Seventh Mississippi State - UAB Conference on Differential Equations and Computational Simulations, *Electronic Journal of Differential Equations*, Conf. 17 (2009), pp. 207–212.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND NONEXISTENCE RESULTS FOR QUASILINEAR SEMIPOSITONE DIRICHLET PROBLEMS

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Abstract. We use the sub/supersolution method to analyze a semipositone Dirichlet problem for the p-Laplacian. To find a positive solution, we therefore focus on a related problem that produces positive subsolutions. We establish a new nonexistence result for this subsolution problem on general domains, discuss the existence of positive radial subsolutions on balls, and then apply our results to problems involving particular semipositone nonlinearities.

1. Introduction

A model problem in nonlinear analysis is the Dirichlet problem

$$\begin{gathered} -\Delta_p u = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p > 1$, $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. We make no attempt to review the many solvability and multiplicity results for (1.1) when the nonlinearity f is nonnegative; we are concerned, instead, with the fact that far less is known when f satisfies the *semipositone* condition

$$f(0) < 0\,, \tag{1.2}$$

which we henceforth assume. This condition arises naturally in resource management models, for example [1], leading to a problem of the form (1.1) for which only positive solutions are meaningful. In a similar vein, this note focuses on determining when the semipositone problem (1.1) has at least one positive solution, i.e., a positive function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx = \int_\Omega f(u)\varphi\,dx$$

for all test functions $\varphi \in C_0^\infty(\Omega)$.

To answer this question, we rely on the well-known sub/supersolution method (see, for instance, the seminal paper [5] or the more recent [3]). We therefore seek positive functions $\underline{u} \in W_0^{1,p}(\Omega)$ and $\overline{u} \in W_0^{1,p}(\Omega)$ such that

2000 *Mathematics Subject Classification.* 34B10, 35J20.
Key words and phrases. semipositone problems; p-Laplacian; positive solutions.

Published April 15, 2009.

- $\underline{u}$ is a subsolution : $\Delta_p(\underline{u}) + f(\underline{u}) \geq 0$,
- $\overline{u}$ is a supersolution : $\Delta_p(\overline{u}) + f(\overline{u}) \leq 0$, and
- $\underline{u} \leq \overline{u}$.

With such an ordered pair of sub- and supersolutions in hand, it follows that problem (1.1) has a positive solution.

Finding a positive subsolution $\underline{u}$ is the real difficulty in semipositone problems ([2],[7]), since the semipositone condition (1.2) precludes the constant function 0 from being a subsolution. As shown below, a positive solution of

$$\begin{gathered} -\Delta_p u = \sigma(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

will be a positive subsolution of (1.1) for certain nonlinearities f, where $\sigma : \mathbb{R} \to \mathbb{R}$ is the step function

$$\sigma(t) := \begin{cases} K, & \text{for } t > 1\,, \\ L, & \text{for } t \leq 1\,, \end{cases}$$

defined by constants $K > 0$ and $L \leq 0$; in [4], Drábek and Robinson used an auxiliary problem of this form to analyze *positone* problems. These constants cannot be given indiscriminately, so we must first determine those values of K and L such that (1.3) has a positive solution.

2. A nonexistence result: Auxiliary problem

We begin with a new nonexistence result for problem (1.3). Since Ω is a smooth domain, it is well-known that $-\Delta_p$ has a simple principal eigenvalue $\lambda_1 > 0$ on Ω. A direct calculation shows that λ_1 provides a lower bound on admissible values of K in problem (1.3).

Proposition 2.1. *Equation* (1.3) *cannot have a positive solution if $K < \lambda_1$.*

Proof. If u is a positive solution of (1.3), then multiplying both sides of the equation by u and integrating yields

$$\begin{aligned} \int_\Omega |\nabla u|^p\,dx &= \int_{\{u>1\}} Ku\,dx + \int_{\{0\leq u\leq 1\}} Lu\,dx \\ &\leq K\int_\Omega |u|^p\,dx \\ &\leq \frac{K}{\lambda_1}\int_\Omega |\nabla u|^p\,dx\,, \end{aligned}$$

where we have used the definition of $\sigma(u)$, the fact that $L \leq 0$, and the Poincaré inequality with its optimal constant. It follows, as claimed, that K cannot be less than λ_1. $\square$

A natural problem, then, is to determine the smallest value of $K \geq \lambda_1$ corresponding to a given $L \leq 0$ such that (1.3) does have a positive solution. Having found such a value of K, a subsequent question is to determine or estimate the number of positive solutions of (1.3).

