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VARIATIONAL METHODS AND LINEARIZATION TOOLS TOWARDS THE SPECTRAL ANALYSIS OF THE P-LAPLACIAN, ESPECIALLY FOR THE FREDHOLM ALTERNATIVE

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ABSTRACT. We look for weak solutions $u \in W_0^{1,p}(\Omega)$ of the degenerate quasilinear Dirichlet boundary value problem

$$-\Delta_p u = \lambda |u|^{p-2} u + f(x) \quad \text{in } \Omega\,; \qquad u = 0 \quad \text{on } \partial\Omega\,. \tag{P}$$

It is assumed that $1 < p < \infty$, $p \neq 2$, $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, Ω is a bounded domain in $\mathbb{R}^N$, $f \in L^\infty(\Omega)$ is a given function, and λ stands for the (real) spectral parameter. Such weak solutions are precisely the critical points of the corresponding energy functional on $W_0^{1,p}(\Omega)$,

$$\mathcal{J}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega f(x)\, u \,\mathrm{d}x\,, \quad u \in W_0^{1,p}(\Omega)\,. \tag{J}$$

I.e., problem (P) is equivalent with $\mathcal{J}'_\lambda(u) = 0$ in $W^{-1,p'}(\Omega)$. Here, $\mathcal{J}'_\lambda(u)$ stands for the (first) Fréchet derivative of the functional $\mathcal{J}_\lambda$ on $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ denotes the (strong) dual space of the Sobolev space $W_0^{1,p}(\Omega)$, $p' = p/(p-1)$.

We will describe a global minimization method for this functional provided $\lambda < \lambda_1$, together with the (strict) convexity of the functional for $\lambda \leq 0$ and possible "nonconvexity" if $0 < \lambda < \lambda_1$. As usual, λ_1 denotes the first (smallest) eigenvalue λ_1 of the positive p-Laplacian $-\Delta_p$. Strict convexity will force the uniqueness of a critical point (which is then the global minimizer for $\mathcal{J}_\lambda$), whereas "nonconvexity" will be shown by constructing a saddle point which is different from any local or global minimizer. These methods are well-known and can be found in many textbooks on Nonlinear Functional Analysis or Variational Calculus.

The problem becomes quite difficult if $\lambda = \lambda_1$ or $\lambda > \lambda_1$, even in space dimension one ($N = 1$). We will restrict ourselves to the case $\lambda = \lambda_1$, the *Fredholm alternative for the p-Laplacian* at the first eigenvalue. Even if the functional $\mathcal{J}_{\lambda_1}$ is no longer coercive on $W_0^{1,p}(\Omega)$, for $p > 2$ we will show that it is bounded from below and does possess a global minimizer. For $1 < p < 2$ the functional $\mathcal{J}_{\lambda_1}$ is unbounded from below and one can find a pair of sub- and supersolutions to problem (P) by a variational method (a simplified minimax principle) performed in the orthogonal decomposition $W_0^{1,p}(\Omega) = \operatorname{lin}\{\varphi_1\} \oplus W_0^{1,p}(\Omega)^\top$ induced by the inner product in $L^2(\Omega)$. First, the minimum is taken in $W_0^{1,p}(\Omega)^\top$, and then (possibly only local) maximum in $\operatorname{lin}\{\varphi_1\}$. The "sub-" and "supercritical" points thus obtained provide a pair of sub- and supersolutions to problem (P). Then a topological (Leray-Schauder) degree has to be employed to obtain a solution to problem (P) by a standard fixed point argument.

Finally, we will discuss the *existence* and *multiplicity* of a solution for problem (P) when f "nearly" satisfies the orthogonality condition $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$ and $\lambda < \lambda_1 + \delta$ (with $\delta > 0$ small enough). A crucial ingredient in our proofs are rather *precise asymptotic estimates* for possible "large" solutions to problem (P) obtained from the linearization of problem (P) about the eigenfunction φ_1. These will be briefly discussed. Naturally, the (linear selfadjoint) Fredholm alternative for the linearization of problem (P) about φ_1 (with $\lambda = \lambda_1$) appears in the proofs.

CONTENTS

1. Introduction

In this lecture notes we will combine variational and topological methods to establish various results on *existence* (or *nonexistence*), *uniqueness*, and *multiplicity* of solutions corresponding to the *Fredholm alternative* for the nonlinear Dirichlet problem

$$-\Delta_p u = \lambda |u|^{p-2} u + f(x) \quad \text{in } \Omega\,; \qquad u = 0 \quad \text{on } \partial\Omega\,, \tag{1.1}$$

with the p-Laplacian Δ_p defined by $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ for $p \in (1, \infty)$. Here, $\lambda \in \mathbb{R}$ is the spectral (control) parameter taking values near the first (smallest) eigenvalue λ_1 of $-\Delta_p$,

$$\lambda_1 = \inf \Big\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x : u \in W_0^{1,p}(\Omega) \text{ with } \int_\Omega |u|^p \,\mathrm{d}x = 1 \Big\}\,, \tag{1.2}$$

and $f \in L^\infty(\Omega)$ is a given function, $f \not\equiv 0$ in Ω. Using a new variational method introduced in Takáč [32, 35] we will be able to show a number of results on the solvability of problem (1.1) under various conditions on λ and f. We assume that Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with sufficiently smooth boundary $\partial\Omega$; C^2 boundary will do. As far as conditions on f are concerned, we will find it convenient to work with the orthogonal sum $L^2(\Omega) = \operatorname{lin}\{\varphi_1\} \oplus L^2(\Omega)^\top$, where

$$L^2(\Omega)^\top \stackrel{\text{def}}{=} \big\{ f \in L^2(\Omega) : \langle f, \varphi_1 \rangle = 0 \big\}\,, \tag{1.3}$$

$\langle f, \varphi_1 \rangle \stackrel{\text{def}}{=} \int_\Omega f \varphi_1 \,\mathrm{d}x$, so that f splits as $f = f^\parallel \cdot \varphi_1 + f^\top$, where

$$f^\parallel \stackrel{\text{def}}{=} \|\varphi_1\|_{L^2(\Omega)}^{-2} \langle f, \varphi_1 \rangle \quad \text{and} \quad \langle f^\top, \varphi_1 \rangle = 0\,. \tag{1.4}$$

Here, φ_1 denotes the eigenfunction associated with λ_1 which is a simple eigenvalue of $-\Delta_p$, by Anane [2, Théorème 1, p. 727] or Lindqvist [25, Theorem 1.3, p. 157].

We normalize φ_1 by $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_{L^p(\Omega)} = 1$. We have $\varphi_1 \in L^\infty(\Omega)$ by Anane [3, Théorème A.1, p. 96]. More details about λ_1 and φ_1 will be presented in Section 5.

We formulate our solvability conditions on f in terms of $f^\parallel$ and $f^\top$ where the ratio between $|f^\parallel|$ and $\|f^\top\|_{L^\infty(\Omega)}$ plays a decisive role. For λ near λ_1, say, $|\lambda-\lambda_1| \leq \delta$, this ratio, combined with the signs of $\lambda - \lambda_1$ and $p-2$, determines the *existence* (or *nonexistence*) and *multiplicity* of weak solutions to problem (1.1) in $W_0^{1,p}(\Omega)$. It is not surprising that we look for solutions to (1.1) coming also in the form $u = u^\parallel \cdot \varphi_1 + u^\top$ defined in (1.4). We will establish strong relations between $f^\parallel$ and $u^\parallel$, and $f^\top$ and $u^\top$, as well, for any $1 < p < \infty$, in analogy with the case $p = 2$ when these two relations are decoupled by the spectral decomposition of the (linear) Laplace operator Δ in $L^2(\Omega)$.

In accordance with (1.3) we write

$$
\begin{aligned}
L^\infty(\Omega)^\top &\stackrel{\text{def}}{=} \big\{ u \in L^\infty(\Omega) : \langle u, \varphi_1 \rangle = 0 \big\}\,, \\
W_0^{1,p}(\Omega)^\top &\stackrel{\text{def}}{=} \big\{ u \in W_0^{1,p}(\Omega) : \langle u, \varphi_1 \rangle = 0 \big\}\,,
\end{aligned}
$$

and take advantage of the orthogonal sums

$$
L^\infty(\Omega) = \mathrm{lin}\{\varphi_1\} \oplus L^\infty(\Omega)^\top\,, \quad W_0^{1,p}(\Omega) = \mathrm{lin}\{\varphi_1\} \oplus W_0^{1,p}(\Omega)^\top\,.
$$

The *main idea* in our variational approach is to look for the unknowns $u^\parallel$ and $u^\top$ separately; first for $u^\top \in W_0^{1,p}(\Omega)^\top$ and then for $u^\parallel \in \mathbb{R}$ (depending on $u^\top$). More precisely, we look for *critical points* of the energy functional

$$
\mathcal{J}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega f(x) u \,\mathrm{d}x\,, \quad u \in W_0^{1,p}(\Omega)\,, \tag{1.5}
$$

corresponding to problem (1.1) as follows.

First, we fix $u^\parallel = \tau \in \mathbb{R}$ arbitrary and minimize the restricted energy functional $u^\top \mapsto \mathcal{J}_\lambda(\tau\varphi_1 + u^\top)$ over the orthogonal complement $W_0^{1,p}(\Omega)^\top$, thus obtaining a global minimizer $u_\tau^\top \in W_0^{1,p}(\Omega)^\top$,

$$
j_\lambda(\tau) \stackrel{\text{def}}{=} \mathcal{J}_\lambda(\tau\varphi_1 + u_\tau^\top) \leq \mathcal{J}_\lambda(\tau\varphi_1 + u^\top) \quad \text{for all } u^\top \in W_0^{1,p}(\Omega)^\top\,.
$$

This is possible provided the functional $u^\top \mapsto \mathcal{J}_\lambda(\tau\varphi_1 + u^\top)$ is coercive on $W_0^{1,p}(\Omega)^\top$ which, indeed, is the case if $\lambda < \Lambda_\infty$. The number

$$
\Lambda_\infty \stackrel{\text{def}}{=} \inf\Big\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x : u \in W_0^{1,p}(\Omega)^\top \text{ with } \int_\Omega |u|^p \,\mathrm{d}x = 1 \Big\} \tag{1.6}
$$

satisfies $\Lambda_\infty > \lambda_1$, by the simplicity of eigenvalue λ_1. Using the uniform convexity of $W_0^{1,p}(\Omega)$, we will be able to show that $j_\lambda : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Now we vary $\tau \in \mathbb{R}$ and look for local extrema (minima or maxima) of the function $j_\lambda(\tau)$ in order to determine possible critical points of the functional $\mathcal{J}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$. Clearly, if τ_0 is a *local minimizer* for j_λ, then also $u_0 = \tau_0\varphi_1 + u_{\tau_0}^\top$ is a local minimizer for $\mathcal{J}_\lambda$ and hence a critical point of $\mathcal{J}_\lambda$. A more complicated situation occurs if j_λ has a *local maximum* at $\tau_0 \in \mathbb{R}$. In general, we even do not know if j_λ is differentiable at τ_0. However, we are still able to show that problem (1.1) possesses a pair of *sub-* and *supersolutions*,

$$
\underline{u} = \tau_0\varphi_1 + \underline{u}^\top \quad \text{and} \quad \overline{u} = \tau_0\varphi_1 + \overline{u}^\top,
$$

respectively, such that

$$\left.\begin{aligned} -\Delta_p \underline{u} &= \lambda |\underline{u}|^{p-2}\underline{u} + f(x) + \underline{\zeta}\cdot\varphi_1 \quad \text{in } \Omega;\\ \underline{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.7}$$

$$\left.\begin{aligned} -\Delta_p \overline{u} &= \lambda |\overline{u}|^{p-2}\overline{u} + f(x) + \overline{\zeta}\cdot\varphi_1 \quad \text{in } \Omega;\\ \overline{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.8}$$

for some $\underline{\zeta}, \overline{\zeta} \in \mathbb{R}$ satisfying $\underline{\zeta} \le 0 \le \overline{\zeta}$. Fortunately, another method ([6, Theorem 8.2, p. 448] or [14, Lemma 2.4, p. 191]), which combines the existence of such a pair of sub- and supersolutions with the topological (Leray-Schauder) degree, renders the existence of a (weak) *solution* $u = \tau\varphi_1 + u^\top$ of problem (1.1) with τ "close enough" to τ_0, relative to the magnitude of $|\tau_0|$. In this way we are able to distinguish the solution u from those other local minima or maxima of j_λ whose absolute value is of a different order of magnitude than $|\tau_0|$ (meaning either much smaller or much larger) or whose sign is opposite to $\operatorname{sgn}\tau_0$. If the sub- and supersolutions coincide, then $u_0 = \underline{u} = \overline{u}$ is a critical point of $\mathcal{J}_\lambda$ and, moreover, it is an easy exercise to show that j_λ is differentiable at τ_0 with vanishing derivative. Employing this procedure we are able to obtain multiple solutions of the form $u = \tau\varphi_1 + u^\top$ to problem (1.1) which can be distinguished *either* by the order of magnitude of $|\tau|$ *or* by the sign of parameter $\tau \in \mathbb{R}$.

The variational method sketched above is somewhat different from Rabinowitz' "Saddle Point Theorem" [30, Theorem 4.6, p. 24] applied to the functional $\mathcal{J}_\lambda$. We actually work with a "maximin" expression,

$$\max_{a<\tau<b} j_\lambda(\tau) = \max_{a<\tau<b} \min_{u^\top \in W_0^{1,p}(\Omega)^\top} \mathcal{J}_\lambda(\tau\varphi_1 + u^\top) \tag{1.9}$$

with $-\infty \le a < b \le \infty$ suitably chosen, and thus obtain a local maximizer τ_0 of the function $j_\lambda : \mathbb{R} \to \mathbb{R}$ which, in turn, induces a sub- and a supersolution to problem (1.1). This method originates in an earlier work of the author [32, Sect. 7] for $p < 2$ and $\lambda_1 - \delta \le \lambda \le \lambda_1$. A number of results on the solvability (existence, or nonexistence, and multiplicity of solutions) of problem (1.1) have been obtained by other methods for any $1 < p < \infty$ and $|\lambda - \lambda_1| \le \delta$; see [7, 9, 28, 40, 41] with further significant progress being made recently in [10, 11, 12, 13, 14, 20, 22, 26, 27, 29, 32, 33, 34, 35]. We will follow the recent work [35] in order to prove (or at least explain) some of these results here employing our variational method. A result from [35, Theorem 2.7, p. 702] concerning the case $p > 2$ and $\lambda_1 < \lambda \le \lambda_1 + \delta$ is of special interest, for it features at least *three* (pairwise distinct) solutions of problem (1.1): two critical points of the functional $\mathcal{J}_\lambda$, which are probably not local minimizers, and a local minimizer "between" them. Occasionally, variational proofs provide generalizations of earlier results. For these reasons we feel that the method is worth of being further explored in order to derive existence and multiplicity results for (1.1) and related problems, with $f(x)$ replaced by a more general function $f(x,u)$, see suggestions in Section 12.

We will see that problem (1.1) possesses multiple solutions in some cases provided $|\langle f, \varphi_1\rangle| > 0$ is small enough relative to $\|f\|_{L^\infty(\Omega)}$: at least two solutions if $\lambda = \lambda_1$, and at least three solutions if $|\lambda - \lambda_1| > 0$ is small enough. We start from the basic case $\langle f, \varphi_1\rangle = 0$ and $\lambda = \lambda_1$. Our variational method is stable enough for the energy functional $\mathcal{J}_\lambda$, so that we can slightly relax the orthogonality condition $\langle f, \varphi_1\rangle = 0$ (relative to the size of $\|f\|_{L^\infty(\Omega)}$) and/or the condition $\lambda = \lambda_1$. Multiple solutions to problem (1.1) with $N = 1$, any $0 < \lambda < \lambda_1$, and a suitable function f have been constructed earlier in [19] (for $1 < p < 2$) and [28] (for $2 < p < \infty$). Other cases, all with λ near λ_1, have been treated more recently in [10, 11, 12, 13, 22, 33, 34].

These lecture notes are organized as follows. As a "warm-up" exercise, in the next section (Section 2) we give a (simple) variational proof of the Riesz representation theorem for continuous linear functionals on $L^p(\Omega)$ inspired by Adams and Fournier [1, Proof of Theorem 2.44, pp. 46–47]. In Section 3 we introduce some notations and basic hypotheses. The energy functional $\mathcal{J}_\lambda$ and (weak and strong) convergence of a minimizing sequence in $W_0^{1,p}(\Omega)$ are studied in Section 4. A few basic properties of the first eigenvalue of $-\Delta_p$ are stated in Section 5. In Section 6 we explore the convexity of the functional $\mathcal{J}_\lambda$, first for $\lambda \leq 0$ (on all of $W_0^{1,p}(\Omega)$, in §6.1) and then also for $0 < \lambda \leq \lambda_1$ and $f \geq 0$ (on the set of positive functions, in §6.2), and its impact on the uniqueness of a solution to problem (1.1). In Section 7 we present a variational method which is based on minimization with constraint. Elementary analysis in Section 8 suggests how to obtain critical points for $\mathcal{J}_\lambda$. In Section 9 we apply the Leray-Schauder degree to a suitable fixed point mapping in order to establish the existence of a critical point for $\mathcal{J}_\lambda$ (which probably is not a local minimizer). Multiple (and "large") critical points of $\mathcal{J}_\lambda$ are discussed in Section 10. A collection of main results that can be obtained by the variational and topological methods presented in these notes is given in Section 11. Finally, Section 12 is devoted to a brief discussion of the general problem

$$-\Delta_p u = \lambda |u|^{p-2} u + f(x, u(x)) \quad \text{in } \Omega\,; \qquad u = 0 \quad \text{on } \partial\Omega\,, \tag{1.10}$$

when the function $f(x, u)$ is allowed to depend also on the state variable $u \in \mathbb{R}$. The lecture notes conclude with two appendices: Appendix A with some auxiliary functional-analytic results from Takáč [32] and Appendix B which contains some (mostly highly nontrivial) results from Drábek et al. [13, Theorem 4.1, pp. 445–446] and Takáč [32, 33].

2. The Riesz representation theorem in $L^p(\Omega)$

A given (equivalent) norm $\|\cdot\|_X$ on a Banach space X (and, hence, also X itself endowed with this norm) is called *uniformly convex* if for every $\varepsilon > 0$ there exists some $\delta \equiv \delta(\varepsilon) > 0$ such that, for all $u, v \in X$, one has

$$\|u\|_X,\ \|v\|_X \leq 1 \quad \text{and} \quad \Big\|\frac{u+v}{2}\Big\|_X > 1 - \delta \quad \Longrightarrow \quad \|u - v\|_X < \varepsilon\,.$$

The uniform convexity of the standard norm on $L^p(\Omega)$ for $1 < p < \infty$,

$$\|u\|_{L^p(\Omega)} \stackrel{\text{def}}{=} \Big(\int_\Omega |u(x)|^p \,\mathrm{d}x \Big)^{1/p}, \quad u \in L^p(\Omega), \tag{2.1}$$

is a direct consequence of *Clarkson's inequalities* (see e.g. Adams and Fournier [1, Theorem 2.38, p. 44] for a proof):

Lemma 2.1 (Clarkson's inequalities)**.** *Let $1 < p < \infty$ and $p' = p/(p-1)$. For $u, v \in L^p(\Omega)$ (real or complex-valued, and vector-valued functions from $[L^p(\Omega)]^N$, as well) we have*

(a) *if* $2 \le p < \infty$ *then*

$$\left\|\frac{u+v}{2}\right\|_{L^p(\Omega)}^{p} + \left\|\frac{u-v}{2}\right\|_{L^p(\Omega)}^{p} \le \frac{1}{2}\left(\|u\|_{L^p(\Omega)}^{p} + \|v\|_{L^p(\Omega)}^{p}\right), \tag{2.2}$$

$$\left\|\frac{u+v}{2}\right\|_{L^p(\Omega)}^{p'} + \left\|\frac{u-v}{2}\right\|_{L^p(\Omega)}^{p'} \ge \left[\frac{1}{2}\left(\|u\|_{L^p(\Omega)}^{p} + \|v\|_{L^p(\Omega)}^{p}\right)\right]^{p'-1};$$

(b) *if* $1 < p \le 2$ *then*

$$\left\|\frac{u+v}{2}\right\|_{L^p(\Omega)}^{p'} + \left\|\frac{u-v}{2}\right\|_{L^p(\Omega)}^{p'} \le \left[\frac{1}{2}\left(\|u\|_{L^p(\Omega)}^{p} + \|v\|_{L^p(\Omega)}^{p}\right)\right]^{p'-1}, \tag{2.3}$$

$$\left\|\frac{u+v}{2}\right\|_{L^p(\Omega)}^{p} + \left\|\frac{u-v}{2}\right\|_{L^p(\Omega)}^{p} \ge \frac{1}{2}\left(\|u\|_{L^p(\Omega)}^{p} + \|v\|_{L^p(\Omega)}^{p}\right).$$

Notice that if $p = p' = 2$, the inequalities above reduce to the parallelogram law in the Hilbert space $L^2(\Omega)$.

