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EXISTENCE OF SOLUTIONS FOR THERMOELASTIC SEMICONDUCTOR EQUATIONS

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Abstract. We study a model for the semiconductor problem that consists of a system of dynamic thermoelasticity equations of displacement and semiconductor equations. This problem arises from the observation that semiconductor devices are too often cracked and broken because of the thermal stresses. Since the heat source generated by Joule heating is quadratic in the gradient of the electrical potential, this causes some problem even in analysis. We establish the existence theorem of a weak solution. The proof is based on time retarding.

1. Introduction

Semiconductors are electrical devices which are extensively used in industrial world and in our daily life. The so-called "Semiconductor Problem" is a system of a stationary charge conservation equation of electrical current and two parabolic equations of electrons and holes. That is,

$$\begin{gathered} -\operatorname{div}(\sigma\nabla\varphi) = p - n + f \\ n_t - \operatorname{div}(d_n\nabla n - \mu_n\sigma n\nabla\varphi) = g(n,p)(1-np) \\ p_t - \operatorname{div}(d_p\nabla p + \mu_p\sigma p\nabla\varphi) = g(n,p)(1-np) \end{gathered} \tag{1.1}$$

where φ represents the electrostatic potential, n and p the densities of electrons and holes, σ the electrical conductivity, d_n and d_p the diffusion coefficients, μ_n and μ_p the mobilities of electrons and holes, respectively, f the net impurity, $g(n,p)(1-np)$ the general rate of recombination-generation. This model, which appeared in the 1950's, has, since then, received much attention (see [1, 5, 15, 20]).

It is a common phenomenon that electrical devices produce heat or temperature. One of the main reasons for considering the thermoelastic semiconductor problem is the observation that these devices are too often cracked and broken because of the thermal stresses. However, the literature on this problem has not dealt with the thermoelastic aspects of the process, only with the thermal and electrical conductions.

In this paper we fill in this gap and consider a new model for the thermoelastic semiconductor. The thermoelastic behavior is taken into account, resulting in a fully coupled system of equations for the temperature, electrical potential and

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elastic displacements. This model consists of an elliptic equation of charge conservation of electrical current, three hyperbolic equations of displacement of the device, a parabolic equation of the energy caused by Joule's heating, and two parabolic equations of the densities of electrons and holes. For the sake of simplicity, we assume that the material constitutive behavior can be adequately described by linear thermoelasticity. The nonlinearity resides in the electrical conductivity. Moreover, Joule's heating introduces a source term in the heat equation that is quadratic in the gradient of the electrical potential. This feature makes the problem interesting also from the mathematical point of view. In this paper we establish the existence of a weak solution for the problem.

This paper is organized as follows. We present the classical model of thermoelastic semiconductor in Section 2 where we explain in the combined process. The formulation of a weak solution for this model is presented in Section 3 together with the assumptions on the data. The statement of our main result is given in Theorem 3.1. The proof of the existence of a weak solution is presented in Section 4. It is based on a sequence of time retarded problems. The basic idea is the following. First we give a prior estimates, and then construct a sequence of solutions of approximate problems. By passing to the limits of the sequence, we will obtain a weak solution.

2. A Model of Thermoelastic Semiconductor Problem

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$, representing the isothermal reference configuration of the thermoelastic body, the semiconductor in our case. We assume that Γ is divided into two relatively open parts Γ_D and Γ_N such that $\Gamma_D \cap \Gamma_N = \emptyset$ and $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \Gamma$. We denote by $\nu = (\nu_1, \dots, \nu_N)$ the outward unit normal to Γ. We assume that the body is held fixed on Γ_D, and the temperature and potential are prescribed there. On Γ_N the body is free, electrically insulted and exchanging heat with the environment. We choose the Dirichlet conditions on Γ_D for the three fields for the sake of simplicity. We use θ to represent the temperature field, φ the electrical potential and $u = (u_1, \dots, u_N)$ the displacement field. Let $T > 0$ and set $\Omega_T = \Omega \times (0, T)$.

The behavior of the system is governed by the energy equation, the equation of charge conservation and the equations of linear elasticity. In a non-dimensional form we may write the system (see, e.g., [2, 6, 9, 10, 20]) as

$$
\begin{gathered}
-\operatorname{div}(\sigma(\theta)\nabla\varphi) = p - n + f, \\
\frac{\partial n}{\partial t} - \operatorname{div}(d_n \nabla n - \mu_n \sigma(\theta) n \nabla\varphi) = g(n,p)(1 - np), \\
\frac{\partial p}{\partial t} - \operatorname{div}(d_p \nabla p + \mu_p \sigma(\theta) p \nabla\varphi) = g(n,p)(1 - np), \\
\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x_j}\Big(k_{ij}(\theta)\frac{\partial \theta}{\partial x_i}\Big) = \sigma(\theta)|\nabla\varphi|^2 - m_{ij}\frac{\partial^2 u_i}{\partial t \partial x_j}, \\
\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\Big(a_{ijkl}\frac{\partial u_k}{\partial x_l} - m_{ij}\theta\Big) = f_i.
\end{gathered} \tag{2.1}
$$

