2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 125–133.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

NONTRIVIAL SOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEMS VIA MORSE THEORY

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Abstract. We prove the existence of nontrivial solutions for perturbations of p-Laplacian. Our approach combine minimax arguments and Morse Theory, under the conditions on the behaviors of the perturbed function $f(x,t)$ or its primitive $F(x,t)$ near infinity and near zero.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, with some appropriate growth condition to be specified later. We consider the Dirichlet problem

$$\begin{gathered} -\Delta_p u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, is the p-Laplacian operator.

Observe that, if $f(x,0) \equiv 0$, then the constant function $u \equiv 0$ is a trivial solution of the problem (1.1). We are going to seek nontrivial solutions of (1.1) in the usual Sobolev space $W_0^{1,p}(\Omega)$, equipped with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p \Big)^{\frac{1}{p}}.$$

It is known that the p-homogeneous boundary problem

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

has the first eigenvalue $\lambda_1 > 0$ that is simple and has an associated normalized eigenfunction φ_1 which is positive in Ω. It is also known, (see [1]), that there exists a second eigenvalue λ_2 such that $\sigma(-\Delta_p) \cap]\lambda_1, \lambda_2[= \emptyset$. Here, $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$ on $W_0^{1,p}(\Omega)$, which contains at least an increasing eigenvalue sequence obtained by the Lusternik-Schnirelaman theory.

2000 *Mathematics Subject Classification.* 58E05, 35J65, 49B27.
Key words and phrases. Minimax method; p-Laplacian; resonance elliptic equation; critical group; Morse theory.

Published September 20, 2006.

The existence of nontrivial solutions for (1.1) has been widely treated by many authors, under various assumptions on nonlinearity f and its primitive F, see [5, 8, 10] and the references therein.

Throughout this paper, we assume that f satisfies the subcritical growth

(F0) for some $q \in (1, p^*)$, there exists a constant $c > 0$, such that
$$|f(x,t)| \le c(1 + |t|^{q-1}), \quad \forall t \in \mathbb{R}, \quad \text{a.e } x \in \Omega,$$
where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = +\infty$ if $N \le p$.

Recall that, under (F0), the weak solutions of (1.1)) correspond to the critical points of his energy functional Φ, given by
$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(x,u)dx, \quad u \in W_0^{1,p}(\Omega),$$
where $F(x,t) = \int_0^t f(x,s)ds$.

It will be seen that critical groups and Morse Theory, developed by Chang [6] or Mawhin and willem [11], are the main tools used to solve our problem. The main point in this theory is to introduce the critical groups of an isolated critical point. With this aim, we need to suppose a conditions that give us information about the behavior of the perturbed function $f(x,t)$ or its primitive $F(x,t)$ near infinity and near zero. More precisely, the following conditions are assumed.

(F1) $\lim_{|t|\to\infty}[tf(x,t) - pF(x,t)] = \infty$ uniformly for a.e. $x \in \Omega$,
(F2) $\lim_{|t|\to\infty}[tf(x,t) - pF(x,t)] = -\infty$ uniformly for a.e. $x \in \Omega$,
(F3) $\limsup_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} < \lambda_2$ uniformly for a.e. $x \in \Omega$,
(F4) $\lim_{|t|\to\infty}[\int_\Omega F(x,t\varphi_1)dx - \frac{1}{p}|t|^p] = \infty$,
(F5) For some $\mu \in (0,p)$, there are $\tau, C_\tau > 0$ such that
$$F(x,t) \ge C_\tau |t|^\mu, \quad \text{for a.e. } x \in \Omega, \quad 0 < |t| \le \tau, \tag{1.3}$$
$$\liminf_{|t|\to 0} \frac{\mu F(x,t) - tf(x,t)}{|t|^q} \ge \alpha, \quad \text{uniformly for a.e.} x \in \Omega, \tag{1.4}$$
for some $q \in (p, p^*)$ and α be a constant non positive.

Now, we may state the main result.

Theorem 1.1. *Assume (F0), (F3)–(F5) and (F1) or (F2). Then the problem* (1.1) *has at least one nontrivial solution.*

For finding critical points of Φ, by applying minimax methods, we will use the following compactness condition, introduced by Cerami [3], which is a generalization of the classical Plais-Smale type (PS).

