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MODIFIED QUASI-REVERSIBILITY METHOD FOR NONAUTONOMOUS SEMILINEAR PROBLEMS

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ABSTRACT. We prove regularization for the ill-posed, semilinear evolution problem $du/dt = A(t, D)u(t) + h(t, u(t))$, $0 \le s \le t < T$, with initial condition $u(s) = \chi$ in a Hilbert space where D is a positive, self-adjoint operator in the space. As in recent literature focusing on linear equations, regularization is established by approximating a solution $u(t)$ of the problem by the solution of an approximate well-posed problem. The approximate problem will be defined by one specific approximation of the operator $A(t, D)$ which extends a recently introduced, modified quasi-reversibility method by Boussetila and Rebbani. Finally, we demonstrate our theory with applications to a wide class of nonlinear partial differential equations in L^2 spaces including the nonlinear backward heat equation with a time-dependent diffusion coefficient.

1. INTRODUCTION

During the previous several decades, much focus has been placed on approximating solutions of ill-posed problems such as the backward heat equation. In such problems, solutions do not depend continuously on initial data forcing numerical estimates of the solutions to become difficult. In general, many authors have established the "regularization" of the ill-posed Cauchy problem

$$\begin{gathered} \frac{du}{dt} = Au(t) \quad 0 \le t < T \\ u(0) = \chi \end{gathered} \tag{1.1}$$

for some prescribed operator A in a Banach space X, in which a known solution $u(t)$ of (1.1) is estimated to be close to the solution of a corresponding well-posed problem. For instance, first introduced by Lattes and Lions [12] and Miller [15], and later studied by Mel'nikova and Filinkov [14, Chapter 3.1.1] (cf. also [13, 16, 17, 1, 10, 20]), the approximate well-posed problem

$$\begin{gathered} \frac{dv}{dt} = f_\beta(A)v(t) \quad 0 \le t < T \\ v(0) = \chi \end{gathered} \tag{1.2}$$

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is used to regularize (1.1) where for $\beta > 0$, $f_\beta(A)$ is defined by the quasi-reversibility method $f_\beta(A) = A - \beta A^2$. Another well-known approximation introduced by Showalter [19] may be used where $f_\beta(A) = A(I + \beta A)^{-1}$ (cf. also [1, 11]). Under these circumstances, approximation results are obtained in the following manner.

Definition 1.1 ([11, Definition 3.1]). Let $u(t)$ be a solution of (1.1) with initial data $\chi \in X$ and let $v_\beta^\delta(t)$ be the solution of the well-posed problem (1.2) with initial data χ_δ. Problem (1.1) is *regularized* if for any $\delta > 0$, there exists $\beta = \beta(\delta) > 0$ such that

(i) $\beta \to 0$ as $\delta \to 0$,
(i) $\|u(t) - v_\beta^\delta(t)\| \to 0$ as $\delta \to 0$ for $0 \le t \le T$ whenever $\|\chi - \chi_\delta\| \le \delta$.

Regularization results (cf. [13, 14, 1, 10, 11]) and numerical estimates (cf. [20, 21]) for these results have been calculated in both Hilbert space and Banach space, and also for different variations of (1.1). For instance, regularization has been applied to backward or final value problems (cf. [23, 24]), nonlinear problems (cf. [22, 3]), and also nonautonomous problems where the operator A in (1.1) is replaced by the time-dependent operator $A(t)$ (cf. [7, 5]).

This paper extends recent regularization results for nonlinear ill-posed problems in [22] to regularization for the *nonautonomous semilinear evolution problem*

$$\begin{gathered} \frac{du}{dt} = A(t,D)u(t) + h(t,u(t)) \quad 0 \le s \le t < T \\ u(s) = \chi \end{gathered} \tag{1.3}$$

in a Hilbert space H where D is a positive, self-adjoint operator in H and $A(t,D) = \sum_{j=1}^k a_j(t)D^j$ where $a_j \in C([0,T] : \mathbb{R}^+) \cap C^1([0,T])$ for each $1 \le j \le k$. Under certain conditions on the function $h : [s,T] \times H \to H$, we prove that the ill-posed problem (1.3) may be regularized as in Definition 1.1, by considering the approximate well-posed problem

$$\begin{gathered} \frac{dv}{dt} = f_\beta(t,D)v(t) + h(t,v(t)) \quad 0 \le s \le t < T \\ v(s) = \chi \end{gathered} \tag{1.4}$$

where

$$f_\beta(t,D) = -\frac{1}{T-s}\ln(\beta + e^{-(T-s)A(t,D)}), \quad 0 \le t \le T.$$

The approximation $f_\beta(t,D)$ of $A(t,D)$ extends the approximation

$$f_\beta(A) = -\frac{1}{pT}\ln(\beta + e^{-pTA}), \quad \beta > 0,\ p \ge 1$$

recently introduced by Boussetila and Rebanni [2] as a modified quasi-reversibility method and employed by Huang [9] and Trong and Tuan [22] in the case of the autonomous problem (1.1). As is discussed in [2], [9], and [22], one advantage of this more recent approximation is that the amplification factor of the error between the operators A and $f_\beta(A)$ is milder than if the approximations $A - \beta A^2$ or $A(I+\beta A)^{-1}$ were used, both of which induce an error of order $e^{C/\beta}$. Results in the current paper may analogously be compared to regularization estimates recently established for nonautonomous problems in which the approximation $f_\beta(t,D) = A(t,D) - \beta D^{k+1}$ of $A(t,D)$ is used (cf. [6] and also [7]).

