2007 Conference on Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems. *Electronic Journal of Differential Equations*, Conference 18 (2010), pp. 23–31.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ODDNESS OF LEAST ENERGY NODAL SOLUTIONS ON RADIAL DOMAINS

CHRISTOPHER GRUMIAU, CHRISTOPHE TROESTLER

Abstract. In this article, we consider the Lane-Emden problem

$$\Delta u(x) + |u(x)|^{p-2}u(x) = 0, \quad \text{for } x \in \Omega,$$
$$u(x) = 0, \quad \text{for } x \in \partial\Omega,$$

where $2 < p < 2^*$ and Ω is a ball or an annulus in $\mathbb{R}^N$, $N \geqslant 2$. We show that, for p close to 2, least energy nodal solutions are odd with respect to an hyperplane – which is their nodal surface. The proof ingredients are a constrained implicit function theorem and the fact that the second eigenvalue is simple up to rotations.

1. Introduction

Let $N \geqslant 2$. We consider the Lane-Emden problem

$$\begin{gathered} -\Delta u(x) = |u(x)|^{p-2}u(x), \quad \text{for } x \in \Omega, \\ u(x) = 0, \quad \text{for } x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a ball or an annulus and $2 < p < 2^*$ is a subcritical exponent (where $2^* := 2N/(N-2)$ if $N \geqslant 3$ and $2^* = +\infty$ if $N = 2$). In 2004, Aftalion and Pacella [2] showed that least energy nodal solutions cannot be radial. However, as proved by Bartsch, Weth and Willem [3], they have a residual symmetry, namely they possess the Schwarz foliated symmetry; i.e., there exists a direction d (depending on the solution) such that the solution is invariant under the subgroup of rotations leaving d fixed and is non-increasing in the angle with d. Such symmetry however does not imply that the zero set of the solution is an hyperplane passing through the origin as is widely believed.

In this article, we show that this is true for p close to 2. Actually, we prove more: least energy nodal solutions are *unique*, up to rotations and a multiplicative

2000 *Mathematics Subject Classification.* 35J20, 35A30.
Key words and phrases. Variational method; least energy nodal solution; symmetry; oddness; (nodal) Nehari manifold; Bessel functions; Laplace-Beltrami operator on the sphere; implicit function theorem.

Published July 10, 2010.
Supported by a grant from the National Bank of Belgium.

constant ± 1, and *odd* with respect to an hyperplane (depending on the solution) that passes through the origin. This hyperplane is their zero set.

This article is inspired by work of Smets, Su and Willem [11] who showed that, for the Henon problem, the ground states are radial for p close to 2. Another approach to the oddness of solutions of (1.1) was written at the same time as this article by Bonheure, Bouchez, Grumiau and Van Schaftingen [4] (with one author in common). For any domain Ω, they establish that, when p is close enough to 2, least energy nodal solutions possess the same symmetries as their projections on the eigenspace E_2 corresponding to the second eigenvalue λ_2 of $-\Delta$. (Here and in the rest of this paper Δ denotes the Laplacian with Dirichlet boundary conditions on Ω.) Since we show in section 2 that, on a ball or an anulus, all eigenfunctions in E_2 are odd with respect to some hyperplane, we could use their result to obtain the oddness of solutions of (1.1). In our case however, because we are able to show that the degeneration of the second eigenspace is solely due to the invariance of (1.1) under the group rotations, we can use the implicit function theorem. That enables us to establish uniqueness (up to symmetries) in addition to oddness (see theorem 4.2).

The paper is organised as follows. The next section is devoted to the second eigenspace E_2 of $-\Delta$. Thanks to the interlacing properties of zeros of (cross-products of) Bessel functions and results by H. Kalf on the symmetries of spherical harmonics [7], we show that all eigenfunctions of eigenvalue λ_2 have an hyperplane as nodal set with respect to which the function is odd. Moreover $E_2 = \{\alpha e_2(g\,\cdot) \mid \alpha \in \mathbb{R},\ g \in O(N)\}$ for any $e_2 \in E_2 \setminus \{0\}$. These results follow from well know formulae for dimensions 2 and 3 (see e.g. the book of Y. Pinchover and J. Rubenstein [10]) but we could not find a ready-to-use reference for higher dimensions, so we (re)prove them here for the reader convenience.