Definition 1.2. Given $c \in \mathbb{R}$, we say that $\Phi \in C^1(X, \mathbb{R})$ satisfies condition (C_c), if

(i) Every bounded sequence $(u_n) \subset X$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ has a convergent subsequence,
(ii) There is constants $\delta, R, \alpha > 0$ such that
$$\|\Phi'(u)\|_{X'}\|u\|_X \ge \alpha, \quad \forall u \in \Phi^{-1}([c-\delta, c+\delta]) \quad \text{with } \|u\|_X \ge R.$$

If Φ satisfies condition (C_c) for every $c \in \mathbb{R}$, we simply say that Φ satisfies (C).

The present paper is organized as follows. In section 2, we will compute the critical groups at zero and at a mountain pass point. In section 3, we give the proof of theorem 1.1.

2. Critical groups

In this section, we investigate the critical groups at zero and at a mountain pass type. To proceed, some concepts are needed. Let X be a Banach space, given a $\Phi \in C^1(X, \mathbb{R})$. For $\beta \in \mathbb{R}$ and $c \in \mathbb{R}$, we set

$$\Phi^\beta = \{u \in X : \Phi(u) \leq \beta\},$$
$$K = \{u \in X : \Phi^{'}(u) = 0\},$$
$$K_c = \{u \in X : \Phi(u) = c, \Phi^{'}(u) = 0\}.$$

Denote by $H_q(A, B)$ the q-th homology group of the topological pair (A, B) with integer coefficient. The critical groups of Φ at an isolated critical point $u \in K_c$ are defined by

$$C_q(\Phi, u) = H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}),\ q \in \mathbb{Z},$$

where U is a neighborhood of u.

Moreover, it is known that $C_q(\Phi, u)$ is independent of the choice of U due to the excision property of homology. We refer the readers to [6, 11] for more information.

Let denote by B_ρ the closed ball in $W_0^{1,p}(\Omega)$ of radius $\rho > 0$ which is to be chosen later , with the center at the origin. We will show that the critical groups of Φ at zero are trivial.

Theorem 2.1. *Assume* $(F0)$ *and* $(F5)$*. Then,*

$$C_q(\Phi, 0) \cong 0,\ \forall q \in \mathbb{Z}.$$

This result will be proved by constructing a retraction of $B_\rho \setminus \{0\}$ to $B_\rho \cap \Phi^0 \setminus \{0\}$ and by proving that $B_\rho \cap \Phi^0$ is contractible in itself. For this purpose, some technical lemmas must be proved.

Note that the following lemma has been proved in case $p = 2$ [12, Lemma 1.1]).

Lemma 2.2. *Under* $(F0)$ *and* $(F5)$*, zero is a local maximum for the functional* $\Phi(su)$*,* $s \in \mathbb{R}$*, for* $u \neq 0$*.*

Proof. Using $(F0)$ and the hypothesis (1.3), we get

$$F(x, t) \geq C_\tau |t|^\mu - C_1 |t|^q, \quad x \in \Omega, \quad t \in \mathbb{R}, \tag{2.1}$$

for some $q \in (p, p^*)$ and $C_1 > 0$. For $u \in W_0^{1,p}(\Omega), u \neq 0$ and $s > 0$, we have

$$\begin{aligned}\Phi(su) &= \frac{1}{p} s^p \int_\Omega |\nabla u|^p dx - \int_\Omega F(x, su) dx \\ &\leq \frac{s^p}{p} \|u\|^p - \int_\Omega (C_\tau |su|^\mu - C_1 |su|^q) dx \\ &\leq \frac{s^p}{p} \|u\|^p - C_\tau s^\mu \|u\|_\mu^\mu + C_1 s^q \|u\|_q^q.\end{aligned} \tag{2.2}$$

Since $\mu < p < q$, there exists a $s_0 = s_0(u) > 0$ such that

$$\Phi(su) < 0, \quad \text{for all } 0 < s < s_0. \tag{2.3}$$

□

Lemma 2.3. *Assume* $(F0)$ *and* $(F5)$*. Then, there exists* $\rho > 0$ *such that for all* $u \in W_0^{1,p}(\Omega)$ *with* $\Phi(u) = 0$ *and* $0 < \|u\| \leq \rho$*, we have*

$$\frac{d}{ds} \Phi(su)|_{s=1} > 0. \tag{2.4}$$

Proof. For $u \in W_0^{1,p}(\Omega)$ be such that $\Phi(u) = 0$. From $(F0)$ and ((1.4)), we have

$$\mu F(x,u) - f(x,u)u \geq -c|u|^q, \quad \text{a.e. } x \in \Omega,$$

for some $q \in (p, p^*)$ and $c > 0$.

