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EXTINCTION OF WEAK SOLUTIONS OF DOUBLY NONLINEAR NAVIER-STOKES EQUATIONS

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Abstract. In this article we discus the doubly nonlinear incompressible Navier-Stokes equations

$$\frac{\partial b(u)}{\partial t} + \operatorname{div}(b(u) \otimes u) = -d\pi + \operatorname{div}(a(\nabla^{\mathrm{sym}} u)) + f\,, \quad \operatorname{div}(u) = 0\,,$$

where u models the velocity vector field of a homogeneous incompressible non-Newtonian fluid whose momentum $b(u)$ depends nonlinearly on u. Particularly, under certain regularity assumptions it is shown that u becomes extinct in finite time for sufficiently small initial values $u(0)$, if $a(\nabla^{\mathrm{sym}} u) := (1 + |\nabla^{\mathrm{sym}} u|^{p-2})\nabla^{\mathrm{sym}} u$ and $b(u) := |u|^{m-2}u$ with $1 < p < m < \infty$.

1. Introduction

Doubly nonlinear incompressible Navier-Stokes equations (DNNS)

$$\begin{aligned} \frac{\partial b(u)}{\partial t} + \operatorname{div}(b(u) \otimes u) &= -d\pi + \operatorname{div}(a(\nabla^{\mathrm{sym}} u)) + f \\ \operatorname{div}(u) &= 0 \end{aligned} \tag{1.1}$$

are a generalization of incompressible Navier-Stokes equations

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla_u u &= -\operatorname{grad} \pi + \operatorname{div}\big(a(\nabla^{\mathrm{sym}} u)\big)^\flat + f^\flat \\ \operatorname{div}(u) &= 0 \end{aligned} \tag{1.2}$$

for the velocity vector field u of a non-Newtonian fluid with pressure π. In fact, while in (1.2) only the viscous stress tensor a is allowed to depend nonlinearly on the symmetric part $\nabla^{\mathrm{sym}} u$ of the derivative of u, in (1.1) additionally the relation between the velocity vector field u and the momentum $b(u)$ of the fluid is allowed to be nonlinear. Observe that in the case $b(u) = u^\sharp$ equation (1.1) is just the dual of equation (1.2). In fact, the divergence of a smooth $(1,1)$-tensor $\alpha \otimes X$ is the one-form defined by $\operatorname{div}(\alpha \otimes X) := \operatorname{div}(X)\alpha + \nabla_X \alpha$, and $\operatorname{div}(u) = 0$ as well as $(\nabla_u u)^\sharp = \nabla_u u^\sharp$ are valid.

Abstractly (1.1) is a doubly nonlinear evolution equation. Let us refer to [1, 8, 11, 18] for general results about such equations. Usually the map b is assumed to

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have a potential ϕ_b, and in this case (1.1) generalizes (1.2) by allowing the fluid to have kinetic energy $\int_\Omega \phi_b(u)\,dx$ instead of the standard quadratic kinetic energy $\frac{1}{2}\int_\Omega |u|^2\,dx$. For example, ϕ_b may be given by a power law like

$$\phi_b(u) := \frac{1}{m}|u|^m \text{ so that } b(u) = |u|^{m-2}u^\sharp, \text{ or} \tag{1.3}$$

$$\phi_b(u) := \begin{cases} \frac{1}{2}|u|^2 & |u| \le 1 \\ \frac{1}{m}|u|^m - (\frac{1}{m} - \frac{1}{2}) & |u| \ge 1 \end{cases} \text{ so that } b(u) = \begin{cases} u^\sharp & |u| \le 1 \\ |u|^{m-2}u^\sharp & |u| \ge 1 \end{cases}, \text{ or} \tag{1.4}$$

$$\phi_b(u) := \frac{1}{2}|u|^2 + \frac{1}{m}|u|^m \text{ so that } b(u) = (1 + |u|^{m-2})u^\sharp \tag{1.5}$$

with $1 < m < \infty$, $m \neq 2$. In this situation doubly nonlinear incompressible Navier-Stokes equations (1.1) may be considered as model for a fluid in a medium which interacts with the fluid. The kinetic energies (1.3) or (1.4) with $m < 2$ correspond to the case of a porous medium, where the particles of the fluid are retarded for large velocities $|u|$ due to the interaction so that the kinetic energy of the fluid is less than the standard quadratic kinetic energy. In the case $m > 2$ the particles of the fluid are accelerated for large velocities $|u|$, and the kinetic energy is greater than the standard quadratic kinetic energy. Further, in the situations (1.3) or (1.5) with $m < 2$ the particles of the fluid are accelerated for small velocities $|u|$ due to the interaction, while for $m > 2$ they are retarded for small velocities $|u|$ in the case (1.3) resp. accelerated for large velocities $|u|$ in the case (1.5).

Similarly, the viscous stress tensor a is often assumed to depend on the strain tensor $\nabla^{\text{sym}}u$ by a power law like

$$a(\nabla^{\text{sym}}u) := |\nabla^{\text{sym}}u|^{p-2}\nabla^{\text{sym}}u \text{ or} \tag{1.6}$$

$$a(\nabla^{\text{sym}}u) := (1 + |\nabla^{\text{sym}}u|^{p-2})\nabla^{\text{sym}}u\,, \tag{1.7}$$

see e.g. [19, Chapter 1, Example 1.73]. The case $p < 2$ models shear thinning (=pseudoplastic) fluids like blood, while the case $p > 2$ models shear thickening (=dilatant) fluids like a thick slurry of beach sand. Note that in contrast to the case $a(\nabla^{\text{sym}}u) := (1+|\nabla^{\text{sym}}u|)^{p-2}\nabla^{\text{sym}}u$ equation (1.1) becomes singular at points x with $(\nabla^{\text{sym}}u)(x)$ for $p < 2$ and (1.6).

