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THE FUČÍK SPECTRA FOR MULTI-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. We study the structure of the Fučík spectra for the linear multi-point differential operators. We introduce a variational approach in order to obtain a robust and global algorithm which is suitable for the exploration of unknown Fučík spectrum structure. We apply our approach in the case of the four-point selfadjoint differential operator of the fourth order which is closely connected to the nonlinear model of a suspension bridge with two towers. Moreover, we reconstruct the Fučík spectra in the case of four-point non-selfadjoint ordinary differential operators of the second order in order to demonstrate their non-trivial and interesting structure.

1. Introduction

In this paper, we investigate the structure of the Fučík spectrum

$$\Sigma(L) = \{(\alpha, \beta) \in \mathbb{R}^2 : Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\},$$

where L is the linear operator and $u^+ := \max\{u, 0\}$, $u^- := \max\{-u, 0\}$. The Fučík spectra of various differential operators have been investigated by many authors (see [2, 7, 8, 11, 13, 14, 15] and the references therein) and the most of the results are proved for selfadjoint operators. The selfadjointness allows to use variational approach and essentially influences the structure of the Fučík spectrum (see results for general linear selfadjoint operator by Ben-Naoum, Fabry and Smets in [1]). Motivated by [6], we focus on the multi-point differential operators, especially on the non-selfadjoint operators, in order to demonstrate how the non-selfadjointness results in non-standard structures of the Fučík spectrum which are not typical for ordinary differential operators.

The paper is organized as follows. Section 2 is devoted to the selfadjoint multi-point differential operator of the fourth order L^{S}. The corresponding Fučík spectrum $\Sigma(L^{\mathsf{S}})$ is defined in a weak sense and using the embedding theorems, we prove the regularity result for weak solutions. Moreover, due to the selfadjointness of L^{S}, we design an algorithm based on the variational approach in order to explore the structure of the Fučík spectrum $\Sigma(L^{\mathsf{S}})$. In Section 3, the main results of [6] are

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briefly recalled and the complete and precise analytical description of the Fučík spectrum is provided for the non-selfadjoint four-point differential operator L. The structure of $\Sigma(L)$ exhibits the non-standard and interesting phenomena which are not obvious for ordinary differential operators: monotonicity and smoothness of the Fučík branches are lost, the Fučík branches intersect away from the diagonal and the nodal properties of the corresponding Fučík eigenfunctions are not preserved. The last Section 4 concerns the adjoint operator L^* of L and reveals the correspondence between the Fučík spectra $\Sigma(L^*)$ and $\Sigma(L)$.

In this paper, let us adopt the concept of the multi-point differential operators introduced in [10]. Thus, suppose that $\mathcal{P} = \{a = x_0 < x_1 < \dots < x_m = b\}$ is a given partition of the interval $[a,b]$. Let $H^n(\mathcal{P})$ denote the set of all functions $u \in L^2(a,b)$ with the following two properties

(1) On each subinterval $[x_{i-1}, x_i]$, the function $u = u(x)$ possesses both right-hand and left-hand limits at the endpoints x_{i-1} and x_i, respectively. For $i = 1, \dots, m$, let $u_i : [x_{i-1}, x_i] \to \mathbb{R}$ be the function defined by $u_i \equiv u$ on (x_{i-1}, x_i), $u(x_{i-1}) = u(x_{i-1}+)$ and $u(x_i) = u(x_i-)$. The functions $u_1, \dots, u_m$ are called the *components* of u, which is denoted by $u = (u_1, \dots, u_m)$.
(2) The components u_i belong to $H^n([x_{i-1}, x_i])$ for $i = 1, \dots, m$.

Finally, let us note that in the next sections, we follow mainly the paper [6] and using three different approaches, we recover the non-trivial Fučík spectrum structures of the three multi-point operators L^{S}, L and L^*.

2. Selfadjoint multi-point operator

Let us consider the multi-point boundary-value problem

$$\begin{gathered} -u^{\mathrm{IV}}(x) + \alpha u^+(x) - \beta u^-(x) = 0, \quad x \in (0,\xi) \cup (\xi,\eta) \cup (\eta,\pi), \\ u'(0) = u(0) = u(\xi) = u(\eta) = u(\pi) = u'(\pi) = 0, \quad 0 < \xi < \eta < \pi, \end{gathered} \tag{2.1}$$

which represents the simplified mathematical model of a suspension bridge with two towers (see Figure 1). The roadbed of the bridge is modelled as a clamped beam which is suspended by two systems of one-sided springs with stiffnesses α and β. The function $u = u(x)$ describes the steady-state displacement of the roadbed and is measured as positive in downward direction. Let us recall the Lazer-McKenna normalized suspension bridge model (see [9] or [12]) and also the open problem in [4] concerning the explicit form of the resonance set for such a mathematical model.

Let us define the partition of the interval $[0,\pi]$ as $\mathcal{P} = \{0, \xi, \eta, \pi\}$ and let us define the following multi-point boundary values

$$\begin{aligned} &B_1(u) = u(0+) = u_1(0), && B_4(u) = u(\xi+) = u_2(\xi), && B_6(u) = u(\eta+) = u_3(\eta), \\ &B_2(u) = u'(0+) = u_1'(0), && B_5(u) = u(\eta-) = u_2(\eta), && B_7(u) = u(\pi-) = u_3(\pi), \\ &B_3(u) = u(\xi-) = u_1(\xi), && && B_8(u) = u'(\pi-) = u_3'(\pi), \end{aligned}$$

(recall that $u = (u_1, u_2, u_3)$). Let $L^{\mathsf{S}} : \mathrm{dom}(L^{\mathsf{S}}) \subset L^2(0,\pi) \to L^2(0,\pi)$ be the linear multi-point differential operator defined by

$$L^{\mathsf{S}} u(x) := u^{\mathrm{IV}}(x), \quad \mathrm{dom}(L^{\mathsf{S}}) := \left\{ u \in H^4(\mathcal{P}) : B_i(u) = 0,\ i = 1, \dots, 8 \right\}.$$

Using Theorem 2 in [10], it is straightforward to verify that L^{S} is the selfadjoint operator. Due to selfadjointness of L^{S}, results by Ben-Naoum, Fabry and Smets in [1] concerning general linear selfadjoint operator can be applied in order to describe the Fučík spectrum $\Sigma(L^{\mathsf{S}})$. Let us remark that in the case of the Navier and the Dirichlet operators of the fourth order, the qualitative properties of the Fučík spectrum are investigated in [8] and [2].