Here and below, $i, j, k, l = 1, \dots, N$ and summation over repeated indices is employed. The density $\rho > 0$ is a constant; $K = \{k_{ij}\}$ and $M = \{m_{ij}\}$ are the heat conduction and thermal expansion tensors, respectively, and $A = \{a_{ijkl}\}$ is the elasticity tensor; $\mathbf{f} = (f_1, \dots, f_N)$ represents the density of body force. Finally, $\sigma = \sigma(\theta)$ is the temperature-dependent electrical conductivity.

The first equation of (2.1) represents the charge conservation of the electrical potential. The second and third ones represent the exchange of the densities of electrons and holes of the semiconductor. The fourth one represents the heat transfer equation in terms of the deformation of the semiconductor and the energy, produced by the so-called Joule heating

$$J = \sigma(\theta)|\nabla\varphi|^2$$

generated by the electrical current for the temperature θ. The last one of (2.1) is the dynamic thermoelasticity equations of displacement of the semiconductor.

To complete the classical formulation of the problem we have to specify the initial and boundary conditions. We set

$$\begin{gathered}
\varphi = \varphi_b,\ n = n_b,\ p = p_b,\ u = 0,\ \theta = \theta_b, \text{ on } \Gamma_D \times (0,T),\\
\frac{\partial\varphi}{\partial\nu} = 0,\ \frac{\partial n}{\partial\nu} = 0,\ \frac{\partial p}{\partial\nu} = 0,\ \frac{\partial u_i}{\partial\nu} = 0,\ i = 1,\dots,N, \text{ on } \Gamma_N \times (0,T),\\
-k_{ij}\frac{\partial\theta}{\partial x_i}\nu_j = h(\theta - \theta_a), \text{ on } \Gamma_N \times (0,T),\\
n = n_0,\ p = p_0,\ u = u_0,\ u_t = u_1,\ \theta = \theta_0, \text{ in } \Omega \times \{t = 0\},
\end{gathered} \tag{2.2}$$

where n_0, p_0 are the initial densities of electrons and holes, respectively, u_0, u_1 the initial displacements and velocities, respectively, θ_0 the initial temperature, n_b, p_b are the density on Γ_D respectively, θ_b the temperature and φ_b the potential there, θ_a the ambient temperature near Γ_N, and h the heat exchange coefficient.

Our goal for this “thermoelastic semiconductor” problem is:

Find $\{\varphi, n, p, \theta, u\}$ such that (2.1)-(2.2) are satisfied.

The precise assumptions on the data and the weak formulation are given in the next section.

We note that, if (the electrical conductivity) $\sigma = 0$ at large temperature, the first equation of (2.1) for φ degenerates and the heating term in the fourth equation of (2.1) vanishes and This degeneracy makes it necessary to consider a capacity solution ([17, 18]) to this model. Recently, Wu and Xu ([16]) consider a model of thermoelastic thermistor problem with the degenerate electrical conductivity and obtain the existence of its capacity solutions. The authors use time retardation to construct an approximate model whose solution is easily to be obtained and whose limit is the solution of the original model. The method used in this paper is similar to the one in that paper. We must note that when $\rho\partial^2 u_i/\partial t^2$ may be omitted; i.e., small ρ or small accelerations, the resulting problem is quasi-static and we will consider in future.

3. Weak Formulation and Main Result

We present a weak formation for our model, the assumptions on the problem data and the statement of our existence results. For simplicity we extend n_b, p_b, θ_b to functions defined on $\overline{\Omega}_T$, and denote it by n_b, p_b, θ_b. This means that they have to satisfy a compatibility condition

$$\begin{gathered}
n_b, p_b, \theta_b \in H^1(\Omega_T), \\
\operatorname*{ess\,sup}_{(\Gamma_D\times(0,T))\cup(\Omega\times 0)} |n_b|,\ |p_b|,\ |\theta_b| < \infty, \\
\varphi_b \in L^2(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T).
\end{gathered}$$