The paper is organized as follows. In Section 2, we prove that the approximate problem (1.4) is well-posed with unique classical solution for every $\chi \in H$, that is a function $v_\beta : [s,T] \to H$ such that $v_\beta(t) \in \mathrm{Dom}(f_\beta(t,D))$ for $s < t < T$, $v_\beta \in C([s,T]:H) \cap C^1((s,T):H)$, and v_β satisfies (1.4) in H (cf. [18, Chapter 5.1 p. 126]). We also in Section 2 discuss the nature in which the operators $f_\beta(t,D)$ approximate the operators $A(t,D)$ and in Section 3, we use these results to show that the solution of (1.4) may be used to regularize the ill-posed problem (1.3). Finally, in Section 4, we apply our theory to a wide class of nonlinear partial differential equations in $L^2(\mathbb{R}^n)$ with a simple application to the nonlinear backward heat equation with a time-dependent diffusion coefficient.

Below, $\rho(D)$ will denote the resolvent set of the operator D which consists of all complex numbers λ such that $(\lambda I - D)^{-1}$ exists as an everywhere-defined bounded operator. The set $\sigma(D)$ will denote the spectrum of D which is defined as the complement of $\rho(D)$.

2. Semilinear Evolution Equations

Consider the generally ill-posed, semilinear evolution equation (1.3) where D is a positive, self-adjoint operator in a Hilbert space H and $A(t,D) = \sum_{j=1}^{k} a_j(t) D^j$ with $a_j \in C([0,T]:\mathbb{R}^+) \cap C^1([0,T])$ for each $1 \le j \le k$. Also, let $0 < \beta < 1$ and consider the approximate problem (1.4) where

$$f_\beta(t,D) = -\frac{1}{T-s} \ln(\beta + e^{-(T-s)A(t,D)}), \quad 0 \le t \le T. \tag{2.1}$$

We note that for $t \in [0,T]$, $f_\beta(t,D)$ is defined by means of the functional calculus for self-adjoint operators in the Hilbert space H. Specifically, since D is positive, self-adjoint, the spectrum $\sigma(D)$ of D is contained in $[0,\infty)$. Furthermore, for $t \in [0,T]$, since the function $f_\beta(t,\lambda) = -\frac{1}{T-s} \ln(\beta + e^{-(T-s)A(t,\lambda)})$ is a Borel function defined for $\lambda \in [0,\infty)$, the operator $f_\beta(t,D)$ is then defined by $f_\beta(t,D)x = \int_0^\infty f_\beta(t,\lambda) dE(\lambda)x$ for

$$x \in \mathrm{Dom}(f_\beta(t,D)) = \{x \in H : \int_0^\infty |f_\beta(t,\lambda)|^2 d(E(\lambda)x,x) < \infty\} \tag{2.2}$$

where $\{E(\cdot)\}$ denotes the resolution of the identity, that is the unique spectral measure associated with the operator D satisfying the equations $\mathrm{Dom}(D) = \{x \in H : \int_0^\infty |\lambda|^2 d(E(\lambda)x,x) < \infty\}$ and $Dx = \int_0^\infty \lambda dE(\lambda)x$ for $x \in \mathrm{Dom}(D)$ (cf. [4, Theorem XII.2.3, Theorem XII.2.6]).

Note that for $(t,\lambda) \in [0,T] \times [0,\infty)$, since $A(t,\lambda) \ge A(t,0) = 0$,

$$\ln \beta \le \ln(\beta + e^{-(T-s)A(t,\lambda)}) \le \ln(\beta + 1).$$

Multiplying through by $-(T-s)^{-1}$ yields

$$-\frac{1}{T-s} \ln(\beta+1) \le f_\beta(t,\lambda) \le -\frac{1}{T-s} \ln\ \beta. \tag{2.3}$$

Hence by (2.2) and (2.3), we see that $\mathrm{Dom}(f_\beta(t,D)) = H$ and $f_\beta(t,D)$ is a bounded, everywhere-defined operator on H for each $t \in [0,T]$.

Since $f_\beta(t,D)$ is a bounded operator on H for each $t \in [0,T]$, the linear version of (1.4) is easily well-posed meaning that a unique classical solution exists for each χ in a dense subset of X and solutions depend continuously on the initial data (cf. [8, Chapter 2.13, p. 140]). In order to show that the *nonlinear* problem (1.4) is well-posed and to ultimately prove regularization for (1.3), we will also require special conditions on the function $h : [s,T] \times H \to H$. We have

Proposition 2.1. *Let H be a Hilbert space and for $0 < \beta < 1$, let the operators $f_\beta(t,D), 0 \le t \le T$ be defined by (2.1). Assume the function $h : [s,T] \times H \to H$ satisfies the following conditions.*

- (i) *h is uniformly Lipschitz in H, i.e. $\|h(t,x_1) - h(t,x_2)\| \le L\|x_1 - x_2\|$ for some constant $L > 0$ independent of $t \in [s,T]$ and every $x_1, x_2 \in H$,*
- (ii) *for each $x \in H$, $h(t,x)$ is continuous from $[s,T]$ into H.*

Then (1.4) is well-posed, with unique classical solution $v_\beta(t)$ for every $\chi \in H$ satisfying the integral equation

$$v_\beta(t) = e^{\int_s^t f_\beta(\tau,D)d\tau}\chi + \int_s^t e^{\int_r^t f_\beta(\tau,D)d\tau} h(r, v_\beta(r))dr. \tag{2.4}$$

Proof. We first note that, as previously discussed, the inequality (2.3) implies that $f_\beta(t,D)$ is a bounded operator on H for each $t \in [0,T]$. Also, by the assumption on the functions a_j, it may be shown that for each $x \in H$, the function $t \mapsto f_\beta(t,D)x$ is a continuously differentiable function. These properties together imply that the linear, homogeneous version of (1.4) is well-posed, and also provides the existence of an evolution system $V_\beta(t,s) = e^{\int_s^t f_\beta(\tau,D)d\tau}, 0 \le s \le t \le T$ associated with the operators $f_\beta(t,D), 0 \le t \le T$ (cf. [18, Theorem 5.4.8, Theorem 5.4.3], [6, Proposition 2.10]).

