USING RATIONAL LOGARITHMIC BASIS FUNCTIONS TO SOLVE SINGULAR DIFFERENTIAL EQUATIONS

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Abstract. Numerical methods based on polynomial approximation perform poorly when applied to singular initial value problems (IVPs). Hence, we are motivated to derive and implement numerical methods involving non-polynomial basis functions such as logarithmic and rational functions. Specifically, by imbedding a constant parameter into the logarithmic function, we are able to improve any discontinuity issues with the natural logarithm approximant. An efficient method is developed using the Taylor Series expansion to optimize the imbedded parameter. Numerical experiments performed show that this method is more accurate than the improved Euler's method. This method is implemented as a predictor-corrector method.

1. Introduction

We consider the initial value problem (IVP)

$$y' = f(x, y),$$
$$y(a) = y_0, \tag{1.1}$$

where $y, f \in \mathbb{R}^m$, $x \in [a, b]$, $a, b \in \mathbb{R}$.

Most numerical methods proposed in the literature for solving IVPs are based on polynomial approximation. Nevertheless, methods constructed using polynomial basis functions perform poorly when applied to singular IVPs. We define singular IVPs as those whose solution contains a point or points of discontinuity. Hence, several techniques involving non-polynomial basis functions have been proposed to cope with singular IVPs (see [1, 3, 4, 5, 6, 7]).

We propose a new method constructed using rational-logarithmic basis functions which outperforms the Improved Euler Method (IEM). Specifically, by imbedding a constant into the logarithmic function, we can improve the solutions close to the point of discontinuity inherent to singular IVPs. An optimized parameter imbedded in the method is recovered from the principal term of the Local Truncation Error (LTE). We then compare the performance of the method with the performance of the IEM.

The paper is organized as follows. In Section 2 we will examine the method's implementation. In Section 3 we will derive the method. In Section 4 we will display numerical examples. In Section 5 we will discuss the behavior of the imbedded parameter, α, around the point of discontinuity. In Section 6 we will discuss opportunities for further research.

2. Implementation of method

The method is implemented as a predictor-corrector in a step-by-step fashion on the partition H_N. The approximation at x_n is used to obtain the approximation at x_{n+1}, such that

$$H_N : a = x_0 < x_1 < \cdots < x_N = b, x_n = x_0 + nh, n = 0, \ldots, N$$

where $h = (b - a)/N$ is the constant step-size, N is a positive integer, and n is the grid index. We note that y_n is the numerical approximation to the analytical solution $y(x_n)$, and $f_n = f(x_n, y_n)$ is supplied by the differential equation.

3. Derivation of the method

This method is constructed using rational-logarithmic basis functions rather than polynomial basis functions.

3.1. **Derivation of the corrector.** The corrector is derived by assuming that on the interval $[x_n, x_{n+1}]$, the exact solution is approximated by the function

$$u(x) = \frac{1}{a_2(\ln[x + \alpha])^2 + a_1 \ln[x + \alpha] + a_0} \tag{3.1}$$

where a_0, a_1, a_2, are coefficients that are uniquely determined and α is an imbedded parameter that controls the behavior of the method. We note that $\ln[x + \alpha] \in \mathbb{C}$ provided $x \neq -\alpha$. Thus we impose that $x \neq -\alpha$ and it follows that $u(x)$ is well defined. To determine the coefficients in (3.1), we require that the following conditions must be satisfied

$$u(x_n) = y_n,$$
$$u'(x_n) = f_n,$$
$$u'(x_{n+1}) = f_{n+1},$$

This leads to a system of three equations and three unknowns, which is solved to obtain the coefficients in (3.1). Our method is then obtained by evaluating $u(x)$ at x_{n+1} leading to the iterative equation

$$y_{n+1} = \frac{y_n^2\big(2y_{n+1} - f_{n+1}(h + \alpha_n)\ln(\frac{\alpha_n}{h+\alpha_n})\big)}{y_{n+1}\big(f_n\alpha_n \ln(\frac{\alpha_n}{h+\alpha_n}) + 2y_n\big)}. \tag{3.2}$$

Note in equation this equation, $\ln(\frac{\alpha_n}{h+\alpha_n}) \in \mathbb{R}$ provided $0 < \alpha_n$ or $\alpha_n < -h$. Thus by closure if for all $x_n, y_n, f_n, f_{n+1} \in \mathbb{R}$, then $y_{n+1} \in \mathbb{R}$.