For the rest of the paper, we deal with the equation

$$\begin{gathered} -\Delta u(x) = \lambda_2 |u(x)|^{p-2} u(x), \quad \text{in } \Omega, \\ u(x) = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

instead of (1.1). Clearly u is a solution of (1.1) if and only if $\lambda_2^{1/(2-p)} u$ is a solution of (1.2). Weak solutions of (1.2) (in fact strong solutions by regularity) are critical points of the following energy functional:

$$\varphi_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \tfrac{1}{2}\|u\|^2 - \lambda_2 \tfrac{1}{p} |u|_p^p,$$

where $|\cdot|_p$ denotes the norm in $L^p(\Omega)$ and $\|\cdot\| := |\nabla \cdot|_2$ is the usual norm in $H_0^1(\Omega)$. Recall that the Nehari manifold is

$$\mathcal{N}_p := \{u \in H_0^1(\Omega) \setminus \{0\} \mid \partial_u \varphi_p(u)(u) = 0\},$$

where $\partial_u \varphi_p(u)$ denotes the Frechet derivative of φ_p at u. The nodal Nehari set is defined as

$$\mathcal{N}_p^1 := \{u \in H_0^1(\Omega) \mid u^+ \in \mathcal{N}_p \text{ and } u^- \in \mathcal{N}_p\},$$

where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. Notice that $\mathcal{N}_p^1 \subseteq \mathcal{N}_p$. Least energy nodal solutions of (1.2) are minimizers of φ_p on $\mathcal{N}_p^1$ [6].

In sections 3 and 4, we consider a family $(u_p)_{p>2}$ of least energy nodal solutions of (1.2). Let $\mathbb{S}^{N-1}$ be the unit sphere of $\mathbb{R}^N$ and $d \in \mathbb{S}^{N-1}$ be a direction fixed for the rest of this paper. The subgroup of rotations leaving d fixed will be denoted $G \subseteq O(N)$. It acts on $H_0^1(\Omega)$ by means of the usual action $(T_g)_{g\in G}$. Let $\mathrm{Fix}(G)$

be the subspace of functions of $H_0^1(\Omega)$ which are invariant under this action. As said, the solutions u_p possess the Schwarz foliated symmetry, so one can assume, rotating u_p if necessary, that $u_p \in \mathrm{Fix}(G)$.

In section 3, we show that the accumulation points of the family $(u_p)_{p>2}$ must be non-zero functions of E_2.

Section 4 details how we circumvent the degeneration of the limit problem (when $p = 2$) in order to use the implicit function theorem and deduce the uniqueness and oddness of least energy nodal solutions.

2. Symmetries of the functions in the second eigenspace

In this section, we gather some symmetry properties of the eigenfunctions for the second eigenvalue λ_2 on radial domains. More precisely, we will prove the following:

Proposition 2.1. *Let $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 2$ be a ball or an annulus and let $d \in \mathbb{S}^{N-1}$ be a direction. The subspace of eigenfunctions for the second eigenvalue λ_2 of $-\Delta$ on Ω with Dirichlet boundary conditions which are rotationally invariant under rotations around d has dimension 1. Moreover, these eigenfunctions are odd in the direction d.*

Eigenfunctions $u : \Omega \to \mathbb{R}$ of $-\Delta$ are the solutions of

$$\begin{gathered}-\Delta u(x) = \lambda u(x), \quad \text{for } x \in \Omega,\\ u(x) = 0, \quad \text{for } x \in \partial\Omega.\end{gathered}$$

In (hyper)spherical coordinates $x = r\theta$ with $r \in [0, +\infty[$ and $\theta \in \mathbb{S}^{N-1}$, the equation $-\Delta u = \lambda u$ reads (see for example [9, p. 38], [8], or reprove it using a local orthogonal parametrisation of $\mathbb{S}^{N-1}$):

$$\partial_r^2 u + \frac{N-1}{r}\partial_r u - \frac{1}{r^2}\big(-\Delta_{\mathbb{S}^{N-1}} u\big) = -\lambda u,$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$. By the method of separation of variables, we search functions $u(r,\theta) = R(r)S(\theta)$ satisfying

$$\begin{gathered}\partial_r^2 R + \frac{N-1}{r}\partial_r R + \Big(\lambda - \frac{\mu}{r^2}\Big) R = 0\\ -\Delta_{\mathbb{S}^{N-1}} S = \mu S.\end{gathered} \tag{2.1}$$