Denote by $\langle .,. \rangle$ the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. Then, since $\Phi(u) = 0$, we have

$$\begin{aligned}
\langle \Phi'(su), u\rangle|_{s=1} &= \int_\Omega |\nabla u|^p dx - \int_\Omega f(x,u)u dx,\\
&= (1 - \frac{\mu}{p}) \int_\Omega |\nabla u|^p dx + \int_\Omega (\mu F(x,u) - f(x,u)u) dx.
\end{aligned}$$

By the above inequality and the Poincaré's inequality, we write

$$\begin{aligned}
\frac{d}{ds}\Phi(su)|_{s=1} &= \langle \Phi'(su), u\rangle|_{s=1},\\
&\geq (1 - \frac{\mu}{p})\|u\|^p - c\int_\Omega |u|^q dx,\\
&\geq (1 - \frac{\mu}{p})\|u\|^p - C\|u\|^q,
\end{aligned}$$

for some $C > 0$. Since $\mu < p < q$, the inequality (2.4) is verified. $\square$

Lemma 2.4. *For all $u \in W_0^{1,p}(\Omega)$ with $\Phi(u) \leq 0$ and $\|u\| \leq \rho$, we have*

$$\Phi(su) \leq 0, \quad \textit{for all } s \in (0,1). \tag{2.5}$$

Proof. Let $\|u\| \leq \rho$ with $\Phi(u) \leq 0$ and assume by contradiction that there exists some $s_0 \in (0,1]$ such that $\Phi(s_0 u) > 0$. Thus, by the continuity of Φ, there exists an $s_1 \in (s_0, 1]$ such that $\Phi(s_1 u) = 0$. Choose $s_2 \in (s_0, 1]$ such that $s_2 = \min\{s \in [s_0, 1] : \Phi(su) = 0\}$. It is easy to see that $\Phi(su) \geq 0$ for each $s \in [s_0, s_2]$. Taking $u_1 = s_2 u$, one deduces

$$\Phi(su) - \Phi(s_2 u) \geq 0 \quad \text{implies that} \quad \frac{d}{ds}\Phi(su)|_{s=s_2} = \frac{d}{ds}\Phi(su_1)|_{s=1} \leq 0.$$

However, by (2.4)

$$\frac{d}{ds}\Phi(su_1)|_{s=1} > 0.$$

This contradiction shows that (2.5) holds. $\square$

Proof of theorem 2.1. Let us fix $\rho > 0$ such that zero is the unique critical point of Φ in B_ρ. First, by taking the mapping $h : [0,1] \times (B_\rho \cap \Phi^0) \to B_\rho \cap \Phi^0$ as

$$h(s,u) = (1-s)u,$$

$B_\rho \cap \Phi^0$ is contractible in itself.

Now, we prove that $(B_\rho \cap \Phi^0) \setminus \{0\}$ is contractible in itself too. For this purpose, define a mapping $T : B_\rho \setminus \{0\} \to (0,1]$ by

$$\begin{gathered}
T(u) = 1, \text{ for } u \in (B_\rho \cap \Phi^0) \setminus \{0\},\\
T(u) = s, \text{ for } u \in B_\rho \setminus \Phi^0 \quad \text{with } \Phi(su) = 0, s < 1.
\end{gathered}$$

¿From the relations (2.3), (2.4) and (2.5), the mapping T is well defined and if $\Phi(u) > 0$ then there exists an unique $T(u) \in (0,1)$ such that

$$\begin{gathered} \Phi(su) < 0, \quad \forall s \in (0, T(u)), \\ \Phi(T(u)u) = 0, \\ \Phi(su) > 0, \quad \forall s \in (T(u), 1). \end{gathered} \tag{2.6}$$

Thus, using (2.4), (2.6) and the Implicit Function Theorem to get that the mapping T is continuous.

Next, we define a mapping $\eta : B_\rho \setminus \{0\} \to (B_\rho \cap \Phi^0) \setminus \{0\}$ by

$$\begin{gathered} \eta(u) = T(u)u, u \in B_\rho \setminus \{0\} \quad \text{with } \Phi(u) \geq 0, \\ \eta(u) = u, u \in B_\rho \setminus \{0\} \quad \text{with } \Phi(u) < 0. \end{gathered} \tag{2.7}$$

Since $T(u) = 1$ as $\Phi(u) = 0$, the continuity of η follows from the continuity of T.