As $(t,s) \mapsto V_\beta(t,s)$ is continuous in the strong operator topology (cf. [18, Theorem 5.4.8]), and by the assumptions on the function $h : [s,T] \times H \to H$, following [18, Theorem 6.1.7], we define the mapping $F : C([s,T] : H) \to C([s,T] : H)$ by

$$(Fv)(t) = V_\beta(t,s)\chi + \int_s^t V_\beta(t,r)h(r,v(r))dr.$$

Using properties of the bounded operators $V_\beta(t,s)$ and the Lipschitz condition on h, it follows from an application of the Contraction Mapping Theorem that F has a unique fixed point $v_\beta \in C([s,T] : H)$ (cf. [18, Proof of Theorem 6.1.2]).

Next, define $G(t) = h(t, v_\beta(t))$ and consider the linear evolution problem

$$\begin{gathered}\frac{dw}{dt} = f_\beta(t,D)w(t) + G(t) \quad 0 \le s \le t < T \\ w(s) = \chi.\end{gathered} \tag{2.5}$$

Note $G(t)$ is continuous from $[s,T]$ into H by the following calculation which follows from our assumptions on h and continuity of $v_\beta(t)$:

$$\begin{aligned}\|G(t) - G(t_0)\| &= \|h(t,v_\beta(t)) - h(t_0, v_\beta(t_0))\| \\ &\le \|h(t,v_\beta(t)) - h(t,v_\beta(t_0))\| + \|h(t,v_\beta(t_0)) - h(t_0,v_\beta(t_0))\| \\ &\le L\|v_\beta(t) - v_\beta(t_0)\| + \|h(t,v_\beta(t_0)) - h(t_0,v_\beta(t_0))\| \\ &\to 0 \quad \text{as} \quad t \to t_0.\end{aligned}$$

Hence, by [18, Theorem 5.5.2], (2.5) is well-posed with unique classical solution

$$w_\beta(t) = V_\beta(t,s)\chi + \int_s^t V_\beta(t,r)G(r)dr,$$

implying that

$$w_\beta(t) = V_\beta(t,s)\chi + \int_s^t V_\beta(t,r)h(r, v_\beta(r))dr = v_\beta(t)$$

since v_β is a fixed point of F. Because $w_\beta(t)$ is a classical solution of (2.5), $v_\beta(t)$ must then be a classical solution of (1.4). Uniqueness follows from uniqueness of the fixed point since any classical solution of (1.4) satisfies the integral equation (2.4).

Finally, continuous dependence on initial data holds by the following calcluation. By (2.3), consider for $0 \le s \le t \le T$ and $x \in H$,

$$\begin{aligned} \|V_\beta(t,s)x\|^2 &= \|e^{\int_s^t f_\beta(\tau,D)d\tau}x\|^2 \\ &= \int_0^\infty |e^{\int_s^t f_\beta(\tau,\lambda)d\tau}|^2 d(E(\lambda)x,x) \\ &\le \int_0^\infty |e^{-\frac{t-s}{T-s}ln\beta}|^2 d(E(\lambda)x,x) \\ &= \int_0^\infty |\beta^{\frac{s-t}{T-s}}|^2 d(E(\lambda)x,x) \\ &= (\beta^{\frac{s-t}{T-s}}\|x\|)^2 \end{aligned} \tag{2.6}$$

which implies $\|V_\beta(t,s)\| \le \beta^{\frac{s-t}{T-s}}$. Now, let v_1 and v_2 be classical solutions of (1.4) corresponding to initial data χ_1 and χ_2 respectively. Then, as v_1 and v_2 each satisfy (2.4), and since $0 < \beta < 1$, we have

$$\begin{aligned} &\|v_1(t) - v_2(t)\| \\ &\le \|V_\beta(t,s)\chi_1 - V_\beta(t,s)\chi_2\| + \int_s^t \|V_\beta(t,r)h(r,v_1(r)) - V_\beta(t,r)h(r,v_2(r))\|dr \\ &\le \beta^{\frac{s-t}{T-s}}\|\chi_1 - \chi_2\| + \int_s^t \beta^{\frac{r-t}{T-s}}\|h(r,v_1(r)) - h(r,v_2(r))\|dr \\ &\le \beta^{-1}\|\chi_1 - \chi_2\| + L\beta^{-1}\int_s^t \|v_1(r) - v_2(r)\|dr. \end{aligned}$$

Gronwall's Inequality (cf. [18, Proof of Theorem 6.1.2]) then implies

$$\begin{aligned} \|v_1(t) - v_2(t)\| &\le \beta^{-1}\|\chi_1 - \chi_2\|e^{L\beta^{-1}(T-s)} \\ &\to 0 \quad \text{as } \|\chi_1 - \chi_2\| \to 0 \text{ for each } t \in [s,T]. \end{aligned}$$

□

We have shown that under the assumptions of Proposition 2.1, the approximate problem (1.4) is well-posed with unique classical solution $v_\beta(t)$. In order that the solution $v_\beta(t)$ of (1.4) is used to regularize problem (1.3), we will examine the difference between the operators $A(t,D)$ and the approximate operators $f_\beta(t,D)$. The following lemma demonstrates this and is motivated by the approximation condition, Condition A, of Ames and Hughes (cf. [1, Definition 1], and also [22, Definition p. 4]).

