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STEKLOV SPECTRUM AND NONRESONANCE FOR ELLIPTIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. This article is devoted to the solvability of second order elliptic partial differential equations with nonlinear boundary conditions. We prove existence results when the nonlinearity on the boundary interacts, in some sense, with the Steklov spectrum. We obtain nonresonance results below the first Steklov eigenvalue as well as between two consecutive Steklov eigenvalues. Our method of proof is variational and relies mainly on minimax methods in critical point theory.

1. Introduction

This paper concerns existence results for second order elliptic partial differential equations with (possibly) nonlinear boundary conditions

$$\begin{aligned} -\Delta u + c(x)u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class $C^{0,1}$, and $\partial/\partial\nu := \nu \cdot \nabla$ is the outward (unit) normal derivative on $\partial\Omega$.

Throughout this paper we shall assume that $n \geq 2$ and that the function $c : \Omega \to \mathbb{R}$, and the nonlinearity $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions.

- (C1) $c \in L^p(\Omega)$ with $p \geq n/2$ when $n \geq 3$ ($p \geq 1$ when $n = 2$), and $c \geq 0$ a.e. on Ω with strict inequality on a set of positive measure; that is, $\int_\Omega c(x)\,dx > 0$.
- (C2) $g \in C(\overline{\Omega} \times \mathbb{R})$.
- (C3) There exist constants a_1, $a_2 > 0$ such that
$$|g(x,u)| \leq a_1 + a_2|u|^s \quad \text{with } 0 \leq s < \frac{n}{n-2}.$$

The purpose of this paper is to study the existence of weak solutions of problem (1.1) in which the nonlinearity interacts, in some sense, with the Steklov spectrum.

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By a weak solution of (1.1) we mean a function $u \in H^1(\Omega)$ such that

$$\int \nabla u \nabla v + \int c(x) uv = \oint g(x,u)v \quad \text{for any } v \in H^1(\Omega), \tag{1.2}$$

where $\int$ denotes the (volume) integral on Ω, $\oint$ denotes the (surface) integral on $\partial\Omega$, and throughout this paper, $H^1(\Omega)$ denotes the usual real Sobolev space of functions on Ω.

The nonlinear problem, (1.1), has been studied by many authors in the framework of sub and super-solutions method. We refer e.g. to Amann [1], Mawhin and Schmitt [11], and references therein. Restricting the domain of the nonlinearity (through a slightly modified problem) to the sub and super solutions interval, the methods used in that framework reduce the problem to essentially considering bounded nonlinearities and then using a priori estimates and fixed points or topological degree arguments. Since it is based on (the so-called) comparison techniques, the (ordered) sub-super solutions method does not apply when the nonlinearities are compared with higher eigenvalues.

In recent years much work has been devoted to the study of the solvability of elliptic boundary value problems (with linear homogeneous bounday condition) where the reaction nonlinearity in the differential equation interacts with the eigenvalues of the corresponding linear differential equation with linear homogeneous boundary condition (resonance and nonresonance problems). For some recent results in this direction we refer to the papers by Castro [4], de Figueiredo and Gossez [6], Rabinowitz [14], and the bibliography therein.

Concerning Problem (1.1) with boundary eigenparameters, there are some (scattered) results in the literature by several authors. For the linear case, we mention the work by Steklov [15] who initiated the problem on a disk in 1902, Amann [1], Bandle [3], and more recently Auchmuty [2]. To the best of our knowledge, not much has been done for the nonlinear problem (1.1) in the framework of the Steklov spectrum. A few results on a disk ($n = 2$) were obtained by Klingelhöfer [7] and Cushing [5]. (The results in [7] were significantly generalized to higher dimensions in [1] in the framework of sub and super-solutions method as aforementioned.) We also refer to Klingelhöfer [8] where monotonicity methods were used for nonlinearities near the first eigenvalue.

In this paper, we prove existence results when the nonlinearity involved asymptotically interacts, in some sense, with the Steklov spectrum. We derive the so-called nonresonance results below the first Steklov eigenvalue as well as between two consecutive Steklov eigenvalues. This appears to be the first time that the boundary nonlinearity g is compared with *higher* Steklov eigenvalues. Our method of proof is variational and relies mainly on a priori estimates and minimax methods in critical point theory.