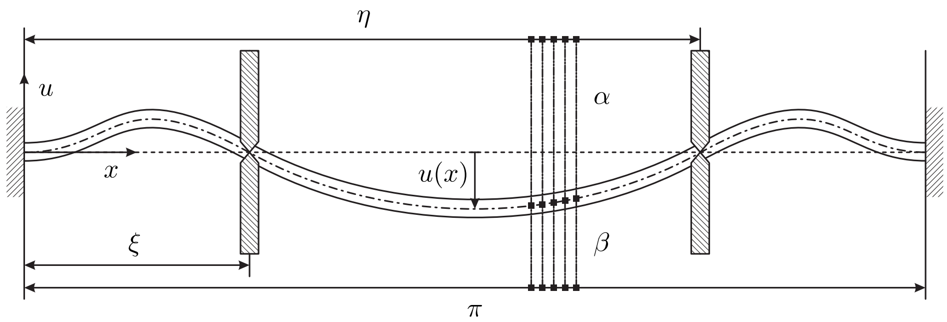

FIGURE 1. The model of a suspension bridge with two towers

To introduce the weak formulation of the main problem (2.1), let us define the space of test functions as $V := \{v \in H^2(\mathcal{P}) : B_i(u) = 0,\ i = 1, \dots, 8\}$ and denote by (u, v) the scalar product on $L^2(0, \pi)$.

Definition 2.1. Let us define a *weak solution* of (2.1) as a function $u \in V$ such that

$$(u'', v'') = (\alpha u^+(x) - \beta u^-(x), v) \quad \forall\, v \in V. \tag{2.2}$$

The following lemma concerns the regularity result for weak solutions of (2.1).

Lemma 2.2. *If u is a weak solution of* (2.1) *then u is the classical solution. Moreover,*

$$u \in C^2([0, \pi]), \quad u_1 \in C^4([0, \xi]),\ u_2 \in C^4([\xi, \eta]),\ u_3 \in C^4([\eta, \pi]),$$

where $u = (u_1, u_2, u_3)$.

Proof. Let $u \in V$ is a weak solution of (2.1),

$$\int_0^\pi u''(x)v''(x)\,\mathrm{d}x - \int_0^\pi g(u(x))v(x)\,\mathrm{d}x = 0 \quad \forall v \in V, \tag{2.3}$$

where $g(u) := \alpha u^+ - \beta u^-$. First of all, since $u \in H^2(0, \pi)$, we have that $u \in C^1([0, \pi])$ due to the compact imbedding $H^2(0, \pi) \subsetneq\subsetneq C^1([0, \pi])$.

If we define $U \subset V$ as

$$U := \{v \in H^2(\mathcal{P}) : v_1 \equiv 0,\ v_2 \in C_0^\infty(\xi, \eta),\ v_3 \equiv 0\}, \tag{2.4}$$

we obtain, using (2.3), that

$$\int_\xi^\eta u_2''(x)v_2''(x)\,\mathrm{d}x - \int_\xi^\eta g(u_2(x))v_2(x)\,\mathrm{d}x = 0 \quad \forall v_2 \in C_0^\infty(\xi, \eta). \tag{2.5}$$

If we integrate by parts two times the second integral in (2.5), we obtain

$$\int_\xi^\eta M(x)v_2''(x)\,\mathrm{d}x = 0 \quad \forall v_2 \in C_0^\infty(\xi, \eta), \tag{2.6}$$

where $M(x) := u_2''(x) - \int_\xi^x \int_\xi^t g(u_2(\tau))\mathrm{d}\tau\mathrm{d}t$. The equality in (2.6) can be interpreted as $\int_\xi^\eta M_2(x)v_2(x)\,\mathrm{d}x = 0$, where $M_2(x)$ denotes the second derivative of $M(x)$ in the sense of distributions. Using the generalized variational lemma, we obtain that $M_2(x) = 0$ in the space of distributions, which means that there exists the constants $c_0, c_1 \in \mathbb{R}$ such that

$$M(x) = c_1 x + c_0 \quad \text{for almost all } x \in (\xi, \eta). \tag{2.7}$$

Now, let us define the function $F : [\xi, \eta] \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, z) = z - \int_\xi^x \int_\xi^t g(u_2(\tau)) \mathrm{d}\tau \mathrm{d}t - c_1 x - c_0.$$

For every $x \in [\xi, \eta]$ there exists exactly one $z \in \mathbb{R}$ such that $F(x, z) = 0$. Due to $u_2 \in C^1([\xi, \eta])$ and g is the continuous function, the function $F(x, z)$ has continuous partial derivatives of the first order on $[\xi, \eta] \times \mathbb{R}$. Thus, using Implicit Theorem, we obtain that the function $z = z(x)$ given implicitely by $F(x, z(x)) = 0$ is continuous together with its first derivative on $[\xi, \eta]$. If we recall the definition of $M(x)$, (2.7) can be written as $F(x, u_2''(x)) = 0$ for almost all $x \in (\xi, \eta)$, which implies that $z(x) = u_2''(x)$ for almost all $x \in (\xi, \eta)$. The continuity of $z(x)$ on $[\xi, \eta]$ immediately gives us that $z(x) = u_2''(x)$ for every $x \in [\xi, \eta]$ (note that for all $x \in [\xi, \eta] : u_2'(x) = u_2'(\xi) + \int_\xi^\eta u_2''(t)\,\mathrm{d}t = u_2'(\xi) + \int_\xi^\eta z(t)\,\mathrm{d}t$). Moreover, due to $z \in C^1([\xi, \eta])$, we conclude that $u_2 \in C^3([\xi, \eta])$.