Lemma 2.2. *Let H be a Hilbert space and for $0 < \beta < 1$, let the operators $f_\beta(t,D), 0 \le t \le T$ be defined by (2.1). Define $B(\lambda) = \sum_{j=1}^k B_j\lambda^j$ where $B_j =$*

$\max_{t\in[0,T]} a_j(t)$ *for each* $1 \le j \le k$. *Then for each* $t \in [0,T]$,

$$\mathrm{Dom}(e^{(T-s)B(D)}) \subseteq \mathrm{Dom}(A(t,D)) \cap \mathrm{Dom}(f_\beta(t,D))$$

and

$$\|(-A(t,D) + f_\beta(t,D))\psi\| \le \frac{\beta}{T-s}\|e^{(T-s)B(D)}\psi\|$$

for all $\psi \in \mathrm{Dom}(e^{(T-s)B(D)})$.

Proof. Let $t \in [0,T]$ and $\psi \in \mathrm{Dom}(e^{(T-s)B(D)})$. Then since $(T-s)A(t,\lambda) \le (T-s)B(\lambda) \le e^{(T-s)B(\lambda)}$ for $\lambda \ge 0$, we have $\psi \in \mathrm{Dom}(A(t,D))$ by (2.2), and so $\psi \in \mathrm{Dom}(A(t,D)) \cap \mathrm{Dom}(f_\beta(t,D))$ since $\mathrm{Dom}(f_\beta(t,D)) = H$. Next,

$$\begin{aligned}
&\|(-A(t,D) + f_\beta(t,D))\psi\|^2\\
&= \int_0^\infty |-A(t,\lambda) + f_\beta(t,\lambda)|^2 d(E(\lambda)\psi,\psi)\\
&= \int_0^\infty |A(t,\lambda) + \frac{1}{T-s}\ln(\beta + e^{-(T-s)A(t,\lambda)})|^2 d(E(\lambda)\psi,\psi)\\
&= \int_0^\infty |\frac{1}{T-s}\ln(e^{(T-s)A(t,\lambda)}) + \frac{1}{T-s}\ln(\beta + e^{-(T-s)A(t,\lambda)})|^2 d(E(\lambda)\psi,\psi)\\
&= \int_0^\infty |\frac{1}{T-s}\ln(\beta e^{(T-s)A(t,\lambda)} + 1)|^2 d(E(\lambda)\psi,\psi)
\end{aligned}$$

and using the fact that $\ln(x+1) \le x$ for $x \ge 0$, then

$$\begin{aligned}
\|(-A(t,D) + f_\beta(t,D))\psi\|^2 &\le \int_0^\infty |\frac{\beta}{T-s}e^{(T-s)A(t,\lambda)}|^2 d(E(\lambda)\psi,\psi)\\
&\le \int_0^\infty |\frac{\beta}{T-s}e^{(T-s)B(\lambda)}|^2 d(E(\lambda)\psi,\psi)\\
&= \|\frac{\beta}{T-s}e^{(T-s)B(D)}\psi\|^2
\end{aligned}$$

proving the desired result. □

In light of the inequality in Lemma 2.2, define for $(t,\lambda) \in [0,T] \times [0,\infty)$,

$$g_\beta(t,\lambda) = -A(t,\lambda) + f_\beta(t,\lambda).$$

Note,

$$\ln(\beta + e^{-(T-s)A(t,\lambda)}) \ge \ln(e^{-(T-s)A(t,\lambda)}) = -(T-s)A(t,\lambda)$$

which, after multiplying through by $-(T-s)^{-1}$, yields $f_\beta(t,\lambda) \le A(t,\lambda)$ and hence

$$g_\beta(t,\lambda) \le 0 \quad \text{for } (t,\lambda) \in [0,T] \times [0,\infty). \tag{2.7}$$

For each natural number n, set $e_n = \{\lambda \in [0,\infty) : \max_{t\in[0,T]} |A(t,\lambda)| \le n\}$. Note by inequality (2.3),

$$\lambda \in e_n \Rightarrow \max_{t\in[0,T]} |f_\beta(t,\lambda)| \le M_\beta \tag{2.8}$$

for some constant M_β, and by the definition of $g_\beta(t,\lambda)$, then

$$\lambda \in e_n \Rightarrow \max_{t\in[0,T]} |g_\beta(t,\lambda)| \le n + M_\beta. \tag{2.9}$$

Set $E_n = E(e_n)$ and let $\psi \in H$ be arbitrary. Consider the homogeneous evolution problem

$$\begin{gathered}\frac{du}{dt} = A(t,D)E_n u(t) \quad 0 \le s \le t < T \\ u(s) = \psi.\end{gathered} \tag{2.10}$$

Lemma 2.3. *The evolution problem* (2.10) *has a unique classical solution* $t \mapsto U_n(t,s)\psi$ *for every* $\psi \in H$*, where* $U_n(t,s)$*,* $0 \le s \le t \le T$ *is an evolution system on* H *such that* $U_n(t,s) = e^{\int_s^t A(\tau,D)d\tau}$ *when acting on* E_nH*.*