1.1. **Outline.** In the first part of this article existence of weak solutions to doubly nonlinear incompressible Navier-Stokes equations (1.1) is discussed. The book [17] is an excellent reference for the theory of incompressible Navier-Stokes equations (1.2) with linear a. Existence of weak solutions to these classical equations was first proved by [15, 13]. For nonlinear monotone a existence of weak solutions was shown by [16] and [14] under the condition $p \ge \frac{3n+2}{n+2}$. In [19] and [20, Chapter 5] existence of weak solutions to (1.2) under periodic boundary conditions was shown for $p > \frac{3n}{n+2}$ and even for non-monotone a (see [20, Chapter 5, Theorem 3.97]). Quasimonotone viscous stress tensors a were discussed in [12].

A first attempt to prove existence of weak solutions to doubly nonlinear incompressible Navier-Stokes equations (1.1) in the case $p < n$ under the condition $mp' < p^*$ [1] can be found in [21]. In section 2, existence of weak solutions of (1.1)

[1]Note that in the case $p < n$ the condition $mp' < p^*$ is equivalent to $p > \frac{(m+1)n}{m+n}$ and thus to $p > \frac{3n}{n+2}$ in the special case $m = 2$.

is established using pseudomonotonicity, and the limit passage in the viscous stress tensor is rigorously justified.

The main aim of this article is to discuss extinction of the velocity vector field u in finite time. For doubly nonlinear reaction-diffusion equations it is well-known that in the case $p < m$ of fast diffusion solutions become extinct in finite time, see [9]. However, to prove a similar result for solutions u of (1.1) or (1.2), some information about the regularity of the pressure is needed. Therefore, in section 3 the Poisson problem for the pressure is investigated, and in section 4 under some regularity assumptions extinction of u is proved for small initial velocities under the condition $p < m$. Particularly, in the case $m = 2$ this result indicates that sufficiently slow shear thinning fluids come to rest in finite time. This property agrees with the observation that fluids like blood or honey stop flowing, if no external energy is used to accelerate them. In contrast to the series of articles [2, 3, 4, 5, 6, 7], where the body force $f = f(b(u))$ is assumed to be a feedback dissipative field satisfying a condition like $f(v)v \le -C|v|^2$, here the much more general case

$$f(x, v) = g(v) - dU(x)$$

with a continuous map g satisfying $g(0) = 0$ and $g(v) \cdot v \le C|v|^2$ for a constant $C < \infty$ and a potential U on Ω is discussed. This condition implies that $f(\cdot, 0)$ is potential and $f(x, \cdot)$ has at most linear growth, but f is not restricted to point into the opposite direction of v.

2. Existence of weak solutions

In this section doubly nonlinear incompressible Navier-Stokes equations (1.1) are considered either under periodic boundary conditions on the fundamental domain $\Omega := (0, L)^n \subset \mathbb{R}^n$ of the periodic lattice $\mathbb{R}^n/(L\mathbb{Z})^n$, $L > 0$, or on the whole space $\Omega = \mathbb{R}^n$, or on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ under Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. The aim of this section is to prove existence of weak solutions for the prototypical power laws, i.e. for (1.3), (1.4) or (1.5) determining the kinetic energy ϕ_b resp. the relation b between velocity and momentum, and for (1.6) or (1.7) determining the viscous stress tensor a, where the parameters $1 < m, p < \infty$ are fixed.

To realize (1.1) as an equation in a Banach space, for the power law (1.6) we consider the Banach space

$$X_{p,\mathrm{per}} := \{u \in W^{1,p}(\mathbb{R}^n/(L\mathbb{Z})^n, \mathbb{R}^n) : \mathrm{div}(u) = 0\,, \int_\Omega u\,dx = 0\}$$

in case of periodic boundary conditions on $\Omega = (0, L)^n$,

$$X_{p,\mathrm{space}} := \{u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n) : \mathrm{div}(u) = 0\}$$

in case of the whole space $\Omega = \mathbb{R}^n$, and

$$X_{p,0} := \{u \in W_0^{1,p}(\Omega, \mathbb{R}^n) : \mathrm{div}(u) = 0\}$$

in case of a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions. These Banach spaces are identical with the closure of the corresponding (periodic with zero average resp. compactly supported) smooth divergence-free vector fields on Ω w.r.t. the norm $\|u\|_{X_p} := \|\nabla u\|_p$. Equivalently, the norm $\|\nabla^{\mathrm{sym}} u\|_p$ can be chosen, where $\nabla^{\mathrm{sym}} u$ denotes the symmetric part of the derivative of u, see [19, 5.1.1]. For the power law (1.7) we replace $W^{1,p}$ by $W^{1,p} \cap W^{1,2}$ in the former definitions and use the norm $\max(\|\nabla u\|_p, \|\nabla u\|_2)$. In the following, denote by X_p one of these spaces.