Clarkson's inequalities (2.2) (if $2 \le p < \infty$) and (2.3) (if $1 < p \le 2$) can be used to give a simple *variational proof* of the Riesz representation theorem for continuous linear functionals on $L^p(\Omega)$; see e.g. the monograph by Adams and Fournier [1, Theorem 2.44, p. 47]. We are now ready to give a similar proof. Our approach is even more "variational" than the one presented in [1, Proof of Theorem 2.44, pp. 46–47]. We apply a standard minimization procedure to the functional

$$\mathcal{E}(u) \equiv \mathcal{E}(u;\ell) = \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x - \mathfrak{Re}\,\ell(u), \quad u \in L^p(\Omega), \tag{2.4}$$

where $\ell : L^p(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) is a continuous linear functional defined on the real (or complex, respectively) Lebesgue space $L^p(\Omega)$, $1 < p < \infty$, which is given. Hence, we will show that this functional has a global minimizer $\tilde{u}$ in $L^p(\Omega)$; see Lemma 2.2 below. Moreover, $\mathcal{E}$ being strictly convex, by Clarkson's inequalities, we can conclude that it possesses a unique critical point. This point will provide us with the desired representation function from the dual space $L^{p'}(\Omega)$, $p' = p/(p-1)$.

In other words, we will apply Clarkson's inequalities (Lemma 2.1) to show

The Riesz representation theorem. *Given a continuous linear functional ℓ on $L^p(\Omega)$, there exists a unique function $f \in L^{p'}(\Omega)$ such that*

$$\ell(u) = \int_\Omega uf \,\mathrm{d}x \quad \textit{holds for all } u \in L^p(\Omega). \tag{2.5}$$

Clearly, the standard complexification procedure for the real Lebesgue space $L^p_{\mathbb{R}}(\Omega)$ (over the field $\mathbb{R}$) yields the complex Lebesgue space $L^p_{\mathbb{C}}(\Omega)$ (over the field $\mathbb{C}$), so that $L^p_{\mathbb{C}}(\Omega) = L^p_{\mathbb{R}}(\Omega) \oplus \mathrm{i}L^p_{\mathbb{R}}(\Omega)$ is a direct sum over the field $\mathbb{R}$ ($\mathrm{i}^2 = -1$). As a consequence, both, the real and imaginary parts of the continuous linear functional ℓ are also continuous linear functionals on the direct sum $L^p_{\mathbb{R}}(\Omega) \oplus \mathrm{i}L^p_{\mathbb{R}}(\Omega)$. Hence, it suffices to verify the Riesz representation theorem for the real Lebesgue space $L^p(\Omega) = L^p_{\mathbb{R}}(\Omega)$ and $\ell : L^p_{\mathbb{R}}(\Omega) \to \mathbb{R}$. To this end, observe that if $f \in L^{p'}(\Omega)$ satisfies (2.5) then $v = |f|^{p'-2}f$ is a critical point of the functional

$$\mathcal{E}(u) = \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x - \ell(u), \quad u \in L^p(\Omega).$$

Vice versa, if $v \in L^p(\Omega)$ is a critical point of $\mathcal{E}$ then $f = |v|^{p-2}v$ satisfies (2.5). Notice that the Fréchet derivative $\mathcal{E}'(v) : L^p(\Omega) \to \mathbb{R}$ of the functional $\mathcal{E}$ at a given point $v \in L^p(\Omega)$ is given by the formula

$$\mathcal{E}'(v)u = \int_\Omega u\,|v|^{p-2} v\,\mathrm{d}x - \ell(u)\,, \quad u \in L^p(\Omega)\,.$$

It remains to establish the following lemma which is valid in both, the real and complex versions of the Lebesgue space $L^p(\Omega)$ (over the field $\mathbb{R}$ or $\mathbb{C}$), that is, also for the functional $\mathcal{E}$ defined in (2.4).

Lemma 2.2. *The functional $\mathcal{E} : L^p(\Omega) \to \mathbb{R}$ defined in (2.4) is continuous, strictly convex, and coercive on $L^p(\Omega)$, $1 < p < \infty$. Moreover, every minimizing sequence for $\mathcal{E}$ converges* strongly *in $L^p(\Omega)$ to the (unique) global minimizer $\tilde{u}$ of $\mathcal{E}$.*

Proof. The continuity and coercivity of $\mathcal{E}$ are obvious. As mentioned above, $\mathcal{E}$ is strictly convex, by Clarkson's inequalities (2.2) (if $2 \le p < \infty$) and (2.3) (if $1 < p \le 2$).

Set $\mu \stackrel{\text{def}}{=} \inf_{u \in L^p(\Omega)} \mathcal{E}(u)$ and consider any minimizing sequence $\{u_n\}_{n=1}^\infty \subset L^p(\Omega)$ for $\mathcal{E}$, i.e., $\mathcal{E}(u_n) \to \mu$ as $n \to \infty$. We wish to show that $u_n \to \tilde{u}$ strongly in $L^p(\Omega)$ as $n \to \infty$. Since the space $L^p(\Omega)$ is complete, this is equivalent to $\{u_n\}_{n=1}^\infty$ being a Cauchy sequence, i.e.,

$$\lim_{m,n\to\infty} \|u_n - u_m\|_{L^p(\Omega)} \to 0 \quad \text{as } m, n \to \infty\,. \tag{2.6}$$

The last claim follows from the facts that

$$\mathcal{E}\left(\tfrac{1}{2}(u_n + u_m)\right) \ge \mu \quad \text{and} \quad \ell\left(\tfrac{1}{2}(u_n + u_m)\right) = \tfrac{1}{2}\left(\ell(u_n) + \ell(u_m)\right) \tag{2.7}$$

combined with Clarkson's inequalities (2.2) and (2.3) which force (2.6), thanks to $\mathcal{E}(u_n) \to \mu$ as $n \to \infty$.

To see this, for $p \ge 2$ one applies inequality (2.2) directly to the functional $\mathcal{E}$ as follows:

$$\begin{aligned}
\mu + \frac{1}{p}\Big\|\frac{u_n - u_m}{2}\Big\|_{L^p(\Omega)}^p &\le \mathcal{E}\Big(\frac{u_n + u_m}{2}\Big) + \frac{1}{p}\Big\|\frac{u_n - u_m}{2}\Big\|_{L^p(\Omega)}^p \\
&\le \frac{1}{2}\Big(\mathcal{E}(u_n) + \mathcal{E}(u_m)\Big)\,.
\end{aligned}$$

Using $\mathcal{E}(u_n) \to \mu$ as $n \to \infty$ we arrive at (2.6).

For $1 < p < 2$ the proof is less direct; this proof works for any $p \in (1, \infty)$. Since the functional $\mathcal{E}$ is coercive, the minimizing sequence $\{u_n\}_{n=1}^\infty$ must be bounded in $L^p(\Omega)$. Consequently, passing to a suitable subsequence if necessary, we may assume

$$\|u_n\|_{L^p(\Omega)} \to \xi \in \mathbb{R}_+ \quad \text{and} \quad \ell(u_n) \to \eta \in \mathbb{C} \quad \text{as } n \to \infty\,;$$

hence, $\frac{1}{p}\xi^p + \mathfrak{Re}\,\eta = \mu$. From Minkowski's inequality (convexity of a norm) and (2.7) we deduce

$$\begin{aligned}
\mu &\le \mathcal{E}\Big(\frac{u_n + u_m}{2}\Big) = \frac{1}{p}\Big\|\frac{u_n + u_m}{2}\Big\|_{L^p(\Omega)}^p - \mathfrak{Re}\,\ell\Big(\frac{u_n + u_m}{2}\Big) \\
&\le \frac{1}{p}\left(\frac{\|u_n\|_{L^p(\Omega)} + \|u_m\|_{L^p(\Omega)}}{2}\right)^p - \frac{1}{2}\left(\mathfrak{Re}\,\ell(u_n) + \mathfrak{Re}\,\ell(u_m)\right) \\
&\le \frac{1}{2}\left(\mathcal{E}(u_n) + \mathcal{E}(u_m)\right)\,.
\end{aligned}$$

Using $\mathcal{E}(u_n) \to \mu$ as $n \to \infty$, we arrive at $\mathcal{E}\big(\frac{u_n+u_m}{2}\big) \to \mu$ as $m, n \to \infty$. Since also $\ell\big(\frac{u_n+u_m}{2}\big) \to \eta$ and $\frac{1}{p}\xi^p + \mathfrak{Re}\,\eta = \mu$, we conclude that $\big\|\frac{u_n+u_m}{2}\big\|_{L^p(\Omega)} \to \xi$

$(m,n \to \infty)$. Finally, we combine this convergence result and $\|u_n\|_{L^p(\Omega)} \to \xi$ $(n \to \infty)$ with Clarkson's inequalities (2.2) (if $2 \le p < \infty$) and (2.3) (if $1 < p \le 2$) to obtain (2.6). □

3. Notation and basic hypotheses

All Banach and Hilbert spaces used in these lecture notes are real. We work with the standard inner product in $L^2(\Omega)$ defined by $\langle u, v\rangle \stackrel{\text{def}}{=} \int_\Omega uv \,\mathrm{d}x$ for $u, v \in L^2(\Omega)$. The orthogonal complement in $L^2(\Omega)$ of a set $\mathcal{M} \subset L^2(\Omega)$ is denoted by $\mathcal{M}^{\perp,L^2}$,

$$\mathcal{M}^{\perp,L^2} \stackrel{\text{def}}{=} \{u \in L^2(\Omega) : \langle u, v\rangle = 0 \text{ for all } v \in \mathcal{M}\}.$$

The inner product $\langle \cdot\,, \cdot\rangle$ in $L^2(\Omega)$ induces a duality between the Lebesgue spaces $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $1 \le p < \infty$ and $1 < p' \le \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and between the Sobolev space $W_0^{1,p}(\Omega)$ and its (strong) dual space $W^{-1,p'}(\Omega)$, as well. We keep the same notation also for the duality between the Cartesian products $[L^p(\Omega)]^N$ and $[L^{p'}(\Omega)]^N$. The closure, interior, and boundary of a set $S \subset \mathbb{R}^N$ are denoted by $\overline{S}$, $\mathrm{int}(S)$, and ∂S, respectively, and the characteristic function of S by $\chi_S : \mathbb{R}^N \to \{0, 1\}$. We write $|S|_N \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \chi_S(x)\,\mathrm{d}x$ if S is also Lebesgue measurable.

We always assume the following hypothesis:

(H1) If $N \ge 2$ then Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact manifold of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, and Ω satisfies also the interior sphere condition at every point of $\partial\Omega$. If $N = 1$ then Ω is a bounded open interval in $\mathbb{R}^1$.

For $N \ge 2$, (H1) is satisfied if $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary.

4. The energy functional

Recall that the energy functional defined in (1.5),

$$\mathcal{J}_\lambda(u) \equiv \mathcal{J}_\lambda(u; f) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega f(x)u \,\mathrm{d}x \tag{1.5}$$

for $u \in W_0^{1,p}(\Omega)$, corresponds to problem (1.1). This problem is equivalent to $\mathcal{J}'_\lambda(u) = 0$ in $W^{-1,p'}(\Omega)$, where $\mathcal{J}'_\lambda(u)$ denotes the (first) Fréchet derivative of the functional $\mathcal{J}_\lambda$ on $W_0^{1,p}(\Omega)$.

If $\lambda < \lambda_1$, the functional $\mathcal{J}_\lambda$ is coercive on $W_0^{1,p}(\Omega)$ which means that $\|u\|_{W_0^{1,p}(\Omega)} \to \infty$ forces $\mathcal{J}_\lambda(u) \to +\infty$. Thus, the Sobolev space $W_0^{1,p}(\Omega)$ being reflexive, there exists a weakly convergent (minimizing) sequence $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$, such that

$$\mathcal{J}_\lambda(u_n) \to \mathcal{J}_\lambda(u_0) = \inf_{u \in W_0^{1,p}(\Omega)} \mathcal{J}_\lambda(u) > -\infty \quad \text{as } n \to \infty\,.$$

Notice that this weak convergence implies only

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \,\mathrm{d}x \ge \int_\Omega |\nabla u_0|^p \,\mathrm{d}x\,.$$

Since the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, by the Rellich--Kondrachov embedding theorem, we have also $u_n \to u_0$ strongly in $L^p(\Omega)$ as

$n \to \infty$. We combine these convergence results with the definition of the functional $\mathcal{J}_\lambda$ to conclude that we must have also

$$\int_\Omega |\nabla u_n|^p \, \mathrm{d}x \longrightarrow \int_\Omega |\nabla u_0|^p \, \mathrm{d}x \quad \text{as } n \to \infty\,.$$

But this and the weak convergence $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$ force the strong convergence

$$\|u_n - u\|_{W_0^{1,p}(\Omega)} = \|\nabla(u_n - u)\|_{L^p(\Omega)} \to 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty,$$

owing to the fact that $L^p(\Omega)$ (and similarly $[L^p(\Omega)]^N$) is a *uniformly convex* Banach space, by Clarkson's inequalities (Lemma 2.1).

Lemma 4.1. *Let $\lambda \in \mathbb{R}$ be arbitrary. If $u_n \rightharpoonup u_0$ weakly in $W_0^{1,p}(\Omega)$ and $\mathcal{J}_\lambda(u_n) \to \mathcal{J}_\lambda(u_0)$ as $n \to \infty$, then also $u_n \to u_0$ strongly in $W_0^{1,p}(\Omega)$.*

We will use such convergence reasoning often throughout the entire lecture notes. In what follows we show a few applications of the reasoning from our proof of Lemma 4.1.

5. The first eigenvalue of $-\Delta_p$

Consider the Rayleigh quotient (1.2) for the first (smallest) eigenvalue λ_1 of $-\Delta_p$,

$$\lambda_1 = \inf \left\{ \frac{\int_\Omega |\nabla u|^p \, \mathrm{d}x}{\int_\Omega |u|^p \, \mathrm{d}x} : 0 \neq u \in W_0^{1,p}(\Omega) \right\}. \tag{1.2}$$

The embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ being compact, there exists a function $\varphi_1 \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\varphi_1|^p \, \mathrm{d}x = 1 \quad \text{and} \quad \int_\Omega |\nabla \varphi_1|^p \, \mathrm{d}x = \lambda_1\,.$$

Hence, $0 < \lambda_1 < \infty$. Next, we use the polar decomposition $u = u^+ - u^-$ of $u \in L^p(\Omega)$, $u^+ \stackrel{\text{def}}{=} \max\{u, 0\}$ and $u^- \stackrel{\text{def}}{=} \max\{-u, 0\}$, with $\nabla u = \nabla u^+ - \nabla u^-$ a.e. in Ω for $u \in W_0^{1,p}(\Omega)$; see, e.g., Gilbarg and Trudinger [21, Theorem 7.8, p. 153]. We observe that also the positive and negative parts $\varphi_1^\pm$ of φ_1 must satisfy

$$\textstyle\int_\Omega |\nabla \varphi_1^\pm|^p \, \mathrm{d}x = \lambda_1 \int_\Omega (\varphi_1^\pm)^p \, \mathrm{d}x\,.$$

We will see in the next section, §6.2, Theorem 6.8, that $\varphi_1 \in W_0^{1,p}(\Omega)$ is determined *uniquely* by the conditions

(i_{φ_1}) $\varphi_1 \geq 0$ almost everywhere in Ω;
(ii_{φ_1}) $\int_\Omega \varphi_1^p \, \mathrm{d}x = 1$ and $\int_\Omega |\nabla \varphi_1|^p \, \mathrm{d}x = \lambda_1$.

This uniqueness result is due to Anane [2, Théorème 1, p. 727] and Lindqvist [25, Theorem 1.3, p. 157]. Hence, either $\varphi_1^+ \equiv 0$ or else $\varphi_1^- \equiv 0$ in Ω, and so we may assume $\varphi_1 \geq 0$ a.e. in Ω.

It follows that λ_1 is a *simple* eigenvalue of the positive Dirichlet p-Laplacian $-\Delta_p$ with a nonnegative eigenfunction φ_1 satisfying

$$-\Delta_p \varphi_1 = \lambda_1 |\varphi_1|^{p-2} \varphi_1 \quad \text{in } \Omega; \qquad \varphi_1 = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

Using this equation, Anane [3, Théorème A.1, p. 96] has derived also $\varphi_1 \in L^\infty(\Omega)$.

6. Convexity and uniqueness

Convexity of a function (or functional) is a standard tool for proving the *uniqueness* of a critical point for that function (or functional), provided the convexity is strict in some sense; cf. §6.1 below. We will see in §6.2 that if the strictness in the convexity is lost in some sense (partially), so may be the uniqueness of critical points (only partially, as well). The following simple example explains the key tool for this method.

Example 6.1. Let X be a Banach space, $x_0 \in X$ a point, and $v \in X \setminus \{0\}$ a direction. Consider the straight line $L = \{x = x_0 + tv \in X : t \in \mathbb{R}\}$ in X. Given a functional $\Phi : X \to \mathbb{R}$ on X, consider its restriction $\Phi|_L$ to the line L or, equivalently, consider the function $\phi(t) = \Phi(x_0 + tv)$ of the variable $t \in \mathbb{R}$. If ϕ happens to be convex and differentiable on $\mathbb{R}$ then the set of all critical points of ϕ coincides with a closed interval in $\mathbb{R}$ (which may be empty or unbounded). This interval coincides, in turn, with the set of all global minimizers for ϕ. If ϕ is also strictly convex then it may possess at most one critical point in $\mathbb{R}$; this point is the global minimizer for ϕ. Consequently, if Φ is convex and Gâteaux-differentiable on X, it suffices to investigate the strict convexity of Φ on every line L in X in order to determine the set of all critical points for Φ in X. This set is convex and consists precisely of all global minimizers for Φ; it may be empty or unbounded in X.

Now let us treat the energy functional defined in (1.5),

$$\mathcal{J}_\lambda(u) \equiv \mathcal{J}_\lambda(u; f) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega f(x) u \,\mathrm{d}x \tag{1.5}$$

for $u \in W_0^{1,p}(\Omega)$. Recall that problem (1.1) is equivalent to $\mathcal{J}'_\lambda(u) = 0$ in $W^{-1,p'}(\Omega)$, where $\mathcal{J}'_\lambda(u)$ denotes the (first) Fréchet derivative of the functional $\mathcal{J}_\lambda$ on $W_0^{1,p}(\Omega)$.

From Section 4 we know that $\mathcal{J}_\lambda$ is coercive on $W_0^{1,p}(\Omega)$ provided $\lambda < \lambda_1$. Hence, if $\lambda < \lambda_1$ then $\mathcal{J}_\lambda$ has a global minimizer in $W_0^{1,p}(\Omega)$. We investigate its uniqueness (or multiplicity) in the next two paragraphs, for $\lambda \le 0$ and $0 < \lambda \le \lambda_1$, respectively. From the end of Section 7 we recall $j_{\lambda_1}(\tau) = -c \cdot |\tau|^{2-p} + o(|\tau|^{2-p})$ as $\tau \to \pm\infty$, where $c = c(f^\top) > 0$ is a constant depending on $f^\top$ ($f^\top \not\equiv 0$ in Ω). The function j_{λ_1} being continuous, it attains a global minimum at $\tau_0 \in \mathbb{R}$ if $p > 2$. It follows that, even if $\mathcal{J}_{\lambda_1}$ is not coercive on $W_0^{1,p}(\Omega)$, for $p > 2$ it possesses a global minimizer $u_0 = \tau_0\varphi_1 + u_0^\top$ with some $u_0^\top \in W_0^{1,p}(\Omega)^\top$.

6.1. **Convexity for $\lambda \le 0$ and any f.** Let $\lambda \le 0$ and $f \in W^{-1,p'}(\Omega)$. Then both nonlinear terms on the right-hand side in (1.5) are nonnegative and convex, the first one even strictly convex on $W_0^{1,p}(\Omega)$. The uniqueness of the global minimizer for $\mathcal{J}_\lambda$ in $W_0^{1,p}(\Omega)$ now follows from Example 6.1 above. Moreover, this global minimizer is the unique critical point for $\mathcal{J}_\lambda$. Thus, we have proved the following theorem.