This paper is organized as follows. In Section 2 we have collected some relevant preliminary results on linear Steklov eigenproblems which are needed for our purposes. (The proofs of these auxiliary results may be found in a recent paper of Auchmuty [2].) We also state our main results which consist of relating the asymptotic behavior of the nonlinearity involved with the Steklov spectrum. In Section 3 we first provide some auxiliary results on Critical Point Theory that are needed for the proofs of our main results. Then we give the proofs of our main results, and a few remarks to relate our results to the previous ones in the literature.

Unlike some previous approaches to problems with nonlinear boundary conditions, all of our results are based upon minimax methods in Critical Point Theory (see e.g. Rabinowitz [14] and references therein).

2. Preliminaries on Steklov problems and main results

To put our results into context, we have collected some relevant results on linear Steklov eigenproblems needed for our purposes. We refer to a very recent and interesting paper of Auchmuty [2] for the proofs of the results regarding Steklov eigenproblems. We then state the main results which consist of relating, in some sense, the asymptotic behavior of the nonlinearity involved with the first Steklov eigenvalue, then subsequently with two consecutive higher Steklov eigenvalues.

Consider the linear problem

$$\begin{gathered} -\Delta u + c(x)u = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \mu u \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

The Steklov eigenproblem is to find a pair $(\mu,\varphi) \in \mathbb{R} \times H^1(\Omega)$, $\varphi \not\equiv 0$, such that

$$\int \nabla\varphi\nabla v + \int c(x)\varphi v = \mu \oint \varphi v \quad \text{for any } v \in H^1(\Omega).$$

Picking $v = \varphi$, and subsequently $v \in H_0^1(\Omega)$ one immediately sees that if there is such an eigenpair, then $\mu > 0$ and $\varphi \perp H_0^1(\Omega)$ in the H^1-c-inner product defined by

$$(u,v)_c = \int \nabla u\nabla v + \int c(x)uv, \tag{2.2}$$

with the associated norm denoted by $\|u\|_c$; which is equivalent to the standard norm on $H^1(\Omega)$. This implies that one can split

$$H^1(\Omega) = H_0^1(\Omega) \oplus_c [H_0^1(\Omega)]^\perp \tag{2.3}$$

as a direct orthogonal sum (in the sense of H^1-c-inner product).

Besides the Sobolev spaces, we shall make use, in what follows, of the real Lebesgue spaces $L^q(\partial\Omega)$, $1 \le q \le \infty$, and the compactness of the trace operator $\Gamma : H^1(\Omega) \to L^q(\partial\Omega)$ for $1 \le q < \frac{2(n-1)}{n-2}$ (see e.g. Kufner, John and Fučík [9, Chap. 6] and references therein). Sometimes we will just use u in place of Γu when considering the trace of a function on $\partial\Omega$. Throughout this paper we denote the $L^2(\partial\Omega)$-inner product by

$$(u,v)_\partial = \oint uv \tag{2.4}$$

and the associated norm by $\|u\|_\partial$.

Assuming that the above assumptions are satisfied, Auchmuty [2] recently proved that, for $n \ge 2$, the Steklov eigenproblem (2.1) has a sequence of real eigenvalues

$$0 < \mu_1 \le \mu_2 \le \ldots \le \mu_j \le \ldots \to \infty, \quad \text{as } j \to \infty,$$

each eigenvalue has a finite-dimensional eigenspace. The eigenfunctions φ_j corresponding to these eigenvalues form a complete orthonormal family in $[H_0^1(\Omega)]^\perp$, which is also complete and orthogonal in $L^2(\partial\Omega)$. Moreover, the trace inequality

$$\mu_1 \oint (\Gamma u)^2 \le \int |\nabla u|^2 + \int c(x)u^2 \tag{2.5}$$

holds for all $u \in H^1(\Omega)$, where $\mu_1 > 0$ is the least Steklov eigenvalue for (2.1). If equality holds in (2.5), then u is a multiple of an eigenfunction of (2.1) corresponding to μ_1.

Theorem 2.1 (Below the first Steklov eigenvalue)**.** *Assume* (C1)–(C3) *hold. Let the potential* $G(x,u) = \int_0^u g(x,s)\,ds$ *be such that the following condition holds.*

(C4) *There exists* $\mu \in \mathbb{R}$ *such that*

$$\limsup_{|u|\to\infty} \frac{2G(x,u)}{u^2} \le \mu < \mu_1$$

uniformly for $x \in \overline{\Omega}$.

Then the nonlinear equation (1.1) *has at least one solution* $u \in [H_0^1(\Omega)]^\perp$.