First, we introduce the following function spaces:

$$V_0 = H_D^1(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_D\},\ U_0 = H_D^1(\Omega)^N$$

and denote by V_0' and U_0' their dual spaces, respectively. Finally, we assume that $\rho > 0$ is constant and that

$$\begin{gathered}
u_0(x) = (u_{01}, \dots, u_{0N}) \in U_0, \\
u_1(x) = (u_{11}, \dots, u_{1N}) \in (L^2(\Omega))^N, \\
f = (f_1, \dots, f_N) \in (L^2(\Omega_T))^N,\ h \in (0,\infty), \\
a_{ijkl}, \frac{\partial}{\partial t} a_{ijkl} \in L^\infty(\Omega_T), a_{ijkl} = a_{jikl} = a_{klij}, \\
a_{ijkl}\eta_{ij}\eta_{kl} \geq \lambda|\eta|^2 = \lambda \sum_{i,j=1}^N \eta_{ij}^2,\ \forall \eta = (\eta_{ij}), \\
\eta_{ij} = \eta_{ji}, \lambda > 0, \\
m_{ij}, \frac{\partial}{\partial t} m_{ij} \in W^{1,\infty}(\Omega_T),\ m_{ij} = m_{ji}, \\
k_{ij} \in L^\infty(\Omega_T),\ k_{ij} = k_{ji}, \\
k_{ij}\xi_i\xi_j \geq \lambda_1|\xi|^2,\ \forall \xi = (\xi_{ij}), \lambda_1 > 0, \\
0 < M_1 \leq \sigma(s) \leq M_2, \text{where } M_1, M_2 \text{ are constants}, \\
d_n, d_p, \mu_n, \mu_p \text{ are positive constants.}
\end{gathered}$$

This completes the assumptions. Now we give the definition of weak solutions.

Definition 3.1. $\{\varphi, n, p, \theta, u, v\}$ is said to be a weak solution of problem (2.1)-(2.2) if $\{\varphi, n, p, \theta, u, v\}$ satisfies

$$\begin{gathered}
\varphi - \varphi_b \in L^\infty(0,T;V_0),\ n - n_b \in L^\infty(0,T;V_0) \cap \in L^2(0,T;V_0), \\
p - p_b \in L^\infty(0,T;V_0) \cap \in L^2(0,T;V_0),\ \theta - \theta_b \in L^2(0,T;V_0), \\
n, p, \varphi \in L^\infty(\Omega_T),\ n_t \in L^2(0,T;V_0'),\ p_t \in L^2(0,T;V_0'),\ \theta_t \in L^2(0,T;V_0'), \\
u \in L^\infty(0,T;U_0),\ v \in L^\infty(0,T;L^2(\Omega)^N),\ v_t \in L^2(0,T;U_0'), \\
n(x,0) = n_0,\ p(x,0) = p_0,\ \theta(x,0) = \theta_0, \\
u(x,0) = u_0,\ v(x,0) = u_1, x \in \Omega,
\end{gathered}$$

such that, for all $\eta \in V_0, \gamma \in V_0 \cap L^\infty(\Omega), w \in U_0$, there hold

$$
\begin{gathered}
\int_\Omega \sigma(\theta)\nabla\varphi\nabla\eta dx = \int_\Omega (n-p+f)\eta dx, \\
\langle n_t, \eta\rangle + \int_\Omega (d_n\nabla n - \mu_n\sigma(\theta)n\nabla\varphi)\nabla\eta dx = \int_\Omega g(n,p)(1-np)\eta dx, \\
\langle p_t, \eta\rangle + \int_\Omega (d_p\nabla p + \mu_p\sigma(\theta)p\nabla\varphi)\nabla\eta dx = \int_\Omega g(n,p)(1-np)\eta dx, \\
\langle \theta_t, \gamma\rangle + \int_\Omega k_{ij}(\theta)\frac{\partial\theta}{\partial x_i}\frac{\partial\gamma}{\partial x_j}dx + \int_{\Gamma_N} h(\theta-\theta_b)\gamma\, dS \\
= \int_\Omega \sigma(\theta)|\nabla\varphi|^2\gamma dx + \int_\Omega m_{ij}\frac{\partial v_i}{\partial x_j}\gamma dx\,, \\
\langle v_t, w\rangle + \int_\Omega \Big(a_{ijkl}\frac{\partial u_k}{\partial x_l} - m_{ij}\theta\Big)\frac{\partial w_i}{\partial x_j}dx = \int_\Omega fw\, dx, \\
v = u_t,
\end{gathered}
\tag{3.1}
$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V_0 and V_0' or U_0 and U_0'.