The eigenvalues μ_k of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ are well known (see for example [12] or [8]):

$$\mu_k = k(k + N - 2), \quad \text{for } k \in \mathbb{N}.$$

The corresponding eigenfunctions are called spherical harmonics. These are restrictions to the unit sphere, $S = P\restriction_{\mathbb{S}^{N-1}}$, of homogeneous polynomials $P : \mathbb{R}^N \to \mathbb{R}$ satisfying $\Delta P = 0$. The eigenfunctions of eigenvalue μ_k are the restrictions of the homogeneous polynomials of degree k among those [9, p. 39].

In order for $R(r) =: r^{-\frac{N-2}{2}} B(\sqrt{\lambda}\, r)$ to be solution of the first equation of (2.1) with $\mu = \mu_k$, it is necessary and sufficient that the function $s \mapsto B(s)$ satisfies:

$$\partial_s^2 B + \frac{1}{s}\partial_s B + \Big(1 - \frac{\nu^2}{s^2}\Big) B = 0 \tag{2.2}$$

where $\nu^2 := \mu_k + \frac{(N-2)^2}{4} = \bigl(k + \frac{N-2}{2}\bigr)^2$, for $k \in \mathbb{N}$. Solutions of equation (2.2) are linear combinations of the Bessel functions of the first kind J_ν and of the second kind Y_ν. Therefore, the solutions of the first equation of (2.1) with $\mu = \mu_k$ are

$$R(r) = r^{-\frac{N-2}{2}} \Bigl(aJ_\nu\bigl(\sqrt{\lambda}\, r\bigr) + bY_\nu\bigl(\sqrt{\lambda}\, r\bigr)\Bigr), \quad a, b \in \mathbb{R}, \tag{2.3}$$

where $\nu = k + \frac{N-2}{2}$.

Let us now distinguish two cases.

If Ω is a ball—which can be assumed to be of radius one without loss of generality—, the function Y_ν cannot appear in (2.3) because $\lim_{r\to 0} Y_\nu(r) = -\infty$ and $R(0)$ must be finite. Imposing the Dirichlet boundary condition $R(1) = 0$, we obtain that the eigenvalue λ must be the square of a positive root of J_ν. As is customary, let us denote $0 < j_{\nu,1} < j_{\nu,2} < \dots$ the infinitely many positive roots of J_ν. The interlacing property of the roots (see e.g. M. Abramowitz and A. Segun [1, § 9.5.2, p. 370]) says,

$$\forall \nu \geqslant 0, \quad j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < \dots \tag{2.4}$$

So, in particular, we obtain that $j^2_{\frac{N-2}{2},1}$ is the first eigenvalue of $-\Delta$ and $j^2_{\frac{N}{2},1}$ its second. Therefore, the eigenfunctions for the second eigenvalue of $-\Delta$ are given by:

$$r^{-\frac{N-2}{2}} J_{N/2}\bigl(\sqrt{\lambda}\, r\bigr) S(\theta)$$

where S is a spherical harmonic of eigenvalue μ_1. (It is well known [1, § 9.1.7, p. 360] that $J_\mu(r) \sim (\frac{1}{2}r)^\nu / \Gamma(\nu+1)$ as $r \to 0$ and therefore the eigenfunction has no singularity at 0.)

To conclude it suffices to use the fact that, for all directions $d \in \mathbb{S}^{N-1}$ and $k \in \mathbb{N}$, there exists exactly one (apart from a multiplicative constant) homogeneous polynomial S of degree k, solution to $\Delta S = 0$ and invariant under rotations around d (see [7, p. 365] and [9, p. 8]). Therefore, there exists one and only one spherical harmonic of eigenvalue μ_1 that is invariant under rotations around a given direction d. Moreover, this spherical harmonic is the restriction to the sphere of an homogeneous polynomial of degree 1 — i.e. a linear functional — and is consequently odd in the direction d.