Consequently, we derive nonresonance below the first eigenvalue associated with the Steklov problem. Notice that we impose conditions on the potential of the boundary nonlinearity g rather than on g itself as was done in the previous papers [1, 5, 7, 8].

In the next result, we are concerned with the case where the asymptotic behavior of the nonlinearity is related to two consecutive Steklov eigenvalues. We impose conditions on the asymptotic behavior of the nonlinearity $g(x,u)$ directly. These conditions imply similar ones on the asymptotic behavior of the potential $G(x,u)$.

Theorem 2.2 (Between consecutive Steklov eigenvalues)**.** *Assume* (C1)–(C3) *are met, and that the following condition holds.*

(C5) *There exist constants* $a, b \in \mathbb{R}$ *such that*

$$\mu_j < a \le \liminf_{|u|\to\infty} \frac{g(x,u)}{u} \le \limsup_{|u|\to\infty} \frac{g(x,u)}{u} \le b < \mu_{j+1},$$

uniformly for $x \in \overline{\Omega}$.

Then the nonlinear equation (1.1) *has at least one solution* $u \in [H_0^1(\Omega)]^\perp$.

We will use a variational approach to prove Theorems 2.1–2.2.

3. Proofs of main results

We first state some auxiliary results which will be needed in the sequel. The following result on the continuity of the Nemytskĭi operator on the boundary readily follows from the arguments similar to those used in the proof of [14, Proposition B.1]. For the last two auxiliary results, we refer to [10] for the proofs.

Proposition 3.1. *Suppose that* g *satisfies* (C2) *and there are constants* $p, q \ge 1$ *and* a_1, a_2 *such that for all* $x \in \overline{\Omega}, \xi \in \mathbb{R}$,

$$|g(x,\xi)| \le a_1 + a_2|\xi|^{p/q},$$

Then the Nemytskĭi operator $\varphi(x) \to g(x,\varphi(x))$ *is continuous from* $L^p(\partial\Omega)$ *to* $L^q(\partial\Omega)$.

Proposition 3.2. *Assume that* (C1)–(C3) *hold. Then the functional associated with* (1.1) *defined by* $I : H^1(\Omega) \to \mathbb{R}$ *with*

$$I(u) = \frac{1}{2}\Big[\int |\nabla u|^2 + \int c(x)u^2\Big] - \oint G(x,u), \tag{3.1}$$

is well-defined and of class C^1 with Fréchet derivative $I'(u)$ in $(H^1(\Omega))^*$ given by

$$I'(u)v = \int \nabla u \nabla v + \int c(x)uv - \oint g(x,u)v \quad \textit{for every } v \in H^1(\Omega). \tag{3.2}$$

Moreover, $J(u) = \oint G(x,u)$ is weakly continuous, and J' is compact.

The next result concerns the Palais-Smale condition (PS) which builds some "compactness" into the functional I. It requires that any sequence $\{u_m\}$ in $H^1(\Omega)$ such that (i) $\{I(u_m)\}$ is bounded, (ii) $\lim_{m\to\infty} I'(u_m) = 0$, be precompact.

Owing to the next proposition, to get (PS) it suffices to show that (i)–(ii) imply that $\{u_m\}$ is a bounded sequence.

Proposition 3.3. *Assume that* (C1)–(C3) *hold. If $\{u_m\}$ is a bounded sequence in $H^1(\Omega)$ such that* $\lim_{m\to\infty} I'(u_m) = 0$*, then $\{u_m\}$ has a convergent subsequence.*

Now we prove Theorems 2.1–2.2. We will use of the Saddle Point Theorem and its variant proved in [14].

Proof of Theorem 2.1. Observe that condition (C4) implies that for every $\epsilon > 0$ there is $r = r(\epsilon) > 0$ such that

$$\frac{2G(x,u)}{u^2} \le \mu + \epsilon \tag{3.3}$$

for all $x \in \overline{\Omega}$ and all $u \in \mathbb{R}$ with $|u| > r$. Combining (3.3) and (C3), there exists a constant $M_\epsilon > 0$ such that

$$\forall x \in \overline{\Omega},\ \forall u \in \mathbb{R},\ G(x,u) \le \frac{1}{2}(\mu+\epsilon)u^2 + M_\epsilon. \tag{3.4}$$

To prove that (1.1) has at least one solution, it suffices, according to [14, Theorem 2.7], to show that the functional I is bounded below and that it satisfies the (PS) condition. Under the assumptions of Theorem 2.1, we shall show that the functional I is coercive on $H^1(\Omega)$; that is,

$$I(u) \to \infty \quad \text{as } \|u\|_c \to \infty, \tag{3.5}$$

which would imply that I is bounded below and that the Palais-Smale is satisfied.