Now, using integration by parts to both integrals in (2.5), we obtain

$$\int_\xi^\eta \Big[u_2'''(x) - \int_\xi^x g(u_2(t))\,\mathrm{d}t \Big] v_2'(x)\,\mathrm{d}x = 0 \quad \forall v_2 \in C_0^\infty(\xi, \eta).$$

If we proceed in the analog way as in the case of (2.6), we obtain that $u_2 \in C^4([\xi, \eta])$. This enables us to integrate by parts two times the first integral in (2.5) to get

$$\int_\xi^\eta \left[u_2^{\mathrm{IV}}(x) - g(u_2(x)) \right] v_2(x)\,\mathrm{d}x = 0 \quad \forall v_2 \in C_0^\infty(\xi, \eta),$$

which implies that $u_2^{\mathrm{IV}}(x) = g(u_2(x))$ for almost all $x \in (\xi, \eta)$ due to the density of $C_0^\infty(\xi, \eta)$ in $L^2(\xi, \eta)$. The continuity of g and u_2^{IV} ensures that $u_2^{\mathrm{IV}}(x) = g(u_2(x))$ for every $x \in [\xi, \eta]$.

Finally, if we replace the definition of U in (2.4) as

$$U := \{ v \in H^2(\mathcal{P}) : v_1 \in C_0^\infty(0, \xi),\ v_2 \equiv 0,\ v_3 \equiv 0 \},$$

or as

$$U := \{ v \in H^2(\mathcal{P}) : v_1 \equiv 0,\ v_2 \equiv 0,\ v_3 \in C_0^\infty(\eta, \pi) \},$$

we obtain in the analog way that $u^{\mathrm{IV}}(x) = g(u(x))$ for all $x \in (0, \xi) \cup (\eta, \pi)$. Thus, u is the classical solution of (2.1). Moreover, if we integrate by parts two times all three integrals in $\int_0^\xi u_1'' v_1''\,\mathrm{d}x + \int_\xi^\eta u_2'' v_2''\,\mathrm{d}x + \int_\eta^\pi u_3'' v_3''\,\mathrm{d}x$ then (2.3) can be written in the following form

$$[u_1''(\xi) - u_2''(\xi)]\, v'(\xi) + [u_2''(\eta) - u_3''(\eta)]\, v'(\eta) = 0 \quad \forall v \in V,$$

which implies that $u''(\xi-) = u''(\xi+)$ and $u''(\eta-) = u''(\eta+)$. Thus, we conclude that $u \in C^2([0, \pi])$. □

To explore the Fučík spectrum for the problem (2.1), let us focus first on the corresponding eigenvalue problem in the case of $\alpha = \beta$. Let us consider the space

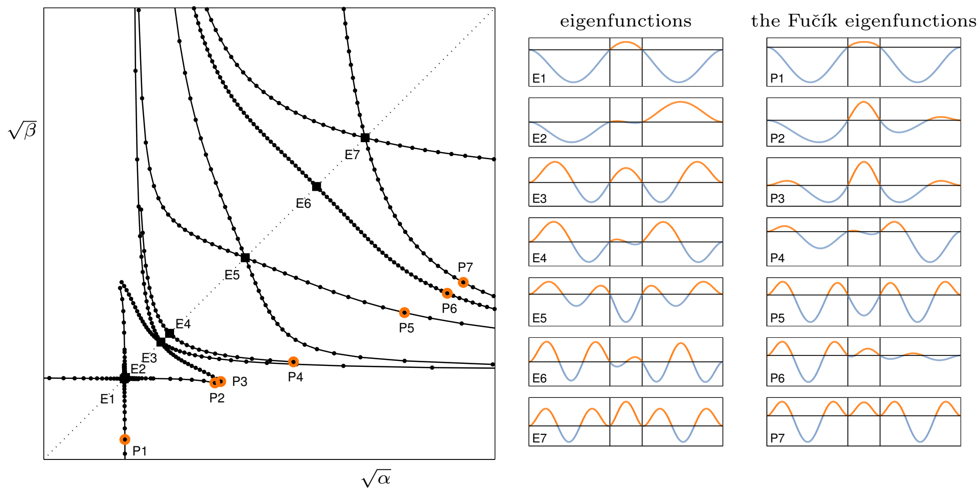

FIGURE 2. The approximation of $\Sigma(L^{\mathsf{S}})$ for $\xi = \frac{5\pi}{12} < \frac{7\pi}{12} = \eta$ and the corresponding Fučík eigenfunctions.

V as the Hilbert space with the inner product $(u,v)_V := \int_0^\pi u''(x)v''(x)\,\mathrm{d}x$ and the norm $\|u\|_V := \sqrt{(u,u)_V}$ and let us define the operator $A : V \to V$ as

$$(A(u),v)_V := \int_0^\pi u(x)v(x)\,\mathrm{d}x$$

for all $u, v \in V$. Thus, in the case of $\alpha = \beta =: \lambda$, the weak formulation of the problem (2.1) has the form of the following *weak eigenvalue problem*

$$\mu u = A(u),$$

where $\mu = \frac{1}{\lambda}$. The operator A is positive, self-adjoint and compact (recall the compact embedding $H^2(0,\pi) \subsetneq\subsetneq L^2(0,\pi)$). Thus, using Courant-Fisher Principle, we obtain that A has a countable set of *weak eigenvalues* $\{\mu_k\}$ of the finite multiplicity such that $\mu_1 \geq \mu_2 \geq \mu_3 \geq \dots > 0$ and $\lim\limits_{k\to+\infty} \mu_k = 0$. Hence, using $\lambda = \frac{1}{\mu}$, we obtain the non-decreasing sequence $\{\lambda_n\}$ such that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \quad \lim_{n\to+\infty} \lambda_n = +\infty,$$

and that

$$\int_0^\pi \varphi_n''(x)v''(x)\,\mathrm{d}x = \lambda_n \int_0^\pi \varphi_n(x)v(x)\,\mathrm{d}x \quad \forall v \in V,$$

where the *weak eigenfunction* $\varphi_n \in V$ satisfies $\mu_n\varphi_n = A(\varphi_n)$. Finally, using Lemma 2.2, we conclude that φ_n is the nontrivial classical solution of (2.1) with $\alpha = \beta = \lambda_n$, $n \in \mathbb{N}$. Thus, we have

$$\forall n \in \mathbb{N} : (\lambda_n, \lambda_n) \in \Sigma(L^{\mathsf{S}}).$$