We note that if $\varphi \in L^2(0,T;H^1(\Omega))$, then

$$
\begin{aligned}
\int_\Omega \operatorname{div}(\sigma(\theta)\varphi\nabla\varphi)\xi\, dx &= \int_\Omega \operatorname{div}(\sigma(\theta)\nabla\varphi\cdot\varphi)\xi\, dx \\
&= \int_\Omega \sigma(\theta)|\nabla\varphi|^2\xi\, dx + \int_\Omega \operatorname{div}(\sigma(\theta)\nabla\varphi)\varphi\xi\, dx \\
&= \int_\Omega \sigma(\theta)|\nabla\varphi|^2\xi\, dx + \int_\Omega (n-p+f)\varphi\xi\, dx
\end{aligned}
$$

for all $\xi \in V_0$. Thus

$$
\sigma(\theta)|\nabla\varphi|^2 = \operatorname{div}(\sigma(\theta)\varphi\nabla\varphi) - (n-p+f)\varphi
$$

in the sense of distributions. Hence the forth equation of (3.1) can be written as

$$
\begin{aligned}
&\langle \theta_t, \gamma\rangle + \int_\Omega k_{ij}(\theta)\frac{\partial\theta}{\partial x_i}\frac{\partial\gamma}{\partial x_j}dx + \int_{\Gamma_N} h(\theta-\theta_b)\gamma\, dS \\
&= -\int_\Omega \sigma(\theta)\varphi\nabla\varphi\nabla\gamma dx - \int_\Omega \varphi(n-p+f)\gamma dx + \int_\Omega m_{ij}\frac{\partial v_i}{\partial x_j}\gamma dx.
\end{aligned}
$$

Theorem 3.1. *The problem stated above has a solution.*

Formally, we may obtain a priori estimates by multiplying the first equation of (2.1) by $\varphi-\varphi_b$, the second equation by $n-n_b$, the third by $p-p_b$, the forth by $\theta-\theta_b$, the fifth by $\partial u_i/\partial t$, and integrating over Ω. For the necessary calculations to be valid, u has to be sufficiently regular. To achieve this we construct an approximation scheme in which θ, n, p, u and φ possess the needed regularities. This is based on a retardation scheme in the time variable. The approximate problems are considered next.

4. Approximate Problems and Proof of Theorem 3.1

In this section we use the time-retardation method to construct a sequence of approximate problems that will lead us to the proof of existence for our problem.

To do that, for fixed function g defined in $[0,T]$ and $m>0$, let $\varepsilon=\frac{T}{m}$ and define the time-retarded function g_ε of g by

$$g_\varepsilon=\begin{cases} g(t-\varepsilon) & \text{if } t>\varepsilon,\\ g_0 & \text{if } t\in[0,\varepsilon],\end{cases}$$

where g_0 will be given. Consider the following approximation problem: for all $\psi\in V_0, \gamma\in V_0\cap L^\infty(\Omega), w\in U_0$

$$\begin{gathered}
\int_\Omega \sigma(\theta^\varepsilon_\varepsilon)\nabla\varphi^\varepsilon\nabla\psi dx=\int_\Omega (p^\varepsilon-n^\varepsilon+f)\psi dx,\\
\langle n^\varepsilon_t,\psi\rangle+\int_\Omega (d_n\nabla n^\varepsilon-\mu_n\sigma(\theta^\varepsilon_\varepsilon)n^\varepsilon\nabla\varphi^\varepsilon)\nabla\psi dx=\int_\Omega g(n^\varepsilon,p^\varepsilon)(1-n^\varepsilon p^\varepsilon)\psi dx,\\
\langle p^\varepsilon,\psi\rangle+\int_\Omega (d_p\nabla p^\varepsilon+\mu_p\sigma(\theta^\varepsilon_\varepsilon)p^\varepsilon\nabla\varphi^\varepsilon)\nabla\psi dx=\int_\Omega g(n^\varepsilon,p^\varepsilon)(1-n^\varepsilon p^\varepsilon)\psi dx,\\
\langle \theta^\varepsilon,\gamma\rangle+\int_\Omega k_{ij}(\theta^\varepsilon_\varepsilon)\frac{\partial\theta^\varepsilon}{\partial x_i}\frac{\partial\gamma}{\partial x_j}dx+\int_{\Gamma_N} h(\theta^\varepsilon-\theta_b)\gamma\, dS\\
=-\int_\Omega \sigma(\theta^\varepsilon_\varepsilon)\varphi^\varepsilon\nabla\varphi^\varepsilon\nabla\gamma dx-\int_\Omega \varphi^\varepsilon(n^\varepsilon-p^\varepsilon+f)\gamma dx-\int_\Omega m_{ij}\frac{\partial v^\varepsilon_i}{\partial x_j}\gamma dx,\\
\rho\langle v^\varepsilon_t,w\rangle+\int_\Omega a_{ijkl}\frac{\partial u^\varepsilon_k}{\partial x_l}\frac{\partial w_i}{\partial x_j}dx-\int_\Omega m_{ij}\theta^\varepsilon\frac{\partial w_i}{\partial x_j}dx=\int_\Omega f_i w_i dx,\\
v^\varepsilon=u^\varepsilon_t
\end{gathered}\tag{4.1}$$