Now let us prove that I is coercive on $H^1(\Omega)$. Assume $\|u\|_c \to \infty$, then by using the continuity of the trace operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$, we get that either $\|u\|_\partial \to \infty$ or $\|u\|_\partial < K$, where K is a positive constant. We claim that in either case $I(u) \to \infty$.

First, suppose that $\|u\|_\partial < K$. Since $I(u) = \frac{1}{2}\|u\|_c^2 - \oint G(x,u)$, using (3.4), one has that $I(u) \ge \frac{1}{2}\|u\|_c^2 - \frac{1}{2}(\mu+\epsilon)\|u\|_\partial^2 - C$, where C is a positive constant. Hence $I(u) \to \infty$ as $\|u\|_c \to \infty$.

Now, suppose $\|u\|_\partial \to \infty$. Using (2.5), one has $I(u) \ge \frac{1}{2}\left(\mu_1 - (\mu+\epsilon)\right)\|u\|_\partial^2 - C$. Since $\mu_1 > \mu$, one gets that $I(u) \to \infty$ as $\|u\|_c \to \infty$. Thus I is coercive.

By combining condition (C3) and the coercivity of I, one deduces that I is bounded from below; that is, there exists $K \in \mathbb{R}$ such that $I(u) \ge K$, for all $u \in H^1(\Omega)$. It follows immediately from the coercivity of I in (3.5) and Proposition 3.3 that I satisfies (PS). By [14, Theorem 2.7], it follows that I has a critical point $u \in H^1(\Omega)$, that is, $I'(u) = 0$. Hence, u satisfies (1.2), and thus (1.1) has at least one solution. Note that $u \in [H_0^1(\Omega)]^\perp$ by the definition (1.2). The proof is complete. □

Proof of Theorem 2.2. Under the assumptions of Theorem 2.2 we need to show that the conditions of the Saddle Point Theorem [14] are fulfilled. Let

$$V = \operatorname{span}\{\varphi_k |\, k \leq j\}, \quad X = Y \oplus_c H_0^1(\Omega), \text{ where } Y = \overline{\operatorname{span}\{\varphi_k : k \geq j+1\}}. \tag{3.6}$$

It follows from (2.3) and (3.6) that

$$H^1(\Omega) = V \oplus_c X. \tag{3.7}$$

We need to prove that there exists a constant $r > 0$ such that

$$\sup_{\partial D} I < \inf_X I, \tag{3.8}$$

where $D = \{v \in V : \|u\|_c \leq r\}$. Assuming that this is the case, and the Palais-Smale condition is satisfied, we deduce by the Saddle Point Theorem [14] that I has a critical point. Therefore, (1.1) has at least one solution.

We shall show that the functional I is coercive on X and $-I$ is coercive on V which would imply that (3.8) is satisfied by choosing $r > 0$ sufficiently large.

Notice that condition (C5) implies a similar condition on the potential G; that is, the constants $a, b \in \mathbb{R}$ are such that for all $x \in \overline{\Omega}$,

$$a \leq \liminf_{|u|\to\infty} \frac{2G(x,u)}{u^2} \leq \limsup_{|u|\to\infty} \frac{2G(x,u)}{u^2} \leq b. \tag{3.9}$$

Combining (C3) and (3.9), one gets that for every $\epsilon > 0$, all $x \in \overline{\Omega}$, all $u \in \mathbb{R}$, we have

$$(a-\epsilon)\frac{u^2}{2} - C \leq G(x,u) \leq (b+\epsilon)\frac{u^2}{2} + C\ , \tag{3.10}$$

where $C = C(\epsilon)$ is a positive constant.