Now let us turn to the second case where Ω is an annulus. Without loss of generality, one can assume that its internal radius is 1 and its external radius is $\rho \in \,]1, +\infty[$. Imposing the Dirichlet boundary conditions on (2.3) leads to the system

$$aJ_\nu(\sqrt{\lambda}) + bY_\nu(\sqrt{\lambda}) = 0$$
$$aJ_\nu(\sqrt{\lambda}\rho) + bY_\nu(\sqrt{\lambda}\rho) = 0$$

A non-trivial solution (a, b) of this system exists if and only if

$$\sqrt{\lambda} \text{ is a root of the function } s \mapsto J_\nu(s)Y_\nu(s\rho) - Y_\nu(s)J_\nu(s\rho).$$

It is known that this function possesses infinitely many positive zeros that we will note $0 < \chi_{\nu,1} < \chi_{\nu,2} < \dots$ Again an interlacing theorem for these zeros holds [5, p. 1736]: for all $\nu \geqslant 0$, $\chi_{\nu,1} < \chi_{\nu+1,1} < \chi_{\nu,2} < \chi_{\nu+1,2} < \dots$ As before, we deduce that the first eigenvalue happens for $k = 0$ (constant spherical harmonic) and $\nu = (N-2)/2$, while the second is when $k = 1$ and $\nu = N/2$. We then conclude in the same way as for the ball.

3. Convergence to a non-zero second eigenfunction

Let $(u_p)_{p>2} \subseteq \mathrm{Fix}(G)$ be a family of least energy nodal solutions of (1.2). In this section, we show that the accumulation points of u_p as $p \to 2$ are non-zero functions of the second eigenspace of $-\Delta$.

Let $e_2 \in \mathrm{Fix}(G)$ be one of the two second eigenfunctions such that $\|e_2\| = 1$ (whose existence was shown in proposition 2.1).

Lemma 3.1. *For any $q \in]2, 2^*[$, $\sup_{2<p<q} t_p^*$ is finite, where t_p^* is the unique positive real such that $t_p^* e_2$ belongs to the Nehari manifold $\mathcal{N}_p$ (i.e. belongs to $\mathcal{N}_p^1$).*

Proof. First of all, since $v := t_p^* e_2$ is odd, $\partial\varphi_p(v)v^+ = -\partial\varphi_p(v)v^- = \frac{1}{2}\partial\varphi_p(v)v$ and thus $v \in \mathcal{N}_p \Leftrightarrow v \in \mathcal{N}_p^1$.

Let $q \in]2, 2^*[$. For all $p \in]2, q[$, the fact that $t_p^* e_2 \in \mathcal{N}_p$ reads $(t_p^*)^2\|e_2\|^2 = (t_p^*)^p \lambda_2 |e_2|_p^p$. So,

$$t_p^* = \Big(\frac{\|e_2\|^2}{\lambda_2|e_2|_p^p}\Big)^{1/(p-2)} > 0.$$

Because $p \mapsto t_p^*$ is continuous, it is enough to show that t_p^* converges as $p \to 2$. We have,

$$\lim_{p\to 2} \ln\Big(\frac{\|e_2\|^2}{\lambda_2|e_2|_p^p}\Big)^{1/(p-2)} = \lim_{p\to 2} \frac{1}{p-2}\big(\ln(1) - \ln(\lambda_2|e_2|_p^p)\big) = \lim_{p\to 2} -\frac{\ln(\lambda_2|e_2|_p^p)}{p-2}.$$

Note that $|e_2|_2^2 = 1/\lambda_2$. In order to be able to apply l'Hospital rule, set $B := \Omega \backslash \{x \in \Omega : e_2(x) = 0\}$ (which makes sense because e_2 is continuous in $\overline{\Omega}$) and notice that $\partial_p |u_2|_p^p = \partial_p \int_B |e_2|^p = \int_B \ln|e_2||e_2|^p$ by Lebesque dominated convergence theorem and the fact that $\ln|t|\,|t| = \mathrm{o}(1)$ as $t \to 0$. Thus,

$$\lim_{p\to 2} -\frac{\ln(\lambda_2|e_2|_p^p)}{p-2} = \lim_{p\to 2} -\frac{\int_B \ln|e_2|\,|e_2|^p}{\int_B |e_2|^p} = -\frac{\int_B \ln|e_2|\,|e_2|^2}{\int_B |e_2|^2}$$

and so t_p^* converges to $\exp\big(-\int_B \ln|e_2|\,|e_2|^2 \,/\, \int_B |e_2|^2\big)$ as $p \to 2$. $\square$