Further, for $\Omega = \mathbb{R}^n$ or bounded $\Omega \subset \mathbb{R}^n$ and the kinetic energies (1.3) or (1.4) we consider the Banach space $Y_m := L^m(\Omega, \mathbb{R}^n)$. Note that the closure of $X_p \cap Y_m$ in Y_m is $Y_{m,\mathrm{div}} := \{u \in Y_m : \mathrm{div}(u) = 0\}$, where $\mathrm{div}(u)$ is understood in the sense of distributions [2]. In the periodic case or for a bounded domain Ω the space $X_p \cap Y_m$ is compactly embedded into Y_m if and only if $p \geq n$, or $p < n$ and $m < p^*$, where p^* is the Sobolev conjugate of p [3] Thus, for $p < n$ at least $m < p^*$ should be assumed. In case of the kinetic energy (1.5) we replace L^m by $L^m \cap L^2$ and use the norm $\max(\|u\|_m, \|u\|_2)$. The Banach space Y_m plays the role of the Hilbert space $L^2(\Omega, \mathbb{R}^n)$ in the usual setting for equation (1.2).

It remains to realize every term of (1.1) as an operator. Assume that the viscous stress tensor a is given in dependence of the strain tensor $\nabla^{\mathrm{sym}} u$ by (1.6) or (1.7), then $-\mathrm{div}(a(\nabla^{\mathrm{sym}} u))$ can be realized as an operator $A : X_p \to X_p^*$ by

$$\langle Au, v\rangle := \int_\Omega a(\nabla^{sym} u) \cdot \nabla^{\mathrm{sym}} v \, dx\,.$$

The operator A is bounded in the sense that

$$\|Au\|_{X_p^*} \leq \|\nabla^{sym} u\|_p^{p-1} \quad \text{in case of (1.6)} \tag{2.1}$$

$$\|Au\|_{X_p^*} \leq \|\nabla^{sym} u\|_p^{p-1} + \|\nabla^{sym} u\|_2 \quad \text{in case of (1.7)} \tag{2.2}$$

coercive in the sense that

$$\langle Au, u\rangle \geq \|\nabla^{sym} u\|_p^p \text{ in case of (1.6)} \tag{2.3}$$

$$\langle Au, u\rangle \geq \|\nabla^{sym} u\|_p^p + \|\nabla^{sym} u\|_2^2 \text{ in case of (1.7)} \tag{2.4}$$

and monotone. In the same way, a induces a monotone operator $A : L^p(0, T; X_p) \to L^{p'}(0, T; X_p^*)$ on the space $L^p(0, T; X_p)$ of time-dependent functions.

Similarly, assume that the kinetic energy ϕ_b and the corresponding momentum $b = d\phi_b$ are given in dependence of velocity u by the power laws (1.3), (1.4) or (1.5), then b induces a superposition operator $B : Y_m \to Y_m^*$ by

$$\langle Bu, v\rangle := \int_\Omega b(u) \cdot v \, dx\,.$$

This operator B is continuous, bounded in the sense that

$$\|Bu\|_{Y_m^*} \leq \|u\|_m^{m-1} \quad \text{in case of (1.3),} \tag{2.5}$$

$$\|Bu\|_{Y_m^*} \leq C(1 + \|u\|_m^{m-1}) \quad \text{in case of (1.4),} \tag{2.6}$$

$$\|Bu\|_{Y_m^*} \leq \|u\|_m^{m-1} + \|u\|_2 \quad \text{in case of (1.4),} \tag{2.7}$$

coercive in the sense that

$$\langle Bu, u\rangle \geq \|u\|_m^m \quad \text{in case of (1.3),} \tag{2.8}$$

$$\langle Bu, u\rangle \geq \begin{cases} \|u\|_m^m & m \geq 2 \\ C\|u\|_m^2 & m \leq 2 \end{cases} \quad \text{in case of (1.4),} \tag{2.9}$$

$$\|Bu\|_{Y_m^*} \geq \|u\|_m^m + \|u\|_2^2 \quad \text{in case of (1.4),} \tag{2.10}$$

[2] In the periodic case consider $Y_m := L^m(\mathbb{R}^n/(L\mathbb{Z})^n, \mathbb{R}^n)$, then the closure of $X_p \cap Y_m$ in Y_m is $Y_{m,\mathrm{div}} = \{u \in Y_m : \mathrm{div}(u) = 0\,,\ \int_\Omega u \, dx = 0\}$.

[3] In the case $\Omega = \mathbb{R}^n$ (and under the condition $m < p^*$ in the case $p < n$) merely the restriction of functions in $X_p \cap Y_m$ to bounded open subsets $U \subset \Omega$ is compact as an operator into $L^m(U, \mathbb{R}^n)$.

strictly monotone and has the potential $\Phi_B(u) := \int_\Omega \phi_b(u)\,dx$. Particularly, B has a demicontinuous inverse B^{-1}, and $B : L^\infty(0,T;Y_m) \to L^\infty(0,T;Y_m^*)$.

An important role in the discussion of doubly nonlinear evolution equations is played by the Legendre transform $\hat{\phi}_b(u) := b(u)\cdot u - \phi_b(u)$ of ϕ_b in dependence of u and the induced functional $\hat{\Phi}_B(u) := \int_\Omega \hat{\phi}_b(u)\,dx$ on Y_m. In fact, the energy identity

$$\frac{d}{dt}\hat{\Phi}_B(u_k) = \langle \frac{\partial B u_k}{\partial t}, u_k\rangle \tag{2.11}$$

holds in the sense of scalar distributions for $u_k \in L^p(0,T,X_p) \cap L^\infty(0,T;Y_m)$ such that $Bu_k \in L^\infty(0,T;Y_m^*)$ has a weak derivative $\frac{\partial B u_k}{\partial t} \in L^{p'}(0,T,X_p^*)$. Note that (2.11) generalizes the formula $\frac{d}{dt}\frac{1}{2}\|u\|_H^2 = \langle \frac{\partial u}{\partial t}, u\rangle$ for time-dependent functions $u \in L^p(0,T;V) \cap L^2(0,T;H)$ with a weak derivative $\frac{\partial u}{\partial t} \in L^{p'}(0,T;V^*)$, where H is a Hilbert space and $V \subset H \subset V^*$ is a Gelfand triple.