On the one hand, it follows that for every $u \in V$ one has that

$$I(u) = \frac{1}{2}\|u\|_c^2 - \oint G(x,u) \leq \frac{1}{2}\|u\|_c^2 - \frac{1}{2}(a-\epsilon)\oint u^2 + \tilde{C} = \frac{1}{2}\|u\|_c^2 - \frac{1}{2}(a-\epsilon)\|u\|_\partial^2 + \tilde{C}.$$

Using the Parseval identities obtained in [2, p. 331] it follows that

$$I(u) \leq \frac{1}{2}\Big(1 - \frac{a}{\mu_j} + \frac{\epsilon}{\mu_j}\Big)\|u\|_c^2 + \tilde{C}.$$

Since $\mu_j < a$, it follows that $1 - \frac{a}{\mu_j} + \frac{\epsilon}{\mu_j} < 0$, provided $\epsilon > 0$ is sufficiently small. Therefore, by going to the limit as $\|u\|_c \to \infty$, one gets

$$I(u) \to -\infty. \tag{3.11}$$

On the other hand, for every $u \in X$, it follows from (3.6) that $u = u^0 + \overline{u}$, where $u^0 \in H_0^1(\Omega)$ and $\overline{u} \in Y$. Taking into account the c-orthogonality of $\overline{u}$ and u^0 in $H^1(\Omega)$, one has

$$I(u) = \frac{1}{2}\|u^0\|_c^2 + \frac{1}{2}\|\overline{u}\|_c^2 - \oint G(x,u) \geq \frac{1}{2}\|u^0\|_c^2 + \frac{1}{2}\left(\|\overline{u}\|_c^2 - (b+\epsilon)\|\overline{u}\|_\partial^2\right) - \tilde{C}.$$

Therefore, using the Parseval identities obtained in [2, p. 331] it follows that

$$I(u) \geq \frac{1}{2}\|u^0\|_c^2 + \frac{1}{2}\Big(1 - \frac{b}{\mu_{j+1}} - \frac{\epsilon}{\mu_{j+1}}\Big)\|\overline{u}\|_c^2 - \tilde{C}.$$

Since $b < \mu_{j+1}$, it follows that for $\epsilon > 0$ sufficiently small, $1 - \frac{b}{\mu_{j+1}} - \frac{\epsilon}{\mu_{j+1}} > 0$. Therefore,

$$I(u) \geq \frac{1}{2}\Big(1 - \frac{b}{\mu_{j+1}} - \frac{\epsilon}{\mu_{j+1}}\Big)\|u\|_c^2 - \tilde{C}.$$

By going to the limit as $\|u\|_c \to \infty$, on gets

$$I(u) \to \infty \quad \text{as } \|u\|_c \to \infty.$$

Thus, I is coercive on X. Furthermore, it follows from the coercivity of I on X and condition (C3) that I is bounded below by a constant on X. Therefore, using (3.11) we obtain the assertion (3.8) for some constant $r > 0$.

It remains to prove that the functional I satisfies the Palais-Smale condition. It suffices, according to Proposition 3.3, to show that for any sequence $\{u_m\}$ in $H^1(\Omega)$ such that $\{I(u_m)\}$ is bounded and $\lim_{m\to\infty} I'(u_m) = 0$, it follows that $\{u_m\}$ is bounded.

Notice that Condition (C5) implies that for every $\epsilon > 0$ there exists $r > 0$ such that for $|u| \geq r$,

$$a - \epsilon \leq \frac{g(x,u)}{u} \leq b + \epsilon \quad \text{for all } x \in \overline{\Omega}. \tag{3.12}$$

Let us define $\gamma : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma(x,u) = \begin{cases} \frac{g(x,u)}{u} & \text{for } |u| \geq r \\ \frac{g(x,r)+g(x,-r)}{2r^2} u + \frac{g(x,r)-g(x,-r)}{2r} & \text{for } |u| < r. \end{cases}$$

The function γ is continuous in $\overline{\Omega} \times \mathbb{R}$ since g is, moreover by (3.12) one has

$$a - \epsilon \leq \gamma(x,u) \leq b + \epsilon \quad \text{for all } u \in \mathbb{R} \text{ and all } x \in \overline{\Omega}. \tag{3.13}$$

Define $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$h(x,u) = g(x,u) - \gamma(x,u)u, \tag{3.14}$$

then it follows from the continuity of g and γ that

$$|h(x,u)| \leq K, \tag{3.15}$$

for all $(x,u) \in \overline{\Omega} \times \mathbb{R}$, where $K > 0$ is a constant.

