col's theorem (Mountain Pass theorem, A. Ambrossett) , which in turn uses Palais Smalle condition, results given in [8, 9] and [10, Corollary 2.16]. As a consequence of these works Φ has the following properties

(1) Φ has a unique minimum in a non null point of $H_0^1(\Omega)$.

(2) Φ is convex and $\lim_{\|y\|_{H_0^1(\Omega)}\to+\infty} \Phi(y) = +\infty$.

Note that this result implies that there exists $\widetilde{y} \in H_0^1(\Omega)$ such that

$$\min_{y\in H_0^1(\Omega)} \Phi(y) \leq \Phi(\widetilde{y}) < 0$$

Then the continuity of Φ gives the existence of two critical points, which completes the proof. □

3. One-dimensional case

Assume a superconductor as a piece of length for which we can assume that the thermal control. The space variables are reduced to curvilinear coordinates. This will allow us to obtain a one dimensional problem by integrating on a line section that is constant and of of diameter very small with respect to the length. Thus, problem (1.10) and hence (1.12) become the differential equation

$$y_{xx} + \lambda\widetilde{F}(y) = 0 \quad \text{in }]0,1[\text{ with } y(0) = y(1) = 0\,. \tag{3.1}$$

Starting from the integral problem, we show that the existence and the number of solutions of this differential equation depend on the minimum of the function

$$E(\eta) = \eta\sqrt{\frac{2}{\lambda}}\int_0^1 \Big(\sqrt{\int_{\eta t}^{\eta} \widetilde{F}(s)ds}\Big)^{-1} dt, \quad \eta \in D_{\widetilde{F}}\,. \tag{3.2}$$

where $\eta \in D_{\widetilde{F}} =]\eta_0, \eta_\infty[$, η_0 is the unique solution of $\int_0^\eta F(s)ds = 0$, and

$$\eta_\infty = \begin{cases} y_2 & \text{if } \widetilde{F} \in U_{ad}^2 \\ +\infty & \text{if } \widetilde{F} \in U_{ad}^1 \cup U_{ad}^3. \end{cases}$$

Critical value and number of possible normal states.

Lemma 3.1. *If $D_{\widetilde{F}} \neq \emptyset$, then the function $E(\eta)$ has a unique minimum and $\eta_{\min} \in D_{\widetilde{F}}$.*

Theorem 3.2. *Set $\lambda_c^* = E(\eta_{\min})$. Then for every $\widetilde{F} \in U_{ad}^1 \cup U_{ad}^2$ we have*

1. *A necessary and sufficient condition for (3.1) to have at least one non-null solution is that*
$$\lambda_c^* \leq 1 \tag{3.3}$$
2. *If $\lambda_c^* = 1$, then (2.11) has a non-null solution $y_{\min}$ with $\max_{x\in]0,1[} y_{\min}(x) = \eta_{\min}$*
3. *If $\lambda_c^* < 1$, then (2.11) has 2 solutions y_a and y_b so that*
$$\max_{x\in]0,1[} y_a(x) = \eta_a \quad \text{and} \quad \max_{x\in]0,1[} y_b(x) = \eta_b$$
with η_a and η_b solutions to $\lambda_c^* = E(\eta)$
4. *For all $\widetilde{F} \in U_{ad}^3$, there is $\eta_* > \eta_0$ with $\widetilde{F}(\eta_*) > \sqrt{\widetilde{F}_{\max}}$ then we have the same results as in 1, 2 and 3 above; otherwise there is at most a non-null solution.*

Optimal criterion of unconditional stability.

Theorem 3.3 ([2, 4]). *Let $F \in U_{ad}^{G}$. A necessary and sufficient condition for the equilibrium problem* (3.1) *to have only the fundamental state $y \equiv 0$ as a solution is*

$$\lambda_c^* > \sup\Big(1, \pi\sqrt{\gamma_0^{-1}}\Big) \tag{3.4}$$

The proof of this theorem and hence of the optimal criterion is very technical. Indeed, one can start by looking at zeros of a differential function or try to study its convexity. By noticing the complexity of $\frac{d^2E}{d\eta^2}$, one can use instead a technic developed by Smoller and Wasserman [11]. To solve a non linear differential equation with $\widetilde{F}$ having a polynomial function of degree three. We justify the existence of at least one extremum η_e of $E(\eta)$. After computing $\frac{dE}{d\eta}$ and $\frac{d^2E}{d\eta^2}$, we look at the sum

$$\Lambda(\eta) = a(\eta)\frac{dE}{d\eta} + b(\eta)\frac{d^2E}{d\eta^2},$$

where $a(\eta)$ and $b(\eta)$ are real functions à priori. Now, we may choose to simplify the expression of $\Lambda(\eta)$, in some extremum η_e which is a zero of $E(\eta)$. Hence, we get a simplified expression of $\frac{dE}{d\eta}$ and $\frac{d^2E}{d\eta^2}$. By studying this sign one may conclude the convexity of $E(\eta)$.

Concluding Remark. In the theory of partial differential equations there is a very strong relationship between dynamic solutions and equilibrium solutions. The study of the equilibrium problem (1.10) is a fundamental step towards the analysis and the study of evolution problem. As application of these results, we can mention the application of the Invariance Principle of Lassalle [5]. Then show, under some regular conditions, that the dynamic solution converges in $H_0^1(\Omega)$ after some time $t \geq t_0$, toward a superconductor state. Hence, we have its stability.

References

[1] B. Bonzi, M. El khomssi: *Pratical criteria for the thermal stability of a unidimensionnal superconducteur*, J Phys. III, France 4. 1994.

[2] M. El khomssi: *Etude des équations paraboliques et elliptiques gérant l'état thermique d'un supraconducteur*, Thèse de Doctorat Nancy INPL 1994.

[3] M. El khomssi: *EDP et Supraconducteur,* Colloque International sur les équations aux dérivées partielles, 4- 8 Mai 1999 Fès Maroc.

[4] Y. Guemouri, M. EL khomssi, C. Meuris: *Stability criterion and stationary states in a superconducting wire of limited length,* International Journal of Engineering Science 40 pp 1285-1295, 2002.

[5] M. A. Jendoubi: *Convergence vers un é quilibre de divers systèmes - gradient multidimensionnels,* small Thèse de Doctorat de l'université de Paris 6., 1997.

[6] C. Meuris: *Contribution à l'étude de la stabilité thermique des supraconducteurs*, Thèse de Doctorat d'Etat es sciences physiques INPL 1982.

[7] A. Mirgaux & J. Saint Jean Paulin: *Asymptotic study of transverse conductivity in superconducting multifilamentary composite*, Int J Sci 20 - 4 1982 pp 587-600.

[8] P. R. Rabinowitz: *Some aspects of critical point theory*, MRC Tech Summary Rep 2645, Jan 1983l.

[9] P. R. Rabinowitz: *Variational methods for nonlinear eigenvalue problems, Eigenvalues of Nonlinear Problems* (G. Prodi. ed.) C.I.M.E, Edizioni Cremonese Rome, 1974 pp 141 - 195.

[10] P. R.Rabinowitz: *Minimax Methods in critical point theory with applications to differential equation.* Conference Board of the Mathematical Sciences, Regional Conferences Series in Mathematics No 65 published by the AMS, 1986.

[11] Smoller & Wasserman: *An existence theorem for positive solution of semi-linear elleptic equation*, ARM 55 No 3 pp 211-266, 1986.

Mohammed El Khomssi
UFR MDA Faculty of Sciences and Technology, Fez, Morocco
E-mail address: elkhomssi@fstf.ma