Finally, the transport term $\operatorname{div}(b(u)\otimes u)$ can be realized as an operator $u \mapsto C(Bu\otimes u)$ from $X_p \cap Y_m$ into X_p^* by

$$\langle C(Bu\otimes u), v\rangle := -\int_\Omega b(u)\cdot \nabla_u v\,dx\,,$$

provided that $p \geq n$, or $p < n$ and $mp' \leq p^*$ (which is a stronger condition than $m < p^*$). Moreover, $C : X_p \cap Y_m \to X_p^*$ is totally continuous, if $p \geq n$, or $p < n$ and $mp' < p^*$, thus let us assume for $p < n$ in the following $mp' < p^*$. In fact, the estimate

$$|\langle C(Bu\otimes u), v\rangle| \leq \int_\Omega |b(u)||u||\nabla v| \leq \|Bu\|_{m'p'}\|u\|_{mp'}\|\nabla v\|_p$$

is valid by Hölder's inequality due to $\frac{1}{m'p'} + \frac{1}{mp'} + \frac{1}{p} = 1$. The growth condition on b implies $\|Bu\|_{m'p'} \leq C(\|u\|_{mp'}^{m-1} + 1)$ for a constant $C < \infty$, and the Gagliardo-Nirenberg inequality $\|u\|_{mp'} \leq (C\|\nabla u\|_p)^{\frac{p^*}{p(p^*-m)}}\|u\|_m^{\frac{p^*(p-1)-mp}{p(p^*-m)}}$ holds due to the validity of $m \leq mp' \leq p^*$. Thus, an estimate of the norm of $C(Bu\otimes u)$ in X_p^* is given by

$$\|C(Bu\otimes u)\|_{X_p^*} \leq (C\|\nabla u\|_p)^{\frac{mp^*}{p(p^*-m)}}\|u\|_m^{m\frac{p^*(p-1)-mp}{p(p^*-m)}}\,,$$

and an estimate in a space of time-dependent functions is

$$\|C(Bu\otimes u)\|_{L^{p^2(\frac{1}{m}-\frac{1}{p^*})}(0,T;X_p^*)}^{p^2(\frac{1}{m}-\frac{1}{p^*})} \leq C^p\|u\|_{L^p(0,T;X_p)}^p\|u\|_{L^\infty(0,T;Y_m)}^{p\frac{p^*(p-1)-mp}{p^*}}\,. \tag{2.12}$$

Note that in the case $mp' < p^*$ the embedding obtained from Gagliardo-Nirenberg inequalities is compact and hence C is totally continuous [4]. Further, note that the space $L^{p^2(\frac{1}{m}-\frac{1}{p^*})}(0,T;X_p^*)$ is continuously embedded into $L^{p'}(0,T;X_p^*)$ if and only if $p(p-1)(1-\frac{m}{p^*}) \geq m$, else it is a weaker space.

As all terms of (1.1) have been realized as operators, the following definition of a weak solution of (1.1) is appropriate for an inhomogeneity $f \in L^{p'}(0,T;X_p^*)$.

Definition 2.1. A time-dependent vector field $u \in L^p(0,T;X_p) \cap L^\infty(0,T;Y_m)$ is called a weak solution of (1.1) to the initial value $u_0 \in Y_m$ with $\operatorname{div}(u_0) = 0$, if $Bu \in L^\infty(0,T;Y_m)$ has the initial value $Bu_0 \in Y_m^*$ and a weak derivative $\frac{\partial Bu}{\partial t} \in$

[4] at least in the case of periodic boundary conditions or Dirichlet boundary conditions on a bounded domain, for the whole space problem use an exhaustion by bounded subdomains

$L^{p^2(\frac{1}{m}-\frac{1}{p^*})}(0,T;X_p^*)$ in the case $p(p-1)(1-\frac{m}{p^*})<m$ (resp. $\frac{\partial Bu}{\partial t}\in L^{p'}(0,T;X_p^*)$ in the case $p(p-1)(1-\frac{m}{p^*})\geq m$) such that

$$\frac{\partial Bu}{\partial t}+Au+C(Bu\otimes u)=f \tag{2.13}$$

is valid as an equation in $L^{p^2(\frac{1}{m}-\frac{1}{p^*})}(0,T;X_p^*)$ in the case $p(p-1)(1-\frac{m}{p^*})<m$ (resp. as an equation in $L^{p'}(0,T;X_p^*)$ in the case $p(p-1)(1-\frac{m}{p^*})\geq m$).

Existence of weak solutions is guaranteed by the following theorem.