3.2. **Derivation of the predictor.** The predictor is derived by assuming that on the interval $[x_n, x_{n+1}]$, the exact solution is approximated by the function

$$u(x) = \frac{1}{a_2(\ln[x+\alpha])^2 + a_1 \ln[x+\alpha] + a_0}, \tag{3.3}$$

where a_0, a_1, a_2, are coefficients that are uniquely determined and α is an imbedded parameter that controls the behavior of the method. Following the same assumptions for α as for the corrector the predictor is well defined. In order to determine the coefficients in (3.4), we demand that the following conditions must be satisfied

$$u(x_n) = y_n,$$
$$u'(x_n) = f_n,$$
$$u''(x_n) = g_n,$$

where $g_n = \frac{df(x,y(x))}{dx}\big|_{x_n}$ This leads to a system of three equations and three unknowns, which is solved to obtain the coefficients in (3.4). The predictor is then obtained by evaluating $u(x)$ at x_{n+1} to give

$$\begin{aligned} y_{n+1} = 2y_n^3 \Big/ \Big(&-\alpha_n y_n \ln(\frac{\alpha_n}{h+\alpha_n})\big[f_n(\ln(\frac{\alpha_n}{h+\alpha_n})-2) + g_n\alpha_n \ln(\frac{\alpha_n}{h+\alpha_n})\big] \\ &+ 2f_n^2\alpha_n^2\big(\ln(\frac{\alpha_n}{h+\alpha_n})\big)^2 + 2y_n^2\Big) \end{aligned} \tag{3.4}$$

3.3. **Error analysis - local truncation error.** We carry out a Taylor series expansion about the point x_n to determine the local truncation error as follows:

$$\begin{aligned} LTE(y(x_n), h) &= y(x_n+h) - \frac{y_n^2\big(2y_{n+1} - f_{n+1}(h+\alpha_n)\ln(\frac{\alpha_n}{h+\alpha_n})\big)}{y_{n+1}\big(f_n\alpha_n \ln(\frac{\alpha_n}{h+\alpha_n}) + 2y_n\big)} \\ &= Ah^3 + \mathcal{O}(h^4). \end{aligned} \tag{3.5}$$

This yields a principal truncation error of Ah^3, where

$$\begin{aligned} A = &-\frac{y(x_n)^2 y^{(3)}(x_n)}{6} - \frac{y(x_n)^2 y''(x_n)}{2\alpha} - \frac{y(x_n)^2 y'(x_n)}{6\alpha^2} \\ &+ \frac{y(x_n)y'(x_n)^2}{\alpha} - y'(x_n)^3 + y(x_n)y'(x_n)y''(x_n) \end{aligned} \tag{3.6}$$

Thus, the method is order $p = 2$.

3.4. **Optimization of α.** To optimize α, we utilize the principal truncation error. By setting $A = 0$ and solving for α we obtain

$$\alpha_n = \frac{-3y_n^2 y_n'' + 6y_n(y_n')^2 \pm \sqrt{9y_n^4(y_n'')^2 + 12y_n^2(y_n')^4 - 4y_n^{(3)}y^4 y_n' - 12y_n^3(y_n')^2 y_n''}}{2\big(y_n^2 y_n^{(3)} + 6(y_n')^3 - 6y_n y_n' y_n''\big)},$$

where α_n is the value of $\alpha(x_n)$. This method yields two possible solutions for α_n. Taking the harmonic mean of the two terms yields

$$\alpha_n = \frac{2f_n y_n}{6f_n^2 - 3g_n y_n}. \tag{3.7}$$

Note the arithmetic mean can be calculated as $(a+b)/2$ and the harmonic mean, $2ab/(a+b)$, is the reciprocal of the arithmetic mean multiplied by ab. The optimization of the Rational-Logarithmic Method (RLM) is found via the reciprocal of the optimization of the logarithmic method, which utilizes the arithmetic mean.

4. Numerical Examples

In this section, we give numerical examples to illustrate the accuracy of the method. All computations were carried out using our written code in Mathematica 10.0. For the sake of brevity we will not include all evaluated points, but will include many to show the overall trend of the approximation.

Let $y(x_n)$ be the exact solution and y_n the approximate solution on the partition H_N. We find the error of the approximate solution as $|y(x_n) - y_n|$. We define $E_N = |y(x_n) - y_n|$ as the maximum absolute error on H_N, and E_n as the error at x_n. We calculate the rate of convergence (ROC) using the formula ROC $= \log_2(E_N/E_{2N})$.

Example 4.1. We consider the non-singular IVP

$$y' = \frac{1}{2} - x + 2y,$$
$$y(0) = 1,$$

where $x \in [0,1]$ and $N = 20$. The exact solution is

$$y(x) = e^{2x} + \frac{x}{2}$$

In our first example we will compare the performance of the rational-logarithmic method (RLM) with the improved Euler method (IEM). This is a continuous example with no discontinuous points. Since both methods are of order $p = 2$, we may consider this a fair comparison. Table 1 highlights the results of this simulation. It contains the values of α, the approximate solutions given by both methods, the exact value, and the error at each step of the approximation. Notice that the improved euler method (IEM) initially outperforms this method, but as $x \to 1$ the method becomes more accurate. In this example the difference in accuracy between the two methods is marginal and there is no clear advantage in this method as compared with the improved Euler method (IEM).