Obviously, $\eta(u) = u$ for $u \in (B_\rho \cap \Phi^0) \setminus \{0\}$. Thus, η is retraction of $B_\rho \setminus \{0\}$ to $(B_\rho \cap \Phi^0) \setminus \{0\}$. Since $W_0^{1,p}(\Omega)$ is infinite-dimensional, $B_\rho \setminus \{0\}$ is contractible in itself. By the fact that retracts of contractible space are also contractible, $(B_\rho \cap \Phi^0) \setminus \{0\}$ is contractible in itself.

From the homology exact sequence, one deduces

$$H_q(B_\rho \cap \Phi^0, (B_\rho \cap \Phi^0) \setminus \{0\}) = 0, \quad \forall q \in \mathbb{Z}.$$

Hence

$$C_q(\Phi, 0) = H_q(B_\rho \cap \Phi^0, (B_\rho \cap \Phi^0) \setminus \{0\}) = 0, \quad \forall q \in \mathbb{Z}.$$

The proof of theorem 2.1 is completed. $\square$

Recall that we have the following Morse relation between the critical groups and homological characterization of sublevel sets. For details of the proof, we refer readers to [7, 13] for example.

Theorem 2.5. *Suppose $\Phi \in C^1(X, \mathbb{R})$ and satisfies (C) condition. If $c \in \mathbb{R}$ is an isolated critical value of Φ, with $K_c = \{u_j\}_{j=1}^n$, then, for every $\varepsilon > 0$ sufficiently small, we have*

$$H_q(\Phi^{c+\epsilon}, \Phi^{c-\epsilon}) = \oplus_{1 \leq j \leq n} C_q(\Phi, u_j).$$

Remark 2.6. ¿From theorem 2.5 follows that if $H_q(\Phi^{c+\epsilon}, \Phi^{c-\epsilon})$ is nontrivial for some q, then there exists a critical point $u \in K_c$ with $C_q(\Phi, u) \ncong 0$. Furthermore, when $C_q(\Phi, 0) \cong 0$ for all q, we get that $u \neq 0$.

We will use the following theorem, which is proved with (PS) condition see for example [11].

Theorem 2.7. *Assume that $\Phi \in C^1(X, \mathbb{R})$, there exists $u_1 \in X, u_2 \in X$ and a bounded open neighborhood Ω of u_0 such that $u_1 \in X \setminus \overline{\Omega}$ and*

$$\inf_{\partial\Omega} \Phi > \max(\Phi(u_0), \Phi(u_1)).$$

Let $\Gamma = \{g \in C([0,1], X) : g(0) = u_0, g(1) = u_1\}$ and

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \Phi(g(s)).$$

If Φ satisfies the (C) condition over X and if each critical point of Φ in K_c is isolated in X, then there exists $u \in K_c$ such that $\dim C_1(\Phi, u) \geq 1$.

Proof. Let $\varepsilon > 0$ be such that $c-\varepsilon > \max(\Phi(u_0), \Phi(u_1))$ and c is the only critical value of Φ in $[c-\varepsilon, c+\varepsilon]$. Consider the exact sequence

$$\dots \to H_1(\Phi^{c+\epsilon}, \Phi^{c-\epsilon}) \xrightarrow{\partial} H_0(\Phi^{c-\epsilon}, \emptyset) \xrightarrow{i_*} H_0(\Phi^{c+\epsilon}, \emptyset) \to \dots$$

where ∂ is the boundary homomorphism and i_* is induced by the inclusion mapping $i : (\Phi^{c-\epsilon}, \emptyset) \to (\Phi^{c+\epsilon}, \emptyset)$. The definition of c implies that u_0 and u_1 are path connected in $\Phi^{c+\varepsilon}$ but not in $\Phi^{c-\epsilon}$. Thus, $\ker i_* \neq \{0\}$ [6, 11] and, by exactness, $H_1(\Phi^{c+\epsilon}, \Phi^{c-\epsilon}) \neq \{0\}$. It follows from theorem 2.5 that $\dim C_1(\Phi, u) \geq 1$. $\square$

3. Proof of main result

The proof is based on the following minimax theorem due to the second author [9, Theorem 3.5]), with Cerami condition. For this, we recall the Krasnoselskii genus.