Now, let us adopt the variational approach introduced in [14] in order to explore numericaly the points of the Fučík spectrum $\Sigma(L^{\mathsf{S}})$. Let $\mu \in \mathbb{R} \setminus \sigma(L^{\mathsf{S}})$ and $\delta \in \mathbb{R}$ be such that $(\mu + \delta) \notin \sigma(L^{\mathsf{S}})$, where $\sigma(L^{\mathsf{S}})$ denotes the spectrum of L^{S}. If we take into account the transformation

$$\mathcal{T}_{\mu,\delta} = \mathcal{T}_{\mu,\delta}(\alpha,\beta,u) = (m,\tilde\lambda,v), \quad \mathcal{T}^{-1}_{\mu,\delta} = \mathcal{T}^{-1}_{\mu,\delta}(m,\tilde\lambda,v) = (\alpha,\beta,u),$$

$$\mathcal{T}_{\mu,\delta} : \begin{cases} m = \frac{\beta-\alpha}{\beta+\alpha-2\mu}, \\ \tilde\lambda = \frac{2\mu-\alpha-\beta}{2(\mu-\alpha)(\mu-\beta)+\delta(2\mu-\alpha-\beta)}, \\ v = (\mu I - L^{\mathsf{S}})\, u, \end{cases} \qquad \mathcal{T}^{-1}_{\mu,\delta} : \begin{cases} \alpha = \mu - \frac{1-\delta\tilde\lambda}{\tilde\lambda(1+m)}, \\ \beta = \mu - \frac{1-\delta\tilde\lambda}{\tilde\lambda(1-m)}, \\ u = (\mu I - L^{\mathsf{S}})^{-1}\, v, \end{cases}$$

then the Fučík spectrum problem $L^{\mathsf{S}}u = \alpha u^+ - \beta u^-$ reads as the nonlinear problem

$$\left((\mu+\delta)I - L^{\mathsf{S}}\right)^{-1} v = \tilde\lambda \left(v + m\left(I - \delta\left[(\mu+\delta)I - L^{\mathsf{S}}\right]^{-1}\right)|v|\right). \tag{2.8}$$

Due to homogenity of the nonlinear eigenpair problem (2.8), the critical points of the corresponding Rayleigh quotient $J : L^2(0,\pi) \to \mathbb{R}$

$$J(v) = \frac{F(v)}{G(v)}, \quad F(v) = \frac{1}{2}\int_0^\pi \left((\mu+\delta)I - L^{\mathsf{S}}\right)^{-1} v \cdot v \,\mathrm{d}x,$$

$$G(v) = \frac{1}{2}\int_0^\pi v^2 + m\left(I - \delta\left[(\mu+\delta)I - L^{\mathsf{S}}\right]^{-1}\right)|v|v \,\mathrm{d}x,$$

together with their critical values $\tilde\lambda = J(v)$ are in one to one correspondence with eigenpairs $(\tilde\lambda, v)$ of (2.8).

In order to simplify the following notation, let us consider the subsequence $\{\lambda_{n_k}\}$ such that $0 < \lambda_{n_1} < \lambda_{n_2} < \lambda_{n_3} < \dots$ and that $\bigcup_{n=1}^{+\infty}\{\lambda_n\} = \bigcup_{k=1}^{+\infty}\{\lambda_{n_k}\}$ and denote it again by $\{\lambda_n\}$. First, let us note that for $m = 0$, the equation (2.8) can be written in the form $L^{\mathsf{S}}v = (\mu + \delta - \frac{1}{\tilde\lambda})v$. Thus, if we set $\mu < 0$ and $\delta = 0$ then we can characterize the first eigenvalue of L^{S} as $\lambda_1 = \mu - \frac{1}{\tilde\lambda_1}$, where $\tilde\lambda_1 = J(\psi_1)$ with $\psi_1 = \arg\min J(v)$. The eigenfunction φ_1 corresponding to λ_1 is given by $\varphi_1 = (\mu I - L^{\mathsf{S}})^{-1}\psi_1$. Moreover, if we take $\mu < 0$ and $\delta > 0$ such that $(\mu+\delta) \in (\lambda_1, \lambda_2)$ then $\lambda_1 = \mu + \delta - \frac{1}{\max J(v)}$ and $\lambda_2 = \mu + \delta - \frac{1}{\min J(v)}$.

Second, in order to obtain the approximation of $\Sigma(L^{\mathsf{S}})$ depicted in Figure 2, let us proceed in the following way. Let us fix $\mu < 0$, $\delta = \frac{\lambda_i + \lambda_{i+1}}{2} - \mu$, $i \in \mathbb{N}$, and $m \in [-1, 1]$. Then the minimization and maximization process of the functional J gives us two critical values $\tilde\lambda = J(v)$ and the inverse transformation $\mathcal{T}^{-1}_{\mu,\delta}(m,\tilde\lambda,v) = (\alpha,\beta,u)$ recovers two points of the Fučík spectrum $\Sigma(L^{\mathsf{S}})$.