Theorem 2.2. *If $p\geq n$, or $p<n$ and $mp'<p^*$, if b is given by* (1.3), (1.4) *or* (1.5), *if a is given by* (1.6) *or* (1.7), *and if $f\in L^{p'}(0,T;X_p^*)$, then to an initial value $u_0\in Y_m$ with $\operatorname{div}(u_0)=0$ there exists a weak solution u of doubly nonlinear Navier-Stokes equations* (1.1) *in the sense of definition 2.1.*

Proof. The proof is very similiar to the proof of [21, Theorem 2.2], where a Faedo-Galerkin method is used and structural assumptions (A1-A4),(B1-B5) on a,b are required. These assumptions are mainly satisfied, if b is given by (1.3), (1.4) or (1.5) and a is given by (1.6) or (1.7). For example, in the case (1.7) and for Dirichlet boundary conditions on a bounded domain not only $a(e)\cdot e\geq|e|^p$ is satisfied, but even $a(e)\cdot e\geq|e|^2+|e|^p$. Thus, in a Faedo-Galerkin method the first energy estimates obtained by testing (2.13) with approximate solutions u_k and using (2.11) have the form

$$\begin{aligned}
&\frac{d}{dt}\hat{\Phi}_B(u_k)+\|u_k\|^2_{W_0^{1,2}}+\|u_k\|^p_{X_p}\\
&\leq\langle\frac{\partial Bu_k}{\partial t},u_k\rangle+\langle Au_k,u_k\rangle+\langle C(Bu_k\otimes u_k),u_k\rangle\\
&=\langle f,u_k\rangle\leq\frac{1}{p'\epsilon^{p'}}(\|f\|^{p'}_{X_p^*}+C_2^{p'})+2\frac{\epsilon^p}{p}\|u_k\|^p_{X_p}\,.
\end{aligned}\tag{2.14}$$

Thus, beneath an a priori estimate of $\hat{\Phi}_B(u_k)$ in $L^\infty(0,T)$ not only an a priori estimate of u_k in $L^p(0,T;X_p)$ can be derived, but additionally an a priori estimate in $L^2(0,T;W_0^{1,2}(\Omega))$ holds. As $\hat{\Phi}_B(u_k)$ dominates $\|Bu_k\|_{Y^*}$, also Bu_k is uniformly bounded in $L^\infty(0,T;Y_m^*)$ and u_k is uniformly bounded in $L^\infty(0,T;Y_m)$. Finally, Au_k is uniformly bounded in $L^2(0,T;W^{-1,2})+L^{p'}(0,T;X_p^*)$, and by (2.12) $C(Bu_k\otimes u_k)$ is uniformly bounded in $L^{p^2(\frac{1}{m}-\frac{1}{p^*})}(0,T;X_p^*)$ (resp. in $L^{p'}(0,T;X_p^*)$ provided that $p(p-1)(1-\frac{m}{p^*})\geq m$). As a consequence, $\frac{\partial Bu_k}{\partial t}$ is uniformly bounded in $L^{p^2(\frac{1}{m}-\frac{1}{p^*})}(0,T;X_p^*)$ (resp. in $L^{p'}(0,T;X_p^*)$). Thus, a subsequence of u_k can be extracted such that

$$\begin{gathered}
u_k\overset{*}{\rightharpoonup}u\quad\text{in }L^p(0,T;X_p)\cap L^\infty(0,T;Y_m)\\
Au_k\rightharpoonup(Au)_{ex}\quad\text{in }L^2(0,T;W^{-1,2})+L^{p'}(0,T;X_p^*)\\
Bu_k\overset{*}{\rightharpoonup}(Bu)_{ex}\quad\text{in }L^\infty(0,T;Y_m^*)\\
C(Bu_k\otimes u_k)\rightharpoonup(C(Bu\otimes u))_{ex}\quad\text{in }L^{p^2(\frac{1}{m}-\frac{1}{p^*})}(0,T;X_p^*)(\text{resp. in }L^{p'}(0,T;X_p^*))\\
\frac{\partial Bu_k}{\partial t}\rightharpoonup(\frac{\partial Bu}{\partial t})_{ex}\quad\text{in }L^{p^2(\frac{1}{m}-\frac{1}{p^*})}(0,T;X_p^*)\ (\text{resp. in }L^{p'}(0,T;X_p^*))\,.
\end{gathered}$$

It merely remains to show that the values $(\cdot)_{ex}$ coincide with their expected values, then the limit passage $k \to \infty$ in the approximate equation shows that u is a weak solution of (1.1).

By time-compactness $Bu_k \to (Bu)_{ex}$ is valid in $L^1(0,T;Y_m^*)$, thus a limit passage in $\int_0^T \langle Bu_k - Bv, u_k - v\rangle \, dt$ implies $\int_0^T \langle (Bu)_{ex} - Bv, u - v\rangle \, dt \geq 0$ for arbitrary $v \in L^\infty(0,T;Y_m)$, and Minty's trick can be applied to conclude $(Bu)_{ex} = Bu$. Then $(\frac{\partial Bu}{\partial t})_{ex} = \frac{\partial Bu}{\partial t}$ eventually is a consequence of the definition of weak time derivatives. Further, once the weak convergence $C(Bu_k \otimes u_k) \rightharpoonup C(Bu \otimes u)$ has been shown, due to pseudomonotonicity of $u \mapsto Au + C(Bu \otimes u)$ from $X_p \cap Y_m$ to X_p^* (recall that $C(Bu \otimes u)$ is totally continuous and satisfies appropriate bounds) a standard argument allows to prove $Au_k \rightharpoonup Au$, and hence $(Au)_{ex} = Au$, see [23, Theorem 8.27, Remark 8.29].