Now, let $\{u_m\} \subset H^1(\Omega)$ be such that $\{I(u_m)\}$ is bounded and $\lim_{m\to\infty} I'(u_m) = 0$. By (3.7) one has $u_m = v_m + x_m$, where $v_m \in V$ and $x_m \in X$. Moreover, by (3.6) $x_m = x_m^0 + \overline{x}_m$, where $x_m^0 \in H_0^1(\Omega)$ and $\overline{x}_m \in Y$.
Since $\lim_{m\to\infty} I'(u_m) = 0$, it follows that for every $\epsilon > 0$, there exists $N > 0$ such that for all $m \geq N$,

$$\sup_{\varphi \neq 0} \frac{|I'(u_m)\varphi|}{\|\varphi\|_c} < \epsilon.$$

Set $\varphi = x_m - v_m$ for m large. Then, $I'(u_m)(x_m - v_m) < \epsilon\|x_m - v_m\|_c$. Taking into account the c-orthogonality of x_m and v_m in $H^1(\Omega)$ and (3.14), one gets from the definition of I' that

$$\begin{aligned} &\|x_m\|_c^2 - \|v_m\|_c^2 - \oint \gamma(x,u_m)\overline{x}_m^2 + \oint \gamma(x,u_m)v_m^2 \\ &< \epsilon(\|x_m\|_c + \|v_m\|_c) + \oint h(x,u_m)\overline{x}_m - \oint h(x,u_m)v_m. \end{aligned}$$

By using (3.13), (3.15) and the continuity of the trace operator, one obtains

$$\begin{aligned} &\big(\|x_m^0\|_c^2 + \|\overline{x}_m\|_c^2\big) - \|v_m\|_c^2 - (b+\epsilon)\|\overline{x}_m\|_\partial^2 + (a-\epsilon)\|v_m\|_\partial^2 \\ &< \epsilon(\|x_m\|_c + \|v_m\|_c) + \widetilde{K}\|\overline{x}_m\|_c + \widetilde{K}\|v_m\|_c, \end{aligned}$$

Now, using the Parseval identities obtained in [2, p. 331] it follows that

$$\begin{aligned}&\|x_m^0\|_c^2 + \big(1 - \frac{b}{\mu_{j+1}} - \frac{\epsilon}{\mu_{j+1}}\big)\|\overline{x}_m\|_c^2 + \big(\frac{a}{\mu_j} - \frac{\epsilon}{\mu_j} - 1\big)\|v_m\|_c^2\\ &< K_0(\|x_m\|_c + \|v_m\|_c).\end{aligned}$$

Since $b < \mu_{j+1}$ and $\mu_j < a$, one has that, for $\epsilon > 0$ sufficiently small,

$$\delta\left(\|x_m^0\|_c^2 + \|\overline{x}_m\|_c^2 + \|v_m\|_c^2\right) < K_0(\|x_m\|_c + \|v_m\|_c),$$

where $0 < \delta < \min\left\{\big(1 - \frac{b}{\mu_{j+1}} - \frac{\epsilon}{\mu_{j+1}}\big), \big(\frac{a}{\mu_j} - \frac{\epsilon}{\mu_j} - 1\big)\right\}$. Hence,

$$\|u_m\|_c^2 < \widetilde{K}_0\|u_m\|_c,$$

which implies that $\{u_m\}$ is bounded in $H^1(\Omega)$. Therefore, by Proposition 3.3, I satisfies the Palais-Smale condition. The proof is complete. □

Remark 3.4. This appears to be the first time that the boundary nonlinearity g is compared with higher Steklov eigenvalues (also see e.g. [10]). Even at the first Steklov eigenvalue, we impose conditions on the potential of the boundary nonlinearity g rather than on g itself as was done in previous papers. Notice that, in this case, we do not require a (one-sided) linear growth on g as was done in [1, 7, 8] nor do we require monotonicity conditions as was done in [7, 8]. This work is still in progress, and we hope that more general results will appear elsewhere.

Remark 3.5. Our results remain valid if one considers nonlinear equations with a more general linear part (in divergence form) with variable coefficients.

$$\begin{gathered}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) + c(x)u = 0 \quad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} + \sigma(x)u = g(x,u) \quad \text{on } \partial\Omega,\end{gathered} \tag{3.16}$$

where $\sigma \in L^\infty(\partial\Omega)$ with $\sigma(x) \geq 0$ a.e. on $\partial\Omega$, and $\partial/\partial\nu := \nu \cdot A\nabla$ is the outward (unit) conormal derivative. The matrix $A(x) := \big(a_{ij}(x)\big)$ is symmetric with $a_{ij} \in L^\infty(\Omega)$ such that there is a constant $\gamma > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$\langle A(x)\xi, \xi\rangle \geq \gamma|\xi|^2 \quad \text{a.e. on } \Omega.$$

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