with the following boundary conditions:

$$\begin{gathered}
\varphi^\varepsilon=\varphi_b,\ n^\varepsilon=n_b,\ p^\varepsilon=p_b,\ u^\varepsilon=0,\ \theta=\theta_b,\ \text{on } \Gamma_D\times(0,T),\\
\frac{\partial\varphi^\varepsilon}{\partial\nu}=0,\ \frac{\partial n^\varepsilon}{\partial\nu}=0,\ \frac{\partial p^\varepsilon}{\partial\nu}=0,\ \frac{\partial u^\varepsilon_i}{\partial\nu}=0,\ i=1,\dots,N,\ \text{on } \Gamma_N\times(0,T),\\
-k_{ij}\frac{\partial\theta^\varepsilon}{\partial x_i}\nu_j=h(\theta^\varepsilon-\theta_a),\ \text{on } \Gamma_N\times(0,T),\\
n^\varepsilon=n_0,\ p^\varepsilon=p_0,\ u^\varepsilon=u_0,\ v^\varepsilon=u_1,\ \theta^\varepsilon=\theta_0,\ \text{in } \Omega\times\{t=0\},
\end{gathered}$$

Here, according to the definition above,

$$\theta^\varepsilon_\varepsilon=\theta^\varepsilon(t-\varepsilon)=\theta_0,\ \text{for } t\in[0,\varepsilon]$$

is given. With $\sigma(\theta^\varepsilon_\varepsilon)$ given and the fact that σ is bounded between positive constants, the first three equations of (4.1) are the system of standard elliptic and parabolic equations of semiconductor devices. The existence and uniqueness of the solutions of these three equations in $[0,\varepsilon]$ are well known (e.g., see [20]). Once we obtain $(\varphi^\varepsilon,n^\varepsilon,p^\varepsilon)$, the rest three equations of (4.1) are standard linear thermoelasticity equations whose existence and uniqueness of the solution $(\theta^\varepsilon,u^\varepsilon,v^\varepsilon)$ in $[0,\varepsilon]$ are also well known (e.g., see [14]). We proceed inductively on each time interval $[k\varepsilon,(k+1)\varepsilon]$ for $0\le k\le m$ to obtain the unique solution of (4.1) in $[0,T]$.

To take limit as $\varepsilon\to0$ in (4.1) we need some a priori estimates. The following result is cited from ([20]).

Lemma 4.1. *There is a constant C independent of ε such that*

$$\begin{gathered}
0\le n^\varepsilon,\ p^\varepsilon\le C,\ \|\varphi^\varepsilon\|_{L^\infty(\Omega_T)}\le C,\ \|\varphi^\varepsilon\|_{H^1(\Omega)}\le C,\\
\|n^\varepsilon\|_{L^2(0,T;H^1(\Omega))}+\|p^\varepsilon\|_{L^2(0,T;H^1(\Omega))}\le C.
\end{gathered}$$

With the estimates of this lemma and using the same method in [14], we have the following a priori estimate results for the last two equations of (4.1).

Lemma 4.2. *The following estimate holds*

$$\begin{aligned}&\sup_{0\le t\le T}\|\theta^\varepsilon(t)\|_{L^2}^2+\int_0^T\|\nabla\theta^\varepsilon(\tau)\|_{L^2}^2d\tau+\int_0^T\int_{\Gamma_N}h|\theta^\varepsilon-\theta_b|^2\,ds\\&+\sup_{0\le t\le T}\|u^\varepsilon(x,t)\|_{U_0}^2+\sup_{0\le t\le T}\|v^\varepsilon(x,t)\|_{L^2}^2\,dx\le C,\end{aligned}$$

where C is a positive constant that depends on the data but is independent of m.

Proof. Using $\theta^\varepsilon-\theta_b$ as a test function in the forth equation of (4.1), we have

$$\begin{aligned}&\int_\Omega\frac{\partial\theta^\varepsilon}{\partial t}(\theta^\varepsilon-\theta_b)dx+\int_\Omega k_{ij}(\theta_\varepsilon^\varepsilon)\frac{\partial\theta^\varepsilon}{\partial x_i}\frac{\partial(\theta^\varepsilon-\theta_b)}{\partial x_j}dx+\int_{\Gamma_N}h|\theta^\varepsilon-\theta_b|^2\,ds\\&=-\int_\Omega\sigma(\theta_\varepsilon^\varepsilon)\varphi^\varepsilon\nabla\varphi^\varepsilon\nabla(\theta^\varepsilon-\theta_b)dx-\int_\Omega\varphi^\varepsilon(n^\varepsilon-p^\varepsilon+f)(\theta^\varepsilon-\theta_b)dx\\&\quad-\int_\Omega m_{ij}\frac{\partial v_i^\varepsilon}{\partial x_j}(\theta^\varepsilon-\theta_b)dx,\end{aligned}$$