Thus, it merely remains to prove weak convergence of $C(Bu_k \otimes u_k)$ to $C(Bu \otimes u)$ in $L^{p^2(\frac{1}{m} - \frac{1}{p^*})}(0,T;X_p^*)$ (resp. in $L^{p'}(0,T;X_p^*)$). Therefore, consider a function $v \in L^\infty(0,T;\{\tilde{v} \in W_0^{1,\infty} \mid \operatorname{div}(\tilde{v}) = 0\})$, then strong convergence $Bu_k \to Bu$ in $L^1(0,T;Y_m^*)$ and weak*-convergence $u_k \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;Y_m)$ allow to conclude

$$
\begin{aligned}
&\int_0^T \langle C(Bu_k \otimes u_k) - C(Bu \otimes u), v\rangle \, dt \\
&= \int_0^T \langle C((Bu_k - Bu) \otimes u_k), v\rangle \, dt + \int_0^T \langle C(Bu \otimes (u_k - u)), v\rangle \, dt \to 0 \,.
\end{aligned}
$$

Hence by denseness $C(Bu_k \otimes u_k) \rightharpoonup C(Bu \otimes u)$ is even valid in $L^{p^2(\frac{1}{m} - \frac{1}{p^*})}(0,T;X_p^*)$, because $C(Bu_k \otimes u_k)$ and $C(Bu \otimes u)$ both lie in this space. □

Remark 2.3. If f is not an inhomogeneity but a nonlinearity, then existence of weak solutions can be proved similarly provided that f satisfies an appropriate growth condition. Particularly, existence can be shown for $f = f(x, b(u))$, where $f(x,v) := g(v) - dU(x)$ with $U \in L^{p'}(\Omega)$ and g satisfying $g(v)v \leq C|v|^2$, i.e. g has at most linear growth.

Remark 2.4. Note that in [19] due to second energy estimates even strong convergence of ∇u_k can be established, which allow to conclude $C(Bu \otimes u) = \nabla_u Bu$ (with $B = \mathrm{Id}$ in the case $m = 2$, $b(u) = u$) and existence can be shown even in the case of non-monotone A, see [19, 5.3.3]. For a nonlinear B the validity of second energy estimates seems to be an open problem.

3. Regularity of the pressure

An important problem in fluid dynamics is the regularity of the pressure function π which is implicitly defined by the incompressibility condition $\operatorname{div}(u) = 0$. Here some information about the pressure is needed, because in our proof of extinction in finite time equation (1.1) is tested by a function with nonvanishing divergence. For results about the regularity of pressure in case of the incompressible Navier-Stokes equations (1.2) with linear a let us refer to [17, Theorem 3.1-3.4, Remark 3.1] and [26, III.1 Proposition 1.1 + 1.2]. Similar as in the linear case, by definition (2.1) the weak solution u of doubly nonlinear Navier-Stokes equations obtained in Theorem (2.2) satisfies

$$
\frac{\partial Bu}{\partial t} + Au + C(Bu \otimes u) = f \tag{2.13}
$$

as an equation in $L^{p^2(\frac{1}{m}-\frac{1}{p^*})}(0,T;X_p^*)$ in the case $p(p-1)(1-\frac{m}{p^*}) < m$ (resp. as an equation in $L^{p'}(0,T;X_p^*)$ in the case $p(p-1)(1-\frac{m}{p^*}) \geq m$). However, it is not directly obvious in which sense (1.1) is solved by a weak solution u.

To answer this question, note that the dual space of X_p can be identified with the quotient

$$W^{-1,p'}(\mathbb{R}^n,(\mathbb{R}^n)^*)/\{d\pi \mid \pi \in L^{p'}(\mathbb{R}^n,\mathbb{R})\}$$

of the dual space $W^{-1,p'}(\mathbb{R}^n,(\mathbb{R}^n)^*)$ of $W_0^{1,p}(\mathbb{R}^n,\mathbb{R}^n)$ w.r.t. the closed subspace given by distributional derivatives of functions from $L^{p'}(\mathbb{R}^n,\mathbb{R})$. Similarly, $X_{p,\mathrm{per}}$ can be identified with the quotient of $W^{-1,p'}(\mathbb{R}^n/(L\mathbb{Z})^n,(\mathbb{R}^n)^*)$ by distributional derivatives $d\pi$ of functions $\pi \in L^{p'}(\mathbb{R}^n/(L\mathbb{Z})^n,\mathbb{R})$ and constant vectors [5]. Hence, due to this characterization of the dual and the validity of (2.13) as an equation in X_p^* for a.e. time $t \in (0,T)$, there exists a pressure function $\pi(t) \in L^{p'}(\Omega)$ (which is unique up to a time-dependent constant) such that the first equation of (1.1) is valid in the sense of distribution.

Moreover, the pressure function π is a very weak solution of an elliptic equation (possibly with degenerate or singular coefficients in the doubly nonlinear case) under Neumann boundary conditions. In fact, if Ω is a smooth bounded domain with Dirichlet boundary and the external force f is given by $f(x,v) := g(v) - dU(x)$, then multiply the first equation of (1.1) from the left by $db^{-1}(b(u))$ (which may be degenerate or singular) and apply afterwards the divergence operator div to obtain

$$\begin{aligned}
&-\operatorname{div}(db^{-1}(b(u))\operatorname{grad}(\pi+U)) \\
&= \operatorname{div}(db^{-1}(u)\operatorname{div}(b(u)\otimes u)) - \operatorname{div}(db^{-1}(u)\operatorname{div}(a(\nabla^{\mathrm{sym}}u))) \\
&- \operatorname{div}(db^{-1}(u)g(b(u)))\,.
\end{aligned} \tag{3.1}$$

Further, multiply the first equation of (1.1) from the right by (an extension of) the exterior outer normal vector field ν on $\partial\Omega$ to obtain

$$\frac{\partial(\pi+U)}{\partial\nu} = -\operatorname{div}(b(u)\otimes u)\cdot\nu + \operatorname{div}(a(\nabla^{\mathrm{sym}}u))\cdot\nu + g(b(u))\cdot\nu \text{ on } \partial\Omega \tag{3.2}$$

on $\partial\Omega$. Hence $\pi(t)+U \in L^{p'}(\Omega)$ is for a.e. $t \in (0,T)$ a very weak solution of (3.1) under Neumann boundary conditions (3.2) [6].