Remark 2.3. To reduce the problem of finding the critical points of the functional $J : L^2(0,\pi) \to \mathbb{R}$ onto a finite-dimensional space, we consider the following descretization. Denote $h := \frac{\pi}{n}$, $n \in \mathbb{N}$, and let $x_i := ih$, $i = 0, 1, \dots, n$, be the equidistant mesh of the interval $[0,\pi]$. Moreover, let $\xi = x_{p_1} < x_{p_2} = \eta$ with $0 < p_1 < p_2 < n$ and denote by U_i the approximation of $u(x_i)$, $i = 0, 1, \dots, n$. Since $U_0 = U_{p_1} = U_{p_2} = U_n = 0$, we take

$$U := [U_1, U_2, \dots, U_{p_1-1}, U_{p_1+1}, \dots, U_{p_2-1}, U_{p_2+1}, \dots, U_{n-1}]$$

as the approximation of u on the interval $[0,\pi]$. Finally, the approximation of the selfadjoint operator L^{S} has the form of the following five-diagonal symmetric matrix of order $n-3$

$$\frac{1}{h^4}\cdot\begin{array}{c} \begin{array}{cccccccccccccc} & & & & & & & p_1-1 & & & & & & \end{array} \\ \begin{bmatrix}
7 & -4 & 1 & & & & & 0 & & & & & & \\
-4 & 6 & -4 & 1 & & & & 0 & & & & & & \\
1 & -4 & 6 & -4 & 1 & & & 0 & & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & & & & & & \\
 & & 1 & -4 & 6 & -4 & 1 & 0 & & & & & & \\
 & & & 1 & -4 & 6 & -4 & 1 & & & & & & \\
 & & & & 1 & -4 & 6 & -4 & 0 & & & & & \\
 & & & & & 1 & -4 & 6 & 1 & 0 & 0 & 0 & 0 & \dots \\
 & & & & & & 0 & 1 & 6 & -4 & 1 & & & \\
 & & & & & & & 0 & -4 & 6 & -4 & 1 & & \\
 & & & & & & & & 1 & -4 & 6 & -4 & 1 & \\
 & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} \end{array} . \quad p_1-1$$

To conclude this section, let us formulate the following conjecture based on the numerical experiments (see Figure 2).

Conjecture 2.4.

(1) *Each point $E_n := (\lambda_n, \lambda_n)$, $n \in \mathbb{N}$, gives arise to exactly one or two Fučík curves of $\Sigma(L^{\mathrm{S}})$.*

(2) *Any Fučík curve that goes through the point E_n with $n \geq 3$ can be described by a strictly decreasing function $\alpha \mapsto \beta(\alpha)$ which is not necessarily convex. Compare it with the Fučík spectrum for the Dirichlet operator of the second order in [3] which is given by explicit analytic formulas and the nontrivial Fučík curves are described as strictly decreasing convex functions.*

(3) *In the case of the symmetrical settings of ξ and η with respect to the mid-point of the interval $[0, \pi]$, we have for every eigenfunction φ_n that $\varphi_n(x+\frac{\pi}{2})$ is an even or an odd function. But, this statement is not necessarily valid for the Fučík eigenfunctions when $\alpha \neq \beta$ (see Figure 2 with the emphasize to the points P_2, P_4 and P_6).*

3. Non-selfadjoint multi-point operator

In this section, let us describe the Fučík spectrum of the following four-point boundary value problem

$$\begin{gathered} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, \quad x \in (0, \pi), \\ u'(0) = u'(\xi), \quad u(\pi) = u(\eta), \quad \xi \in (0, \pi),\ \eta \in (0, \pi), \end{gathered} \tag{3.1}$$

where we assume in addition that $\xi > \frac{\pi+\eta}{2}$ in order to simplify the following notation. Let us denote by $\mathcal{P} := \{0, \eta, \xi, \pi\}$ the partition of the interval $[0, \pi]$ and let us define the following multi-point boundary values (note that $u = (u_1, u_2, u_3)$)

$$\begin{aligned} B_1(u) &:= u_2(\xi) - u_3(\xi), \quad & B_3(u) &:= u_2'(\xi) - u_3'(\xi), \quad & B_5(u) &:= u_1'(0) - u_3'(\xi), \\ B_2(u) &:= u_1(\eta) - u_2(\eta), \quad & B_4(u) &:= u_1'(\eta) - u_2'(\eta), \quad & B_6(u) &:= u_1(\eta) - u_3(\pi). \end{aligned}$$

Let us associate to (3.1) the linear multi-point differential operator $L : \operatorname{dom}(L) \subset L^2(0, \pi) \to L^2(0, \pi)$ defined by

$$Lu(x) := -u''(x), \quad \operatorname{dom}(L) := \left\{u \in H^2(\mathcal{P}) : B_i(u) = 0,\ i = 1, \dots, 6\right\}.$$

Using Theorem 2 in [10], it is possible to verify that the operator L is non-selfadjoint operator.

The complete description of the Fučík spectrum $\Sigma(L)$ is provided in [6], thus, in the following text, let us only recall the main results. Let us note that the non-selfadjointness of the operator L results in nontrivial structure of the Fučík spectrum with interesting patterns (see Figure 3).

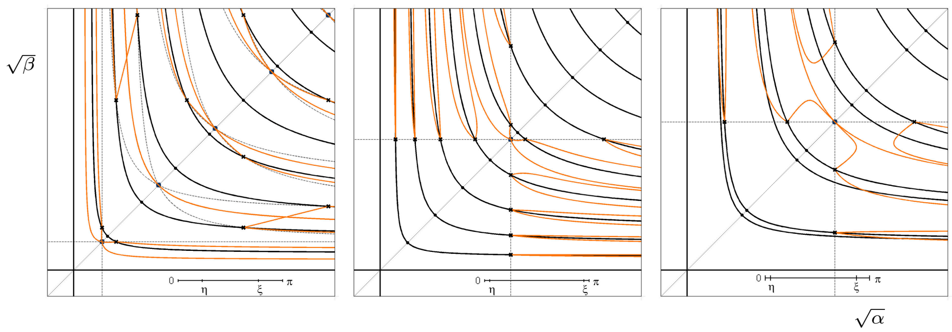

FIGURE 3. The decomposition of the Fučík spectrum $\Sigma(L)$ into $\Sigma(L^{\mathtt{P}\xi})$, $\Sigma(L^{\mathtt{P}\eta})$ (black curves) and $\Sigma(L^{\mathtt{3P}})$ (orange curves) for $\xi = \pi - \eta$ (left, middle) and for $\xi \neq \pi - \eta$ (right).