Proof. For each $t \in [0,T]$, $A(t,D)E_n$ is a bounded operator on H since $|A(t,\lambda)| \le n$ for $(t,\lambda) \in [0,T]\times e_n$. Also, the function $t \mapsto A(t,D)E_n$ is continuous in the uniform operator topology since each a_j is a continuous function. This implies the existence of a solution operator $U_n(t,s), 0 \le s \le t \le T$ such that $t \mapsto U_n(t,s)\psi$ is a unique classical solution of the homogeneous problem (2.10) for every $\psi \in H$ (cf. [18, Theorem 5.1.1]). It may also be shown that $U_n(t,s)$ is an evolution system with the property that $U_n(t,s) = e^{\int_s^t A(\tau,D)d\tau}$ when acting on E_nH (cf. [18, Theorem 5.1.2] and [6, Lemma 3.2]). □

Note by replacing $A(t,D)E_n$ with either $f_\beta(t,D)E_n$ or $g_\beta(t,D)E_n$ in (2.10), we obtain by (2.8) and (2.9), evolution systems $V_{\beta,n}(t,s)$ or $W_{\beta,n}(t,s)$, respectively, similarly as in Lemma 2.3. We have the following corollary.

Corollary 2.4. *Let* $\psi_n \in E_nH$*. Then*

$$U_n(t,s)W_{\beta,n}(t,s)\psi_n = V_{\beta,n}(t,s)\psi_n = W_{\beta,n}(t,s)U_n(t,s)\psi_n$$

for all $0 \le s \le t \le T$*.*

Proof. Note that just as $U_n(t,s) = e^{\int_s^t A(\tau,D)d\tau}$ when acting on E_nH, we have $V_{\beta,n}(t,s) = e^{\int_s^t f_\beta(\tau,D)d\tau}$ and $W_{\beta,n}(t,s) = e^{\int_s^t g_\beta(\tau,D)d\tau}$ when acting on E_nH as well. The identity then follows from the relation $g_\beta(t,\lambda) = -A(t,\lambda) + f_\beta(t,\lambda)$ and from properties of the functional calculus for self-adjoint operators (cf. [4, Corollary XII.2.7]). □

3. Regularization for problem (1.3)

In this section, we use the results from Section 2 to prove regularization for the ill-posed problem (1.3) (Theorem 3.4 below).

Lemma 3.1. *Let* $u(t)$ *and* $v_\beta(t)$ *be classical solutions of* (1.3) *and* (1.4) *respectively where the operators* $f_\beta(t,D)$*,* $0 \le t \le T$ *are defined by* (2.1) *and* $h : [s,T] \times H \to H$ *satisfies the hypotheses of Proposition 2.1. Also, set* $\chi_n = E_n\chi$ *and* $h_n(t,x) = E_nh(t,x)$ *for all* $(t,x) \in [s,T] \times H$*. Then*

$$\begin{gathered}E_nu(t) = U_n(t,s)\chi_n + \int_s^t U_n(t,r)h_n(r,u(r))dr, \\ E_nv_\beta(t) = V_{\beta,n}(t,s)\chi_n + \int_s^t V_{\beta,n}(t,r)h_n(r,v_\beta(r))dr\end{gathered}$$

for all $t \in [s,T]$*.*

Proof. The first identity follows from uniqueness of solutions since both sides of the equation are classical solutions of the linear inhomogeneous problem

$$\begin{gathered}\frac{dw}{dt} = A(t,D)E_n w(t) + h_n(t,u(t)) \quad 0 \le s \le t < T \\ w(s) = \chi_n.\end{gathered} \tag{3.1}$$

The second identity holds by a similar argument with $A(t,D)E_n$ replaced by the function $f_\beta(t,D)E_n$ in (3.1). □

As in Lemma 2.2, set $B(\lambda) = \sum_{j=1}^{k} B_j\lambda^j$ where $B_j = \max_{t\in[0,T]} a_j(t)$ for each $1 \le j \le k$. We have the following result.

Theorem 3.2. *Let $u(t)$ and $v_\beta(t)$ be classical solutions of* (1.3) *and* (1.4) *respectively where the operators $f_\beta(t,D)$, $0 \le t \le T$ are defined by* (2.1) *and $h : [s,T] \times H \to H$ satisfies the hypotheses of Proposition 2.1. Then if there exist constants $M', M'' \ge 0$ such that*

$$\begin{gathered}\|e^{(T-s)B(D)}e^{\int_s^t A(\tau,D)d\tau}\chi\| \le M', \\ \|e^{(T-s)B(D)}e^{\int_s^t A(\tau,D)d\tau}h(t,u(t))\| \le M''\end{gathered}$$

for all $t \in [s,T]$, then there exist constants C and L independent of β such that

$$\|u(t) - v_\beta(t)\| \le \beta^{\frac{T-t}{T-s}} C e^{L(T-s)} \quad \text{for } 0 \le s \le t < T.$$

Proof. As in Lemma 3.1, set $\chi_n = E_n\chi$ and $h_n(t,x) = E_n h(t,x)$ for all $(t,x) \in [s,T] \times H$. From Lemma 3.1, for $0 \le s \le t < T$,

$$\begin{aligned}
&\|E_n u(t) - E_n v_\beta(t)\| \\
&\le \|U_n(t,s)\chi_n - V_{\beta,n}(t,s)\chi_n\| + \int_s^t \|U_n(t,r)h_n(r,u(r)) - V_{\beta,n}(t,r)h_n(r,v_\beta(r))\|dr \\
&\le \|U_n(t,s)\chi_n - V_{\beta,n}(t,s)\chi_n\| &(3.2)\\
&\quad + \int_s^t \|U_n(t,r)h_n(r,u(r)) - V_{\beta,n}(t,r)h_n(r,u(r))\|dr &(3.3)\\
&\quad + \int_s^t \|V_{\beta,n}(t,r)h_n(r,u(r)) - V_{\beta,n}(t,r)h_n(r,v_\beta(r))\|dr. &(3.4)
\end{aligned}$$