Since Example 4.1 is continuous on $[0,1]$ we may consider the rate of convergence of the rational-logarithmic method (RLM). In Table 2 we compare the rate of convergence of this method with that of improved Euler method (IEM). Notice that this method converges to two confirming that it is of order $p = 2$. Though both converge to two it is clear from the table that this method converges more quickly than the improved Euler method (IEM). Also this method's convergence exceeds two and then converges from the right instead of the left.

Example 4.2. We consider the singular IVP

$$y' = 1 + y^2,$$
$$y(0) = 1,$$

where $x \in [0,1]$ and $N = 20$. The exact solution is

$$y(x) = \tan\left(x + \frac{\pi}{4}\right).$$

In Example 4.2 we consider a singular example. It follows that $x \neq \pi/4$ since $\tan(\pi/2)$ is undefined. For this example we will compare the rational-logarithmic method (RLM) with both the improved Euler method (IEM) and Runge-Kutta of order 4 (RK4). In Table 3 we see the rational-logarithmic Method's (RLM) performance separate from the other methods. In Table 4 we see only the error of each method's approximation. We note that until the discontinuous point this

x_n	α_n	RLM	IEM	Exact	RLM E_n	IEM E_n
0	0.196078	1	1	1	0	0
0.1	0.224511	1.27088	1.27103	1.2714	5.21362×10^{-4}	3.77758×10^{-4}
0.2	0.250209	1.59069	1.5909	1.59182	1.13924×10^{-3}	9.22647×10^{-4}
0.3	0.272716	1.97021	1.97043	1.97212	1.90678×10^{-3}	1.69012×10^{-3}
0.4	0.291788	2.42266	2.42279	2.42554	2.88351×10^{-3}	2.752×10^{-3}
0.5	0.307406	2.96414	2.96408	2.96828	4.14095×10^{-3}	4.20098×10^{-3}
0.6	0.319753	3.61435	3.61396	3.62012	5.76783×10^{-3}	6.15636×10^{-3}
0.7	0.329155	4.39732	4.39643	4.4052	7.87584×10^{-3}	8.77127×10^{-3}
0.8	0.336024	5.34243	5.34079	5.35303	1.06064×10^{-2}	1.22418×10^{-2}
0.9	0.340797	6.48551	6.48283	6.49965	1.41389×10^{-2}	1.68186×10^{-2}
1.	0.343899	7.87036	7.86623	7.88906	1.87011×10^{-2}	2.28213×10^{-2}

TABLE 1. Results for example 4.1

method outperforms IEM by at least an order of magnitude. RK4 outperforms this method until the point of discontinuity. However, when $x_n > \frac{\pi}{2}$ this method is able to accurately step over the discontinuity with $E_n \approx 10^{-1}$. This is a clear advantage over both of the other methods. This is because the other two methods diverge at the point of discontinuity. This method not only outperforms another method of order $p = 2$, but furthermore it outperforms a method of order $p = 4$.

N	IEM	E_N	ROC	RLM	E_N	ROC
20	7.86623	2.28213×10^{-2}	-	7.87036	1.87011×10^{-2}	-
40	7.88313	5.92887×10^{-3}	1.94455	7.88435	4.71061×10^{-3}	1.98914
80	7.88755	1.51066×10^{-3}	1.97258	7.88788	1.17902×10^{-3}	1.99832
160	7.88867	3.81247×10^{-4}	1.98638	7.88876	2.94705×10^{-4}	2.00025
320	7.88896	9.57612×10^{-5}	1.99321	7.88898	7.36551×10^{-5}	2.00042
640	7.88903	2.39966×10^{-5}	1.99661	7.88904	1.84102×10^{-5}	2.00028

TABLE 2. Rate of convergence for example 4.1

5. BEHAVIOR OF α ABOUT THE DISCONTINUITY

The method's ability to step over the discontinuity is due to the imbedded parameter α. By examining Figure 1 it is clear that the method is able to accurately step over the point of discontinuity. If we note the optimized function

$$\alpha_n = \frac{2f_n y_n}{6f_n^2 - 3g_n y_n}$$

immediately prior to the point of discontinuity, the evaluation of g_n is sufficient to change α_n from a positive value to a negative value. That is $3g_n y_n > 6f_n^2$. This is confirmed in the α_n column of Table 3.