Theorem 6.2. *Let $1 < p < \infty$, $-\infty < \lambda \le 0$, and $f \in W^{-1,p'}(\Omega)$. Then the functional $\mathcal{J}_\lambda$ defined in* (1.5) *has a unique critical point in $W_0^{1,p}(\Omega)$. This critical point is its (unique) global minimizer. Moreover, $\mathcal{J}_\lambda$ is strictly convex on $W_0^{1,p}(\Omega)$.*

6.2. **Convexity for $0 < \lambda \le \lambda_1$ and $f \ge 0$.** In this paragraph we restrict ourselves to the special case $0 \le f \in L^\infty(\Omega)$ and investigate the functional $\mathcal{J}_\lambda(u)$ for $u \in W_0^{1,p}(\Omega)$, $u > 0$ a.e. in Ω. We follow the approach used in Fleckinger et al. [19, Sect. 6 (Appendix)]; cf. also Girg and Takáč [22, §4.1], Takáč [34, Sect. 3], or Takáč, Tello, and Ulm [36, Sect. 2] for some generalizations.

Lemma 6.3. *Let $0 \leq f \in L^\infty(\Omega)$ and $-\infty < \lambda \leq \lambda_1$. If $\lambda = \lambda_1$, assume also $f \not\equiv 0$ in Ω. Then every critical point for $\mathcal{J}_\lambda$ is nonnegative.*

Proof. Consider any critical point $u \in W_0^{1,p}(\Omega)$ for $\mathcal{J}_\lambda$. On the contrary, assume $u^- \not\equiv 0$ in Ω. Then we have

$$
\begin{aligned}
0 &= \langle \mathcal{J}_\lambda'(u),\, u^- \rangle \\
&= \int_\Omega |\nabla u|^{p-2} (\nabla u \cdot \nabla u^-)\, \mathrm{d}x - \lambda \int_\Omega |u|^{p-2} u\, u^-\, \mathrm{d}x - \int_\Omega f(x)\, u^-\, \mathrm{d}x \\
&= - \int_\Omega |\nabla u^-|^p\, \mathrm{d}x + \lambda \int_\Omega |u^-|^p\, \mathrm{d}x - \int_\Omega f(x)\, u^-\, \mathrm{d}x \\
&\leq (\lambda - \lambda_1) \int_\Omega |u^-|^p\, \mathrm{d}x - \int_\Omega f(x)\, u^-\, \mathrm{d}x \leq 0\,.
\end{aligned}
$$

Since $u^- \not\equiv 0$ and $f \geq 0$ in Ω, these inequalities force $\lambda = \lambda_1$, u^- is a minimizer for λ_1 in eq. (1.2), and $u^-(x) = 0$ almost everywhere in the set $\{x \in \Omega : f(x) > 0\}$. On the other hand, from the strong maximum principle of Tolksdorf [37, Prop. 3.2.2, p. 801] or Vázquez [39, Theorem 5, p. 200] we deduce $u^- > 0$ a.e. in Ω. This forces $f = 0$ a.e. in Ω, a contradiction. The lemma is proved. $\square$

We say that a functional $\mathcal{K}$ is *ray-strictly convex* if it is convex on a convex set $\mathcal{C}$ (say, $\mathcal{C} \subset L^1_{\mathrm{loc}}(\Omega)$), $\mathcal{K} : \mathcal{C} \to \mathbb{R} \cup \{+\infty\}$, and if $v_1, v_2 \in \mathcal{C}$ obey the equality

$$\mathcal{K}((1-\theta)v_1 + \theta v_2) = (1-\theta)\mathcal{K}(v_1) + \theta\mathcal{K}(v_2) < \infty$$

for some $\theta \in (0,1)$, then v_1 and v_2 are linearly dependent.

We now define the functional

$$\mathcal{K}(v) \stackrel{\mathrm{def}}{=} \int_\Omega |\nabla(v^{1/p})|^p\, \mathrm{d}x = p^{-p} \int_\Omega v\, |\nabla(\log v)|^p\, \mathrm{d}x$$

for all functions $v : \Omega \to (0,\infty)$ such that $v^{1/p} \in W_0^{1,p}(\Omega)$. More generally, we may replace $\boldsymbol{v} \mapsto |\boldsymbol{v}|^p : \mathbb{R}^N \to \mathbb{R}_+$ by any continuous, strictly convex function $K : \mathbb{R}^N \to \mathbb{R}_+$ and define (up to the constant factor p^{-p})

$$\mathcal{K}(v) = \int_\Omega v\, K(\nabla(\log v))\, \mathrm{d}x \in \mathbb{R} \cup \{+\infty\}$$

for all functions $v : \Omega \to (0,\infty)$ such that $v^{1/p} \in W_0^{1,p}(\Omega)$, cf. Girg and Takáč [22, §4.1], Takáč [34, Sect. 3], or Takáč, Tello, and Ulm [36, Sect. 2]. (Notice that $\nabla(v^{1/p}) = p^{-1} v^{1/p}\, \nabla(\log v)$.)

Lemma 6.4. *The functional $\mathcal{K}$ defined above is ray-strictly convex.*

Proof. For $\theta \in (0,1)$, $u_1 > 0$, $u_2 > 0$ and $\xi_1, \xi_2 \in \mathbb{R}^N$, we compute

$$
\begin{aligned}
&((1-\theta)u_1 + \theta u_2) K \left(\frac{(1-\theta)\xi_1 + \theta\xi_2}{(1-\theta)u_1 + \theta u_2} \right) \\
&= ((1-\theta)u_1 + \theta u_2) K \left(\frac{(1-\theta)u_1}{(1-\theta)u_1 + \theta u_2}\, \frac{\xi_1}{u_1} + \frac{\theta u_2}{(1-\theta)u_1 + \theta u_2}\, \frac{\xi_2}{u_2} \right) \\
&\leq (1-\theta)u_1\, K\left(\frac{\xi_1}{u_1}\right) + \theta u_2\, K\left(\frac{\xi_2}{u_2}\right),
\end{aligned}
$$

where equality holds if and only if $\xi_1/u_1 = \xi_2/u_2$. Putting $u_i = v_i(x)$ and $\xi_i = \nabla v_i(x)$ for a.e. $x \in \Omega$ and $i = 1, 2$, and integrating the last inequality over Ω, we obtain

$$\mathcal{K}((1-\theta)v_1 + \theta v_2) \le (1-\theta)\mathcal{K}(v_1) + \theta\mathcal{K}(v_2), \tag{6.1}$$

where equality holds if and only if $\nabla v_1/v_1 = \nabla v_2/v_2$ a.e. in Ω. The latter equality is equivalent to v_1 and v_2 being linearly dependent. $\square$

We combine Lemmas 6.3 and 6.4 to obtain the following theorem:

Theorem 6.5. *Let $0 \le f \in L^\infty(\Omega)$ with $f \not\equiv 0$ in Ω. If $\lambda \in (0, \lambda_1)$ then the functional $\mathcal{J}_\lambda$ possesses a unique critical point $u \in W_0^{1,p}(\Omega)$. This critical point is the (unique) minimizer for $\mathcal{J}_\lambda$ and satisfies $u > 0$ a.e. in Ω. If, in addition, the boundary $\partial\Omega$ of Ω is of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, then u satisfies also the* Hopf maximum principle

$$u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} < 0 \quad \text{on } \partial\Omega. \tag{6.2}$$

Proof. By our assumption $0 < \lambda < \lambda_1$, the functional $\mathcal{J}_\lambda$ is coercive on $W_0^{1,p}(\Omega)$. Thus, it has a minimizer (which is a critical point). From Lemma 6.3 we know that every critical point $u \in W_0^{1,p}(\Omega)$ for $\mathcal{J}_\lambda$ must be nonnegative throughout Ω. Moreover, the strong maximum principle of Tolksdorf [37, Prop. 3.2.2, p. 801] or Vázquez [39, Theorem 5, p. 200] guarantees $u > 0$ a.e. in Ω. For such u, let us now consider the functional $u \mapsto \mathcal{J}_\lambda(u^{1/p})$ which is strictly convex, by Lemma 6.4 combined with our hypotheses on f. Therefore, the critical point of this new functional and, consequently, also the critical point of $\mathcal{J}_\lambda$ are unique. Both are minimizers of the corresponding functionals.

Finally, assume that $\partial\Omega$ is of class $C^{1,\alpha}$. Then we have $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,\alpha)$, by a regularity result which is due to DiBenedetto [8, Theorem 2, p. 829] and Tolksdorf [38, Theorem 1, p. 127] (interior regularity, shown independently), and to Lieberman [24, Theorem 1, p. 1203] (regularity near the boundary). More precisely, in order to apply their regularity result, one needs to invoke the boundedness of $u \in L^\infty(\Omega)$ due to Anane [3, Théorème A.1, p. 96]. Thus, the Hopf maximum principle [37, Prop. 3.2.1 and 3.2.2, p. 801] or [39, Theorem 5, p. 200] can be applied to obtain (6.2) as desired. $\square$

Applying similar arguments as in the proof above, one can verify the following complementary result for $\lambda = \lambda_1$:

Theorem 6.6. *Let $0 \le f \in L^\infty(\Omega)$ with $f \not\equiv 0$ in Ω. Then the functional $\mathcal{J}_{\lambda_1}$ possesses* no *critical point $u \in W_0^{1,p}(\Omega)$. Furthermore, $\mathcal{J}_{\lambda_1}$ is unbounded from below with*

$$\mathcal{J}_{\lambda_1}(t\varphi_1) = -t \int_\Omega f\varphi_1 \,\mathrm{d}x \longrightarrow -\infty \quad \text{as } t \to +\infty\,. \tag{6.3}$$

Proof. On the contrary, assume that $\mathcal{J}_{\lambda_1}$ has a critical point $u_0 \in W_0^{1,p}(\Omega)$. In analogy with the proof of Theorem 6.5 above, one shows that $u_0 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,\alpha)$, together with the Hopf maximum principle (6.2) for u_0. The functional $u \mapsto \mathcal{J}_\lambda(u^{1/p})$ being strictly convex on the cone of all functions $u \in C^1(\overline{\Omega})$ satisfying (6.2), we conclude that u_0 is a global minimizer for $\mathcal{J}_{\lambda_1}$ over that cone. However, this conclusion contradicts (6.3). The theorem is proved. $\square$

Remark 6.7. If $p \neq 2$, $f \in L^\infty(\Omega)$ changes sign, and $0 < \lambda < \lambda_1$, then the functional $\mathcal{J}_\lambda$ may possess two or more distinct critical points in $W_0^{1,p}(\Omega)$. Examples of a suitable function f, such that $\mathcal{J}_\lambda$ exhibits multiple critical points, have been constructed in Fleckinger et al. [19, Example 2, p. 148] for $1 < p < 2$ and del Pino, Elgueta, and Manásevich [28, Eq. (5.26), p. 12] for $2 < p < \infty$. There, $\Omega \subset \mathbb{R}^1$ is a bounded open interval and $\mathcal{J}_\lambda$ has a global minimizer together with a saddle point.

Finally, concerning minimizers in the variational formula (1.2) for λ_1,

$$\lambda_1 = \inf\left\{ \frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\int_\Omega |u|^p \,\mathrm{d}x} : 0 \neq u \in W_0^{1,p}(\Omega) \right\}, \tag{1.2}$$

we have the following result:

Theorem 6.8. *A minimizer $u \in W_0^{1,p}(\Omega)$ for λ_1 is unique up to a constant multiple, that is, $u = c\varphi_1$ for some constant $c \in \mathbb{R}$, where $\varphi_1 \in W_0^{1,p}(\Omega)$ is a minimizer for λ_1 satisfying $\varphi_1 > 0$ a.e. in Ω and $\int_\Omega \varphi_1^p \,\mathrm{d}x = 1$. If, in addition, the boundary $\partial\Omega$ of Ω is of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, then φ_1 satisfies also the Hopf maximum principle* (6.2).

This theorem is due to Anane [2, Théorème 1, p. 727] and Lindqvist [25, Theorem 1.3, p. 157].

Proof. Let u be any (nontrivial) minimizer for λ_1. First, suppose that u changes sign in Ω. Then we have

$$\begin{aligned} \lambda_1 &= \frac{\int_\Omega (u^+)^p}{\int_\Omega |u|^p} \cdot \frac{\int_\Omega |\nabla u^+|^p \,\mathrm{d}x}{\int_\Omega (u^+)^p \,\mathrm{d}x} + \frac{\int_\Omega (u^-)^p}{\int_\Omega |u|^p} \cdot \frac{\int_\Omega |\nabla u^-|^p \,\mathrm{d}x}{\int_\Omega (u^-)^p \,\mathrm{d}x} \\ &\geq \lambda_1 \left(\frac{\int_\Omega (u^+)^p}{\int_\Omega |u|^p} + \frac{\int_\Omega (u^-)^p}{\int_\Omega |u|^p} \right) = \lambda_1 \,. \end{aligned}$$

Consequently, both u^+ and u^- are (nontrivial) minimizers for λ_1. Hence, we must have $u^+ > 0$ and $u^- > 0$ a.e. in Ω, by the strong maximum principle [37, Prop. 3.2.2, p. 801] or [39, Theorem 5, p. 200]. But this is impossible. We conclude that either $u > 0$ a.e. in Ω or else $u < 0$ a.e. in Ω.

Second, recall the proof of Theorem 6.5 above and take there $f \equiv 0$ in Ω. Replacing u by $-u$ if necessary, we may assume that $u > 0$ a.e. in Ω. For such u, the functional $u \mapsto \mathcal{J}_\lambda(u^{1/p})$ is ray-strictly convex, by Lemma 6.4. Therefore, the critical point of this new functional and, consequently, also the critical point of $\mathcal{J}_\lambda$ are unique up to a constant multiple.

Finally, the Hopf maximum principle (6.2) for φ_1 is proved in the same way as in Theorem 6.5. ∎

7. Minimization with constraint

Recall the orthogonal decomposition $W_0^{1,p}(\Omega) = \mathrm{lin}\{\varphi_1\} \oplus W_0^{1,p}(\Omega)^\top$ induced by the inner product in $L^2(\Omega)$. For $u \in W_0^{1,p}(\Omega)$ we write $u = \tau\varphi_1 + u^\top$ with $\tau \in \mathbb{R}$ and $\int_\Omega u^\top \varphi_1 \,\mathrm{d}x = 0$. In analogy with the Rayleigh quotient in (1.2), we have introduced another quotient in (1.6),

$$\Lambda_\infty \stackrel{\text{def}}{=} \inf\left\{ \frac{\int_\Omega |\nabla u^\top|^p \,\mathrm{d}x}{\int_\Omega |u^\top|^p \,\mathrm{d}x} : 0 \neq u \in W_0^{1,p}(\Omega)^\top \right\}. \tag{1.6}$$

Clearly, $\lambda_1 < \Lambda_\infty < \infty$, because λ_1 is a simple eigenvalue of $-\Delta_p$. (If Ω has reflection symmetry, one can show $\Lambda_\infty = \lambda_2$, the second eigenvalue of $-\Delta_p$.)

Let $\lambda < \Lambda_\infty$ and consider the functional $\mathcal{J}_\lambda(u) = \mathcal{J}_\lambda(\tau\varphi_1 + u^\top)$ with $\tau \in \mathbb{R}$ being *fixed* (but arbitrary) and $u^\top \in W_0^{1,p}(\Omega)^\top$ *variable*. Then the restricted functional $u^\top \mapsto \mathcal{J}_\lambda(\tau\varphi_1 + u^\top)$ is coercive on $W_0^{1,p}(\Omega)^\top$ and thus possesses a global minimizer $u_\tau^\top \in W_0^{1,p}(\Omega)^\top$. Such a global minimizer satisfies the boundary value problem

$$\begin{gathered} -\Delta_p(\tau\varphi_1 + u^\top) = \lambda\, |\tau\varphi_1 + u^\top|^{p-2}(\tau\varphi_1 + u^\top) + f(x) + \zeta\cdot\varphi_1(x) \quad \text{in } \Omega; \\ u^\top = 0 \quad \text{on } \partial\Omega; \\ \langle u^\top, \varphi_1\rangle = 0, \end{gathered} \tag{7.1}$$

where $\zeta \in \mathbb{R}$ is a Lagrange multiplier (which is unknown). In particular, when investigating the solvability of this problem, without loss of generality we may assume that $f = f^\top \in L^\infty(\Omega)^\top$, by simply substituting ζ for $f^\parallel + \zeta$. Notice that, for $u \equiv \tau\varphi_1 + u^\top$ and $f \equiv \zeta\varphi_1 + f^\top$, with $\tau, \zeta \in \mathbb{R}$, $u^\top \in W_0^{1,p}(\Omega)^\top$, and $f^\top \in L^\infty(\Omega)^\top$, we have

$$\mathcal{J}_\lambda(u; f) = \mathcal{J}_\lambda(\tau\varphi_1 + u^\top; f^\top) - \tau\zeta\, \|\varphi_1\|^2_{L^2(\Omega)}. \tag{7.2}$$

It is now clear that it suffices to determine the properties of $\mathcal{J}_\lambda$ in the special case $f = f^\top \in L^\infty(\Omega)^\top$.

By arguments used in the proof of Lemma 4.1 above, we conclude that *any* minimizing sequence for the restricted functional $u^\top \mapsto \mathcal{J}_\lambda(\tau\varphi_1 + u^\top)$ on $W_0^{1,p}(\Omega)^\top$ contains a strongly convergent subsequence $u_{\tau,n}^\top \to u_\tau^\top$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$, which converges to a global minimizer $u_\tau^\top \in W_0^{1,p}(\Omega)^\top$. Taking advantage of this technique, it is not difficult to see that

$$j_\lambda(\tau) \stackrel{\text{def}}{=} \inf\Big\{\mathcal{J}_\lambda(\tau\varphi_1 + u^\top) : u^\top \in W_0^{1,p}(\Omega)^\top\Big\} = \mathcal{J}_\lambda(\tau\varphi_1 + u_\tau^\top) \tag{7.3}$$

is a *continuous* function of $\tau \in \mathbb{R}$; a complete proof is in Takáč [32, Lemma 7.2, p. 222].

We notice that eq. (7.1) yields $\mathcal{J}'_\lambda(\tau\varphi_1 + u^\top) = \zeta\cdot\varphi_1$ in $W^{-1,p'}(\Omega)$, which entails

$$\begin{aligned} \tau\zeta\, \|\varphi_1\|^2_{L^2(\Omega)} &= \zeta\,\langle \varphi_1, \tau\varphi_1 + u^\top\rangle = \langle \mathcal{J}'_\lambda(\tau\varphi_1 + u^\top), \tau\varphi_1 + u^\top\rangle \\ &= p\cdot\mathcal{J}_\lambda(\tau\varphi_1 + u^\top) + (p-1)\,\langle f, \tau\varphi_1 + u^\top\rangle\,. \end{aligned}$$

Thus, if $f = f^\top \in L^\infty(\Omega)^\top$ and if $u_\tau^\top \in W_0^{1,p}(\Omega)^\top$ is any global minimizer for the restricted energy functional $u^\top \mapsto \mathcal{J}_\lambda(\tau\varphi_1 + u^\top)$ on $W_0^{1,p}(\Omega)^\top$, then the Lagrange multiplier $\zeta = \zeta_\tau \in \mathbb{R}$ from the boundary value problem (7.1) satisfies

$$\begin{aligned} \tau\zeta_\tau\, \|\varphi_1\|^2_{L^2(\Omega)} &= p\cdot\mathcal{J}_\lambda(\tau\varphi_1 + u_\tau^\top) + (p-1)\,\langle f^\top, u_\tau^\top\rangle \\ &= p\cdot j_\lambda(\tau) + (p-1)\,\langle f^\top, u_\tau^\top\rangle\,. \end{aligned}$$

Unfortunately, we do *not* know whether

$$\tau \mapsto \langle f^\top, u_\tau^\top\rangle = \int_\Omega f^\top u_\tau^\top\,\mathrm{d}x : \mathbb{R} \to \mathbb{R} \tag{7.4}$$

is a continuous function of τ or whether this function is independent from the choice of the global minimizer $u_\tau^\top$ for the restricted energy functional. Consequently, also $\tau \mapsto \tau\zeta_\tau$ might not be continuous. Fortunately, and this fact is very important for us, we will be able to determine the *asymptotic behavior* of both, the global minimizer $u_\tau^\top$ and the Lagrange multiplier ζ_τ rather precisely, depending on τ, as $\tau \to \pm\infty$. Applying these asymptotic estimates to eq. (7.4) we will easily derive the asymptotic behavior of the (continuous) function $j_\lambda(\tau)$ as $\tau \to \pm\infty$. It will depend on whether $1 < p < 2$, $p = 2$, or $2 < p < \infty$.