Let us define the following multi-point differential operators

$$L^{\mathtt{P}\xi}u := L^{\mathtt{P}\eta}u := L^{\mathtt{DN}}u := L^{\mathtt{3P}}u := -u'',$$

$$\begin{aligned} \operatorname{dom}(L^{\mathtt{P}\xi}) &:= \left\{u \in H^2(\mathcal{P}) : B_i(u) = 0,\ i = 1, \dots, 4, 5, 7\right\}, \\ \operatorname{dom}(L^{\mathtt{P}\eta}) &:= \left\{u \in H^2(\mathcal{P}) : B_i(u) = 0,\ i = 1, \dots, 4, 6, 8\right\}, \end{aligned}$$

where

$$B_7(u) = u_1(0) - u_3(\xi), \quad B_8(u) = u_1'(\eta) - u_3'(\pi),$$

and

$$\begin{aligned} \operatorname{dom}(L^{\mathtt{DN}}) &:= \left\{u \in H^2([0, \pi]) : u\left(\tfrac{\xi}{2}\right) = u'\left(\tfrac{\pi+\eta}{2}\right) = 0\right\}, \\ \operatorname{dom}(L^{\mathtt{3P}}) &:= \left\{u \in H^2([0, \pi]) : u'(0) - u'(\xi) = u'\left(\tfrac{\pi+\eta}{2}\right) = 0,\ u(0)u(\xi) \le 0\right\}. \end{aligned}$$

According to Theorem 6 in [6], we have that

$$\Sigma(L) = \Sigma(L^{\mathtt{P}\xi}) \cup \Sigma(L^{\mathtt{P}\eta}) \cup \Sigma(L^{\mathtt{3P}}) \neq \Sigma(L^{\mathtt{P}\xi}) \cup \Sigma(L^{\mathtt{P}\eta}) \cup \Sigma(L^{\mathtt{DN}}) \tag{3.2}$$

in contrast to

$$\sigma(L) = \sigma(L^{\mathtt{P}\xi}) \cup \sigma(L^{\mathtt{P}\eta}) \cup \sigma(L^{\mathtt{3P}}) = \sigma(L^{\mathtt{P}\xi}) \cup \sigma(L^{\mathtt{P}\eta}) \cup \sigma(L^{\mathtt{DN}}),$$

where the spectra $\sigma(L^{\mathtt{P}\xi})$, $\sigma(L^{\mathtt{P}\eta})$ and $\sigma(L^{\mathtt{3P}}) = \sigma(L^{\mathtt{DN}})$ are identified as pure point discrete spectra made only of the real eigenvalues

$$\lambda_k^{\mathtt{P}\xi} := \left(\frac{2k\pi}{\xi}\right)^2, \quad \lambda_m^{\mathtt{P}\eta} := \left(\frac{2m\pi}{\pi - \eta}\right)^2 \quad \text{and} \quad \lambda_l^{\mathtt{DN}} := \left(\frac{(2l+1)\pi}{\pi + \eta - \xi}\right)^2, \ k, l, m \in \mathbb{N}_0.$$

(see [6], [3] and [5] for the explicit analytical description of $\Sigma(L^{\mathtt{P}\xi})$, $\Sigma(L^{\mathtt{P}\eta})$ and $\Sigma(L^{\mathtt{DN}})$ and note that $\Sigma(L^{\mathtt{P}\xi}) = \Sigma(L^{\mathtt{P}\eta})$ for $\xi = \pi - \eta$). Moreover, the Fučík spectrum $\Sigma(L^{\mathtt{DN}})$ determines the intersection of $\Sigma(L^{\mathtt{3P}})$ and $\Sigma(L^{\mathtt{P}\xi})$

$$\Sigma(L^{\mathtt{3P}}) \cap \Sigma(L^{\mathtt{P}\xi}) = \Sigma(L^{\mathtt{DN}}) \cap \Sigma(L^{\mathtt{P}\xi}) \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^+,$$

which is illustrated in Figure 3 for two symmetric and one non-symmetric settings of ξ and η with respect to the midpoint of $[0, \pi]$.

Finally, to describe explicitely the Fučík spectrum $\Sigma(L^{\mathtt{3p}})$, let us denote for $k, l \in \mathbb{N}_0$

$$S_k^{\mathtt{P}\xi} := \left\{(\alpha,\beta) \in \mathbb{R}^2 : 0 \le \frac{\xi}{\pi} - \frac{k}{\sqrt{\alpha}} - \frac{k}{\sqrt{\beta}} \le \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}}\right\},$$

$$\begin{aligned} S_l^{\mathtt{DN}} := \big\{(\alpha,\beta) \in \mathbb{R}^2 : &\frac{1}{\max\{\sqrt{\alpha},\sqrt{\beta}\}} \\ &\le \frac{\pi+\eta-\xi}{\pi} - \frac{l}{\sqrt{\alpha}} - \frac{l}{\sqrt{\beta}} \le \frac{1}{\min\{\sqrt{\alpha},\sqrt{\beta}\}}\big\}. \end{aligned}$$

Proposition 3.1. *The Fučík spectrum of the three-point operator $L^{\mathtt{3p}}$ on $\mathbb{R}^+ \times \mathbb{R}^+$ is given by*

$$\Sigma(L^{\mathtt{3p}}) = \bigcup_{l \in \mathbb{N}_0} \mathcal{C}_l^{\mathtt{3p}}, \quad \textit{where} \quad \mathcal{C}_l^{\mathtt{3p}} := \bigcup_{j \in \mathbb{N}_0} \mathcal{C}_{j,l}^{\mathtt{3p}\pm}, \quad l \in \mathbb{N}_0,$$

and for $k, l \in \mathbb{N}_0$,

$$\mathcal{C}_{k,l}^{\mathtt{3p}\pm} := \left\{(\alpha,\beta) \in S_k^{\mathtt{P}\xi} \cap S_l^{\mathtt{DN}} : \left(F_{k,l}(\alpha,\beta) - \frac{\pi+\eta}{2\pi}\right)\left(F_{k,l}(\beta,\alpha) - \frac{\pi+\eta}{2\pi}\right) = 0\right\},$$

$$F_{k,l}(\alpha,\beta) := \frac{\xi}{\pi}\frac{\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}} + \frac{l-k}{2\sqrt{\alpha}} + \frac{l+k+1}{2\sqrt{\beta}}.$$

For the proof of the above proposition see [6].