3. Existence results: Auxiliary problem

There seem to be no existence results for problem (1.3) on *general* domains Ω; one inherent difficulty is the free boundary $\{u = 1\}$. When Ω is a ball, however, conditions on L and K can be found that guarantee the existence of a positive radial solution $u = u(r)$, where $r = |x|$.

3.1. **Positive radial subsolutions.** Looking for radial solutions of (1.3) on the unit ball, $\Omega = B_1(0) \subset \mathbb{R}^N$, leads to consideration of the boundary value problem

$$\begin{gathered} r^{-\gamma}\left(r^{\alpha}|u'|^{\beta}u'\right)' + \sigma(u) = 0, \quad 0 < r < 1\,, \\ u'(0) = u(1) = 0, \end{gathered} \tag{3.1}$$

for parameters $\alpha \geq 0$, $\gamma + 1 > \alpha$, and $\beta > -1$. This framework includes the radial versions of operators other than the p-Laplacian, as indicated in the following table [6]:

α	β	γ	Operator
$N-1$	0	$N-1$	Laplacian
$N-1$	$p-2$	$N-1$	p-Laplacian
$N-k$	$k-1$	$N-1$	k-Hessian

It is not difficult to show that u solves (3.1) if and only if it is a fixed point of the operator $T : C[0,1] \to C[0,1]$ defined by

$$(T(v))(r) := \int_r^1 \left(t^{-\alpha}\int_0^t s^{\gamma}\sigma(v(s))\,ds\right)^{\frac{1}{\beta+1}} dt\,. \tag{3.2}$$

By analyzing this explicit operator carefully, [9] established the following existence result.

Theorem 3.1. *Given any $\mu \in (0,1)$, suppose that $L < 0$ and $K > 0$ satisfy*

$$\left(\frac{\mu K - L}{K - L}\right)^{\frac{1}{\gamma+1}} + \left(\frac{\gamma+1}{\mu K}\right)^{\frac{1}{\beta+1}} \leq 1\,. \tag{3.3}$$

Then (3.1) *has at least one positive solution.*

In short, the proof of Theorem 3.1 in [9] proceeds by finding $\rho \in (0,1)$ such that C_ρ is invariant under T, where $C_\rho \subset C[0,1]$ is the set

$$C_\rho := \{v : v \geq 1 \text{on} [0,\rho], 0 \leq v < 1 \text{on} (\rho,1]\}\,.$$

Working with the set C_ρ therefore simplifies the problem by specifying the free boundary from the outset (at the point ρ). It is important here to note that, for a fixed value of ρ, T maps all of C_ρ onto a single function.

We next outline an alternative approach to proving an existence result along the lines of Theorem 3.1. This technique can be applied in other situations, as shown in detail in [10] where it yields existence results for singular problems of the form

$$\begin{gathered} r^{-\gamma}\left(r^{\alpha}|u'|^{\beta}u'\right)' + \sigma(u)u^{-\delta} = 0, \quad 0 < r < 1\,, \\ u'(0) = 0, \quad u(1) = 0 \end{gathered} \tag{3.4}$$

for exponents $\delta \in (0,1)$. As in the present paper, positive solutions of (3.4) serve as positive subsolutions of related boundary value problems.