For the expression (3.2), we have by Corollary 2.4 and standard properties of evolution systems ([18, Theorem 5.1.2]),

$$\begin{aligned}
\|U_n(t,s)\chi_n - V_{\beta,n}(t,s)\chi_n\| &= \|(I - W_{\beta,n}(t,s))U_n(t,s)\chi_n\| \\
&= \|(W_{\beta,n}(t,t) - W_{\beta,n}(t,s))U_n(t,s)\chi_n\| \\
&= \Big\|\int_s^t \frac{\partial}{\partial q}W_{\beta,n}(t,q)U_n(t,s)\chi_n dq\Big\| \\
&= \Big\|\int_s^t (-W_{\beta,n}(t,q)g_\beta(q,D)E_n)U_n(t,s)\chi_n dq\Big\|.
\end{aligned}$$

Recall $W_\beta(t,s) = e^{\int_s^t g_\beta(\tau,D)d\tau}$ when acting on E_nH (Proof of Corollary 2.4). Also, since e_n is a bounded subset of $[0,\infty)$ and $E_n = E(e_n)$ is a projection operator, it is clear from (2.2) that $U_n(t,s)\chi_n \in \mathrm{Dom}(e^{(T-s)B(D)})$. Therefore, we have by (2.7) and Lemma 2.2,

$$\begin{aligned}
(3.2) &\le \int_s^t \|W_{\beta,n}(t,q)g_\beta(q,D)U_n(t,s)\chi_n\|dq \\
&\le \int_s^t \|g_\beta(q,D)U_n(t,s)\chi_n\|dq \\
&\le \frac{\beta}{T-s}(t-s)\|e^{(T-s)B(D)}U_n(t,s)\chi_n\|.
\end{aligned}$$

Similarly, the second expression

$$\begin{aligned}
(3.3) &= \int_s^t \|(I-W_{\beta,n}(t,r))U_n(t,r)h_n(r,u(r))\|dr \\
&\le \int_s^t \frac{\beta}{T-s}(t-r)\|e^{(T-s)B(D)}U_n(t,r)h_n(r,u(r))\|dr.
\end{aligned}$$

Combining these calculations for (3.2) and (3.3), we have

$$\begin{aligned}
&\|U_n(t,s)\chi_n - V_{\beta,n}(t,s)\chi_n\| + \int_s^t \|U_n(t,r)h_n(r,u(r)) - V_{\beta,n}(t,r)h_n(r,u(r))\|dr \\
&\le \beta C
\end{aligned} \tag{3.5}$$

where C is a constant independent of β and also independent of n and t by our stabilizing constants M' and M''. Finally, by (2.6) and the Lipschitz condition on h, the third expression is

$$\begin{aligned}
(3.4) &= \int_s^t \|V_{\beta,n}(t,r)(h_n(r,u(r)) - h_n(r,v_\beta(r)))\|dr \\
&\le \int_s^t \beta^{\frac{r-t}{T-s}}\|h_n(r,u(r)) - h_n(r,v_\beta(r))\|dr \qquad (3.6) \\
&\le L\int_s^t \beta^{\frac{r-t}{T-s}}\|u(r)-v_\beta(r)\|dr.
\end{aligned}$$

Combining (3.5) and (3.6), we have shown that $\|E_nu(t) - E_nv_\beta(t)\| \le \beta C + L\int_s^t \beta^{\frac{r-t}{T-s}}\|u(r)-v_\beta(r)\|dr$, and since all constants on the right are independent of n, we may let $n\to\infty$ to obtain

$$\|u(t)-v_\beta(t)\| \le \beta C + L\int_s^t \beta^{\frac{r-t}{T-s}}\|u(r)-v_\beta(r)\|dr.$$

Next, note since $0<\beta<1$, we have

$$\|u(t)-v_\beta(t)\| \le \beta^{\frac{T-t}{T-s}}C + L\int_s^t \beta^{\frac{r-t}{T-s}}\|u(r)-v_\beta(r)\|dr$$

which yields

$$\beta^{\frac{t-T}{T-s}}\|u(t)-v_\beta(t)\| \le C + L\int_s^t \beta^{\frac{r-T}{T-s}}\|u(r)-v_\beta(r)\|dr.$$

By Gronwall's inequality, then

$$\beta^{\frac{t-T}{T-s}}\|u(t)-v_\beta(t)\| \le Ce^{L(T-s)}$$

from which the desired result follows. $\square$

The inequality in Theorem 3.2 establishes continuous dependence on modeling for problems (1.3) and (1.4) meaning that as $\beta \to 0$, the operators $f_\beta(t, D)$ approach the operators $A(t, D)$ and also the difference in solutions $u(t)$ and $v_\beta(t)$ tends to 0 for each $t \in [s, T)$. Since convergence to 0 in the inequality $\|u(t) - v_\beta(t)\| \leq C\beta^{\frac{T-t}{T-s}} e^{L(t-s)}$ is lost when $t = T$, we provide an alternate estimate for this specific case. Following calculations in [22], we have

Lemma 3.3. *Let $u(t)$ and $v_\beta(t)$ be the solutions of* (1.3) *and* (1.4) *respectively as in Theorem 3.2 and let the hypotheses of Theorem* 3.2 *be satisfied. Then there exists $t_\beta \in (s, T)$ and constants C and M independent of β such that*

$$\|u(T) - v_\beta(t_\beta)\| \leq M\sqrt{-\frac{1}{\ln \beta}} + \beta^{\frac{1}{T-s}\sqrt{-\frac{1}{ln\beta}}} Ce^{L(T-s)}.$$