Define the class of closed symmetric subsets of X as

$$\Sigma = \{A \subset X : A \text{ is closed and } A = -A\}.$$

Definition 3.1. *For a non empty set A in Σ, following Coffman* [4]*, we define the Krasnoselskii genus as*

$$\gamma(A) = \begin{cases} \inf\{m : \exists h \in C(A, \mathbb{R}^m \backslash \{0\}); h(-x) = -h(x)\}, \\ \infty, \ \textit{if } \{\dots\} \textit{ is empty, in particular if } 0 \in A. \end{cases}$$

For A empty we define $\gamma(A) = 0$.

Note that $A_k = \{C \in \Sigma : C \text{ is compact }, \gamma(C) \geq k\}$.

Theorem 3.2. *Let Φ be a C^1 functional on X satisfying (C), let Q be a closed connected subset of X such that $\partial Q \cap \partial(-Q) \neq \emptyset$ and $\beta \in \mathbb{R}$. Assume that*

1. *for every $K \in \mathcal{A}_2$, there exists v_K such that $\Phi(v_K) \geq \beta$ and $\Phi(-v_K) \geq \beta$,*
2. *$a = \sup_{\partial Q} \Phi < \beta$,*
3. *$\sup_{\partial Q} \Phi < \infty$.*

Then Φ has a critical value $c \geq \beta$ given by

$$c = \inf_{h \in \Gamma} \sup_{x \in Q} \Phi(h(x)),$$

where $\Gamma = \{h \in C(X, X) : h(x) = x \text{for every } x \in \partial Q\}$.

We will establish the compactness condition under the conditions $(F0)$, $(F3)$ and $(F1)$. The proof is similar for $(F0)$, $(F3)$ and $(F2)$.

Lemma 3.3. *Assume $(F0)$, $(F3)$ and $(F1)$. Then Φ satisfies the condition(C).*

Proof. (i) First, we verify that the Palais- Small condition is satisfied on the bounded subsets of $W_0^{1,p}(\Omega)$. Let $(u_n) \subset W_0^{1,p}(\Omega)$ be bounded such that

$$\Phi'(u_n) \to 0 \quad \text{and} \quad \Phi(u_n) \to c, \quad c \in \mathbb{R}. \tag{3.1}$$

Passing if necessary to a subsequence, we may assume that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u \quad \text{strongly in } L^p(\Omega), \\ u_n(x) \to u(x) \quad \text{a.e.in } \Omega. \end{gathered} \tag{3.2}$$

From (3.1) and (3.2), we have $\langle \Phi'(u_n), u_n - u\rangle \to 0$, or equivalently

$$\int_\Omega |\nabla u|^p \nabla u_n \nabla(u_n - u)dx - \int_\Omega f(x,u_n)(u_n - u)dx \to 0. \tag{3.3}$$

Applying the Hölder inequality, we deduce that

$$\int_\Omega f(x,u_n)(u_n - u)dx \to 0. \tag{3.4}$$

Thus, it follows from (3.3) and (3.4) that $\langle -\Delta_p u_n, u_n - u\rangle \to 0$. Since, $-\Delta_p$ is of type S^+ (see [2]), we conclude that

$$u_n \to u \quad \text{strongly in } W_0^{1,p}(\Omega).$$

Now, by contradiction, we will show that (ii) is satisfied for every $c \in \mathbb{R}$. Let $c \in \mathbb{R}$ and $(u_n) \subset W_0^{1,p}(\Omega)$ such that

$$\Phi(u_n) \to c, \quad \langle \Phi'(u_n), u_n\rangle \to 0 \quad \text{and } \|u_n\| \to +\infty. \tag{3.5}$$

Therefore,

$$\lim_n \int_\Omega g(x,u_n)dx = pc, \tag{3.6}$$

where $g(x,u_n) = u_n f(x,u_n) - pF(x,u_n)$.