Returning to problem (3.1), suppose that $L < 0$ has been given. We first determine $\rho \in (0,1)$ such that

$$\begin{gathered} r^{-\gamma}\left(r^{\alpha}|u_1'|^{\beta}u_1'\right)' + L = 0, \quad \rho < r < 1\,, \\ u_1(\rho) = 1, \quad u_1(1) = 0 \end{gathered} \tag{3.5}$$

has a positive solution u_1; solutions are of the form

$$u_1(r) = \int_r^1 \Big(\frac{L}{\gamma+1}s^{\gamma+1-\alpha} + cs^{-\alpha}\Big)^{\frac{1}{\beta+1}}\,ds\,, \quad \text{for } c \ge \frac{-L}{\gamma+1},$$

and the value of ρ follows from choosing the constant c. In particular, the smallest possible value of ρ corresponds to $c = (-L)/(\gamma+1)$, and increasing c increases ρ.

Having found ρ, we identify $K = K(L) > 0$ such that the solution u_2 of

$$\begin{gathered} r^{-\gamma}\left(r^{\alpha}|u_2'|^{\beta}u_2'\right)' + K = 0, \quad 0 < r < \rho\,, \\ u_2'(0) = 0, \quad u_2(\rho) = 1 \end{gathered}$$

matches the solution of (3.5) at ρ; i.e., satisfies $u_2'(\rho) = u_1'(\rho)$; the solution is

$$u_2(r) = 1 + \int_r^{\rho} \Big(\frac{K}{\gamma+1}s^{\gamma+1-\alpha}\Big)^{\frac{1}{\beta+1}}\,ds\,.$$

As noted above, increasing the choice of c increases ρ and thereby increases K as well. Combining these solutions produces a solution of (3.1), namely

$$u(r) := \begin{cases} u_2(r), & \text{for } 0 \le r \le \rho\,, \\ u_1(r), & \text{for } \rho \le r \le 1\,. \end{cases}$$

Letting $K^* > 0$ denote the value of K corresponding to the choice $c = (-L)/(\gamma+1)$, we have thus sketched the proof.

Theorem 3.2. *Let $L < 0$ be given. There is a corresponding $K^* > 0$ such that problem* (3.1) *has a positive, strictly decreasing solution for any $K \ge K^*$, and this value of K^* increases as $|L|$ increases.*

This approach of piecing together solutions of subproblems to solve (3.1) determines an admissible region of pairs (L,K) such that (3.1) has a positive solution; the resulting region has the same qualitative shape as that described by the inequality (3.3). (This region can be described explicitly in the semilinear case; see [8].) A benefit of this new approach is that it clarifies the relationship between L, K, and the free boundary ρ. It would be very interesting to implement this technique on more general domains Ω, for which problem (1.3) does not reduce to the simple form (3.1).

4. Applications

The pair (L,K) is an *admissible point* if the corresponding problem (1.3) has a positive solution. Having determined a region of admissible points in the previous section, the following theorems illustrate how to use a positive solution of (1.3) to solve the original problem (1.1) when Ω is the unit ball. Although these theorems only concern radial solutions, their proofs are intentionally written in a form that would apply on a more general domain if a positive subsolution were available. This highlights the fact stated earlier that finding such a subsolution is the central difficulty. Finally, we emphasize that, modulo the bounds stated below, the nonlinearity f can behave arbitrarily.

Theorem 4.1. *Let* (L, K) *be an admissible point, let* ψ *be a positive solution of the corresponding problem* (1.3), *and let* M *be its maximum value. If* $0 < a < b^{p-1} < c/K$, $\Omega = B_1(0)$, *and* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous function such that*

- $f(t) > b^{p-1}K$ *for* $b \le t \le Mb$ *and*
- $aL < f(t) < c$ *for all* t,

then (1.1) *has a positive radial solution.*

Proof. Define $\underline{u} := b\psi$; this function is clearly positive. Since

$$\Delta_p \underline{u} + f(\underline{u}) = b^{p-1}\Delta_p \psi + f(b\psi) = -b^{p-1}\sigma(\psi) + f(b\psi)\,,$$

it follows from the definition of σ and the given bounds on f that

$$\Delta_p \underline{u} + f(\underline{u}) \ge 0\,.$$

Thus, $\underline{u}$ is a positive subsolution of (1.1).