from which we derive

$$\begin{aligned}&\frac12\frac{d}{dt}\|\theta^\varepsilon-\theta_b\|_{L^2}^2dx+\int_\Omega k_{ij}(\theta_\varepsilon^\varepsilon)\frac{\partial(\theta^\varepsilon-\theta_b)}{\partial x_i}\frac{\partial(\theta^\varepsilon-\theta_b)}{\partial x_j}dx+\int_{\Gamma_N}h|\theta^\varepsilon-\theta_b|^2\,ds\\&=-\int_\Omega\frac{\partial\theta_b}{\partial t}(\theta^\varepsilon-\theta_b)dx-\int_\Omega k_{ij}(\theta_\varepsilon^\varepsilon)\frac{\partial\theta_b}{\partial x_i}\frac{\partial(\theta^\varepsilon-\theta_b)}{\partial x_j}dx\\&\quad-\int_\Omega\sigma(\theta_\varepsilon^\varepsilon)\varphi^\varepsilon\nabla\varphi^\varepsilon\nabla(\theta^\varepsilon-\theta_b)dx-\int_\Omega\varphi^\varepsilon(n^\varepsilon-p^\varepsilon+f)(\theta^\varepsilon-\theta_b)dx\\&\quad-\int_\Omega m_{ij}\frac{\partial v_i^\varepsilon}{\partial x_j}(\theta^\varepsilon-\theta_b)dx\\&\le C\int_\Omega|\theta^\varepsilon-\theta_b|dx+\frac12\int_\Omega k_{ij}(\theta_\varepsilon^\varepsilon)\frac{\partial\theta_b}{\partial x_i}\frac{\partial\theta_b}{\partial x_j}dx+\frac12\int_\Omega k_{ij}(\theta_\varepsilon^\varepsilon)\frac{\partial(\theta^\varepsilon-\theta_b)}{\partial x_i}\frac{\partial(\theta^\varepsilon-\theta_b)}{\partial x_j}dx\\&\quad-\int_\Omega\sigma(\theta_\varepsilon^\varepsilon)\varphi^\varepsilon\nabla\varphi^\varepsilon\nabla(\theta-\theta_b)dx-\int_\Omega\varphi^\varepsilon(n^\varepsilon-p^\varepsilon+f)(\theta^\varepsilon-\theta_b)dx\\&\quad-\int_\Omega m_{ij}\frac{\partial v_i^\varepsilon}{\partial x_j}(\theta^\varepsilon-\theta_b)dx.\end{aligned}$$

With the estimates in Lemma 4.1 and by the Poincaré inequality, we have

$$\begin{aligned}&\frac{d}{dt}\|\theta^\varepsilon-\theta_b\|_{L^2}^2+C_1\|\nabla(\theta^\varepsilon-\theta_b)\|_{L^2}^2+\int_{\Gamma_N}h|\theta^\varepsilon-\theta_b|^2\,ds\\&\le C_2-\int_\Omega m_{ij}\frac{\partial v_i^\varepsilon}{\partial x_j}(\theta^\varepsilon-\theta_b)dx,\end{aligned}$$

Using v_i as a test function in the fifth equation of (4.1), we have

$$\frac12\rho\frac{d}{dt}\|v\|_{L^2}^2+\int_\Omega a_{ijkl}\frac{\partial u_k^\varepsilon}{\partial x_l}\frac{\partial v_i^\varepsilon}{\partial x_j}dx-\int_\Omega m_{ij}\theta^\varepsilon\frac{\partial v_i^\varepsilon}{\partial x_j}dx=\int_\Omega f_iv_idx$$

Noting that $v_i^\varepsilon = \frac{\partial u_i^\varepsilon}{\partial t}$, we have

$$\begin{aligned}
&\frac{1}{2}\rho\frac{d}{dt}\|v\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega a_{ijkl}\frac{\partial u_k^\varepsilon}{\partial x_l}\frac{\partial u_i^\varepsilon}{\partial x_j}dx\\
&= \frac{1}{2}\int_\Omega \frac{\partial a_{ijkl}}{\partial t}\frac{\partial u_k^\varepsilon}{\partial x_l}\frac{\partial u_i^\varepsilon}{\partial x_j}dx + \int_\Omega m_{ij}\theta^\varepsilon\frac{\partial v_i^\varepsilon}{\partial x_j}dx + \int_\Omega f_i v_i dx\\
&\le C + C\|\nabla u^\varepsilon\|_{L^2}^2 + \int_\Omega m_{ij}\theta^\varepsilon\frac{\partial v_i^\varepsilon}{\partial x_j}dx + C\|v\|_{L^2}^2.
\end{aligned}$$