In the classical case of incompressible Navier-Stokes equations (1.2) for a Newtonian fluid with standard quadratic energy and a force depending linearly on momentum, i.e. in the case $a(\nabla^{\mathrm{sym}}u) = \mu\nabla^{\mathrm{sym}}u$ with $\mu > 0$, $b(u) = u^\sharp$ and $f(x,v) := \alpha v - dU(x)$ with $\alpha \in \mathbb{R}$, equation (3.1) simplifies to the classical Poisson problem

$$-\Delta(\pi+U) = (\nabla u)^T\cdot(\nabla u)\,. \tag{3.3}$$

Note that due to $(\nabla u)(t) \in L^2(\Omega,\mathbb{R}^{n\times n})$ for a.e. $t \in (0,T)$ (as $p = 2$) the right hand side lies in $L^1(\Omega,\mathbb{R})$. In the case of (1.2) with a given by (1.6) and $f(x,v) := \alpha v - dU(x)$ with $\alpha \in \mathbb{R}$, the corresponding Poisson problem reads as

$$-\Delta(\pi+U) = (\nabla u)^T\cdot(\nabla u) + (p-2)|\nabla^{\mathrm{sym}}u|^{p-1}$$

[5] which can be hidden in the external force f

[6] i.e. for every smooth test function ϕ such that $\operatorname{div}(db^{-1}(b(u))\operatorname{grad}\phi)$ exists as a function in $L^p(\Omega)$ and $\frac{\partial\phi}{\partial\nu} = 0$ holds, the integral $\int(\pi+U)\operatorname{div}(db^{-1}(b(u))\operatorname{grad}\phi)\,dx$ obtained by multiplying the left hand side of (3.1) by ϕ and shifting both derivatives onto ϕ is the same as the corresponding integral on the right hand side.

under Neumann boundary conditions

$$\frac{\partial(\pi+U)}{\partial\nu}=\alpha b(u)\cdot\nu+\operatorname{div}((\nabla^{\mathrm{sym}}u)^{p-1})\cdot\nu$$

on $\partial\Omega$. The regularity of very weak solutions of $-\Delta v=f$ with right hand side $f\in L^1(\Omega)$ and Dirichlet boundary conditions $v=0$ on $\partial\Omega$ is well-studied. For example, in [10] the validity of

$$\|v\|_{L^{N',\infty}}\le c\|f\|_{L^1(\Omega,\delta)} \tag{3.4}$$

is shown for very weak solutions, where near the boundary f is even allowed to grow less than $\frac{1}{\delta}$ with the distance δ to the boundary. However, there are only few articles about the corresponding Neumann problem, e.g. [22], so that in this case a regularity result like (3.4) seems to be an open problem.

Remark 3.1. For sufficiently smooth solutions [24] and [25] even show the equivalence of classical incompressible Navier-Stokes equations and the equations solved when using the pressure Poisson method with appropriate boundary conditions.

4. Extinction

It is generally known (see e.g. [9]) that weak solutions u of doubly nonlinear diffusion equations $\frac{\partial u^{m-1}}{\partial t}-\Delta_p u=0$ to small initial values become extinct in finite time in the case $p<m$ of fast diffusion, i.e. there exists a time $T>0$ such that $u(t)\equiv 0$ for a.e. $t\ge T$. For doubly nonlinear Navier-Stokes equations, additionally the transport term, the pressure term and the body force term have to be handeled to obtain a similar result. Note that in the case $f(x,v)=g(v)-dU(x)$ with $g(0)=0$ the function $(u,\pi)=(0,-U)$ is a solution of (1.1).

Theorem 4.1. *If $1<m,p<\infty$, $mp'<p^*$ and $p<m$, if b is given by* (1.3), (1.4) *or* (1.5)*, if a is given by* (1.6) *or* (1.7)*, if the f has the form $f(x,v)=g(v)-dU(x)$ with $U\in L^{p'}(\Omega)$ and a continuous g satisfying $g(0)=0$ and $g(v)v\le C|v|^2$, then weak solutions of* (1.1) *on a bounded domain Ω under Dirichlet boundary conditions $u=0$ on $\partial\Omega$ to initial values $u_0\neq 0$ become extinct in finite time, provided that $\|Bu_0\|_r$ is sufficiently small for some sufficiently large index $1<r<\infty$ and the very weak solution $\pi+U$ of* (3.1) *under Neumann boundary conditions* (3.2) *satisfies an estimate of the form*

$$\|\pi+U\|_{r\frac{p'}{m'}}^{p'}\le c\|Bu\|_r^{m'}\ \textit{with a constant}\ c<\infty\,. \tag{4.1}$$

Proof. Test equation (1.1) by $(Bu)^{r-1}$ to obtain

$$\begin{aligned}&\frac{d}{dt}\frac{1}{r}\|Bu\|_r^r+\langle Au,(Bu)^{r-1}\rangle+\langle C(Bu\otimes u),(Bu)^{r-1}\rangle\\&=\int_\Omega(\pi+U)\operatorname{div}((Bu)^{r-1})\,dx+\langle Gu,(Bu)^{r-1}\rangle\,,\end{aligned}$$

where G denotes the superposition operator associated with g. Note that due to $\operatorname{div}(u)=0$ the transport term vanishes, as

$$\begin{aligned}\langle C(Bu\otimes u),(Bu)^{r-1}\rangle&=-\int_\Omega b(u)\nabla_u b(u)^{r-1}\,dx=-\frac{1}{r'}\int_\Omega\nabla_u|b(u)|^r\,dx\\&=-\frac{1}{r'}\int_{\partial\Omega}|b(u)|^r u\cdot\nu\,dS+\frac{1}{r'}\int_\Omega|b(u)|^r\operatorname{div}(u)\,dx=0\,.\end{aligned}$$