Now let $\overline{u}$ be the (radial) solution of

$$\begin{gathered} -\Delta_p \overline{u} = \overline{C} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered}$$

If $\overline{C} > c$, it is easy to see that $\overline{u}$ is a supersolution of (1.1). Moreover, $\underline{u} \le \overline{u}$ when $\overline{C}$ is large enough that $\overline{u}(0) \ge b\psi(0)$, so applying the sub/supersolution theorem completes the proof. $\square$

Imposing further conditions on f yields a multiplicity result.

Theorem 4.2. *Let* (L, K) *be an admissible point, let* ψ *be a positive solution of the corresponding problem* (1.3), *and let* M *be its maximum value. If* $0 < a < b^{p-1} < c/K$, $\Omega = B_1(0)$, *and* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous function such that*

- $f(t) < 0$ *for* $t \le 0$,
- $f(t) > b^{p-1}K$ *for* $b \le t \le Mb$, *and*
- $aL < f(t) < c$ *for all* t,

then (1.1) *has at least three distinct radial solutions, one of which is positive and one of which is negative.*

Proof. Let $\underline{u}_1$ and $\overline{u}_2$ solve

$$-\Delta_p \underline{u}_1 = \underline{C}, \quad -\Delta_p \overline{u}_2 = \overline{C} \quad \text{in } \Omega,$$

respectively, with $\underline{u}_1 = \overline{u}_2 = 0$ on $\partial\Omega$,

$$\underline{C} < aL, \quad \text{and} \quad \overline{C} > c\,.$$

Finally, define $\overline{u}_1 \equiv 0$ and $\underline{u}_2 := b\psi$. Calculating directly shows that $\underline{u}_1$ and $\underline{u}_2$ are subsolutions and that $\overline{u}_1$ and $\overline{u}_2$ are supersolutions, with $\underline{u}_1 < \overline{u}_1 < \underline{u}_2 < \overline{u}_2$. The sub/supersolution theorem immediately yields two distinct solutions, and a standard argument (cf. [11]) guarantees the existence of a distinct third solution. $\square$

References

[1] A. Castro, C. Maya, and R. Shivaji, *Nonlinear eigenvalue problems with semipositone structure*, Proceedings of the Conference on Nonlinear Differential Equations (Coral Gables, FL 1999), Electron. J. Differ. Equ. Conf., Conf. 5, 2000, pp. 33–49.

[2] M. Chhetri and R. Shivaji, *Existence of a positive solution for a p-Laplacian semipositone problem*, Boundary Value Problems 3 (2005), pp. 323–327.

[3] P. Drábek and J. Hernández, *Existence and uniqueness of positive solutions for some quasilinear elliptic problems*, Nonlinear Anal. Ser. A: Theory Methods 44 (2001), no. 2, pp. 189–204.

[4] P. Drábek and S. Robinson, *Multiple positive solutions for elliptic boundary value problems*, Rocky Mountain J. Math 36 (2006), no. 1, pp. 97–113.

[5] P. Hess, *On the solvability of nonlinear elliptic boundary value problems*, Indiana Univ. Math J. 25 (1976), pp. 461–466.

[6] J. Jacobsen and K. Schmitt, *Radial solutions of quasilinear elliptic differential equations*, Handbook of Differential Equations, Vol. 1 (2004), pp. 359–435.

[7] S. Oruganti and R. Shivaji, *Existence results for classes of p-Laplacian semipositone equations*, Boundary Value Problems (2006), pp. 1–7.

[8] S. Robinson and M. Rudd, *Multiplicity results for semipositone problems on balls*, Dynam. Systems Appl. 15 (2006), pp. 133–146.

[9] M. Rudd, *Radial solutions of quasilinear semipositone boundary-value problems*, Comm. Appl. Nonlinear Anal. 14 (2007), no. 1, 113–119.

[10] M. Rudd and C. C. Tisdell, *Positive symmetric solutions of singular semipositone boundary value problems*, Preprint.

[11] R. Shivaji, *A remark on the existence of three solutions via sub–super solutions*, vol. 109 of Lecture Notes in Pure and Applied Mathematics, 1987, pp. 561–566.

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