Lemma 3.2. *For any $q \in]2, 2^*[$, the family $(u_p)_{p\in]2,q[}$ is bounded in $H_0^1(\Omega)$.*

Proof. Let us start by remarking that, for any $v \in \mathcal{N}_p$, $\big(\frac{1}{2} - \frac{1}{p}\big)\|v\|^2 = \varphi_p(v)$. Therefore, as $u_p \in \mathcal{N}_p^1 \subseteq \mathcal{N}_p$, one has

$$\begin{aligned}\Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_p\|^2 &= \varphi_p(u_{p,2}) = \inf_{u\in\mathcal{N}_p^1} \varphi_p(u) \\ &\leqslant \varphi_p(t_p^* e_2) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\|t_p^* e_2\|^2,\end{aligned} \tag{3.1}$$

where the inequality and the last equality result from the fact that $t_p^* e_2 \in \mathcal{N}_p^1$. Using (3.1) and lemma 3.1, we conclude that $(u_p)_{p\in]2,q[}$ is bounded in $H_0^1(\Omega)$. $\square$

Proposition 3.3. *All weak accumulation points of the family $(u_p)_{p>2}$ as $p \to 2$ are invariant under rotations leaving d fixed and have the form αe_2 for some $\alpha \in \mathbb{R}$.*

Proof. Let u^* be a weak accumulation point of (u_p). Thus, there exists a sequence $(p_n)_{n\in\mathbb{N}}$ such that $p_n \to 2$ and $u_{p_n} \rightharpoonup u^*$. Since, for all $g \in G$ and p, $T_g u_p = u_p$, it is clear that the same is true for u^*. In view of proposition 2.1, it remains to justify that $\partial\varphi_2(u^*) = 0$ to conclude.

For all $v \in H_0^1(\Omega)$ and all $n \in \mathbb{N}$, one has

$$\partial\varphi_{p_n}(u_{p_n})(v) = \int_\Omega \nabla u_{p_n} \nabla v - \lambda_2 \int_\Omega |u_{p_n}|^{p_n-2} u_{p_n} v = 0.$$

On one side, as $(u_{p_n})_{n\in\mathbb{N}}$ weakly converges in $H_0^1(\Omega)$ to u^*, $\int_\Omega \nabla u_{p_n} \nabla v$ converges to $\int_\Omega \nabla u^* \nabla v$. On the other side, by Rellich embedding theorem, u_{p_n} converges to u^* in $L^q(\Omega)$ with $q := \max\{p_n : n \in \mathbb{N}\} \in]2, 2^*[$, and thus, taking if necessary a subsequence still denoted u_{p_n}, u_{p_n} converges almost everywhere to u^* and there exists a function $f \in L^q(\Omega)$ such that, for all $n \in \mathbb{N}$, $|u_{p_n}| \leqslant f$ almost everywhere (see for example [13]). We conclude using Lebesgue dominated convergence theorem and the fact that, for all n, we have

$$\big||u_{p_n}|^{p_n-2} u_{p_n} v\big| \leqslant |f|^{p_n-1}|v| \leqslant \big|\max\{f, 1\}\big|^{q-1}|v| \in L^1(\Omega). \qquad \square$$

To conclude this section, we show that the accumulation points stay away from zero.

Lemma 3.4. *For any $p \in]2, 2^*[$ and $u \in H_0^1(\Omega) \setminus \{0\}$ such that $u^+ \neq 0$ and $u^- \neq 0$, there exist $t^+ > 0$ and $t^- > 0$ such that $t^+u^+ + t^-u^-$ belongs to $\mathcal{N}_p$ and is orthogonal to e_1 in $L^2(\Omega)$ where $e_1 > 0$ is a first eigenfunction of $-\Delta$.*

Proof. We consider the line segment

$$T : [0, 1] \to H_0^1(\Omega) \setminus \{0\} : \alpha \mapsto (1 - \alpha)u^+ + \alpha u^-.$$