For instance, if $\lambda = \lambda_1$ and $f = f^\top \in L^\infty(\Omega)$, $f^\top \not\equiv 0$ in Ω, it has been shown in Takáč [34, Lemma 9.7, p. 466] that $j_{\lambda_1}(\tau) = -c \cdot |\tau|^{2-p} + o(|\tau|^{2-p})$ as $\tau \to \pm\infty$, where $c = c(f^\top) > 0$ is a constant depending on $f^\top$. (Of course, the symbol "$o(\,\cdot\,)$" means $o(t)/t \to 0$ as either $t \to \pm\infty$ or $t \to 0$ in $\mathbb{R}$, $t \neq 0$.) The proof of this result relies on formula (B.18) from Proposition B.4 in the Appendix, §B.2. For $p > 2$ it follows that the function j_{λ_1} possesses a global minimizer $\tau_0 \in \mathbb{R}$, even though j_{λ_1} is not coercive. For $1 < p < 2$ it possesses a global maximizer $\tau_0 \in \mathbb{R}$ and tends to $-\infty$ as $\tau \to \pm\infty$. For $p = 2$ it is an easy exercise to see that $j_{\lambda_1}(\tau) \equiv \text{const} < 0$ is a constant function.

8. Some elementary analysis

We know that $j_\lambda : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let us consider the following two standard cases:

(i) If j_λ has a local minimum at $\tau = \tau_0$, then $u_{\tau_0} = \tau_0\varphi_1 + u_{\tau_0}^\top$ is a local minimizer for $\mathcal{J}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$. (This claim is easy to verify.)

(ii) If j_λ has a local maximum ($= \beta_\lambda$) at $\tau = \tau_0$, then we do *not* know if $u_{\tau_0} = \tau_0\varphi_1 + u_{\tau_0}^\top$ is a critical point for the energy functional $\mathcal{J}_\lambda$. However, again by arguments used in the proof of Lemma 4.1, it is not difficult to show the following lemma about the existence of a pair of sub- and supersolutions.

Lemma 8.1. *Let $-\infty < \lambda < \Lambda_\infty$ and $f \in L^\infty(\Omega)$, $f \not\equiv 0$. Assume that $j_\lambda : \mathbb{R} \to \mathbb{R}$ attains a local maximum β_λ at some point $\tau_0 \in \mathbb{R}$. Then there exist two functions*
$$\underline{u} = \tau_0\varphi_1 + \underline{u}^\top, \quad \overline{u} = \tau_0\varphi_1 + \overline{u}^\top \quad \textit{with} \quad \underline{u}^\top, \overline{u}^\top \in W_0^{1,p}(\Omega)^\top,$$
such that
$$\mathcal{J}_\lambda(\tau_0\varphi_1 + \underline{u}^\top) = \mathcal{J}_\lambda(\tau_0\varphi_1 + \overline{u}^\top) = j_\lambda(\tau_0) = \beta_\lambda \tag{8.1}$$
and
$$\mathcal{J}'_\lambda(\tau_0\varphi_1 + \underline{u}^\top) = \underline{\zeta} \cdot \varphi_1, \quad \mathcal{J}'_\lambda(\tau_0\varphi_1 + \overline{u}^\top) = \overline{\zeta} \cdot \varphi_1 \tag{8.2}$$
hold for some (Lagrange multipliers) $\underline{\zeta}, \overline{\zeta} \in \mathbb{R}$ with $\underline{\zeta} \le 0 \le \overline{\zeta}$.

A proof of this lemma is given in Takáč [35, Lemma 4.6, p. 715]. It is based on a construction of two arbitrary sequences $\{\tau'_n\}_{n=1}^\infty$ and $\{\tau''_n\}_{n=1}^\infty$, satisfying
$$-\infty < \tau'_1 < \tau'_2 < \cdots < \tau'_n < \cdots < \tau_0 < \cdots < \tau''_n < \cdots < \tau''_2 < \tau''_1 < \infty$$
with $\tau'_n \nearrow \tau_0$ and $\tau''_n \searrow \tau_0$ as $n \to \infty$, such that for some functions $u_\tau^\top \in W_0^{1,p}(\Omega)^\top$ indexed by $\tau \in \{\tau'_n\}_{n=1}^\infty \cup \{\tau''_n\}_{n=1}^\infty$ we have
$$\mathcal{J}_\lambda(\tau\varphi_1 + u_\tau^\top) = j_\lambda(\tau) \quad (\le \beta_\lambda) \tag{8.3}$$
and
$$\mathcal{J}'_\lambda(\tau\varphi_1 + u_\tau^\top) = \zeta_\tau \cdot \varphi_1 \quad \text{with some} \quad \zeta_\tau \in \mathbb{R}, \tag{8.4}$$
for every $\tau \in \{\tau'_n\}_{n=1}^\infty \cup \{\tau''_n\}_{n=1}^\infty$, where
$$\zeta_{\tau'_n} \ge 0 \quad \text{and} \quad \zeta_{\tau''_n} \le 0 \quad \text{for all} \quad n = 1, 2, \ldots. \tag{8.5}$$

If $\underline{\zeta} = 0$ and/or $\overline{\zeta} = 0$, then we have a critical point for $\mathcal{J}_\lambda$; one may call it a *simple saddle point* for $\mathcal{J}_\lambda$. Otherwise, $\underline{u}$ is called a *strict subsolution* of the problem $\mathcal{J}'_\lambda(u) = 0$, because of $\underline{\zeta} < 0$, and similarly, $\overline{u}$ is called a *strict supersolution*, because of $\overline{\zeta} > 0$. In order *to deduce* the existence of a solution to $\mathcal{J}'_\lambda(u) = 0$ *from* the existence of a pair of (strict) sub- and supersolutions, $\underline{u}$ and $\overline{u}$, we will apply a topological method (Leray-Schauder degree theory). Needless to say, we will lose all information about the "geometry" of the functional $\mathcal{J}_\lambda$ near such a critical point.

9. Existence by a topological degree

Let us recall that the first (smallest) eigenvalue λ_1 of the positive Dirichlet p-Laplacian $-\Delta_p$ is simple with the associated eigenfunction φ_1 normalized by $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_{L^p(\Omega)} = 1$, by Anane [2, Théorème 1, p. 727] or Lindqvist [25, Theorem 1.3, p. 157]. We have $\varphi_1 \in L^\infty(\Omega)$ by Anane [3, Théorème A.1, p. 96]. Consequently, recalling hypothesis (H1), we get even $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, \alpha)$, by a regularity result due to DiBenedetto [8, Theorem 2, p. 829] and Tolksdorf [38, Theorem 1, p. 127] (interior regularity), and to Lieberman [24, Theorem 1, p. 1203] (regularity near the boundary). The constant β depends solely on α, N, and p. We keep the meaning of the constants α and β throughout the entire lecture notes. Finally, the Hopf maximum principle (see Tolksdorf [37, Prop. 3.2.1 and 3.2.2, p. 801] or Vázquez [39, Theorem 5, p. 200]) renders

$$\varphi_1 > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \varphi_1}{\partial \nu} < 0 \quad \text{on } \partial\Omega\,. \tag{9.1}$$

As usual, $\partial/\partial\nu$ denotes the outer normal derivative on $\partial\Omega$. We set

$$U \stackrel{\text{def}}{=} \{x \in \Omega : \nabla\varphi_1(x) \neq \mathbf{0}\}, \quad \text{hence } \ \Omega \setminus U = \{x \in \Omega : \nabla\varphi_1(x) = \mathbf{0}\},$$

and observe that $\Omega \setminus U$ is a compact subset of Ω, by (9.1).

As we have already mentioned at the end of Section 7, if $f = f^\top \in L^\infty(\Omega)$, $f^\top \not\equiv 0$ in Ω, then for $1 < p < 2$ the function $j_{\lambda_1}(\tau)$ possesses a global maximizer $\tau_0 \in \mathbb{R}$ and tends to $-\infty$ as $|\tau| \to \infty$. As this topological method is typical for the case $1 < p < 2$ and $\lambda = \lambda_1$, to which it has been originally applied in Drábek and Holubová [14, Theorem 1.1, p. 184], we will explain it for this parameter setting. Of course, it works similarly for any $p > 1$ and any $\lambda < \Lambda_\infty$; see Takáč [35].

So fix $1 < p < 2$ and $\lambda = \lambda_1$. In the (resonant) Dirichlet problem

$$-\Delta_p u = \lambda_1 |u|^{p-2} u + f(x) \quad \text{in } \Omega\,; \quad u = 0 \quad \text{on } \partial\Omega\,, \tag{9.2}$$

assume $f = f^\top + \zeta\varphi_1$, where $f^\top \in L^\infty(\Omega)^\top$, $f^\top \not\equiv 0$ in Ω, and $\zeta \in \mathbb{R}$. If $f^\top$ is continuous in a an open neighborhood of the (compact) set $\{x \in \Omega : \nabla\varphi_1(x) = \mathbf{0}\}$ and if $|\zeta|$ is small enough, then it is possible to show that the functional $\mathcal{J}_{\lambda_1}$ has a "saddle point geometry" (Drábek and Holubová [14, Lemma 2.1, p. 185]). If also $\zeta \neq 0$, then $\mathcal{J}_{\lambda_1}$ satisfies the so-called "Palais-Smale condition" ([14, Lemma 2.2, p. 188]). Thus, if $|\zeta| > 0$ is small enough, a "saddle point theorem" from Rabinowitz [30, Theorem 4.6, p. 24] guarantees the existence of a critical point $u_0 \in W_0^{1,p}(\Omega)$ for $\mathcal{J}_{\lambda_1}$. If $\zeta = 0$, the validity of the Palais-Smale condition for $\mathcal{J}_{\lambda_1}$ is still an open question and, as indicated in Drábek and Takáč [15], it might not be satisfied at all; see [15, Theorem 4.1, pp. 47–48] for difficulties and hints to ramifications. This means that we will not be able to apply the saddle point theorem [30, Theorem 4.6, p. 24] to treat the most natural case $\zeta = 0$. Taking $|\zeta| > 0$ small enough one can apply this theorem to construct only a pair of strict sub- and supersolutions, $\underline{u}$ and $\overline{u}$, as in the previous section (Section 8). Notice that the construction in the previous section does not require $f^\top$ to be continuous in a an open neighborhood of the (compact) set $\Omega \setminus U$ defined above. Besides, it gives

$$\langle \underline{u}, \varphi_1 \rangle = \langle \overline{u}, \varphi_1 \rangle = \tau_0 \, \|\varphi_1\|^2_{L^2(\Omega)} \quad \text{with some } \tau_0 \in \mathbb{R}. \tag{9.3}$$

However, it uses the asymptotic behavior $j_{\lambda_1}(\tau) \to -\infty$ as $\tau \to \pm\infty$ which is not easy to establish (except for the case when $f^\top$ is continuous in an open neighborhood of $\Omega \setminus U$).

We will therefore use a topological (hence, nonvariational) method to handle this situation. It has been introduced in De Coster and Henrard [6, Theorem 8.2, p. 448] for semilinear elliptic boundary value problems. Given a pair of *unordered* sub- and supersolutions, a fixed point mapping is constructed first and then its Leray-Schauder degree is computed. We follow the presentation given in Takáč [35, §4.5, pp. 727–733].

Theorem 9.1. *Let $1 < p < \infty$, $\lambda = \lambda_1$, and $f = \zeta\varphi_1 + f^\top$ with some $\zeta \in \mathbb{R}$ and $f^\top \in L^\infty(\Omega)^\top$, $f^\top \not\equiv 0$ in Ω. Assume that $\underline{u}, \overline{u} \in W_0^{1,p}(\Omega)$ is a pair of strict sub- and supersolutions of the problem $\mathcal{J}'_{\lambda_1}(u) = 0$ as described in* Lemma 8.1. *(In particular, $\underline{u}$ and $\overline{u}$ are unordered, by eq.* (9.3)*.) Then the Dirichlet problem $\mathcal{J}'_{\lambda_1}(u) = 0$ possesses a weak solution $u \in W_0^{1,p}(\Omega)$.*

We remark that $\underline{u}, \overline{u} \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, \alpha)$, by the regularity result mentioned above [3, 8, 24, 38]. We use this fact repeatedly in an essential way.

We will prove Theorem 9.1 using the topological (Leray-Schauder) degree. In the proof of the next lemma we obtain another pair of sub- and supersolutions, ordered by "$\leq$", which provides lower and upper bounds for the unordered pair.

Recalling $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$ and the Hopf maximum principle (9.1), we introduce the space X of all functions $\phi \in C(\overline{\Omega})$ such that

$$\|\phi\|_X \stackrel{\text{def}}{=} \sup_{\Omega} \, (|\phi|/\varphi_1) < \infty \, . \tag{9.4}$$

Then X endowed with the norm $\|\cdot\|_X$ is a Banach space. Notice that the embeddings $C_0^1(\overline{\Omega}) \hookrightarrow X \hookrightarrow C(\overline{\Omega})$ are continuous, where

$$C_0^1(\overline{\Omega}) \stackrel{\text{def}}{=} \{\phi \in C^1(\overline{\Omega}) : \phi = 0 \text{ on } \partial\Omega\}$$

is a closed linear subspace of $C^1(\overline{\Omega})$.

We denote $\mathbb{R}_+ = [0, \infty)$. Given any $R > 0$, let us define the function $\gamma_R : \mathbb{R}_+ \to [0, 1]$ by

$$\gamma_R(\xi) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } 0 \leq \xi \leq R\,; \\ 2 - (\xi/R) & \text{if } R < \xi \leq 2R\,; \\ 0 & \text{if } \xi > 2R\,. \end{cases} \tag{9.5}$$

Notice that γ_R is a monotone decreasing, Lipschitz-continuous function. Next, for $u \in X$ we define $G_R(u) : \Omega \to \mathbb{R}$ by

$$[G_R(u)](x) \stackrel{\text{def}}{=} \begin{cases} \lambda_1 |u|^{p-2} u + \gamma_R\left(\frac{|u(x)|}{\varphi_1(x)}\right) f(x) & \text{if } \frac{|u(x)|}{\varphi_1(x)} \leq 2R\,; \\ -\lambda_1 \, [2R\,\varphi_1(x)]^{p-1} & \text{if } \frac{u(x)}{\varphi_1(x)} < -2R\,; \\ \lambda_1 \, [2R\,\varphi_1(x)]^{p-1} & \text{if } \frac{u(x)}{\varphi_1(x)} > 2R\,, \end{cases} \tag{9.6}$$

at every $x \in \Omega$. As $\gamma_R(2\varepsilon) = 0$, $f \in L^\infty(\Omega)$, and $u/\varphi_1 \in L^\infty(\Omega)$, it is easy to see that the mapping

$$G_R : u \mapsto G_R(u) : X \to L^\infty(\Omega)$$

is continuous.

Lemma 9.2. *Assume that f and the sub- and supersolutions $\underline{u}$ and $\overline{u}$ are exactly as in* Lemma 8.1. *Let $\{\eta_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{R_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfy $\eta_n \to 0$ and $R_n \to \infty$ as $n \to \infty$. Finally, for each $n = 1, 2, \dots$, let $u_n \in W_0^{1,p}(\Omega)$ have the following properties:*

(i) $u_n \in C_0^1(\overline{\Omega})$;
(ii) u_n *is a weak solution of the boundary value problem*

$$-\Delta_p u_n = [G_{R_n}(u_n)](x) \quad \text{in } \Omega\,; \quad u_n = 0 \quad \text{on } \partial\Omega\,; \tag{9.7}$$

(iii) *there exist points $x_n', x_n'' \in \Omega$ such that*

$$u_n(x_n') \le \underline{u}(x_n') + \eta_n R_n\, \varphi_1(x_n')\,, \tag{9.8}$$

$$u_n(x_n'') \ge \overline{u}(x_n'') - \eta_n R_n\, \varphi_1(x_n'')\,. \tag{9.9}$$

Then, given any $0 < \varepsilon \le 1$, there exists an integer $n_\varepsilon \ge 1$ such that inequalities

$$\|u_n\|_{C^1(\overline{\Omega})} + \sup_{\Omega} (|u_n|/\varphi_1) \le \varepsilon R_n \tag{9.10}$$

hold for every $n \ge n_\varepsilon$. In particular, for every $n \ge n_\varepsilon$, u_n is a weak solution of problem (9.2) *as well, i.e., $\mathcal{J}_{\lambda_1}'(u_n) = 0$.*

In fact, we wish to prove Theorem 9.1 by computing the Leray-Schauder degree in subsets of

$$\mathcal{U}_n' = \{u \in X : u(x) > \underline{u}(x) + \eta_n\, \varphi_1(x)\}\,,$$
$$\mathcal{U}_n'' = \{u \in X : u(x) < \overline{u}(x) - \eta_n\, \varphi_1(x)\}\,.$$

Here, we have dropped the factor R_n (≥ 1 for all n sufficiently large) from the product $\eta_n R_n$ in (9.8) and (9.9); the (weaker) inequalities above will do. Lemma 9.2 is needed to take care of the complement (in X) of the union of these two sets, $X \setminus (\mathcal{U}_n' \cup \mathcal{U}_n'')$. Clearly, the "pointwise boundedness" conditions (9.8) and (9.9) are very generous as η_n and R_n are not related.

Proof of Lemma 9.2. The "normalized" sequence $v_n \stackrel{\text{def}}{=} R_n^{-1} u_n$ ($n = 1, 2, \dots$) satisfies

$$-\Delta_p v_n = R_n^{-(p-1)}\, [G_{R_n}(R_n v_n)](x) \quad \text{in } \Omega; \quad v_n = 0 \quad \text{on } \partial\Omega. \tag{9.11}$$

Since $\partial\Omega$ is assumed to be of class $C^{1,\alpha}$, for some $0 < \alpha < 1$, we conclude that $v_n \in C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0,\alpha)$, and the sequence $\{v_n\}_{n=1}^{\infty}$ is bounded in $C^{1,\beta}(\overline{\Omega})$, by the regularity result mentioned above [3, 8, 24, 38]. Now, for any fixed $\beta' \in (0,\beta)$, we can apply Arzelà-Ascoli's theorem in $C^{1,\beta'}(\overline{\Omega})$ to the sequence $\{v_n\}_{n=1}^{\infty}$ to obtain a convergent subsequence $v_n \to v^*$ in $C^{1,\beta'}(\overline{\Omega})$ as $n \to \infty$. Letting $n \to \infty$ in the weak formulation of problem (9.11), and recalling our notation from (9.5), we arrive at

$$-\Delta_p v^* = [H(v^*)](x) \quad \text{in } \Omega\,; \quad v^* = 0 \quad \text{on } \partial\Omega\,, \tag{9.12}$$

where $H(v) : \Omega \to \mathbb{R}$ is defined by

$$[H(v)](x) \stackrel{\text{def}}{=} \begin{cases} \lambda_1 |v|^{p-2} v & \text{if } \frac{|v(x)|}{\varphi_1(x)} \le 2\,; \\ -\lambda_1\, [2\, \varphi_1(x)]^{p-1} & \text{if } \frac{v(x)}{\varphi_1(x)} < -2\,; \\ \lambda_1\, [2\, \varphi_1(x)]^{p-1} & \text{if } \frac{v(x)}{\varphi_1(x)} > 2\,, \end{cases}$$

at every $x \in \Omega$, for $v \in X$. As both functions $v_\pm = \pm 2\varphi_1$ are solutions of problem (9.12) and $H(v_-) \le H(v^*) \le H(v_+)$ in Ω, we conclude that v^* itself must satisfy $v_- \le v^* \le v_+$ in Ω, by the weak comparison principle (see e.g. [37, Lemma 3.1, p. 800]). Consequently, eq. (9.12) reads

$$-\Delta_p v^* = \lambda_1 |v^*|^{p-2} v^* \quad \text{in } \Omega; \quad v^* = 0 \quad \text{on } \partial\Omega.$$

Eigenvalue λ_1 of $-\Delta_p$ being simple, this equation forces $v^* = \kappa\varphi_1$ in Ω for some $\kappa \in \mathbb{R}$ with $|\kappa| \le 2$.

Next, we observe that (9.8) and (9.9), respectively, are equivalent to

$$v_n(x_n') \le R_n^{-1}\, \underline{u}(x_n') + \eta_n \varphi_1(x_n'), \tag{9.13}$$

$$v_n(x_n'') \ge R_n^{-1}\, \overline{u}(x_n'') - \eta_n \varphi_1(x_n''). \tag{9.14}$$

From $v_n \to v^* = \kappa\varphi_1$ in $C^{1,\beta'}(\overline{\Omega})$ as $n \to \infty$ we deduce that $v_n/\varphi_1 \to v^*/\varphi_1 = \kappa$ in $L^\infty(\Omega)$. We claim that $\kappa = 0$ which implies the conclusion of our lemma immediately. So, on the contrary, suppose that $\kappa \ne 0$.

If $\kappa > 0$ then there exists an integer $n_0 \ge 1$ such that $v_n/\varphi_1 \ge \frac{1}{2}\kappa$ in Ω for all $n \ge n_0$. Combining this inequality with (9.13) we arrive at

$$R_n^{-1}\, \frac{\underline{u}(x_n')}{\varphi_1(x_n')} + \eta_n \ge \frac{v_n(x_n')}{\varphi_1(x_n')} \ge \frac{\kappa}{2} > 0 \quad \text{for all } n \ge n_0.$$

But this contradicts $R_n \to \infty$ and $\eta_n \to 0$ as $n \to \infty$.