Integrating by parts and adding the above two estimates we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\theta^\varepsilon - \theta_b\|_{L^2}^2 + \rho\|v\|_{L^2}^2 + \int_\Omega a_{ijkl}\frac{\partial u_k^\varepsilon}{\partial x_l}\frac{\partial u_i^\varepsilon}{\partial x_j}dx\Big)\\
&+ C_1\|\nabla(\theta^\varepsilon - \theta_b)\|_{L^2}^2 + \int_{\Gamma_N} h|\theta^\varepsilon - \theta_b|^2\,ds\\
&\le C + C\|v^\varepsilon\|_{L^2}^2 + C\|\nabla u^\varepsilon\|_{L^2}^2
\end{aligned}$$

The assumption on a_{ijkl} and the Gronwall inequality derive the desired estimates. ∎

To use the energy estimates in Lemmas 4.1-4.2 to extract convergence subsequences of $\{\varphi^\varepsilon\}, \{n^\varepsilon\}, \{p^\varepsilon\}, \{\theta^\varepsilon\}, \{u^\varepsilon\}$ and $\{v^\varepsilon\}$, we need the following lemma.

Lemma 4.3. *Let B_0, B, B_1 be Banach spaces with $B_0 \subset B \subset B_1$; assume $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ is continuous. Let $1 < p < \infty$, $1 < q < \infty$, let B_0 and B_1 be reflexive, and define*

$$W \equiv \{u \in L^p(0,T;B_0), \frac{du}{dt} \in L^q(0,T;B_1)\}.$$

Then the inclusion $W \hookrightarrow L^p(0,T;B)$ is compact.

The Proof of Theorem 3.1 follows from the following lemmas.

Lemma 4.4. *There holds $\theta^\varepsilon \to \theta$ strongly in $L^2(0,T;L^2(\Omega)) \equiv L^2(\Omega_T)$ as $\varepsilon \to 0$.*

Proof. From Lemmas 4.1-4.2, there are subsequences, still denoted by the original ones, such that

$$\begin{gathered}
\theta^\varepsilon \to \theta \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)),\\
\theta^\varepsilon \to \theta \text{ weakly in } L^2(0,T;H^1(\Omega)),\\
\frac{d\theta^\varepsilon}{dt} \to \frac{d\theta}{dt} \text{ weakly in } L^2(0,T;H^1(\Omega)'),
\end{gathered}$$

as $\varepsilon \to 0$. By Lemma 4.3,

$$\theta^\varepsilon \to \theta \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \equiv L^2(\Omega_T),$$

and hence $\theta^\varepsilon \to \theta$ a.e. in Ω_T as $\varepsilon \to 0$. Noting that $\theta_\varepsilon^\varepsilon(t) = \theta^\varepsilon(t-\varepsilon)$, we also have $\theta_\varepsilon^\varepsilon \to \theta$ a.e. in Ω_T. ∎

Using this lemma, we can extract convergence subsequences of $\{\varphi^\varepsilon\}$, $\{v^\varepsilon\}$ and $\{u^\varepsilon\}$ in suitable spaces.

Lemma 4.5. *There is a subsequence of $\{\varphi^\varepsilon\}$, still denoted by $\{\varphi^\varepsilon\}$, such that $\varphi^\varepsilon \to \varphi$ strongly in $H^1(\Omega)$ as $\varepsilon \to 0$.*