We project it on $\mathcal{N}_p$: for all $\alpha \in [0, 1]$, there exists a unique $t_\alpha > 0$ such that $t_\alpha T(\alpha) \in \mathcal{N}_p$. For $\alpha = 0$, we have $\int_\Omega t_\alpha u^+ e_1 > 0$ and, for $\alpha = 1$, we have $\int_\Omega t_\alpha u^- e_1 < 0$. So, by continuity, there exists $\alpha^* \in]0, 1[$ such that $\int_\Omega t_{\alpha^*} T(\alpha^*)e_1 = 0$ and $t_{\alpha^*} T(\alpha^*) \in \mathcal{N}_p$. We just set $t^+ := t_{\alpha^*}(1 - \alpha^*)$ and $t^- := t_{\alpha^*}\alpha^*$ to conclude. $\square$

Proposition 3.5. *All weak accumulation points of u_p as $p \to 2$ are non-zero functions.*

Proof. By the preceding lemma, for all $p \in]2, 2^*[$, there exist $t_p^\pm > 0$ such that $v_p := t_p^+ u_p^+ + t_p^- u_p^-$ belongs to $\mathcal{N}_p$ and is orthogonal to e_1 in $L^2(\Omega)$.

We claim that $|v_p|_p \leqslant |u_p|_p$. As $u_p \in \mathcal{N}_p^1$, $u_p^+ \in \mathcal{N}_p$ maximizes the energy functional φ_p in the direction of u_p^+ and, similarly, $u_p^- \in \mathcal{N}_p$ maximizes φ_p in the direction of u_p^-. As the energy is the sum of the energy of the positive and negative parts, u_p maximizes the energy in the cone $K := \{t^+u_p^+ + t^-u_p^- : t^+ > 0 \text{ and } t^- > 0\}$. Since $v \in \mathcal{N}_p$ implies $\lambda_2\big(\frac{1}{2} - \frac{1}{p}\big)|v|_p^p = \varphi_p(v)$ and given that $v_p \in \mathcal{N}_p \cap K$, we deduce

$$\lambda_2\Big(\frac{1}{2} - \frac{1}{p}\Big)|v_p|_p^2 = \varphi_p(v_p) \leqslant \varphi_p(u_p) = \lambda_2\Big(\frac{1}{2} - \frac{1}{p}\Big)|u_p|_p^2.$$

Thus the claim is proved.

Let us now prove that v_p stays away from zero. By Hölder inequality, we have

$$|v_p|_p^2 \leqslant |v_p|_2^{2-2\lambda}|v_p|_{2^*}^{2\lambda},$$

where $\lambda := \frac{2^*}{2^*-2}\frac{p-2}{p}$. (In dimension 2, $2^* = +\infty$. In this case, we can replace 2^* by a sufficiently large q in the last inequality and use the same argument as below.)

As v_p is orthogonal to e_1 in $L^2(\Omega)$, $\lambda_2 \int_\Omega v_p^2 \leqslant \int_\Omega |\nabla v_p|^2$. By Sobolev embedding theorem, there exists a constant $S > 0$ such that

$$|v_p|_p^2 \leqslant \big(\lambda_2^{-1}\|v_p\|^2\big)^{1-\lambda}\big(S^{-1}\|v_p\|^2\big)^{\lambda}.$$

As v_p belongs to $\mathcal{N}_p$, $\|v_p\|^2 = \lambda_2 |v_p|_p^p$ and so

$$|v_p|_p^2 \leqslant \big(|v_p|_p^p\big)^{1-\lambda}\big(|v_p|_p^p\big)^{\lambda}(S^{-1}\lambda_2)^{\lambda}$$

or, equivalently,

$$|v_p|_p \geqslant \big(S\lambda_2^{-1}\big)^{\lambda/(p-2)} = \big(S\lambda_2^{-1}\big)^{\frac{2^*}{2^*-2}\frac{1}{p}}.$$

Therefore, if u^* is the weak limit of a sequence (u_{p_n}) in $H_0^1(\Omega)$ for some sequence $p_n \xrightarrow{>} 2$, by using Rellich embedding theorem, $|u^*|_2 = \lim_{n\to\infty} |u_{p_n}|_{p_n} \geqslant \liminf_{n\to\infty} |v_{p_n}|_{p_n} > 0$. □