Similarly, if $\kappa < 0$ then $v_n/\varphi_1 \le \frac{1}{2}\kappa$ in Ω for all $n \ge n_0$. Combining this inequality with (9.14) we get

$$R_n^{-1}\, \frac{\overline{u}(x_n'')}{\varphi_1(x_n'')} - \eta_n \le \frac{v_n(x_n'')}{\varphi_1(x_n'')} \le \frac{\kappa}{2} < 0 \quad \text{for all } n \ge n_0.$$

Again, this contradicts $R_n \to \infty$ and $\eta_n \to 0$. The lemma follows from $R_n^{-1} u_n = v_n \to \kappa\varphi_1 = 0$ in $C_0^1(\overline{\Omega})$. $\square$

Let us denote by X_+ the positive cone in X, that is,

$$X_+ \stackrel{\text{def}}{=} \{\phi \in X : \phi \ge 0 \text{ in } \Omega\},$$

and by $\mathring{X}_+$ its (topological) interior,

$$\mathring{X}_+ = \{\phi \in X : \phi \ge \kappa\varphi_1 \text{ in } \Omega \quad \text{for some } \ \kappa \in (0,\infty)\}.$$

Given any $a, b \in X$, we write $a \ll b$ (or, equivalently, $b \gg a$) if and only if $b - a \in \mathring{X}_+$. We denote

$$[a,b] \stackrel{\text{def}}{=} \{\phi \in X : a \le \phi \le b \text{ in } \Omega\} \text{ and } [[a,b]] \stackrel{\text{def}}{=} \{\phi \in X : a \ll \phi \ll b\}.$$

Notice that $[[a,b]]$ is the (topological) interior of $[a,b]$ in X.

Proof of Theorem 9.1. With regard to Lemma 9.2 above, it suffices to construct a weak solution $u_n \in W_0^{1,p}(\Omega)$ of problem (9.7) with properties (i), (ii), and (iii) of Lemma 9.2, for each $n = 1, 2, \dots$. The sequences $\{\eta_n\}_{n=1}^\infty \subset (0,1)$ and $\{R_n\}_{n=1}^\infty \subset (0,\infty)$ may be chosen arbitrarily with $\eta_n \to 0$ and $R_n \to \infty$ as $n \to \infty$.

Given $0 < r < \infty$, we denote by

$$\mathcal{B}_r = \{u \in X : \|u\|_X < r\}$$

the open ball in X with radius r centered at the origin, and by

$$\overline{\mathcal{B}}_r = \{u \in X : \|u\|_X \le r\}$$

its closure in X. Set

$$R_0 = \max\{\|\underline{u}\|_X,\, \|\overline{u}\|_X\} + 1\,.$$

In particular, recalling (9.5), for $R \ge R_0$ we have $\gamma_R(|\underline{u}|/\varphi_1) = 1$ and $\gamma_R(|\overline{u}|/\varphi_1) = 1$ throughout Ω. Hence, from now on, we may assume $R_n \ge R_0$ for all $n \ge 1$. Let us fix any $n \ge 1$ and recall $0 < \eta_n < 1 \le R_n < \infty$.

Now we follow Drábek and Holubová [14], proof of Lemma 2.4, pp. 191–192. First, fix any number $\varrho > 3R_n$. We define the (fixed point) mapping

$$\mathcal{T} : \overline{\mathcal{B}}_\varrho \to X : u \mapsto \mathcal{T}u \overset{\text{def}}{=} \tilde{u}\,,$$

where $\tilde{u} \in C_0^1(\overline{\Omega})$ is the unique weak solution of

$$-\Delta_p \tilde{u} = [G_{R_n}(u)](x) \quad \text{in } \Omega\,; \quad \tilde{u} = 0 \quad \text{on } \partial\Omega\,. \tag{9.15}$$

We claim that $\mathcal{T}$ is compact, i.e., $\mathcal{T}$ is continuous and its image is contained in a compact set. Indeed, it is easy to see that $\mathcal{T} : \overline{\mathcal{B}}_\varrho \subset X \to W_0^{1,p}(\Omega)$ is continuous with the image $\mathcal{T}(\overline{\mathcal{B}}_\varrho)$ being a bounded set in $C^{1,\beta}(\overline{\Omega})$, by regularity [3, 8, 24, 38]. Consequently, for any fixed $\beta' \in (0,\beta)$,

$$\mathcal{T} : \overline{\mathcal{B}}_\varrho \subset X \to C^{1,\beta'}(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$$

is continuous with $\mathcal{T}(\overline{\mathcal{B}}_\varrho)$ having compact closure in each of the spaces

$$C^{1,\beta'}(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \hookrightarrow C_0^1(\overline{\Omega}) \hookrightarrow X,$$

by Arzelà-Ascoli's theorem. Thus, $\mathcal{T} : \overline{\mathcal{B}}_\varrho \to X$ is compact.

If there exists a fixed point $u_n \in \overline{\mathcal{B}}_\varrho$ of $\mathcal{T}$, i.e., $\mathcal{T}u_n = u_n$, such that both inequalities (9.8) and (9.9) are satisfied, then we are done. So let us assume the contrary, that is, if $u_n \in \overline{\mathcal{B}}_\varrho$ satisfies $\mathcal{T}u_n = u_n$ then *at least one* of the following two inequalities must be valid:

$$u_n(x) > \underline{u}(x) + \eta_n\,\varphi_1(x) \quad \text{for all } x \in \Omega\,, \tag{9.16}$$

$$u_n(x) < \overline{u}(x) - \eta_n\,\varphi_1(x) \quad \text{for all } x \in \Omega\,. \tag{9.17}$$

Notice that we have dropped the factor R_n (≥ 1) from the product $\eta_n R_n$ in (9.8) and (9.9); the (weaker) inequalities above will suffice to get a contradiction. Consequently, we get $u_n \gg \underline{u}$ or $u_n \ll \overline{u}$ in X. Moreover, both inequalities cannot hold simultaneously; for otherwise we would have $\underline{u} \le \overline{u}$ throughout Ω which forces $\underline{u} = \overline{u}$ in Ω, owing to eq. (9.3). Hence, there exists a point $y \in \Omega$ such that $\overline{u}(y) < \underline{u}(y)$. From either inequality $u_n \gg \underline{u}$ or $u_n \ll \overline{u}$ in X we deduce easily that $u_n \in X \setminus \overline{\mathcal{S}}$ where $\overline{\mathcal{S}}$ denotes the closure in X of the set

$$\mathcal{S} = \left\{u \in X : u(x') < \underline{u}(x') \text{ and } u(x'') > \overline{u}(x'') \text{ for some } x', x'' \in \Omega\right\}.$$

Clearly, $\mathcal{S}$ is open in X with the complement

$$X \setminus \mathcal{S} = \{u \in X : u \ge \underline{u} \text{ in } \Omega\} \cup \{u \in X : u \le \overline{u} \text{ in } \Omega\}\,.$$

Next, we introduce the functions $A_\pm \overset{\text{def}}{=} \pm\, 3R_n\varphi_1$. We have $A_\pm \in \mathcal{B}_\varrho$, together with $A_- \ll \underline{u} \ll A_+$ and $A_- \ll \overline{u} \ll A_+$. Hence, also $\underline{u}, \overline{u} \in \mathcal{B}_\varrho$. Observe that both

A_- and $\underline{u}$ (A_+ and $\overline{u}$) are strict subsolutions (supersolutions, respectively) of the boundary value problem (9.7); more precisely, they satisfy

$$\begin{gathered} -\Delta_p u - [G_{R_n}(u)](x) \le -\phi_1(x) < 0 \quad (\ge \phi_1(x) > 0) \quad \text{in } \Omega\,; \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{9.18}$$

where $\phi_1 = c_{R_n} \cdot \min\{\varphi_1,\, \varphi_1^{p-1}\}$ with some constant $c_{R_n} > 0$. This in turn implies $u \ll \mathcal{T}u$ ($u \gg \mathcal{T}u$) in X, by the strong comparison principle due to Cuesta and Takáč [4, Theorem 1, p. 81] (see also [5, Theorem 2.1, p. 725]). We remark that the connectedness of the boundary $\partial\Omega$, assumed in [4, Theorem 1], is not needed here owing to the strict inequality in eq. (9.18) throughout the domain Ω. Hence, in the proof of [4, Theorem 1, p. 81], one may apply [4, Prop. 3, p. 82] on any connected component of the boundary $\partial\Omega$.

Finally, let $\deg[I - \mathcal{T}; \mathcal{U}, 0]$ denote the Leray-Schauder degree of the mapping $I - \mathcal{T} : \overline{\mathcal{U}} \to X$ relative to the origin $0 \in X$, where $\mathcal{U}$ is any open set in X such that $\mathcal{U} \subset \mathcal{B}_\varrho$ and $0 \notin (I - \mathcal{T})(\partial\mathcal{U})$. As usual, I stands for the identity mapping and $\partial\mathcal{U}$ for the boundary of $\mathcal{U}$ in X. We compute this degree in the sets $[[A_-, A_+]]$, $[[A_-, \overline{u}_n]]$, $[[\underline{u}_n, A_+]]$, and $\mathcal{S} \cap [[A_-, A_+]]$, all of which are open in X. Clearly, the last three sets, $[[A_-, \overline{u}_n]]$, $[[\underline{u}_n, A_+]]$, and $\mathcal{S} \cap [[A_-, A_+]]$, are pairwise disjoint and the union of their closures equals $[A_-, A_+] \subset \overline{\mathcal{B}}_\varrho$. Using the excision property of the Leray-Schauder degree, we compute

$$\begin{aligned} &\deg[I - \mathcal{T};\, [[A_-, A_+]],\, 0] \\ &= \deg[I - \mathcal{T};\, [[A_-, \overline{u}_n]],\, 0] + \deg[I - \mathcal{T};\, [[\underline{u}_n, A_+]],\, 0] \\ &\quad + \deg[I - \mathcal{T};\, \mathcal{S} \cap [[A_-, A_+]],\, 0]. \end{aligned} \tag{9.19}$$

Recalling the fact that $\mathcal{T}$ has no fixed point $u_n \in \overline{\mathcal{B}}_\varrho$ on the boundary of any of the sets $[[A_-, A_+]]$, $[[A_-, \overline{u}_n]]$, and $[[\underline{u}_n, A_+]]$, we may apply [14, Lemma 2.3, p. 190] to conclude that

$$\begin{aligned} \deg[I - \mathcal{T};\, [[A_-, A_+]],\, 0] &= \deg[I - \mathcal{T};\, [[A_-, \overline{u}_n]],\, 0] \\ &= \deg[I - \mathcal{T};\, [[\underline{u}_n, A_+]],\, 0] = 1. \end{aligned}$$

Furthermore, since $\mathcal{T}$ has no fixed point in $\overline{\mathcal{S}}$, we must have also

$$\deg[I - \mathcal{T};\, \mathcal{S} \cap [[A_-, A_+]],\, 0] = 0.$$

Inserting these results into eq. (9.19) we arrive at a contradiction, $1 = 1 + 1 + 0$.

Hence, we have proved that, indeed, there exists a fixed point $u_n \in \overline{\mathcal{B}}_\varrho$ of $\mathcal{T}$ such that both inequalities (9.8) and (9.9) are valid. So Lemma 9.2 can be applied and Theorem 9.1 is proved. □

10. Large critical points of $\mathcal{J}_\lambda$

In the previous two sections we have shown how to obtain a (weak) solution to the resonant problem (9.2) (i.e., when $\lambda = \lambda_1$), first for $p > 2$ (in Section 8) and then for $1 < p < 2$ (in Section 9). When $\lambda < \Lambda_\infty$ and $f \equiv \zeta\varphi_1 + f^\top$ with some $\zeta \in \mathbb{R}$ and $f^\top \in L^\infty(\Omega)^\top$, $f^\top \not\equiv 0$ in Ω, a refinement of the techniques from Sections 8 and 9 yields *multiple* critical points of $\mathcal{J}_\lambda$ for *suitable* combinations of the pair of parameters (λ, ζ) near $(\lambda_1, 0)$. Such critical points $u = \tau(\varphi_1 + v^\top)$ are distinguished from each other by the *size* of $\tau \in \mathbb{R}$; one has $v^\top \to 0$ in $C^1(\overline{\Omega})$ as $\tau \to \pm\infty$. Thus, such "large solutions" of the equation $\mathcal{J}_\lambda'(u) = 0$ need to be obtained. This is done as follows, using the function $j_\lambda \equiv j_\lambda(\,\cdot\,; f)\mathbb{R} \to \mathbb{R}$ and starting from the simpliest case $\lambda = \lambda_1$ and $\zeta = 0$.

The results obtained in Section 8 (Section 9, respectively) apply to this case if $p > 2$ ($1 < p < 2$). Indeed, from the end of Section 7 we recall $j_{\lambda_1}(\tau) = -c \cdot |\tau|^{2-p} + o(|\tau|^{2-p})$ as $\tau \to \pm\infty$, where $c = c(f^\top) > 0$ is a constant depending on $f^\top$. The function j_{λ_1} being continuous, it attains a global minimum (maximum, respectively) at $\tau_0 \in \mathbb{R}$ if $p > 2$ ($1 < p < 2$). We note that this scenario is a special case of what has been described at the beginning of Section 8 in cases (i) and (ii). Next, in $f \equiv \zeta\varphi_1 + f^\top$ the original choice of $\zeta = 0$ is perturbed to $\zeta \neq 0$ with $|\zeta|$ small enough. Since formula (7.2) yields

$$j_\lambda(\tau; f) = j_\lambda(\tau; f^\top) - \tau\zeta \, \|\varphi_1\|^2_{L^2(\Omega)}, \tag{10.1}$$

this means that for $\lambda = \lambda_1$ the second term on the right-hand side above, the linear function $\tau \mapsto \tau\zeta \, \|\varphi_1\|^2_{L^2(\Omega)}$, determines the asymptotic behavior of $j_{\lambda_1}(\tau; f)$ as $\tau \to \pm\infty$:

$$\begin{aligned} j_{\lambda_1}(\tau; f) &= -\, c(f^\top) \cdot |\tau|^{2-p} - \tau\zeta \, \|\varphi_1\|^2_{L^2(\Omega)} + o(|\tau|^{2-p}) \\ &= -\, |\tau|^{2-p} \Big(c(f^\top) + |\tau|^{p-2}\tau\zeta \, \|\varphi_1\|^2_{L^2(\Omega)} + o(1) \Big) \,. \end{aligned}$$

It is not difficult to see that if $|\zeta| > 0$ is small enough then j_{λ_1} possesses a local minimizer and a local maximizer, such that one of them stays in a bounded interval and the other tends to $+\infty$ or $-\infty$ as $\zeta \to 0$, depending on the signs of $p-2$ and ζ. Consequently, the local minimizer and maximizer are distinguished from each other by the size of their absolute values. Finally, keeping ζ constant (with $|\zeta| > 0$ small enough), we perturb $\lambda = \lambda_1$ to λ near λ_1 (with $|\lambda - \lambda_1| > 0$ small enough) in order to obtain a second local minimizer (local maximizer, respectively) for j_λ if $\lambda > \lambda_1$ ($\lambda < \lambda_1$), whose absolute value is even much larger than the absolute values of the local minimizers and maximizers constructed before for the case $\lambda = \lambda_1$. With some additional caution about the size of critical points, the (multiple) critical points of $\mathcal{J}_\lambda$ corresponding to the local minimizers and maximizers of j_λ are now obtained by the methods presented in Sections 8 and 9. We refer the interested reader to the article Takáč [35, Sect. 4 and 5] for (rather complicated) technical details.

An essential tool in the perturbation process just described is, of course, *continuous dependence* of the function $j_\lambda(\tau; \zeta\varphi_1 + f^\top)$ on *all* (real) variables τ, λ, and ζ, which is proved in Takáč [32, Lemma 7.2, p. 222].

11. A collection of main results

In order to give an idea to the reader interested in what kinds of results can be obtained by the techniques from Sections 8 and 9 applied to the energy functional $\mathcal{J}_\lambda$ (concerning multiple critical points and their classification), below we present a collection of the main results from Takáč [35, Sect. 2, pp. 698–705]. Some of them appeared before in [10, 11, 12, 13, 14, 20, 22, 26, 27, 29, 32, 33, 34].

Recall that we always assume that the domain $\Omega \subset \mathbb{R}^N$ satisfies hypothesis (H1).

The first (smallest) eigenvalue λ_1 of the positive Dirichlet p-Laplacian $-\Delta_p$ for $1 < p < \infty$ is given by formula (1.2). We recall from the beginning of Section 9 that the eigenvalue λ_1 is simple and the eigenfunction φ_1 associated with λ_1 can be normalized by $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_{L^p(\Omega)} = 1$. Furthermore, we have $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$

for some $\beta \in (0, \alpha)$, together with the Hopf maximum principle

$$\varphi_1 > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \varphi_1}{\partial \nu} < 0 \quad \text{on } \partial\Omega. \tag{9.1}$$

There, we have also introduced the set

$$U = \{x \in \Omega : \nabla \varphi_1(x) \neq \mathbf{0}\}.$$

Often, a function $u \in L^1(\Omega)$ will be decomposed as the orthogonal sum $u = u^{\parallel} \cdot \varphi_1 + u^{\top}$ according to (1.4). Given a linear subspace $\mathcal{M}$ of $L^1(\Omega)$ with $\varphi_1 \in \mathcal{M}$, we write

$$\mathcal{M}^{\top} \stackrel{\text{def}}{=} \{u \in \mathcal{M} : \langle u, \varphi_1 \rangle = 0\}.$$

The following concept is tailored for our treatment of the functional $\mathcal{J}_\lambda$ defined in (1.5); we recall

$$j_\lambda(\tau) \stackrel{\text{def}}{=} \min_{u^{\top} \in W_0^{1,p}(\Omega)^{\top}} \mathcal{J}_\lambda(\tau\varphi_1 + u^{\top}) \quad \text{for } \tau \in \mathbb{R}. \tag{7.3}$$

Definition 11.1. $u_0 \in W_0^{1,p}(\Omega)$ will be called a *simple saddle point* for $\mathcal{J}_\lambda$ if $u_0 = \tau_0\varphi_1 + u_0^{\top}$ is a critical point for $\mathcal{J}_\lambda$, $u_0^{\top}$ is a global minimizer for the restricted functional $u^{\top} \mapsto \mathcal{J}_\lambda(\tau_0\varphi_1 + u^{\top})$ on $W_0^{1,p}(\Omega)^{\top}$, and the function $j_\lambda : \mathbb{R} \to \mathbb{R}$ attains a local maximum at τ_0.

A more general type of a saddle point is obtained in Rabinowitz [30, Theorem 4.6, p. 24]. From now on we separate the cases $p > 2$ and $1 < p < 2$, respectively.

11.1. **The degenerate case** $2 < p < \infty$**.** Let $2 < p < \infty$. We introduce a new norm on $W_0^{1,p}(\Omega)$ by

$$\|v\|_{\varphi_1} \stackrel{\text{def}}{=} \left(\int_\Omega |\nabla\varphi_1|^{p-2} |\nabla v|^2 \,\mathrm{d}x\right)^{1/2} \quad \text{for } v \in W_0^{1,p}(\Omega), \tag{11.1}$$

and denote by $\mathcal{D}_{\varphi_1}$ the completion of $W_0^{1,p}(\Omega)$ with respect to this norm. That the seminorm (11.1) is in fact a norm on $W_0^{1,p}(\Omega)$ follows from an inequality in [32, ineq. (4.7), p. 200]. The Hilbert space $\mathcal{D}_{\varphi_1}$ coincides with the domain of the closure of the quadratic form $\mathcal{Q}_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} 2 \cdot \mathcal{Q}_0(\phi) = &\int_\Omega |\nabla\varphi_1|^{p-2} \Big\{ |\nabla\phi|^2 + (p-2) \Big| \frac{\nabla\varphi_1}{|\nabla\varphi_1|} \cdot \nabla\phi \Big|^2 \Big\} \,\mathrm{d}x \\ &- \lambda_1(p-1) \int_\Omega \varphi_1^{p-2} \phi^2 \,\mathrm{d}x, \quad \phi \in W_0^{1,p}(\Omega). \end{aligned} \tag{11.2}$$

We impose the following additional hypothesis on the domain Ω:

(H2) If $N \geq 2$ and $\partial\Omega$ is not connected, then there is *no* function $v \in \mathcal{D}_{\varphi_1}$, $\mathcal{Q}_0(v) = 0$, with the following four properties:
- (i) $v = \varphi_1 \cdot \chi_S$ a.e. in Ω, where $S \subset \Omega$ is Lebesgue measurable, $0 < |S|_N < |\Omega|_N$;
- (ii) $\overline{S}$ is connected and $\overline{S} \cap \partial\Omega \neq \emptyset$;
- (iii) if V is a connected component of U, then either $V \subset S$ or else $V \subset \Omega \setminus S$;
- (iv) $(\partial S) \cap \Omega \subset \Omega \setminus U$. (Recall $\Omega \setminus U = \{x \in \Omega : \nabla\varphi_1(x) = \mathbf{0}\}$.)