Taking $v_n = \frac{u_n}{\|u_n\|}$, clearly v_n is bounded in $W_0^{1,p}(\Omega)$. So, there is a function $v \in W_0^{1,p}(\Omega)$ and a subsequence still denote by (u_n) such that

$$\begin{gathered} v_n \rightharpoonup v \quad \text{weakly in } W_0^{1,p}(\Omega), \\ v_n \to v \quad \text{strongly in } L^p(\Omega), \\ v_n(x) \to v(x) \quad \text{a.e.in } \Omega. \end{gathered} \tag{3.7}$$

On the other hand, in view $(F0)$ and $(F3)$, it follows that

$$F(x,s) \le \frac{\lambda_2}{p}|s|^p + b, \quad \forall s \in \mathbb{R}, \quad b \in L^p(\Omega). \tag{3.8}$$

Combining relations (3.5) and (3.8), we obtain

$$\frac{1}{p}\|u_n\|^p - \frac{\lambda_2}{p}\|u_n\|_{L^p}^p - b \le C, \quad C \in \mathbb{R}.$$

Dividing by $\|u_n\|$ and passing to the limit, we conclude

$$\frac{1}{p} - \frac{\lambda_2}{p}\|v\|_{L^p}^p \le 0,$$

and consequently $v \neq 0$. Let $\Omega_0 = \{x \in \Omega : v(x) \neq 0\}$, via the result above we have $|\Omega_0| > 0$ and

$$|u_n(x)| \to +\infty, \quad \text{a.e. } x \in \Omega_0. \tag{3.9}$$

Furthermore, $(F0)$ and $(F1)$ implies that there exist $M > 0$ and $d \in L^1(\Omega)$ such that

$$sf(x,s) - pF(x,s) \ge -M + d(x), \quad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

Hence,

$$\int_\Omega g(x,u_n)dx \ge \int_{\Omega_0} g(x,u_n)dx - M|\Omega \setminus \Omega_0| - \|d\|_{L^1}.$$

Using (3.9) and Fatou's lemma, one deduces

$$\lim_n \int_\Omega g(x,u_n)d = +\infty.$$

This contradicts (3.6). $\square$

Now, we will prove the geometric conditions of Theorem 3.2. Let denote $E(\lambda_1)$ the eigenspace associated to the eigenvalue λ_1.

Lemma 3.4. *Under the hypothesis* $(F0)$, $(F3)$ *and* $(F4)$, *we have:*

- (i) Φ *is anticoercive on* $E(\lambda_1)$.
- (ii) *For all* $K \in \mathcal{A}_2$, *there exists* $v_K \in K$ *and* $\beta \in \mathbb{R}$ *such that* $\Phi(v_K) \geq \beta$ *and* $\Phi(-v_K) \geq \beta$.

Proof. (i) For each $v \in E(\lambda_1)$, there exist $t \in \mathbb{R}$ such that $v = t\varphi_1$. Therefore, using $(F4)$, we write

$$\begin{aligned}\Phi(v) &= \frac{|t|^p}{p}\int_\Omega |\nabla\varphi_1|^p dx - \int_\Omega F(x,t\varphi_1)dx \\ &= -[\int_\Omega F(x,t\varphi_1)dx - \frac{|t|^p}{p}] \to -\infty, \quad \text{as } |t| \to \infty.\end{aligned}$$

ii) By the Lusternik-Schnirelaman theory, we write

$$\lambda_2 = \inf_{K\in\mathcal{A}_2} \sup\{\int_\Omega |\nabla u|^p dx, \int_\Omega |u|^p dx = 1 \text{ and } u \in K\}.$$

Then, for all $K \in \mathcal{A}_2$, and all $\varepsilon > 0$, there exists $v_K \in K$ such that

$$(\lambda_2 - \varepsilon)\int_\Omega |v_K|^p dx \leq \int_\Omega |\nabla v_K|^p dx. \tag{3.10}$$

Indeed, if $0 \in K$, we take $v_K = 0$.
Otherwise, we consider the odd mapping

$$g : K \to K', v \mapsto \frac{v}{\|v\|_{L^p}}.$$

By the genus properties, we have $\gamma(g(K)) \geq 2$, and by the definition of λ_2, there exist $w_K \in K'$ such that

$$\int |w_K|^p dx = 1 \quad \text{and} \quad (\lambda_2 - \varepsilon) \leq \int_\Omega |\nabla w_K|^p dx.$$

Thus (3.10) is satisfied by setting $v_K = g^{-1}(w_K)$.