4. Oddness

Lemma 4.1. *In dimension $N \geqslant 2$, in $\mathrm{Fix}(G) \times \mathbb{R}$, the problem*

$$\begin{gathered} -\Delta u(x) = \lambda |u(x)|^{p-2}u(x), \quad \text{in } \Omega, \\ u(x) = 0, \quad \text{on } \partial\Omega, \\ \|u\| = 1. \end{gathered} \tag{4.1}$$

possesses a single curve of solutions $p \mapsto (p, u_p^*, \lambda_p^*)$ defined for p close to 2 and starting from $(2, e_2, \lambda_2)$. It also possesses a single curve of solutions starting from $(2, -e_2, \lambda_2)$ which is given by $p \mapsto (p, -u_p^*, \lambda_p^*)$.

Proof. Let us define

$$\begin{gathered} \psi : [2, 2^*[\times \mathrm{Fix}(G) \times \mathbb{R} \to \mathrm{Fix}(G) \times \mathbb{R} \\ (p, u, \lambda) \mapsto \big(u - \lambda(-\Delta)^{-1}(|u|^{p-2}u), \|u\|^2 - 1\big). \end{gathered}$$

The first component is the H_0^1-gradient of the following energy functional

$$\varphi_{p,\lambda} : \mathrm{Fix}(G) \to \mathbb{R} : u \mapsto \tfrac{1}{2}\|u\|^2 - \lambda \tfrac{1}{p}|u|_p^p\,.$$

The function ψ is well defined, thanks to the symmetric criticality principle.

The existence and local uniqueness of a branch emanating from $(2, e_2, \lambda_2)$ follows from the implicit function theorem and the closed graph theorem if we prove that the Frechet derivative of ψ w.r.t. (u, λ) at the point $(2, e_2, \lambda_2)$ is bijective on $\mathrm{Fix}(G)\times\mathbb{R}$. We have,

$$\partial_{(u,\lambda)}\psi(2, e_2, \lambda_2)(v, t) = \Big(v - \lambda_2(-\Delta)^{-1}v - t(-\Delta)^{-1}e_2,\ 2\int_\Omega \nabla e_2 \nabla v\Big). \tag{4.2}$$

For the injectivity, let us start by showing that $\partial_{(u,\lambda)}\psi(2, e_2, \lambda_2)(v, t) = 0$ if and only if

$$\begin{gathered} v - \lambda_2(-\Delta)^{-1}v = 0, \\ t = 0, \\ v \text{ is orthogonal to } e_2 \text{ in } H_0^1(\Omega). \end{gathered} \tag{4.3}$$

It is clear that (4.3) is sufficient. For its necessity, remark that the nullity of second component of (4.2) implies that e_2 is orthogonal to v in $H_0^1(\Omega)$ and thus also in $L^2(\Omega)$ because e_2 is an eigenfunction. Taking the L^2-inner product of the first component of (4.2) with e_2 yields $t = 0$, hence the equivalence is complete. Now, the only solution of (4.3) is $(v, t) = (0, 0)$ because the first equation and the dimention 1 of the trace of the second eigenspace in $\mathrm{Fix}(G)$ (proposition 2.1) imply $v \in \mathrm{span}\{e_2\}$ and then the third property implies $v = 0$. This concludes the proof of the injectivity of $\partial_{(u,\lambda)}\psi(2, e_2, \lambda_2)$.

Let us now show that, for any $(w, s) \in \mathrm{Fix}(G) \times \mathbb{R}$, the equation

$$\partial_{(u,\lambda)}\psi(2, e_2, \lambda_2)(v, t) = (w, s)$$

always possesses at least one solution $(v, t) \in \mathrm{Fix}(G)\times\mathbb{R}$. One can write $w = \bar{w}e_2 + \widetilde{w}$ for some $\bar{w} \in \mathbb{R}$ and $\widetilde{w}$ orthogonal to e_2 in H^1_0. Of course, $\widetilde{w} \in \mathrm{Fix}(G)$ since w and e_2 both are. Similarly, one can decompose $v = \bar{v}e_2 + \widetilde{v}$. Arguing as for the first part, the equation can be written

$$\begin{gathered} \widetilde{v} - \lambda_2(-\Delta)^{-1}\widetilde{v} = \widetilde{w}, \\ t = \lambda_2\bar{w}, \\ \bar{v} = s/2. \end{gathered} \tag{4.4}$$