Hence, the behavior or the imbedded constant α controls the behavior of the method about the point of discontinuity, permitting it to move from a relatively large positive function value to a relatively large negative function value.

x_n	α_n	RLM	Exact	Method E_n
0	0.333333	1	1	0
0.1	0.407675	1.22303	1.22305	2.36822×10^{-5}
0.2	0.50283	1.50849	1.5085	7.68901×10^{-6}
0.3	0.631945	1.89584	1.89577	7.13438×10^{-5}
0.4	0.821748	2.46525	2.46496	2.82619×10^{-4}
0.5	1.13636	3.40907	3.40822	8.50248×10^{-4}
0.6	1.77821	5.33462	5.33186	2.76644×10^{-3}
0.65	2.44873	7.3462	7.34044	5.76176×10^{-3}
0.7	3.89891	11.6967	11.6814	1.53674×10^{-2}
0.75	9.44124	28.3237	28.2383	8.54765×10^{-2}
0.8	-22.8622	-68.5866	-68.4797	1.06918×10^{-1}
0.85	-5.14964	-15.4489	-15.4579	8.96374×10^{-3}
0.9	-2.89397	-8.6819	-8.68763	5.72979×10^{-3}
0.95	-2.00545	-6.01636	-6.0203	3.9352×10^{-3}
1.	-1.52837	-4.58511	-4.58804	2.92595×10^{-3}

TABLE 3. Results for example 4.2

x_n	RLM E_n	IEM E_n	RK4 E_n
0	0	0	0
0.1	2.36822×10^{-5}	2.73043×10^{-4}	2.15331×10^{-8}
0.2	7.68901×10^{-6}	9.50794×10^{-4}	2.79710×10^{-8}
0.3	7.13438×10^{-5}	2.68231×10^{-3}	5.21260×10^{-7}
0.4	2.82619×10^{-4}	7.56687×10^{-3}	3.63068×10^{-6}
0.5	8.50248×10^{-4}	2.42079×10^{-2}	2.59756×10^{-5}
0.6	2.76644×10^{-3}	1.05272×10^{-1}	2.91448×10^{-4}
0.65	5.76176×10^{-3}	2.77873×10^{-1}	1.46124×10^{-3}
0.7	1.53674×10^{-2}	1.01467	1.33594×10^{-2}
0.75	8.54765×10^{-2}	7.94846	5.4355×10^{-1}
0.8	1.06918×10^{-1}	140.98	1392.15
0.85	8.96374×10^{-3}	3031.14	1.39964×10^{26}
0.9	5.72979×10^{-3}	5.23821×10^{9}	2.69341×10^{394}
0.95	3.93520×10^{-3}	4.70556×10^{34}	9.52550×10^{6286}
1.	2.92595×10^{-3}	3.06426×10^{134}	$5.70492 \times 10^{100567}$

TABLE 4. Results for example 4.2

Conclusion. The rational-logarithmic method (RLM) is an efficient method which yields more accurate results than the improved Euler's method. It is able to accurately and efficiently overcome points of discontinuity in the resultant functions. Also, the method is able to outperform methods of higher order in overcoming these discontinuous points. In future papers we will consider implementing this method in singular partial differential equations and systems of singular first order differential equations.
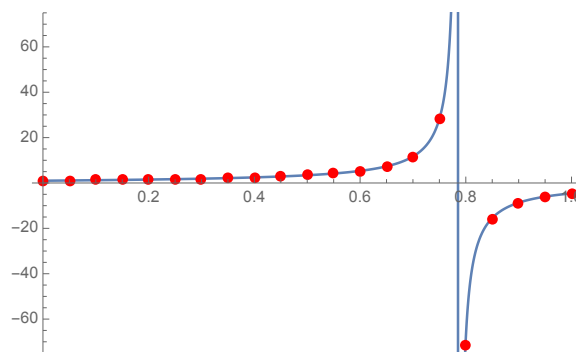
FIGURE 1. Graph of exact solution with the approximate solutions
of the Rational-Logarithmic Method

References

[1] S. O Fatunla; *Nonlinear multistep methods for initial value problems*. Computers & Mathematics with Applications, 8 (1982), 231-239.
[2] E. Hairer, G. Wanner; *Solving Ordinary Differential Equations II*, Springer, New York, 1996 .
[3] M. N. O. Ikhile; *Coefficients for Studying One-Step Rational Schemes for IVPs in ODEs: III. Extrapolation Methods*, Computers & Mathematics with Applications, 47 (2004), 1463-1475.
[4] J. D. Lambert; *Numerical Methods for Ordinary Differential Systems 1991*, England, John Wiley.
[5] F. D. Van Niekerk; *Rational one-step methods for initial value problems*, Computers Math. Applic. 16, (1988), 1035-1039.
[6] M. R. Odekunle, N. D. Oye, S. O. Adee, R. A. Ademiluyi; *A class of inverse Runge-Kutta schemes for the numerical integration of singular problems*, Applied Mathematics and Computation, 158 (2004), 149-158.
[7] H. Ramos, J. Vigo-Aguiar; *A new algorithm appropriate for solving singular and singularly perturbed autonomous initial-value problems*, International Journal of Computer Mathematics 85, (2008) 603-611.

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