It has been *conjectured* in Takáč [32, §2.1] that (H2) always holds true provided (H1) is satisfied. The cases, when Ω is either an interval in $\mathbb{R}^1$ or else $\partial\Omega$ is connected if $N \geq 2$, have been covered within the proof of Proposition 4.4 in [32, pp. 202–205] which claims:

Proposition 11.2. *Let $2 < p < \infty$ and assume both hypotheses* (H1) *and* (H2)*. Then a function $u \in \mathcal{D}_{\varphi_1}$ satisfies $\mathcal{Q}_0(u) = 0$ if and only if $u = \kappa\varphi_1$ for some constant $\kappa \in \mathbb{R}$.*

In particular, there is no function $v \in \mathcal{D}_{\varphi_1}$, $\mathcal{Q}_0(v) = 0$, with properties (i)–(iv). This proposition is the only place where (H2) is needed explicitly. All other results in these notes depend solely on the conclusion of the proposition which, in turn, implies (H2).

We write $f \equiv f^\top + \zeta\varphi_1$ with $f^\top \in K$ and $\zeta \in \mathbb{R}$, where K is as follows:

- **(H3)** K is a nonempty, weakly-star compact set in $L^\infty(\Omega)$ such that $0 \notin K$ and $\langle g, \varphi_1\rangle = 0$ for all $g \in K$.

We begin by stating the following existence result which generalizes the existence part of Takáč [32, Theorem 2.2, p. 194].

Theorem 11.3. *There exist positive constants $\delta \equiv \delta(K)$, $\delta' \equiv \delta'(K)$, and $C(K)$ such that, whenever $\lambda \le \lambda_1 + \delta$, $f^\top \in K$, and $|\zeta| \le \delta'$, the functional $\mathcal{J}_\lambda$ possesses a local minimizer $u_1 \in W_0^{1,p}(\Omega)$ (hence, a weak solution to* (1.1)*) that satisfies $\|u_1\|_{C^{1,\beta}(\overline{\Omega})} \le C(K)$. Furthermore, if $\delta_0, \delta_0' > 0$ (in place of δ and δ') are arbitrary and $\lambda \le \lambda_1 - \delta_0$, $|\zeta| \le \delta_0'$, the same bound (depending also on δ_0 and δ_0') holds for any global minimizer of $\mathcal{J}_\lambda$.*

A variational proof of this theorem relies on the methods described in Sections 7 and 8, case (i). It can be found in [35, §6.1, pp. 738–739] (proof of Theorem 2.2 in [35]). For all $\lambda < \lambda_1$ and $\zeta \in \mathbb{R}$, this result follows from the coercivity of the energy functional $\mathcal{J}_\lambda$, whereas for $0 < \lambda - \lambda_1 \le \delta$ and any $\zeta \in \mathbb{R}$, it can be proved by a well-known argument employing topological degree; see Drábek [9, Theorem 14.18, p. 189]. Finally, for $\lambda = \lambda_1$ it has been established in Fleckinger and Takáč [20, Theorem 3.3] and Takáč [32, Theorem 2.2] if $\zeta = 0$, and in Takáč [33, Theorem 3.1] if $|\zeta| \le \delta'$.

Remark 11.4. The local minimizer $u_1 \in W_0^{1,p}(\Omega)$ described in Theorem 11.3 is the same as the one obtained in Theorems 11.5, 11.7, and 11.8 below.

Our second theorem is a multiplicity result for the resonant value $\lambda = \lambda_1$. Although it has been obtained originally in [33, Theorem 3.1], its present form (taken from [35, Theorem 2.4, p. 701]) is more specific about the qualitative description of solutions.

Theorem 11.5. *There exists a constant $\delta' \equiv \delta'(K) > 0$ such that problem* (1.1) *with $\lambda = \lambda_1$ and $f \equiv f^\top + \zeta\varphi_1$ has at least* two *(distinct) weak solutions u_1, u_2 specified as follows, whenever $f^\top \in K$ and $0 < |\zeta| \le \delta'$: Functional $\mathcal{J}_{\lambda_1}$ (which is unbounded from below) possesses a local minimizer $u_1 \in W_0^{1,p}(\Omega)$ and another critical point $u_2 \in W_0^{1,p}(\Omega)$.*

The proof of this theorem can be derived from that of Theorem 11.3; see [35, §6.2, pp. 739–741] (proof of Theorem 2.4 in [35]).

Remark 11.6. The critical point $u_2 \in W_0^{1,p}(\Omega)$ obtained in Theorem 11.5 above is constructed from a suitable pair of sub- and supersolutions to problem (1.1) satisfying (1.7) and (1.8), respectively, with $\lambda = \lambda_1$. This method is a refinement of the topological degree arguments described in Section 9. If the sub- and supersolutions coincide, then $u_2 = \underline{u} = \overline{u}$ is a simple saddle point for $\mathcal{J}_{\lambda_1}$, $u_2 = \tau_2\varphi_1 + u_2^\top$, and, moreover, j_{λ_1} is differentiable at τ_2 with vanishing derivative, $j_{\lambda_1}'(\tau_2) = 0$.

Analogous remarks apply to *all* critical points (other than local minimizers) obtained in our theorems below, Theorems 11.7 and 11.8 in this paragraph, and Theorems 11.13, 11.15, 11.16, and 11.17 in the next one (§11.2).

The following two theorems on the existence of at least *three* solutions to the Dirichlet problem (1.1) are the main new results for $p > 2$ obtained in [35]. First, we consider the subcritical case $\lambda_1 - \delta \le \lambda < \lambda_1$.

Theorem 11.7. *There exists a constant $\delta' \equiv \delta'(K) > 0$ such that, for any $d \in (0, \delta')$, there is another constant $\delta \equiv \delta(K, d) > 0$ such that problem* (1.1) *with $f \equiv f^\top + \zeta\varphi_1$ has at least* three *(pairwise distinct) weak solutions u_1, u_2, u_3 specified as follows, whenever $\lambda_1 - \delta \le \lambda < \lambda_1$, $f^\top \in K$, and $d \le |\zeta| \le \delta'$: Functional $\mathcal{J}_\lambda$ (which is bounded from below) possesses* two *local minimizers $u_1, u_2 \in W_0^{1,p}(\Omega)$ of which at least one is global, and another critical point $u_3 \in W_0^{1,p}(\Omega)$.*

The proof of this theorem is derived from those of Theorems 11.3 and 11.5; see [35, §6.3, pp. 742–744] (proof of Theorem 2.6 in [35]).

Finally, we treat the supercritical case $\lambda_1 < \lambda \le \lambda_1 + \delta$. Here we obtain the following multiplicity and uniform boundedness results for problem (1.1):

Theorem 11.8. *There exist constants $\delta \equiv \delta(K) > 0$ and $\delta' \equiv \delta'(K) > 0$ such that problem* (1.1) *with $f \equiv f^\top + \zeta\varphi_1$ has at least* three *(pairwise distinct) weak solutions u_1, u_2, u_3 specified as follows, whenever $\lambda_1 < \lambda \le \lambda_1 + \delta$, $f^\top \in K$, and $|\zeta| \le \delta'$: Functional $\mathcal{J}_\lambda$ (which is unbounded from below) possesses a local minimizer $u_1 \in W_0^{1,p}(\Omega)$ and* two *other critical points $u_2, u_3 \in W_0^{1,p}(\Omega)$.*

The proof of this theorem is derived from that of Theorem 11.3; see [35, §6.4, pp. 745–746] (proof of Theorem 2.7 in [35]).

Remark 11.9. In Theorem 11.5, the orthogonal projections of u_1 and u_2 onto $\mathrm{lin}\{\varphi_1\}$ satisfy $u_1^\parallel < u_2^\parallel$ if $\zeta > 0$, and $u_1^\parallel > u_2^\parallel$ if $\zeta < 0$. In Theorem 11.7, $u_3^\parallel$ lies between $u_1^\parallel$ and $u_2^\parallel$, and in Theorem 11.8, $u_1^\parallel$ lies between $u_2^\parallel$ and $u_3^\parallel$.

For the question of *boundedness* of the solution set for problem (1.1) with λ near λ_1, we refer to Drábek et al. [13] and Takáč [32, Sect. 2] and [33, Prop. 6.1]. Although the variational methods developed in our present lecture notes are clearly not suitable for resolving this question, in the proofs of all our theorems one makes essential use of [33, Prop. 6.1] (stated as Proposition B.4 in the Appendix, §B.2) which in turn provides the following answer in the special case $\lambda = \lambda_1$; see [33, Theorem 3.2]:

Theorem 11.10. *Let $f^\top$ be as in* Theorem 11.3 *above. If $\zeta = 0$ then the set of all weak solutions to problem* (1.1) *with $\lambda = \lambda_1$ is bounded in $C^{1,\beta}(\overline{\Omega})$. Given any $\delta > 0$, this set is bounded in $C^{1,\beta}(\overline{\Omega})$ uniformly for all $|\zeta| \ge \delta$ as well.*

This theorem is proved in [35, §6.5, p. 747] (proof of Theorem 2.9 in [35]).

In contrast with Theorem 11.3, if $|\zeta|$ in $f \equiv f^\top + \zeta\varphi_1$ is "too large" relative to the size of $\|f^\top\|_{L^\infty(\Omega)}$, say $|\zeta| \ge \delta > 0$, then problem (1.1) with $\lambda = \lambda_1$ has *no* weak solution; see [32, Corollary 2.4, p. 195] or [33, Theorem 3.1]:

Corollary 11.11. *Given an arbitrary function $g \in L^\infty(\Omega)$ with $0 \le g \not\equiv 0$ in Ω, there exists a constant $\gamma \equiv \gamma(g) > 0$ with the following property: If $f \in L^\infty(\Omega)$, $f \not\equiv 0$, is such that*

$$f = f^g \cdot g + \bar{f}^g \quad \textit{with some } \ f^g \in \mathbb{R} \quad \textit{and } \ \bar{f}^g \in L^\infty(\Omega),$$

and $\|\bar{f}^g\|_{L^\infty(\Omega)} \le \gamma\,|f^g|$, *then problem* (1.1) *with* $\lambda = \lambda_1$ *has no weak solution.*

Equivalently, given g as above, notice that there is an open cone $\mathcal{C}$ in $L^\infty(\Omega)$ with vertex at the origin ($0 \notin \mathcal{C}$) such that $g \in \mathcal{C}$ and problem (1.1) with $\lambda = \lambda_1$ has no weak solution whenever $f \in \mathcal{C}$. This result improves a nonexistence result due to [17, Théorème 1] (see also [31, Theorem 7.2, p. 154]) for $0 \le f \not\equiv 0$ in Ω.

As already mentioned in the Introduction, one needs a number of auxiliary results to prove these theorems. Complete proofs can be found in Takáč [35, Sect. 6, pp. 738–747]. Additional results revealing more details about the structure of solutions to problem (1.1) have been established within these proofs, e.g., the positivity or negativity of solutions with a sufficiently large norm.

11.2. **The singular case** $1 < p < 2$**.** We further require hypothesis (H1); (H2) will be replaced by a hypothesis on f. In fact, hypothesis (H2) always holds true in this case; see Takáč [32, Sect. 8, p. 225].

Remark 11.12. It is not difficult to verify that the conclusion of Proposition 11.2 remains valid also for $1 < p < 2$, by [32, Remark 8.1, p. 225].

The Hilbert space $\mathcal{D}_{\varphi_1}$, endowed with the norm (11.1) for $p > 2$, needs to be redefined for $1 < p < 2$ as follows: We define $v \in \mathcal{D}_{\varphi_1}$ if and only if $v \in W_0^{1,2}(\Omega)$, $\nabla v(x) = \mathbf{0}$ for almost every $x \in \Omega \setminus U = \{x \in \Omega : \nabla\varphi_1(x) = \mathbf{0}\}$, and

$$\|v\|_{\varphi_1} \stackrel{\text{def}}{=} \Big(\int_U |\nabla\varphi_1|^{p-2}|\nabla v|^2 \,\mathrm{d}x\Big)^{1/2} < \infty. \tag{11.3}$$

Consequently, $\mathcal{D}_{\varphi_1}$ endowed with the norm $\|\cdot\|_{\varphi_1}$ is continuously embedded into $W_0^{1,2}(\Omega)$. We *conjecture* that $\mathcal{D}_{\varphi_1}$ is dense in $L^2(\Omega)$. This conjecture would immediately follow from $|\Omega \setminus U|_N = 0$. The latter holds true if Ω is convex; then also $\Omega \setminus U$ is a convex set in $\mathbb{R}^N$ with empty interior, and hence of zero Lebesgue measure, see [18, Lemma 2.6, p. 55].

If the conjecture is false, we need to consider also the orthogonal complement

$$\mathcal{D}_{\varphi_1}^{\perp,L^2} = \{v \in L^2(\Omega) : \langle v, \phi\rangle = 0 \text{ for all } \phi \in \mathcal{D}_{\varphi_1}\}.$$

Notice that $v \in \mathcal{D}_{\varphi_1}^{\perp,L^2}$ implies $v = 0$ almost everywhere in U. This means that $\mathcal{D}_{\varphi_1}^{\perp,L^2}$ is isometrically isomorphic to a closed linear subspace of $L^2(\Omega\setminus U)$. Moreover, $\chi_{\Omega\setminus U} \notin \mathcal{D}_{\varphi_1}^{\perp,L^2}$ since $\Omega \setminus U$ is a compact subset of Ω; hence, there is a C^1 function $\phi \in \mathcal{D}_{\varphi_1}$, $0 \le \phi \le 1$, with compact support in Ω and such that $\phi = 1$ in an open neighborhood of $\Omega \setminus U$.

As above, we write $f \equiv f^\top + \zeta\varphi_1$ with $f^\top \in K$ and $\zeta \in \mathbb{R}$, where our hypothesis on K below admits the possibility $\mathcal{D}_{\varphi_1}^{\perp,L^2} \ne \{0\}$.

- (H3') K is a nonempty, weakly-star compact set in $L^\infty(\Omega)$ such that $K \cap \mathcal{D}_{\varphi_1}^{\perp,L^2} = \emptyset$ and $\langle g, \varphi_1\rangle = 0$ for all $g \in K$.

The condition $g \notin \mathcal{D}_{\varphi_1}^{\perp,L^2}$ is satisfied if g is continuous in Ω and $g \not\equiv 0$. Our first result for $p < 2$ below is an analogue of Theorem 11.3; it generalizes the existence part of Drábek and Holubová [14, Theorem 1.1, p. 184] and Takáč [32, Theorem 2.6, p. 196]. We assume that both hypotheses (H1) and (H3') are satisfied.

Theorem 11.13. *There exist positive constants* $\delta \equiv \delta(K)$, $\delta' \equiv \delta'(K)$, *and* $C(K)$ *such that, whenever* $|\lambda - \lambda_1| \le \delta$, $f^\top \in K$, *and* $|\zeta| \le \delta'$, *the functional* $\mathcal{J}_\lambda$ *possesses a critical point* $u_1 \in W_0^{1,p}(\Omega)$ *(hence, a weak solution to* (1.1)*) that* *satisfies* $\|u_1\|_{C^{1,\beta}(\overline{\Omega})} \leq C(K)$*. Furthermore, if* $\delta_0, \delta_0' > 0$ *(in place of* δ *and* δ'*) are arbitrary and* $\lambda \leq \lambda_1 - \delta_0$*,* $|\zeta| \leq \delta_0'$*, the same bound (depending also on* δ_0 *and* δ_0'*) holds for any global minimizer of* $\mathcal{J}_\lambda$*.*

A proof is given in [35, §7.1, pp. 747–749] (proof of Theorem 2.12 in [35]). It is based on the topological (Leray-Schauder) degree as described in Section 9. It has been originally taken from [32, §8.4, p. 229] and is analogous to that of Theorem 11.3. For all $\lambda \leq \lambda_1 + \delta$, $\lambda \neq \lambda_1$, and $\zeta \in \mathbb{R}$, this result can be proved in the same way as for $p > 2$; see [9, Theorem 14.18, p. 189]. For $\lambda = \lambda_1$ it was established in [14, Theorem 1.1] and [33, Theorem 3.5] if $|\zeta| \leq \delta'$ (by completely different methods), and in [32, Theorem 2.6] if $\zeta = 0$ (by the same variational method we use here).

Remark 11.14. The critical point $u_1 \in W_0^{1,p}(\Omega)$ described in Theorem 11.13 is the same as the one obtained in Theorems 11.15, 11.16, and 11.17 below.

We recall that Remark 11.6 applies also to *all* critical points (other than local minimizers) obtained in our theorems throughout this paragraph, Theorems 11.13, 11.15, 11.16, and 11.17.

Again, our second theorem for $p < 2$ is a multiplicity result for $\lambda = \lambda_1$ taken from [33, Theorem 3.5] in a more specific form obtained in [35, Theorem 2.14, p. 704].

Theorem 11.15. *There exists a constant* $\delta' \equiv \delta'(K) > 0$ *such that problem* (1.1) *with* $\lambda = \lambda_1$ *and* $f \equiv f^\top + \zeta\varphi_1$ *has at least* two *(distinct) weak solutions* u_1, u_2 *specified as follows, whenever* $f^\top \in K$ *and* $0 < |\zeta| \leq \delta'$*: Functional* $\mathcal{J}_{\lambda_1}$ *(which is unbounded from below) possesses a critical point* $u_1 \in W_0^{1,p}(\Omega)$ *and a local minimizer* $u_2 \in W_0^{1,p}(\Omega)$*.*

The proof of this theorem can be derived from that of Theorem 11.13; see [35, §7.2, pp. 749–750] (proof of Theorem 2.14 in [35]).

The following theorem on the existence of at least *three* solutions to the Dirichlet problem (1.1) in the subcritical case $\lambda_1 - \delta \leq \lambda < \lambda_1$ is a generalization of [32, Theorem 2.7, p. 196] where it is established for $\zeta = 0$ only.

Theorem 11.16. *There exist constants* $\delta \equiv \delta(K) > 0$ *and* $\delta' \equiv \delta'(K) > 0$ *such that problem* (1.1) *with* $f \equiv f^\top + \zeta\varphi_1$ *has at least* three *(pairwise distinct) weak solutions* u_1, u_2, u_3 *specified as follows, whenever* $\lambda_1 - \delta \leq \lambda < \lambda_1$*,* $f^\top \in K$*, and* $|\zeta| \leq \delta'$*: Functional* $\mathcal{J}_\lambda$ *(which is bounded from below) possesses a critical point* $u_1 \in W_0^{1,p}(\Omega)$ *and* two *(distinct) local minimizers* $u_2, u_3 \in W_0^{1,p}(\Omega)$ *of which at least one is global.*

Again, the proof of this theorem is derived from that of Theorem 11.13; see [35, §7.3, p. 751] (proof of Theorem 2.15 in [35]).

Our last theorem on the existence of at least *three* solutions to problem (1.1) appeared for the first time in Takáč [35, Theorem 2.16, p. 705]. Here we consider the supercritical case $\lambda_1 < \lambda \leq \lambda_1 + \delta$.

Theorem 11.17. *There exists a constant* $\delta' \equiv \delta'(K) > 0$ *such that, for any* $d \in (0, \delta')$*, there is another constant* $\delta \equiv \delta(K, d) > 0$ *such that problem* (1.1) *with* $f \equiv f^\top + \zeta\varphi_1$ *has at least* three *(pairwise distinct) weak solutions* u_1, u_2, u_3 *specified as follows, whenever* $\lambda_1 < \lambda \leq \lambda_1 + \delta$*,* $f^\top \in K$*, and* $d \leq |\zeta| \leq \delta'$*: Functional* $\mathcal{J}_\lambda$ *(which is unbounded from below) possesses* two *(distinct) critical points* $u_1, u_2 \in W_0^{1,p}(\Omega)$ *and a local minimizer* $u_3 \in W_0^{1,p}(\Omega)$*.*

The proof is derived from those of Theorems 11.13 and 11.16, see [35, §7.4, pp. 751–754] (proof of Theorem 2.16 in [35]).

Remark 11.18. In Theorem 11.15, the orthogonal projections of u_1 and u_2 on $\mathrm{lin}\{\varphi_1\}$ satisfy $u_1^{\|} < u_2^{\|}$ if $\zeta < 0$, and $u_1^{\|} > u_2^{\|}$ if $\zeta > 0$. In Theorem 11.16, $u_1^{\|}$ lies between $u_2^{\|}$ and $u_3^{\|}$, and in Theorem 11.17, $u_3^{\|}$ lies between $u_1^{\|}$ and $u_2^{\|}$.