Thanks to the principle of symmetric criticality, the solution $\widetilde{v}$ is the minimizer of the functional

$$X \to \mathbb{R} : \widetilde{v} \mapsto \int_\Omega |\nabla\widetilde{v}|^2 - \lambda_2|\widetilde{v}|^2 - \int_\Omega \nabla\widetilde{w}\nabla\widetilde{v}$$

where X is the subspace of $\mathrm{Fix}(G)$ orthogonal to e_2. This concludes the proof that $\partial_{(u,\lambda)}\psi(2, e_2, \lambda_2)$ is onto and thus of the existence and uniqueness of the branch emanating from $(2, e_2, \lambda_2)$.

It is clear that $p \mapsto (p, -u^*_p, \lambda^*_p)$ is a branch emanating from $(2, -e_2, \lambda_2)$ and, using as above the implicit function theorem at that point, we know it is the only one. $\square$

Theorem 4.2. *For p close to 2, least energy nodal solutions on a ball or an annulus are unique (up to a rotation and their sign) and odd with respect to a direction.*

Proof. Let $(u_p)_{p>2}$ be a family of solutions of (1.2). Up to a rotation, we can assume that the solution $u_p \in \mathrm{Fix}(G)$. Thanks to lemma 3.2, for any sequence $p_n \xrightarrow{>} 2$, there exists a subsequence, still denoted p_n, such that u_{p_n} weakly converges in H^1_0 to some $u^* \in \mathrm{Fix}(G)$. The Rellich embedding theorem and

$$\begin{aligned} 0 &= \partial\varphi_{p_n}(u_{p_n})(u_{p_n} - u^*) - \partial\varphi_2(u^*)(u_{p_n} - u^*) \\ &= \|u_{p_n} - u^*\|^2 - \lambda_2 \int_\Omega |u_{p_n}|^{p_n-2}u_{p_n}(u_{p_n} - u^*) + \lambda_2 \int_\Omega u^*(u_{p_n} - u^*) \end{aligned}$$

imply that $u_{p_n} \to u^*$ in $H^1_0(\Omega)$. Therefore, propositions 3.3 and 3.5, yield $u^* = \alpha e_2$ for some $\alpha \in \mathbb{R} \setminus \{0\}$.

On the other hand, notice that u is a solution of (1.2) if and only if $\big(u/\|u\|, \lambda_2\|u\|^{p-2}\big)$ is a solution of (4.1). Because (u_{p_n}) stays away from 0, one has

$$\Big(\frac{u_{p_n}}{\|u_{p_n}\|}, \lambda_2\|u_{p_n}\|^{p_n-2}\Big) \xrightarrow[n]{} \big(\mathrm{sign}(\alpha)e_2, \lambda_2\big).$$

Then, when p_n is close enough to 2, lemma 4.1 implies that

$$\frac{u_{p_n}}{\|u_{p_n}\|} = \mathrm{sign}(\alpha)\, u^*_{p_n}.$$

Hence, the claimed uniqueness of u_p up to its sign. This also implies that u_{p_n} is odd in the direction d. To show that, let us consider u'_{p_n} the anti-symmetric of u_{p_n} — defined by $u'_{p_n}(x) := -u_{p_n}\big(x - 2(x \cdot d)d\big)$ where $x \cdot d$ is the inner product in $\mathbb{R}^N$. Because e_2 is odd in the direction d, $u'_{p_n} \to \alpha e_2$ with the same α as for u^*. Arguing as before, we conclude that

$$\frac{u_{p_n}}{\|u_{p_n}\|} = \operatorname{sign}(\alpha)\, u^*_{p_n} = \frac{u'_{p_n}}{\|u'_{p_n}\|}$$

and therefore that u_{p_n} is odd in the direction d. $\square$

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Christopher Grumiau

Institut de Mathématique, Université de Mons, Place du Parc, 20, B-7000 Mons, Belgium

E-mail address: `Christopher.Grumiau@umons.ac.be`

Christophe Troestler

Institut de Mathématique, Université de Mons, Place du Parc, 20, B-7000 Mons, Belgium

E-mail address: `Christophe.Troestler@umons.ac.be`