Let us close this section by summing up the main properties of the Fučík spectrum $\Sigma(L)$ which make the operator L unique and interesting (see Figure 3):

(1) monotonicity and smoothness of the Fučík branches are not preserved;
(2) the intersection points of $\Sigma(L^{\mathtt{P}\xi})$ and $\Sigma(L^{\mathtt{DN}})$ are bifurcation points of new fragments which belong to $\Sigma(L^{\mathtt{3p}}) \subset \Sigma(L)$;
(3) the intersection points of $\Sigma(L^{\mathtt{P}\eta})$ and $\Sigma(L^{\mathtt{DN}})$ are of no importance;
(4) interesting patterns of $\Sigma(L)$ can be observed for different settings of ξ and η, the Fučík branches intersect away from the diagonal and if we continuously change $(\alpha,\beta) \in \Sigma(L)$, the nodal properties of the corresponding Fučík eigenfunctions are not preserved.

4. The Fučík spectrum of the adjoint operator

In this section, let us focus on the Fučík spectrum of the adjoint operator of L from the previous section. Using Theorem 1 in [10], we find that the adjoint operator $L^* : \mathrm{dom}(L^*) \subset L^2(0,\pi) \to L^2(0,\pi)$ of L is the multi-point differential operator given by

$$L^*u(x) = -u''(x), \quad \mathrm{dom}(L^*) = \{u \in H^2(\mathcal{P}) : B_i^*(u) = 0,\ i = 1,\dots,6\},$$

where $\mathcal{P} = \{0,\eta,\xi,\pi\}$ is the partition of the interval $[0,\pi]$, $u = (u_1,u_2,u_3)$, and the adjoint multi-point boundary values are given by

$$\begin{gathered} B_1^*(u) = u_2(\xi) - u_3(\xi) + u_1(0), \quad B_2^*(u) = u_1(\eta) - u_2(\eta), \\ B_3^*(u) = u_2'(\xi) - u_3'(\xi), \quad B_4^*(u) = u_1'(\eta) - u_2'(\eta) + u_3'(\pi), \\ B_5^*(u) = u_1'(0), \quad B_6^*(u) = u_3(\pi). \end{gathered}$$

The Fučík spectrum problem for the adjoint operator $L^*u = \alpha u^+ - \beta u^-$ can be written in the form of the following four-point boundary-value problem

$$
\begin{gathered}
u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, \quad x \in (0,\pi) \setminus \{\xi, \eta\}, \\
u'(0) = u(\pi) = 0, \\
u(\eta-) = u(\eta+), \quad u'(\xi-) = u'(\xi+), \\
u(\xi-) = u(\xi+) + u(0), \quad u'(\eta+) = u'(\eta-) + u'(\pi), \quad \xi \in (0,\pi),\ \eta \in (0,\pi).
\end{gathered} \tag{4.1}
$$

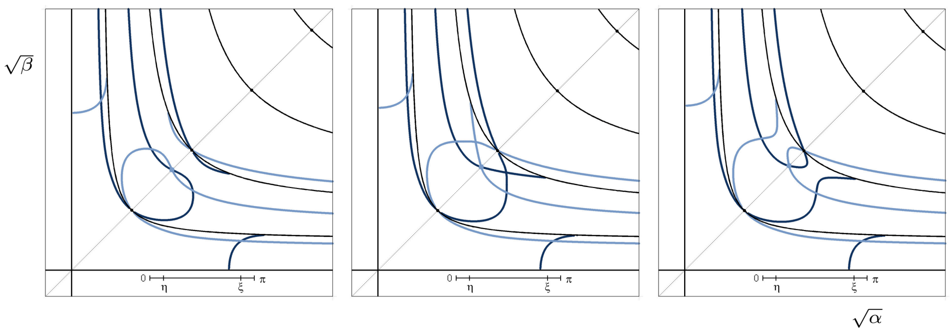

FIGURE 4. The Fučík spectrum $\Sigma(L^*)$ for three different symmetrical settings of ξ and η with respect to the midpoint of $[0,\pi]$.

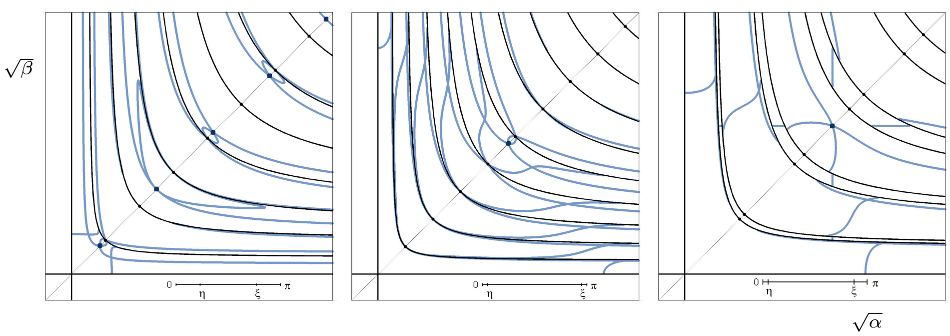

FIGURE 5. The Fučík spectrum $\Sigma(L^*)$ for $\xi = \pi - \eta$ (left, middle) and for $\xi \neq \pi - \eta$ (right).