Under the same hypotheses, we obtain the corresponding uniform boundedness result [33, Theorem 3.6]; we refer to [13], [32, Sect. 2], and [33, Prop. 6.1] for additional results:

Theorem 11.19. *The conclusion of* Theorem 11.10 *is valid also for* $1 < p < 2$.

The proof of this theorem is identical with the proof of Theorem 11.10 above ([35, §7.5, p. 754], proof of Theorem 2.18 in [35]).

Finally, the nonexistence for $f \equiv f^{\top} + \zeta\varphi_1$ with $|\zeta|$ large enough has been proved in [14, Theorem 1.1, p. 184], [32, Corollary 2.9, p. 197], or [33, Theorem 3.5] in various ways:

Corollary 11.20. *Let $g \in L^\infty(\Omega)$ be an arbitrary function such that $g \geq 0$ in Ω and $g \not\equiv 0$. Then the conclusion of* Corollary 11.11 *is valid also for* $1 < p < 2$.

Complete proofs of all these results can be found in Takáč [35, Sect. 7, pp. 747–754].

12. Discussion

The variational method used in this work can be applied to finding critical points of some functionals of the following more general type,

$$\mathcal{J}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega F(x, u(x)) \,\mathrm{d}x \tag{12.1}$$

on $W_0^{1,p}(\Omega)$. This functional corresponds to the "spectral" Dirichlet problem (1.10), considered in the same setting as problem (1.1), with

$$F(x,u) \stackrel{\text{def}}{=} \int_0^u f(x,t)\,\mathrm{d}t \quad \text{for } x \in \Omega \text{ and } u \in \mathbb{R}.$$

The reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function of Carathéodory type with suitable growth or decay properties as $|u| \to \infty$. For instance, one may assume $f(\,\cdot\,, u) \stackrel{*}{\rightharpoonup} f_\infty$ weakly-star in $L^\infty(\Omega)$ as $|u| \to \infty$, where $f_\infty \not\equiv 0$ in Ω.

For $p = 2$ both the resonant and nonresonant cases of problem (1.10) have been studied in numerous works; see [9, 12, 22] for references. In contrast, the more difficult case $p \neq 2$ has been investigated less intensively [7, 9, 12, 22, 28, 29, 31, 40, 41]. For technical reasons (e.g., complicated asymptotic expressions), in the present work we have treated only the special case when $f(x,u)$ is independent from the (unknown) state variable u, i.e., $f(x,u) \equiv f(x)$ for a.e. $x \in \Omega$, where $f \in L^\infty(\Omega)$ with $f \not\equiv 0$ in Ω. The general problem (1.10) can be treated similarly provided f satisfies

$$f(x,u)/|u|^{p-1} \to 0 \quad \text{as } |u| \to \infty \text{ uniformly for } x \in \Omega. \tag{12.2}$$

Our method applies to a number of related problems at resonance for λ near λ_1 that have to be treated individually. We refer the interested reader to [12] and [22].

So let us assume the asymptotic growth condition (12.2). Obviously, all what is needed is the asymptotic behavior of the function $j_{\lambda_1} : \mathbb{R} \to \mathbb{R}$ near $\pm\infty$, i.e., some analogue of the formula

$$|\tau|^{p-2} \cdot j_{\lambda_1}(\tau; f) \to -\mathcal{Q}_0(w, w) \quad \text{as } |\tau| \to \infty\,. \tag{12.3}$$

Here, $w \in \mathcal{D}_{\varphi_1}^{\top}$ is the unique weak solution of problem (B.12) if $p > 2$, and (B.13) if $p < 2$. (We refer to [35], Lemmas 5.2 (p. 736) and Lemmas 5.3 (p. 737), respectively.) With regard to Proposition B.4, this means investigating sequences of large solutions,

$$u_n = t_n^{-1}\varphi_1 + u_n^{\top} = t_n^{-1}(\varphi_1 + v_n^{\top}) \text{ with } t_n \to 0 \text{ and } \|v_n^{\top}\|_{C^{1,\beta'}(\overline{\Omega})} \to 0 \text{ as } n \to \infty,$$

to the following generalized version of problem (B.16), see §B.2 (Appendix):

$$\begin{aligned} &- \Delta_p(t^{-1}\varphi_1 + u^{\top}) - \lambda_1|t^{-1}\varphi_1 + u^{\top}|^{p-2}(t^{-1}\varphi_1 + u^{\top}) \\ &= f(x, u(x))^{\top} + \zeta \cdot \varphi_1(x) \quad \text{in } \Omega; \\ &\qquad\qquad u^{\top} = 0 \quad \text{on } \partial\Omega; \\ &\qquad\qquad \langle u^{\top}, \varphi_1\rangle = 0. \end{aligned} \tag{12.4}$$

After a careful inspection of the proof of Proposition B.4 one finds out that only Theorem B.2 is needed. Although we will not provide formal proofs of our claims in the present lecture notes, all these auxiliary results can be established without major changes provided f satisfies the following two conditions, cf. Girg and Takáč [22, §2.3, hypothesis (H'_∞)]:

- (**f1**) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function of Carathéodory type such that the function $u \mapsto f(\cdot\, u)$ maps bounded intervals in $\mathbb{R}$ into bounded sets in $L^\infty(\Omega)$.
- (**f2**) f satisfies $f(\cdot, u)/\theta(u) \overset{*}{\rightharpoonup} f_{\pm\infty}$ weakly-star in $L^\infty(\Omega)$ as $u \to \pm\infty$, where $f_{\pm\infty} \not\equiv 0$ in Ω, and *either* $\theta(u) \equiv 1$ for $u \in \mathbb{R}$, *or else* $\theta : \mathbb{R} \to \mathbb{R}$ is some C^1 function such that $\theta(0) = 0$, $\theta'(u) \neq 0$ for $u \neq 0$,
$$\lim_{|u|\to\infty} \big(\theta(u)/|u|^{p-1}\big) = 0, \quad \sup_{u\neq 0} |\theta'(u)\, u/\theta(u)| < \infty,$$
and
$$\theta(\tau\varphi_1(x))/\theta(\tau) \to \theta_{\pm\infty}(x) \quad \text{uniformly for } x \in \Omega \text{ as } \tau \to \pm\infty,$$
with $\theta_{\pm\infty} > 0$ in Ω.

In particular, the expression $V_n = t_n^{1-p} v_n^{\top}$ (for $\theta \equiv 1$) in Theorem B.2 becomes

$$V_n = \big[t_n^{p-1}\theta(t_n^{-1})\big]^{-1}\, v_n^{\top}.$$

Notice that $t_n \to 0$ entails $t_n^{p-1}\theta(t_n^{-1}) \to 0$. Of course, first, formula (B.18) has to be recalculated, and then also formula (12.3).

In contrast with the case $\theta \equiv 1$, two obvious nontrivial examples when f satisfies conditions (f1) and (f2) are as follows:

$$f(x, u) = f_\infty(x)\, |u|^{q-1} + f_0(x, u); \quad x \in \Omega,\ u \in \mathbb{R},$$

and

$$f(x, u) = f_\infty(x)\, |u|^{q-2}u + f_0(x, u); \quad x \in \Omega,\ u \in \mathbb{R},$$

where q is a constant, $1 < q < p$, $f_\infty \in L^\infty(\Omega)$, $f_\infty \not\equiv 0$ in Ω, and f_0 satisfies condition (f1) together with

$$f_0(x,u)/|u|^{q-1} \to 0 \quad \text{as } |u| \to \infty \text{ uniformly for } x \in \Omega.$$

Consequently, in condition (f2) we may take $\theta(u) = |u|^{q-1}$ in the former case and $\theta(u) = |u|^{q-2}u$ in the latter one.

The Neumann boundary conditions, $\partial u/\partial\nu = 0$ on $\partial\Omega$, can be treated as well by replacing the underlying space $W_0^{1,p}(\Omega)$ by $W^{1,p}(\Omega)$. Then $\lambda_1 = 0$ and $\varphi_1 \equiv \text{const} = 1/|\Omega|_N^{1/p}$ in Ω. Hence, the energy functional (12.1) becomes

$$\begin{aligned} \mathcal{J}_\lambda(u) &\equiv \mathcal{J}_\lambda(\tau + u^\top) \\ &\stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u^\top|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |\tau + u^\top|^p \,\mathrm{d}x - \int_\Omega F(x, \tau + u^\top(x)) \,\mathrm{d}x \end{aligned} \tag{12.5}$$

for $u \equiv \tau + u^\top \in W^{1,p}(\Omega)$, where $\tau \in \mathbb{R}$ and $u^\top \in W^{1,p}(\Omega)$ satisfies $\int_\Omega u^\top \,\mathrm{d}x = 0$. Consequently, the resonant case $\lambda = \lambda_1 = 0$ is somewhat easier to treat than for the Dirichlet boundary conditions. For instance, setting $u^\top \equiv 0$ in Ω we get

$$j_0(\tau) \le \mathcal{J}_0(\tau) = -\int_\Omega F(x,\tau) \,\mathrm{d}x \quad \text{for } \tau \in \mathbb{R}.$$

Thus, as $|\tau| \to \infty$, if $\int_\Omega F(x,\tau) \,\mathrm{d}x \to +\infty$ then $j_0(\tau) \to -\infty$.

Appendix A. Some auxiliary functional-analytic results

A.1. **Linearization and quadratic forms.** In order to determine the asymptotic behavior of the function $j_{\lambda_1}(\tau)$ as $|\tau| \to \infty$ which has been introduced in eq. (7.3), we will estimate the functional $u^\top \mapsto \mathcal{J}_{\lambda_1}(\tau\varphi_1 + u^\top)$ by suitable quadratic forms. We need to compute the first two Fréchet derivatives of the functional $\mathcal{J}_{\lambda_1}$, see [32, Sect. 3, p. 197]. Define

$$\mathcal{F}(u) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x, \quad u \in W_0^{1,p}(\Omega). \tag{A.1}$$

The first Fréchet derivative $\mathcal{F}'(u)$ of $\mathcal{F}$ at $u \in W_0^{1,p}(\Omega)$ is given by $\mathcal{F}'(u) = -\Delta_p u$ in $W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. The second Fréchet derivative $\mathcal{F}''(u)$ is a bit more complicated; if $1 < p < 2$, it might have to be considered only as a Gâteaux derivative which is not even densely defined: For all $\phi, \psi \in W_0^{1,p}(\Omega)$, one has (if $2 \le p < \infty$)

$$\begin{aligned} &\langle \mathcal{F}''(u)\psi, \phi\rangle = \\ &\int_\Omega |\nabla u|^{p-2} \left\{ (\nabla\phi \cdot \nabla\psi) + (p-2)|\nabla u|^{-2}(\nabla u \cdot \nabla\phi)(\nabla u \cdot \nabla\psi) \right\} \mathrm{d}x \\ &= \int_\Omega |\nabla u|^{p-2} \Big\langle I + (p-2)\,\frac{\nabla u \otimes \nabla u}{|\nabla u|^2},\, \nabla\phi \otimes \nabla\psi \Big\rangle_{\mathbb{R}^{N\times N}} \mathrm{d}x. \end{aligned} \tag{A.2}$$

Here, I is the identity matrix in $\mathbb{R}^{N\times N}$, $\mathbf{a}\otimes\mathbf{b}$ is the $(N\times N)$-matrix $\mathbf{T} = (a_i b_j)_{i,j=1}^N$ for $\mathbf{a} = (a_i)_{i=1}^N$, $\mathbf{b} = (b_i)_{i=1}^N \in \mathbb{R}^N$, and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}^{N\times N}}$ is the Euclidean inner product in $\mathbb{R}^{N\times N}$.

For $\mathbf{a} \in \mathbb{R}^N$ ($\mathbf{a} = \nabla u$ in our case), $\mathbf{a} \ne \mathbf{0} \in \mathbb{R}^N$, we abbreviate

$$\mathbf{A}(\mathbf{a}) \stackrel{\text{def}}{=} |\mathbf{a}|^{p-2}\Big(I + (p-2)\,\frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2}\Big). \tag{A.3}$$

If $p > 2$, we set also $\mathbf{A}(\mathbf{0}) \stackrel{\text{def}}{=} \mathbf{0} \in \mathbb{R}^{N\times N}$. For $\mathbf{a} \neq \mathbf{0}$, $\mathbf{A}(\mathbf{a})$ is a positive definite, symmetric matrix. The spectrum of $|\mathbf{a}|^{2-p}\mathbf{A}(\mathbf{a})$ consists of eigenvalues 1 and $p-1$, whence

$$\min\{1,\, p-1\} \leq \frac{\langle \mathbf{A}(\mathbf{a})\mathbf{v}, \mathbf{v}\rangle_{\mathbb{R}^N}}{|\mathbf{a}|^{p-2}|\mathbf{v}|^2} \leq \max\{1,\, p-1\}, \quad \mathbf{a}, \mathbf{v} \in \mathbb{R}^N \setminus \{\mathbf{0}\}. \tag{A.4}$$

From this point on, until the end of this paragraph, we restrict ourselves to $p \geq 2$. The case $1 < p < 2$ will be taken care of in the next paragraph, §A.2. We rewrite the p-homogeneous part of the functional (1.5) with $\lambda = \lambda_1$ as follows [32, eq. (4.1), p. 198]:

$$\begin{aligned}
&\mathcal{J}_{\lambda_1}(\varphi_1 + \phi) + \langle f, \varphi_1 + \phi\rangle \\
&= \frac{1}{p}\int_\Omega |\nabla(\varphi_1 + \phi)|^p \,\mathrm{d}x - \frac{\lambda_1}{p}\int_\Omega |\varphi_1 + \phi|^p \,\mathrm{d}x \\
&= \int_0^1 \int_\Omega |\nabla(\varphi_1 + s\phi)|^{p-2}\nabla(\varphi_1 + s\phi)\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \lambda_1 \int_0^1 \int_\Omega |\varphi_1 + s\phi|^{p-2}(\varphi_1 + s\phi)\phi \,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{A.5}$$

for all $\phi \in W_0^{1,p}(\Omega)$. Similarly, using (A.2), we get

$$\mathcal{J}_{\lambda_1}(\varphi_1 + \phi) + \langle f, \varphi_1 + \phi\rangle = \mathcal{Q}_\phi(\phi, \phi), \tag{A.6}$$

where $\mathcal{Q}_\phi$ is the symmetric bilinear form on $[W_0^{1,p}(\Omega)]^2$ defined as follows, using (A.3):

$$\begin{aligned}
\mathcal{Q}_\phi(v, w) \stackrel{\text{def}}{=} &\int_\Omega \Big\langle \Big[\int_0^1 \mathbf{A}(\nabla(\varphi_1 + s\phi))(1-s)\,\mathrm{d}s\Big]\nabla v,\, \nabla w\Big\rangle_{\mathbb{R}^N} \mathrm{d}x \\
&- \lambda_1(p-1)\int_\Omega \Big[\int_0^1 |\varphi_1 + s\phi|^{p-2}(1-s)\,\mathrm{d}s\Big] vw \,\mathrm{d}x
\end{aligned} \tag{A.7}$$

for $v, w \in W_0^{1,p}(\Omega)$. In particular, one has (cf. (11.2))

$$2\cdot\mathcal{Q}_0(v, v) = \int_\Omega \langle \mathbf{A}(\nabla\varphi_1)\nabla v, \nabla v\rangle_{\mathbb{R}^N} \,\mathrm{d}x - \lambda_1(p-1)\int_\Omega \varphi_1^{p-2} v^2 \,\mathrm{d}x.$$

Furthermore, equations (1.2) and (A.6) guarantee (see [32, ineq. (4.4), p. 199])

$$\mathcal{Q}_0(\phi, \phi) \geq 0 \quad \text{for all } \phi \in W_0^{1,p}(\Omega). \tag{A.8}$$

Form $\mathcal{Q}_0$ is closable in $L^2(\Omega)$ and the domain of its closure is $\mathcal{D}_{\varphi_1}$ (see [32, Sect. 4, p. 201]).

A.2. **The weighted Sobolev space $\mathcal{D}_{\varphi_1}$.** We set $\mathbb{R}_+ = [0, \infty)$ and begin with a few inequalities from Takáč [32, Lemma A.1, p. 233]. Let $1 < p < \infty$ and $p \neq 2$. Assume that $\Theta \in L^\infty(0,1)$ satisfies $\Theta \geq 0$ in $(0,1)$ and $T = \int_0^1 \Theta(s)\,\mathrm{d}s > 0$. Then there exists a constant $c_p(\Theta) > 0$ such that the following inequalities hold true for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$: If $p > 2$ then

$$\begin{aligned}
c_p(\Theta)^{p-2}\Big(\max_{0\leq s\leq 1} |\mathbf{a} + s\mathbf{b}|\Big)^{p-2} &\leq \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2}\,\Theta(s)\,\mathrm{d}s \\
&\leq T\cdot\Big(\max_{0\leq s\leq 1} |\mathbf{a} + s\mathbf{b}|\Big)^{p-2},
\end{aligned} \tag{A.9}$$

and if $1 < p < 2$ and $|\mathbf{a}| + |\mathbf{b}| > 0$ then

$$\begin{aligned} T \cdot \Big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2} &\le \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2}\, \Theta(s)\, \mathrm{d}s \\ &\le c_p(\Theta)^{p-2} \Big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2}. \end{aligned} \tag{A.10}$$

The inequalities below are a combination of (A.4) with (A.9) (if $p > 2$) and (A.10) (if $p < 2$), see [32, Lemma A.2]: If $p > 2$ then

$$\begin{aligned} c_p(\Theta)^{p-2} \Big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2} |\mathbf{v}|^2 &\le \int_0^1 \langle \mathbf{A}(\mathbf{a} + s\mathbf{b})\mathbf{v}, \mathbf{v}\rangle\, \Theta(s)\, \mathrm{d}s \\ &\le (p-1) T \cdot \Big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2} |\mathbf{v}|^2 \end{aligned} \tag{A.11}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{v} \in \mathbb{R}^N$, and if $1 < p < 2$ and $|\mathbf{a}| + |\mathbf{b}| > 0$ then

$$\begin{aligned} (p-1) T \cdot \Big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2} |\mathbf{v}|^2 &\le \int_0^1 \langle \mathbf{A}(\mathbf{a} + s\mathbf{b})\mathbf{v}, \mathbf{v}\rangle\, \Theta(s)\, \mathrm{d}s \\ &\le c_p(\Theta)^{p-2} \Big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2} |\mathbf{v}|^2. \end{aligned} \tag{A.12}$$

Several important properties of $\mathcal{D}_{\varphi_1}$ established in Takáč [32] are listed below.

The following claim is obvious ([32, Lemma 4.1]): If $1 < p < \infty$, $p \neq 2$, and if (H1) is satisfied, then one has $\mathcal{Q}_0(\varphi_1, \varphi_1) = 0$ and $0 \le \mathcal{Q}_0(v, v) < \infty$ for all $v \in \mathcal{D}_{\varphi_1}$.

Now we need to distinguish between the cases $p > 2$ and $1 < p < 2$. Assume $2 < p < \infty$ together with (H1). Notice that inequality (A.4) entails

$$\|v\|_{\varphi_1}^2 \le \int_\Omega \langle \mathbf{A}(\nabla\varphi_1)\nabla v, \nabla v\rangle_{\mathbb{R}^N}\, \mathrm{d}x \le (p-1)\|v\|_{\varphi_1}^2 \quad \text{for } v \in \mathcal{D}_{\varphi_1}. \tag{A.13}$$

For $0 < \delta < \infty$, we denote by

$$\Omega_\delta \overset{\text{def}}{=} \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\} \tag{A.14}$$

the δ-neighborhood of $\partial\Omega$. Its complement in Ω is denoted by $\Omega'_\delta = \Omega \setminus \Omega_\delta$.

The following compact embedding result is proved in [32, Lemma 4.2, p. 199].