Proof. Following [22, Proof of Theorem 3.1], define the function

$$q(t) = -\frac{1}{(T-t)^2} - \ln\ \beta, \quad s < t < T.$$

It is easy to see that for sufficiently small β, there exists $t_0 \in (s, T)$ such that $q(t_0) > 0$. Since $\lim_{t\to T^-} q(t) = -\infty$, there must then exist $t_\beta \in (t_0, T)$ such that $q(t_\beta) = 0$, that is $-\frac{1}{(T-t_\beta)^2} = \ln \beta$. Hence, we have

$$T - t_\beta = \sqrt{-\frac{1}{\ln \beta}}. \tag{3.7}$$

Now, consider by Lemma 3.1,

$$\begin{aligned}
\|E_n u(T) - E_n u(t_\beta)\| &= \|\int_{t_\beta}^T \frac{d}{dt} E_n u(t)\ dt\| \\
&= \|\int_{t_\beta}^T (A(t, D)E_n u(t) + h_n(t, u(t)))\ dt\| \\
&\leq \int_{t_\beta}^T (\|A(t, D)U_n(t, s)\chi_n\| \\
&\quad + \int_s^t \|A(t, D)U_n(t, r)h_n(r, u(r))\|dr + \|h_n(t, u(t))\|)\ dt \\
&\leq M(T - t_\beta)
\end{aligned}$$

for some constant M independent of β and n where the last inequality follows from the stabilizing constants M' and M'' of Theorem 3.2. Letting $n \to \infty$, we obtain $\|u(T) - u(t_\beta)\| \leq M(T - t_\beta)$. Finally, since $t_\beta \in (s, T)$, we have by Theorem 3.2 and (3.7),

$$\begin{aligned}
\|u(T) - v_\beta(t_\beta)\| &\leq \|u(T) - u(t_\beta)\| + \|u(t_\beta) - v_\beta(t_\beta)\| \\
&\leq M(T - t_\beta) + \beta^{\frac{T-t_\beta}{T-s}} Ce^{L(T-s)} \\
&= M\sqrt{-\frac{1}{\ln\ \beta}} + \beta^{\frac{1}{T-s}\sqrt{-\frac{1}{ln\beta}}} Ce^{L(T-s)},
\end{aligned}$$

as desired. $\square$

We utilize the estimates in Theorem 3.2 and Lemma 3.3 to prove regularization for (1.3) as follows.

Theorem 3.4. *Let $u(t)$ be a classical solution of (1.3) as in Theorem 3.2 and let the hypotheses of Theorem 3.2 be satisfied. Then for any $\delta > 0$, there exists $\beta = \beta(\delta) > 0$ such that*

(i) $\beta \to 0$ *as* $\delta \to 0$,
(ii) $\|u(t) - v_\beta^\delta(t)\| \to 0$ *as* $\delta \to 0$ *for* $s \le t \le T$ *whenever* $\|\chi - \chi_\delta\| \le \delta$

where $v_\beta^\delta(t)$ is the solution of (1.4) with initial data χ_δ.

Proof. Let $\delta > 0$ be given and let $\|\chi - \chi_\delta\| \le \delta$. Also, let $v_\beta(t)$ be the solution of (1.4) as in Theorem 3.2. For $s \le t < T$, by Theorem 3.2, then

$$\begin{aligned} \|u(t) - v_\beta^\delta(t)\| &\le \|u(t) - v_\beta(t)\| + \|v_\beta(t) - v_\beta^\delta(t)\| \\ &\le \beta^{\frac{T-t}{T-s}} C e^{L(T-s)} + \|v_\beta(t) - v_\beta^\delta(t)\|. \end{aligned} \tag{3.8}$$

Consider the second quantity in (3.8). As in previous calculations, by (2.6) and the Lipschitz condition on H, we have

$$\begin{aligned} &\|v_\beta(t) - v_\beta^\delta(t)\| \\ &\le \|V_\beta(t,s)\chi - V_\beta(t,s)\chi_\delta\| + \int_s^t \|V_\beta(t,r)(h(r, v_\beta(r)) - h(r, v_\beta^\delta(r)))\| dr \\ &\le \delta \beta^{\frac{s-t}{T-s}} + L \int_s^t \beta^{\frac{r-t}{T-s}} \|v_\beta(r) - v_\beta^\delta(r)\| dr. \end{aligned}$$

Hence,

$$\beta^{\frac{t-T}{T-s}} \|v_\beta(t) - v_\beta^\delta(t)\| \le \delta \beta^{\frac{s-T}{T-s}} + L \int_s^t \beta^{\frac{r-T}{T-s}} \|v_\beta(r) - v_\beta^\delta(r)\| dr$$

which by Gronwall's Inequality gives us

$$\beta^{\frac{t-T}{T-s}} \|v_\beta(t) - v_\beta^\delta(t)\| \le \delta \beta^{\frac{s-T}{T-s}} e^{L(T-s)}.$$

Therefore,

$$\|v_\beta(t) - v_\beta^\delta(t)\| \le \delta \beta^{\frac{s-t}{T-s}} e^{L(T-s)} \tag{3.9}$$

and choosing $\beta = \delta$ yields

$$\|v_\beta(t) - v_\beta^\delta(t)\| \le \beta^{\frac{T-t}{T-s}} e^{L(T-s)}.$$

Thus, $\beta \to 0$ as $\delta \to 0$, and by (3.8), we have

$$\begin{aligned} \|u(t) - v_\beta^\delta(t)\| &\le \beta^{\frac{T-t}{T-s}} C e^{L(T-s)} + \beta^{\frac{T-t}{T-s}} e^{L(T-s)} \\ &\to 0 \quad \text{as } \delta \to 0. \end{aligned} \tag{3.10}$$