On the other hand, the two assumptions $(F0)$ and $(F4)$ implies

$$F(x,s) \leq (\frac{\lambda_2 - 2\varepsilon}{p})|s|^p + C, \quad \forall s \in \mathbb{R}, \tag{3.11}$$

for some constant $C > 0$. Consequently, one deduces from (3.10) and (3.11) that

$$\begin{aligned}\Phi(w_K) &\geq \frac{1}{p}\int_\Omega |\nabla w_K|^p dx - (\frac{\lambda_2 - 2\varepsilon}{p})\int_\Omega |w_K|^p dx - C|\Omega| \\ &\geq \frac{1}{p}(1 - \frac{\lambda_2 - 2\varepsilon}{\lambda_2 - \varepsilon})\int_\Omega |\nabla w_K|^p dx - C|\Omega|.\end{aligned} \tag{3.12}$$

The argument is similar for

$$\Phi(-w_K) \geq \frac{1}{p}(1 - \frac{\lambda_2 - 2\varepsilon}{\lambda_2 - \varepsilon})\int_\Omega |\nabla w_K|^p dx - C|\Omega|. \tag{3.13}$$

Finally, for every $K \in \mathcal{A}_2$, we have $\Phi(\pm w_K) \geq \beta := -C|\Omega|$, which completes the proof. $\square$

Proof of theorem 1.1. Putting $Q = \{t\varphi_1 : |t| \leq R\}$ for $R > 0$, clearly, Q is closed and compact. In view of lemma 3.3, we can find $t_0 > 0$ such that $\Phi(\pm t_0\varphi_1) < \beta$. In return for lemma 3.4, we may apply Theorem 3.2 to get that Φ has a critical value given by

$$c = \inf_{h\in\Gamma} \sup_{x\in Q} \Phi(h(x)) \geq \beta,$$

where $\Gamma = \{h \in C([0,1], W_0^{1,p}(\Omega)) : h(0) = -t_0\varphi_1, h(1) = t_0\varphi_1\}$. Therefore, there exists at least one critical point u^* of Φ. More precisely, u^* is a Mountain Pass type. However, by theorem 2.7, we have $C_1(\Phi, u^*) \not\cong 0$. Using theorem 2.1, one deduces $u^* \neq 0$. $\square$

References

[1] A.Anane and N.Tsouli, *On the second eigenvalue of the p-Laplacien*, Nonlinear Partiel Differential Equations, Pitmen research Notes 343(1996), 1-9.

[2] J.Berkovits and V.Mustonen, *Nonlinear mappings of monotone type(classification and degree theory)*, Math. Univer. Oulu, Linnanmaa, Finland, 1988

[3] G.Cerami, *Un criterio di esistenza per i punti citici su varieta'illimitate*, Rend. Acad. Sci. Lett. Ist. Lambardo, 112(1978), 332-336.

[4] C.V.Coffman, *A minimum-maximum principle for a class of nonlinear integral equations*, J. Analyse Math. 22(1969), 1401-1412.

[5] D.G.Costa and C.A.Magalh aes, *Existence results for perturbations of the p-Laplacian*, Non. Anal. T. M. A 24(1995), 409-418.

[6] K.Chang, *Infinite Dimensional Morse theory and Multiple Solution Problems*, Birkhäuser, Boston, 1993.

[7] A.R.El Amrouss, *Théorèmes de points critiques et applications aux problèmes variationnels non linéaires*, Thèse de Doctorat, Université Mohamed I, Oujda, 1998.

[8] A.R.El Amrouss et M.Moussaoui, *Minimax principle for critical point theory in applications to quasilinear boundary value problems*, Elec.J.Diff. Equa, 18(2000), 1-9.

[9] A.R.El Amrouss, *Critical Point Theorems and Applications to Differential Equations*, Acta Mathematica Sinica, English Series, Vol.21, No.1 (2005), 129-142.

[10] J.Liu et J.Su, *Remarks on Multiple Nontrivial Solutions for Quasi-Lineat Resonant Problems*, Journal of Mathematical Analysis and Applications ,258(2001), 209-222.

[11] J.Mawhin and M.Willem, *Critical Point Theory and Hamiltonien systems* , Springer-Verlag, New-York, 1989.

[12] V.Moroz, *Solutions of superlinear at zero elliptic equations via Morse Theory*, Topol. Methods in Nonlinear Anal.10(1997),387-397.

[13] E.A.B.Silva, *Critical point theorems and applications to semilnear elliptic problem*, NoDEA 3(1996), 245-261.

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