Lemma A.1. *Let $2 < p < \infty$ and assume that hypothesis* (H1) *is satisfied. Then we have:*

(a) *For every $\delta > 0$ small enough, $\|\cdot\|_{\varphi_1}$ is an equivalent norm on $W_0^{1,2}(\Omega_\delta)$.*
(b) *The embedding $\mathcal{D}_{\varphi_1} \hookrightarrow L^2(\Omega)$ is compact.*

Due to inequality (A.13) combined with Lemma A.1, Part (b) above, we can extend the domain of $\mathcal{Q}_0$ to $\mathcal{D}_{\varphi_1} \times \mathcal{D}_{\varphi_1}$. This extension of $\mathcal{Q}_0$ is unique and closed in $L^2(\Omega)$. We denote by $\mathcal{A}_{\varphi_1}$ the Friedrichs representation of the quadratic form $2 \cdot \mathcal{Q}_0$ in $L^2(\Omega)$; see [23, Theorem VI.2.1, p. 322]. This means that $\mathcal{A}_{\varphi_1}$ is a positive semidefinite, selfadjoint linear operator on $L^2(\Omega)$ with domain $\operatorname{dom}(\mathcal{A}_{\varphi_1})$ dense in $\mathcal{D}_{\varphi_1}$ and

$$\langle \mathcal{A}_{\varphi_1} v, w\rangle = 2 \cdot \mathcal{Q}_0(v, w) \quad \text{for all } v, w \in \operatorname{dom}(\mathcal{A}_{\varphi_1}).$$

Notice that our definition of $\mathcal{Q}_0$ yields $\mathcal{A}_{\varphi_1}\varphi_1 = 0$. Since the embedding $\mathcal{D}_{\varphi_1} \hookrightarrow L^2(\Omega)$ is compact, by Lemma A.1, Part (b), the null space of $\mathcal{A}_{\varphi_1}$ denoted by

$$\ker(\mathcal{A}_{\varphi_1}) = \{v \in \operatorname{dom}(\mathcal{A}_{\varphi_1}) : \mathcal{A}_{\varphi_1} v = 0\}$$

is finite-dimensional, by the Riesz-Schauder theorem [23, Theorem III.6.29, p. 187]. Owing to hypothesis (H2), we have even $\ker(\mathcal{A}_{\varphi_1}) = \mathrm{lin}\{\varphi_1\}$, by Proposition 11.2.

Now we switch to the case $1 < p < 2$ and again require only (H1). We highlight a few places at which the proof of the boundedness part in Theorem 11.13 differs from that in Theorem 11.3. The most substantial difference is that the role of the compact embedding $\mathcal{D}_{\varphi_1} \hookrightarrow L^2(\Omega)$ needs to be replaced by that of $W_0^{1,2}(\Omega) \hookrightarrow \mathcal{H}_{\varphi_1}$, where $\mathcal{H}_{\varphi_1}$ is the Hilbert space defined below, $\mathcal{H}_{\varphi_1} \hookrightarrow L^2(\Omega)$. Let us define another norm on $W_0^{1,2}(\Omega)$ by

$$|v|_{\varphi_1} \stackrel{\text{def}}{=} \Big(\int_\Omega \varphi_1^{p-2} v^2 \,\mathrm{d}x \Big)^{1/2} \quad \text{for } v \in W_0^{1,2}(\Omega), \tag{A.15}$$

and denote by $\mathcal{H}_{\varphi_1}$ the completion of $W_0^{1,2}(\Omega)$ with respect to this norm.

The embeddings below are taken from [32, Lemma 8.2, p. 226].

Lemma A.2. *Let $1 < p < 2$ and let hypothesis* (H1) *be satisfied. Then we have:*

(a) *The embedding $\mathcal{H}_{\varphi_1} \hookrightarrow L^2(\Omega)$ is continuous.*
(b) *The embedding $W_0^{1,2}(\Omega) \hookrightarrow \mathcal{H}_{\varphi_1}$ is compact.*

Appendix B. Auxiliary results for the equation with Δ_p

B.1. **An approximation scheme for a solution.** Here we investigate an approximation scheme for a weak solution to the Dirichlet problem (1.1) in order to compute the asymptotic behavior of its large solutions provided $f \in L^\infty(\Omega)$ satisfies $f \not\equiv 0$ and λ is close to λ_1. The condition $\langle f, \varphi_1 \rangle = 0$ is *not* required in this paragraph.

We study the sequence of Dirichlet problems for $n = 1, 2, \dots$,

$$-\Delta_p u_n = (\lambda_1 + \mu_n)|u_n|^{p-2} u_n + f_n(x) \quad \text{in } \Omega; \quad u_n = 0 \quad \text{on } \partial\Omega, \tag{B.1}$$

that is, in the weak formulation, for all $\phi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \phi \rangle \,\mathrm{d}x = (\lambda_1 + \mu_n) \int_\Omega |u_n|^{p-2} u_n \,\phi \,\mathrm{d}x + \int_\Omega f_n \,\phi \,\mathrm{d}x. \tag{B.2}$$

Here, $\{\mu_n\}_{n=1}^\infty \subset \mathbb{R}$ and $\{f_n\}_{n=1}^\infty \subset L^\infty(\Omega)$ are bounded sequences, and $\{u_n\}_{n=1}^\infty$ is an unbounded sequence of corresponding weak solutions to problem (B.1) in $W_0^{1,p}(\Omega)$.

We assume that these sequences satisfy the following hypotheses:

(S1) $\mu_n \to 0$ as $n \to \infty$.
(S2) $f_n \stackrel{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$ (in the weak-star topology) as $n \to \infty$, where $f \not\equiv 0$ in Ω.
(S3) $\|u_n\|_{L^\infty(\Omega)} \to \infty$ as $n \to \infty$.

By a regularity result [3, Théorème A.1, p. 96], hypothesis (S3) is equivalent to

(S3') $\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$ as $n \to \infty$.

Furthermore, since $\partial\Omega$ is assumed to be of class $C^{1,\alpha}$, for some $0 < \alpha < 1$, we can apply another regularity result, [8, Theorem 2, p. 829] or [38, Theorem 1, p. 127] for interior regularity, and [24, Theorem 1, p. 1203] for regularity near the boundary, to conclude that $u_n \in C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0, \alpha)$, and hypothesis (**S3**) is equivalent to

(S3") $\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \to \infty$ as $n \to \infty$.

We often work with a chain of subsequences of $\{(\mu_n, f_n, u_n)\}_{n=1}^{\infty}$ by passing from the current one to the next, but keeping the index n unchanged if no confusion may arise.

We commence with the asymptotic behavior of the normalized sequence $\tilde{u}_n \stackrel{\text{def}}{=} \|u_n\|_{L^\infty(\Omega)}^{-1} u_n$ as $n \to \infty$. Observe that each $\tilde{u}_n$ satisfies $\|\tilde{u}_n\|_{L^\infty(\Omega)} = 1$ and

$$\begin{gathered} -\Delta_p \tilde{u}_n = (\lambda_1 + \mu_n)|\tilde{u}_n|^{p-2}\tilde{u}_n + \|u_n\|_{L^\infty(\Omega)}^{1-p} f_n(x) \quad \text{in } \Omega; \\ \tilde{u}_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{B.3}$$

Hence, $\{\tilde{u}_n\}_{n=1}^{\infty}$ is bounded in $C^{1,\beta}(\overline{\Omega})$, by regularity [8, 24, 38]. We allow $1 < p < \infty$.

Lemma B.1 ([32, Lemma 5.1])**.** *Let $\beta' \in (0, \beta)$. The sequence $\{\tilde{u}_n\}_{n=1}^{\infty}$ contains a convergent subsequence $\tilde{u}_n \to \kappa\varphi_1$ in $C^{1,\beta'}(\overline{\Omega})$ as $n \to \infty$, where $\kappa \in \mathbb{R}$ is a constant, $|\kappa| \cdot \|\varphi_1\|_{L^\infty(\Omega)} = 1$. In particular, we have $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$, where $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a sequence such that $\kappa t_n > 0$ and $t_n u_n \geq \frac{1}{2}\varphi_1$ in Ω for all n large enough; moreover, $t_n \to 0$ and $v_n^\top \to 0$ in $C^{1,\beta'}(\overline{\Omega})$ as $n \to \infty$, with $\langle v_n^\top, \varphi_1\rangle = 0$ for $n = 1, 2, \dots$.*

As a consequence of this lemma, we can rewrite problem (B.3) as

$$\begin{gathered} -\Delta_p(\varphi_1 + v_n^\top) = (\lambda_1 + \mu_n)|\varphi_1 + v_n^\top|^{p-2}(\varphi_1 + v_n^\top) + t_n^{p-1} f_n(x) \quad \text{in } \Omega; \\ v_n^\top = 0 \quad \text{on } \partial\Omega; \\ \langle v_n^\top, \varphi_1\rangle = 0, \end{gathered} \tag{B.4}$$

with all $t_n > 0$, $t_n \searrow 0$ as $n \to \infty$. Indeed, if $\kappa < 0$, we take advantage of the $(p-1)$-homogeneity of problem (B.1) and replace all functions f_n, f and u_n by $-f_n$, $-f$ and $-u_n$, respectively, thus switching to the case $\kappa > 0$. Hence, without loss of generality, we may assume $t_n > 0$ and $t_n u_n = \varphi_1 + v_n^\top \geq \frac{1}{2}\varphi_1 > 0$ in Ω for all $n \geq 1$.

A useful equivalent form of problem (B.1) is obtained by subtracting equation (5.1) from (B.4) and using the Taylor formula with a help from identity (A.2), for $n = 1, 2, \dots$:

$$\begin{gathered} -\operatorname{div}(\mathbf{A}_n \nabla v_n^\top) = (p-1)(\lambda_1 + \mu_n) a_n v_n^\top + \mu_n \varphi_1^{p-1} + |t_n|^{p-2} t_n f_n(x) \quad \text{in } \Omega; \\ v_n^\top = 0 \quad \text{on } \partial\Omega; \\ \langle v_n^\top, \varphi_1\rangle = 0, \end{gathered} \tag{B.5}$$

with the abbreviations

$$\mathbf{A}_n \stackrel{\text{def}}{=} \int_0^1 \mathbf{A}(\nabla\varphi_1 + s\nabla v_n^\top)\,\mathrm{d}s \quad \text{and} \quad a_n \stackrel{\text{def}}{=} \int_0^1 |\varphi_1 + s v_n^\top|^{p-2}\,\mathrm{d}s. \tag{B.6}$$

Recall that the matrix $\mathbf{A}(\mathbf{a})$ is defined in (A.3). We abbreviate also

$$\mathbf{A}_{\varphi_1} \stackrel{\text{def}}{=} \mathbf{A}(\nabla\varphi_1) \quad \text{and write} \quad \mathbf{A}_{\varphi_1}^{1/2} = \sqrt{\mathbf{A}_{\varphi_1}}\,. \tag{B.7}$$

Equivalently, $V_n \stackrel{\text{def}}{=} t_n^{1-p} v_n^\top \in C^{1,\beta'}(\overline{\Omega})$ satisfies the linear boundary value problem

$$\begin{gathered} -\operatorname{div}(\mathbf{A}_n \nabla V_n) = (p-1)(\lambda_1 + \mu_n) a_n V_n + \frac{\mu_n}{|t_n|^{p-2} t_n}\,\varphi_1^{p-1} + f_n(x) \quad \text{in } \Omega; \\ V_n = 0 \quad \text{on } \partial\Omega; \\ \langle V_n, \varphi_1\rangle = 0. \end{gathered} \tag{B.8}$$

The asymptotic behavior of V_n and $\mu_n/(|t_n|^{p-2}t_n)$ as $n\to\infty$ is determined as follows ([13, Theorem 4.1]).

Theorem B.2. *Let $1<p<\infty$, $p\neq 2$, and let $\{\mu_n\}_{n=1}^{\infty}\subset\mathbb{R}$, $\{f_n\}_{n=1}^{\infty}\subset L^{\infty}(\Omega)$, and $\{u_n\}_{n=1}^{\infty}\subset W_0^{1,p}(\Omega)$ be sequences satisfying hypotheses* (**S1**), (**S2**), *and* (**S3**), *respectively. In addition, assume that they satisfy equation* (B.2) *for all $\phi\in W_0^{1,p}(\Omega)$ and for each $n\in\mathbb{N}$. Then, writing $u_n=t_n^{-1}(\varphi_1+v_n^{\top})$ with $t_n\in\mathbb{R}$, $t_n\neq 0$, and $v_n^{\top}\in W_0^{1,p}(\Omega)^{\top}$, we have $t_n\to 0$ as $n\to\infty$, $V_n=\big(|t_n|^{p-2}t_n\big)^{-1}v_n^{\top}\to V^{\top}$ strongly in $\mathcal{D}_{\varphi_1}$ if $p>2$ and in $W_0^{1,2}(\Omega)$ if $1<p<2$, and*

$$\mu_n=-|t_n|^{p-2}t_n\int_\Omega f\varphi_1\,\mathrm{d}x+o(|t_n|^{p-1}), \tag{B.9}$$

$$\begin{aligned}\mu_n=\;&-|t_n|^{p-2}t_n\int_\Omega f_n\varphi_1\,\mathrm{d}x+(p-2)\,|t_n|^{2(p-1)}\,\mathcal{Q}_0(V^{\top},V^{\top})\\&+(p-1)\,|t_n|^{2(p-1)}\Big(\int_\Omega f\varphi_1\,\mathrm{d}x\Big)\Big(\int_\Omega\varphi_1^{p-1}\,V^{\top}\,\mathrm{d}x\Big)+o(|t_n|^{2(p-1)}).\end{aligned}\tag{B.10}$$

Moreover, the limit function $V^{\top}\in\mathcal{D}_{\varphi_1}\cap\{\varphi_1\}^{\perp,L^2}$ is the (unique) solution to

$$2\cdot\mathcal{Q}_0(V^{\top},\phi)=\int_\Omega f^{\dagger}\,\phi\,\mathrm{d}x\quad\text{for all }\phi\in\mathcal{D}_{\varphi_1}, \tag{B.11}$$

where the symmetric bilinear form $\mathcal{Q}_0$ is given by (11.2) *and*

$$f^{\dagger}=f-\Big(\int_\Omega f\varphi_1\,\mathrm{d}x\Big)\varphi_1^{p-1}.$$

Formula (B.9) is an improvement of Takáč [33, Prop. 6.1, p. 331] whereas (B.10) is established in Drábek et al. [13, Theorem 4.1]. Notice that $\mathcal{Q}_0(V^{\top},V^{\top})>0$ holds by Proposition 11.2 and Remark 11.12. In addition, we have $\int_\Omega f^{\dagger}\varphi_1\,\mathrm{d}x=0$.

The linear degenerate Dirichlet problem (B.11) above has been obtained by linearizing (1.1) with $\lambda=\lambda_1$ about φ_1, that is to say, from

$$\begin{gathered}-\mathrm{div}\,(\mathbf{A}(\nabla\varphi_1)\nabla w)=\lambda_1(p-1)\varphi_1^{p-2}w+f(x)\quad\text{in }\Omega;\\ w=0\quad\text{on }\partial\Omega.\end{gathered}\tag{B.12}$$

It plays a crucial role in our asymptotic formulas as $|\tau|\to\infty$. Its solution set in $\mathcal{D}_{\varphi_1}$ is described in Theorem B.2. If $p<2$ then $\mathcal{D}_{\varphi_1}$ is not necessarily dense in $L^2(\Omega)$, and so equation (B.12) can be satisfied only in the following weak sense, cf. eq. (B.11): For all test functions $\phi\in\mathcal{D}_{\varphi_1}$,

$$\int_U\langle\mathbf{A}(\nabla\varphi_1)\nabla w,\nabla\phi\rangle\,\mathrm{d}x=\lambda_1(p-1)\int_\Omega\varphi_1^{p-2}w\phi\,\mathrm{d}x+\int_\Omega f\phi\,\mathrm{d}x\,. \tag{B.13}$$

The following two special cases of formula (B.10) are of particular interest in the present work; both force $\int_\Omega f\varphi_1\,\mathrm{d}x=0$.

Corollary B.3. *In the situation of* Theorem B.2 *we have: If $\int_\Omega f_n\varphi_1\,\mathrm{d}x=0$ for all $n\in\mathbb{N}$, then*

$$\mu_n=(p-2)\,|t_n|^{2(p-1)}\,\mathcal{Q}_0(V^{\top},V^{\top})+o\big(|t_n|^{2(p-1)}\big). \tag{B.14}$$

On the other hand, if $\mu_n=0$ for all $n\in\mathbb{N}$, then

$$\lim_{n\to\infty}\frac{1}{|t_n|^{p-2}t_n}\int_\Omega f_n\varphi_1\,\mathrm{d}x=(p-2)\,\mathcal{Q}_0(V^{\top},V^{\top}). \tag{B.15}$$

In particular, in both cases we must have $\int_\Omega f\varphi_1\,\mathrm{d}x=0$.

B.2. **Uniform boundedness of the solution set.** Finally, we are ready to establish the asymptotic behavior (blow-up) of every weak solution to the resonant problem (9.2), that is,

$$-\Delta_p u = \lambda_1 |u|^{p-2} u + f(x) \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega, \tag{9.2}$$

with $f \equiv f^\top + \zeta\varphi_1$, where $f^\top \in L^\infty(\Omega)^\top$ and $\zeta \in \mathbb{R}$. It turns out to be convenient to apply Lemma B.1, that is, to fix an arbitrary number $t \in \mathbb{R} \setminus \{0\}$ and consider only those weak solutions to problem (9.2) that take the form $u = t^{-1}\varphi_1 + u^\top$ where $u^\top \in W_0^{1,p}(\Omega)^\top$ is an unknown function. Hence $u^\top \in C^{1,\beta}(\overline{\Omega})$ by regularity [3, 8, 24, 38]. While determining the asymptotic behavior of u as $t \to 0$, we regard $\zeta \in \mathbb{R}$ in eq. (9.2) as a parameter depending on t as well.

Let $F(t)$ denote the set of all pairs $(\zeta, u^\top) \in \mathbb{R} \times W_0^{1,p}(\Omega)^\top$ satisfying the boundary value problem

$$\begin{gathered}
\begin{aligned}
&- \Delta_p(t^{-1}\varphi_1 + u^\top) - \lambda_1 |t^{-1}\varphi_1 + u^\top|^{p-2}(t^{-1}\varphi_1 + u^\top) \\
&= f^\top(x) + \zeta \cdot \varphi_1(x) \quad \text{in } \Omega;
\end{aligned} \\
u^\top = 0 \quad \text{on } \partial\Omega; \\
\langle u^\top, \varphi_1 \rangle = 0.
\end{gathered} \tag{B.16}$$

The asymptotic behavior of $F(t)$ as $t \to 0$ is determined next. The following equivalent form of problem (B.16) will be needed with the new unknown function $v^\top \stackrel{\text{def}}{=} t u^\top$:

$$\begin{gathered}
\begin{aligned}
&- \Delta_p(\varphi_1 + v^\top) - \lambda_1 |\varphi_1 + v^\top|^{p-2}(\varphi_1 + v^\top) \\
&= |t|^{p-2} t \left(f^\top(x) + \zeta \cdot \varphi_1(x) \right) \quad \text{in } \Omega;
\end{aligned} \\
v^\top = 0 \quad \text{on } \partial\Omega; \\
\langle v^\top, \varphi_1 \rangle = 0.
\end{gathered} \tag{B.17}$$

We take a sequence of nonzero real numbers $t_n \to 0$ as $n \to \infty$ and a sequence of pairs $(\zeta_n, u_n^\top) \in F(t_n)$. We substitute $v_n^\top \stackrel{\text{def}}{=} t_n u_n^\top$ and $V_n \stackrel{\text{def}}{=} |t_n|^{-(p-2)} u_n^\top = |t_n|^{-(p-2)} t_n^{-1} v_n^\top$, and abbreviate $f_n \stackrel{\text{def}}{=} f^\top + \zeta_n \varphi_1$. More generally, we can replace function $f^\top \in L^\infty(\Omega)^\top$, $f^\top \not\equiv 0$ in Ω, by a bounded sequence $\{f_n^\top\}_{n=1}^\infty$ of functions from $L^\infty(\Omega)^\top$ satisfying the following hypothesis (cf. (**S2**) in §B.1):

($\mathbf{S2}^\top$) $f_n^\top \stackrel{*}{\rightharpoonup} f^\top$ in $L^\infty(\Omega)$ (weakly-star) as $n \to \infty$. We require $f^\top \notin \mathcal{D}_{\varphi_1}^{\perp, L^2}$.

The following *a priori* asymptotic formula was obtained in Takáč [33, Prop. 6.1]:

Proposition B.4. *Assume* ($\mathbf{S2}^\top$). *If $t_n \neq 0$ and $t_n \to 0$ as $n \to \infty$, then $\zeta_n \to 0$, all conclusions of* Theorem B.2 *and* Corollary B.3 *remain valid with $f = f^\top$, and moreover*

$$\lim_{n\to\infty} \frac{\zeta_n}{|t_n|^{p-2} t_n} = (p-2) \|\varphi_1\|_{L^2(\Omega)}^{-2} \cdot \mathcal{Q}_0(w, w) \neq 0. \tag{B.18}$$

Here, for $p > 2$, $w \in \mathcal{D}_{\varphi_1}$ is the unique weak solution of problem (B.12) with $f = f^\top$ satisfying $\langle w, \varphi_1 \rangle = 0$. For $p < 2$, equation (B.13) replaces (B.12). Finally, Proposition 11.2 and Remark 11.12 guarantee $\mathcal{Q}_0(w, w) > 0$.

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