It is straightforward to verify that the spectrum $\sigma(L^*)$ of the adjoint operator L^* is a countable real discrete spectrum and that

$$\sigma(L^*) = \sigma(L) = \sigma(L^{\mathtt{P}\xi}) \cup \sigma(L^{\mathtt{P}\eta}) \cup \sigma(L^{\mathtt{DN}}).$$

Moreover, we have that $\Sigma(L^{\mathtt{P}\xi}) \cup \Sigma(L^{\mathtt{P}\eta}) \subset \Sigma(L^*)$. But, the complete explicit analytical description of the Fučík spectrum $\Sigma(L^*)$ seems to be still an open problem. On the other hand, we can use the numerical continuation techniques combined with the shooting method in order to explore the structure of the Fučík spectrum $\Sigma(L^*)$ and to formulate new conjectures. Thus, let us consider the following three initial value problems which corresponds to the problem (4.1)

$$
\begin{cases} u_1'' + \alpha u_1^+ - \beta u_1^- = 0, \\ u_1(0) = A, \\ u_1'(0) = 0, \end{cases}
\quad
\begin{cases} u_2'' + \alpha u_2^+ - \beta u_2^- = 0, \\ u_2(\eta) = u_1(\eta), \\ u_2'(\eta) = u_1'(\eta) + B, \end{cases}
\quad
\begin{cases} u_3'' + \alpha u_3^+ - \beta u_3^- = 0, \\ u_3(\xi) = u_2(\xi) + A, \\ u_3'(\xi) = u_2'(\xi), \end{cases}
$$

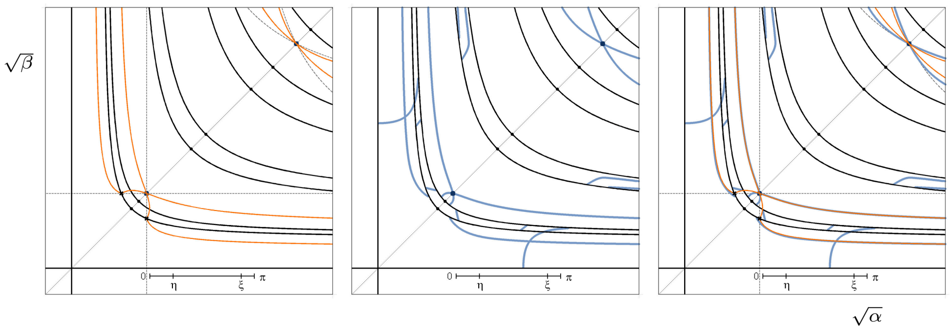

FIGURE 6. The Fučík spectrum $\Sigma(L)$ (left) and $\Sigma(L^*)$ (middle) and their overlapping (right).

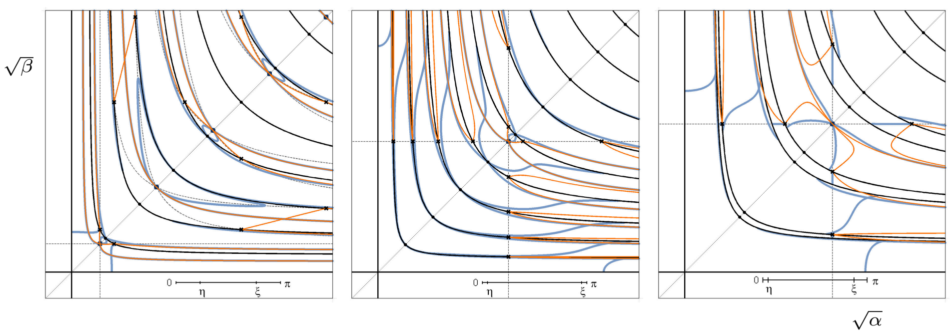

FIGURE 7. The overlapping of the Fučík spectrum $\Sigma(L)$ (orange and black curves) and $\Sigma(L^*)$ (blue and black curves).

where $A, B \in \mathbb{R}$. To obtain a point $(\alpha, \beta) \in \Sigma(L^*)$, it is enough to find $(\alpha, \beta, A, B) \in \mathbb{R}^4$ such that $u_3(\pi) = 0$ and $u_3'(\pi) = B$. Since L^* is the positively homogeneous operator, we can restrict the values of B without any loss of generality to be equal to -1, 0 and 1. Figure 4 illustrates the numerical approximation of three nontrivial patterns of the Fučík spectrum $\Sigma(L^*)$ for $\xi = \pi - \eta$ and $\eta = \eta_1, \eta_2, \eta_3$ with $\eta_1 > \eta_2 > \eta_3$, which enables us to formulate the following conjecture (see Figure 4, middle).

Conjecture 4.1. *For all $i \in \mathbb{N}$ and all $j \in \mathbb{N}_0$ there exist $\xi \in (0, \pi)$ and $\eta = \pi - \xi$ such that*

$$\lambda_j^{\mathtt{P}\xi} < \lambda_i^{\mathtt{DN}} < \lambda_{j+1}^{\mathtt{P}\xi}, \quad \Big\{(\lambda_j^{\mathtt{P}\xi}, \lambda_j^{\mathtt{P}\xi}), (\lambda_i^{\mathtt{DN}}, \lambda_i^{\mathtt{DN}}), (\lambda_{j+1}^{\mathtt{P}\xi}, \lambda_{j+1}^{\mathtt{P}\xi})\Big\} \subset C,$$

where C is the component of $\Sigma(L^*)$.

Figure 5 illustrates the pathological patterns of the Fučík spectrum $\Sigma(L^*)$ with "nonstandard" behavior: the intersection of the Fučík branches away from the diagonal, monotonicity of the Fučík branches is not preserved; some fragments of $\Sigma(L^*)$ connect the consecutive periodic Fučík curves of $\Sigma(L^{\mathtt{P}\xi}) \cup \Sigma(L^{\mathtt{P}\eta})$.

Now, if we compare the numerical approximation of the Fučík spectrum $\Sigma(L^*)$ to the Fučík spectrum $\Sigma(L)$ given explicitly by (3.2) and Proposition 3.1, we can formulate the following conjecture (see Figures 6 and 7).

Conjecture 4.2.
(1) $\Sigma(L^*) \neq \Sigma(L)$ *inspite of the fact that* $\sigma(L^*) = \sigma(L)$;

(2) *there exist pairs* $(\alpha, \beta) \in \Sigma(L) \cap \Sigma(L^*)$ *with* $\alpha \neq \beta$ *such that* $(\alpha, \beta) \notin \Sigma(L^{\mathrm{P}\xi}) \cup \Sigma(L^{\mathrm{P}\eta})$.

Let us note that Figure 7 is the combination of Figures 3 and 5 and illustrates which parts of the Fučík spectra $\Sigma(L)$ and $\Sigma(L^*)$ coincide.

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