For the case that $t = T$ we utilize Lemma 3.3. Since $t_\beta \in (s,T)$, we have by Lemma 3.3 and (3.9),

$$\begin{aligned} &\|u(T) - v_\beta^\delta(T)\| \\ &\le \|u(T) - v_\beta(t_\beta)\| + \|v_\beta(t_\beta) - v_\beta^\delta(t_\beta)\| + \|v_\beta^\delta(t_\beta) - v_\beta^\delta(T)\| \\ &\le \Big(M\sqrt{-\frac{1}{\ln\beta}} + \beta^{\frac{1}{T-s}\sqrt{-\frac{1}{ln\beta}}} C e^{L(T-s)} \Big) + \delta \beta^{\frac{s-t_\beta}{T-s}} e^{L(T-s)} + \|v_\beta^\delta(t_\beta) - v_\beta^\delta(T)\|\,. \end{aligned}$$

Again choosing $\beta = \delta$ and applying (3.7), we have

$$
\begin{aligned}
&\|u(T) - v_\beta^\delta(T)\| \\
&\le M\sqrt{-\frac{1}{\ln\,\beta}} + \beta^{\frac{1}{T-s}\sqrt{-\frac{1}{ln\beta}}} Ce^{L(T-s)} + \beta^{\frac{T-t_\beta}{T-s}} e^{L(T-s)} + \|v_\beta^\delta(t_\beta) - v_\beta^\delta(T)\| \\
&= M\sqrt{-\frac{1}{\ln\beta}} + \beta^{\frac{1}{T-s}\sqrt{-\frac{1}{ln\beta}}} Ce^{L(T-s)} + \beta^{\frac{1}{T-s}\sqrt{-\frac{1}{ln\beta}}} e^{L(T-s)} + \|v_\beta^\delta(t_\beta) - v_\beta^\delta(T)\| \\
&\to 0 \quad \text{as } \delta \to 0,
\end{aligned}
\tag{3.11}
$$

the reason for convergence of the last term being that $t \mapsto v_\beta^\delta(t)$ is continuous and that t_β converges to T as $\beta \to 0$.

Combining (3.10) and (3.11) completes the proof and so the ill-posed problem (1.3) is regularized. □

4. Examples

The theory in Section 3 may be applied to a wide class of ill-posed partial differential equations in the Hilbert space $H = L^2(\mathbb{R}^n)$ with $D = -\Delta$ where Δ denotes the Laplacian defined by $\Delta\psi = \sum_{i=1}^n \frac{\partial^2\psi}{\partial x_i^2}$. The operator $-\Delta$ is a positive, self-adjoint operator in $L^2(\mathbb{R}^n)$ and hence the partial differential equation

$$
\begin{gathered}
\frac{\partial}{\partial t}u(t,x) = A(t,-\Delta)u(t,x) + h(t,x,u(t,x)), (t,x) \in [s,T) \times \mathbb{R}^n \\
u(s,x) = \psi(x), \quad x \in \mathbb{R}^n
\end{gathered}
\tag{4.1}
$$

in $L^2(\mathbb{R}^n)$, where $A(t,-\Delta) = \sum_{j=1}^k a_j(t)(-\Delta)^j$ and $a_j \in C([0,T] : \mathbb{R}^+) \cap C^1([0,T])$ for $1 \le j \le k$, is generally ill-posed. By Theorem 3.4, problem (4.1) is regularized by considering solutions of the approximate well-posed problem (1.4).

We note a simple example of (4.1) is the case that $H = (L^2(\mathbb{R}), \|\cdot\|_2)$ and there is only one term in the sum $A(t,-\Delta)$ (i.e. $k = 1$), yielding regularization for the nonlinear backward heat equation

$$
\begin{gathered}
\frac{\partial}{\partial t}u(t,x) = -a(t)\Delta u(t,x) + tu(t,x) + e^{-(x-t)^2}, (t,x) \in [s,T) \times \mathbb{R} \\
u(s,x) = \psi(x), \quad x \in \mathbb{R}
\end{gathered}
\tag{4.2}
$$

with time-dependent diffusion coefficient $a(t)$ and h defined as $h(t,x,\phi(x)) = t\phi(x) + e^{-(x-t)^2}$ for $\phi \in L^2(\mathbb{R})$. The function h is well-defined since $e^{-(x-t)^2} \in L^2(\mathbb{R})$ for any $t \in [s,T]$. Using a dominated convergence argument, it may also be shown that for each $\phi \in H$, $t \mapsto h(t,x,\phi(x))$ is continuous. Finally, h is clearly uniformly Lipschitz in H by the calculation,

$$
\begin{aligned}
\|h(t,x,\phi_1(x)) - h(t,x,\phi_2(x))\|_2 &= \|(t\phi_1(x) + e^{-(x-t)^2}) - (t\phi_2(x) + e^{-(x-t)^2})\|_2 \\
&= \|t\phi_1(x) - t\phi_2(x)\|_2 \\
&\le T\|\phi_1(x) - \phi_2(x)\|_2.
\end{aligned}
$$

Hence, by Proposition 2.1, the approximate problem (1.4) corresponding to (4.2) is well-posed, and by Theorem 3.4, provides regularization for the ill-posed problem (4.2).

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