Proof. Since $\{\varphi^\varepsilon\}$ is bounded in $L^\infty(0,T;H^1(\Omega))$, there is a subsequence of $\{\varphi^\varepsilon\}$, still denoted by $\{\varphi^\varepsilon\}$, such that
$$\varphi^\varepsilon \to \varphi \quad \text{weakly* in } L^\infty(0,T;H^1(\Omega)).$$
By Sobolev's embedding theorem
$$\varphi^\varepsilon \to \varphi \quad \text{strongly in } L^2(\Omega).$$
Using $\varphi^\varepsilon - \varphi$ as a test function in the first equation of (4.1) to get
$$\int_\Omega \sigma(\theta_\varepsilon^\varepsilon)\nabla\varphi^\varepsilon\nabla(\varphi^\varepsilon-\varphi)dx = \int_\Omega (n^\varepsilon - p^\varepsilon + f)(\varphi^\varepsilon - \varphi)dx$$
from which, by the assumption of σ, we obtain
$$\begin{aligned}
&\int_\Omega |\nabla(\varphi^\varepsilon-\varphi)|^2 dx\\
&\le C\int_\Omega (n^\varepsilon - p^\varepsilon + f)(\varphi^\varepsilon - \varphi)dx - C\int_\Omega \sigma(\theta_\varepsilon^\varepsilon)\nabla\varphi\nabla(\varphi^\varepsilon-\varphi)dx\\
&\le C\int_\Omega |\varphi^\varepsilon-\varphi|dx - C\int_\Omega (\sigma(\theta_\varepsilon^\varepsilon)-\sigma(\theta))\nabla\varphi\nabla(\varphi^\varepsilon-\varphi)dx\\
&\quad - C\int_\Omega \sigma(\theta)\nabla\varphi\nabla(\varphi^\varepsilon-\varphi)dx.
\end{aligned}$$
Note that
$$\begin{aligned}
&\Big|\int_\Omega (\sigma(\theta_\varepsilon^\varepsilon)-\sigma(\theta))\nabla\varphi\nabla(\varphi^\varepsilon-\varphi)dx\Big|\\
&\le \Big(\int_\Omega |\sigma(\theta_\varepsilon^\varepsilon)-\sigma(\theta)|^2|\nabla\varphi|^2dx\Big)^{1/2}\Big(\int_\Omega |\nabla(\varphi^\varepsilon-\varphi)|^2dx\Big)^{1/2}.
\end{aligned}$$
Since
$$|\sigma(\theta_\varepsilon^\varepsilon)-\sigma(\theta)||\nabla\varphi| \le C|\nabla\varphi|$$
we have $|\sigma(\theta_\varepsilon^\varepsilon)-\sigma(\theta)||\nabla\varphi| \in L^2(\Omega)$. Thus by Lebsegue's Convergence Theorem, we derive
$$\int_\Omega |\nabla(\varphi^\varepsilon-\varphi)|^2dx \to 0 \quad \text{as } \varepsilon\to 0.$$
This implies that $\varphi^\varepsilon \to \varphi$ strongly in $H^1(\Omega)$ as $\varepsilon \to 0$. □

With the help of Lemmas 4.4 and 4.5, it is easy to obtain the following corollary.

Corollary 4.1. *As $\varepsilon \to 0$, the first equation of* (4.1) *becomes the first equation of* (3.1).

Lemma 4.6. *There holds $n^\varepsilon \to n, p^\varepsilon \to p$ weakly in $L^2(0,T;H^1(\Omega))$ and strongly in $L^2(0,T;L^2(\Omega))$.*

Proof. From Lemmas 4.1-4.2, there are subsequences, still denoted by the original ones, such that
$$\begin{gathered}
n^\varepsilon \to n, p^\varepsilon \to p \text{ weakly* in } L^\infty(0,T;L^2(\Omega)),\\
n^\varepsilon \to n, p^\varepsilon \to p \text{ weakly in } L^2(0,T;H^1(\Omega)),\\
\frac{dn^\varepsilon}{dt} \to \frac{dn}{dt}, \frac{dp^\varepsilon}{dt} \to \frac{dp}{dt}, \text{ weakly in } L^2(0,T;H^1(\Omega)').
\end{gathered}$$

By Lemma 4.3,

$$n^\varepsilon \to n, p^\varepsilon \to p \text{ strongly in } L^2(0,T;L^2(\Omega)) \equiv L^2(\Omega_T),$$

and hence $n^\varepsilon \to n$ and $p^\varepsilon \to p$ a.e. in Ω_T. □

With the help of Lemmas 4.4-4.6, we obtain

Corollary 4.2. *As $\varepsilon \to 0$, the second and third equations of* (4.1) *go to the second and third equations of* (3.1)*, respectively.*

Lemma 4.7. *There hold:*

$$\begin{gathered}
v^\varepsilon \to v \quad \textit{weakly* in } L^\infty(0,T;L^2(\Omega)),\\
\frac{\partial v^\varepsilon}{\partial t} \to \frac{\partial v}{\partial t} \quad \textit{weakly in } L^2(0,T;H^1(\Omega)'),\\
u^\varepsilon \to u \quad \textit{weakly* in } L^\infty(0,T;H^1(\Omega)).
\end{gathered}$$

Proof. From Lemma 4.2, we conclude that there exist subsequences, still denoted by the original ones, such that

$$v^\varepsilon \to v \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega)),\ u^\varepsilon \to u \quad \text{weakly* in } L^\infty(0,T;V_0).$$

Thus

$$\frac{\partial v^\varepsilon}{\partial t} \to \frac{\partial v}{\partial t} \quad \text{weakly in } L^2(0,T;H^1(\Omega)').$$

□

With the help of Lemmas 4.4-4.7, we obtain

Corollary 4.3. *As $\varepsilon \to 0$, the fourth, fifth and sixth equations of* (4.1) *go to the fourth, fifth and sixth equations of* (3.1)*, respectively.*

Collecting the results in Corollaries 4.1-4.3, we have the proof of Theorem 3.1.

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