On the right hand side use

$$\begin{aligned}\operatorname{div}((Bu)^{r-1}) &= \operatorname{div}(|u|^{(r-1)(m-1)-1}u) \\ &= |u|^{(r-1)(m-1)-1}\operatorname{div}(u) + \nabla_u |u|^{(r-1)(m-1)-1} \\ &= \nabla_u |u|^{(r-1)(m-1)-1} = c|u|^{\frac{(r-1)(m-1)-1}{p'}-1}\nabla_u |u|^{\frac{(p-1)+(r-1)(m-1)}{p}}\end{aligned}$$

and the assumption $\|\pi + U\|^{p'}_{r\frac{p'}{m'}} \le c\|Bu\|_r^{m'}$, to estimate

$$\begin{aligned}&\int_\Omega (\pi+U)\operatorname{div}((Bu)^{r-1})\,dx \\ &\le c\|\pi+U\|_{r\frac{p'}{m'}} \|u^{\frac{(r-1)(m-1)-1}{p'}}\|_{\frac{p'r(m-1)}{(r-1)(m-1)-1}} \|\nabla |u|^{\frac{(p-1)+(r-1)(m-1)}{p}}\|_p \\ &= c\|\pi+U\|_{r\frac{p'}{m'}} \|Bu\|_r^{\frac{(r-1)(m-1)-1}{p'(m-1)}} \|\nabla |u|^{\frac{(p-1)+(r-1)(m-1)}{p}}\|_p \\ &\le \frac{c}{\epsilon^{p'}p'}\|\pi+U\|^{p'}_{r\frac{p'}{m'}}\|Bu\|_r^{r-m'} + \frac{\epsilon^p}{p}\|\nabla |u|^{\frac{(p-1)+(r-1)(m-1)}{p}}\|_p^p \\ &\le \frac{c}{\epsilon^{p'}p'}\|Bu\|_r^r + \frac{\epsilon^p}{p}\|\nabla |u|^{\frac{(p-1)+(r-1)(m-1)}{p}}\|_p^p .\end{aligned} \tag{4.2}$$

Finally, define the exponent $s = s(r) := \frac{(p-1)+(r-1)(m-1)}{r(m-1)}$, then $[m', \infty) \ni r \mapsto s(r)$ is increasing due to $p < m$, and $\frac{1}{r'} < s < 1$ holds for every $m' \le r < \infty$. Using this exponent,

$$\begin{aligned}\langle Au, (Bu)^{r-1}\rangle &= \int_\Omega (\nabla^{\rm sym} u)^{p-1}\cdot \nabla u^{(r-1)(m-1)}\,dx \\ &= c\int_\Omega \Big|\nabla^{\rm sym} u^{\frac{(p-1)+(r-1)(m-1)}{p}}\Big|^p\,dx\end{aligned} \tag{4.3}$$

can be estimated from below by $c\|Bu\|_r^{rs}$ with a constant $c > 0$, where Sobolev's (and Korn's) inequality $\|\nabla^{\rm sym} w\|_p^p \ge c\|w\|^p_{\frac{p}{s}}$ is applied (note that $\frac{p}{s}$ lies between $m < p^*$ and p for $r \ge m'$). However, before applying this estimate use (4.3) to compensate the last term of (4.2) by choosing ϵ sufficiently small. Finally, estimate the remaining body force term by

$$\langle Gu, (Bu)^{r-1}\rangle \le C\|Bu\|_r^r$$

to obtain

$$\frac{d}{dt}\frac{1}{r}\|Bu\|_r^r + c\Big(\frac{1}{r}\|Bu\|_r^r\Big)^s \le C\frac{1}{r}\|Bu\|_r^r .$$

with a constant $C < \infty$. Hence, as $s < 1$ for every $m' \le r < \infty$, for small values of $\frac{1}{r}\|Bu\|_r^r$ the root term $C(\frac{1}{r}\|Bu\|_r^r)^s$ dominates all other terms so that $\frac{1}{r}\|Bu\|_r^r$ becomes extinct in finite time. □

Remark 4.2. The condition $g(v)v \le C|v|^2$ may be replaced by a weaker condition, but generally g has to vanish for $Bu = 0$. Particularly, extinction does not occur for constant body forces (as in this case $u = 0$ is not a solution of (1.1)).

Remark 4.3. In (4.1) the constant c is even allowed to depend on u via terms like $\|\nabla^{\rm sym} u\|_p$ which can be controlled by a priori estimates. However, the validity of (4.1) for very weak solutions of the Neumann problem (3.1), (3.2) and for the index r needed in the proof of theorem 4.1 seems to be an open problem.

Conclusion. In this article extinction of fluid vector fields for non-Newtonian fluids modeled by nonlinear of doubly nonlinear Navier-Stokes equations was discussed. A method was presented which allows to prove that slow shear-thinning fluid come to rest in finite time. However, this method needed some information about the regularity of the very weak solution of the corresponding pressure Poisson problem under Neumann boundary conditions, and the validity of this regularity seems to be an open problem.

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