Tenth MSU Conference on Differential Equations and Computational Simulations,
Electronic Journal of Differential Equations, Conference 23 (2016), pp. 35–45.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLVABILITY OF NONLINEAR DIFFERENTIAL SYSTEMS WITH COUPLED NONLOCAL BOUNDARY CONDITIONS

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Abstract. This article shows the existence of at least one solution to nonlinear differential systems with coupled nonlocal boundary conditions on an infinite interval. Our main tool is the Alternative of Leray-Schauder.

1. Introduction

Boundary value problems on an infinite interval arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and in various applications such as an unsteady flow of gas through a semi-infinite porous media and theory of draining flows (see, e.g., [1, 2, 8]). Nonlocal boundary value problems also represent a very interesting and important class of problems that have multipoint and integral boundary conditions as special cases. The study on nonlocal elliptic boundary value problems was started by Bicadze and Samarskiĭ [3], and later continued by Il'in and Moiseev [13] and Gupta [11]. Since then, the existence of solutions for nonlocal boundary value problems has received a great deal of attention in the literature. Recently, Zhang [26] investigated the existence of positive solutions for multipoint boundary value problems on an infinite interval with uncoupled boundary conditions in view of cone theory with Mönch fixed point theorem and a monotone iterative technique. Cui et al [5] studied the existence and uniqueness of the positive solutions for a singular differential system with coupled integral boundary conditions by using mixed monotone methods. For more recent results, we refer the reader to [6, 7, 9, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] and the references therein.

Inspired by the above results, we consider the nonlinear differential system with coupled nonlocal boundary conditions,

$$
\begin{gathered}
(w_i\varphi_p(u_i'))'(t) + f_i(t, u_1(t), \dots, u_N(t), u_1'(t), \dots, u_N'(t)) = 0, \quad t \in (0,\infty), \\
u_i(0) = \sum_{j=1}^{N} \int_0^\infty k_{ij}(s) u_j(s) ds, \quad \lim_{t\to\infty} (\varphi_p^{-1}(w_i) u_i')(t) = l_i^\infty, \quad 1 \le i \le N,
\end{gathered} \tag{1.1}
$$

2010 *Mathematics Subject Classification.* 34B10, 34B15, 34B40.
Key words and phrases. Solvability; nonlocal boundary condition; p-Laplacian; half-line.

Published March 21, 2016.

where $N \geq 1$, $p > 1$, $\varphi_p(s) = |s|^{p-2}s$ for $s \in (-\infty, \infty)$, $l_i^\infty \in (-\infty, \infty)$, $w_i : (0, \infty) \to (0, \infty)$ is a continuous function, $f_i : [0, \infty) \times (-\infty, \infty)^{2N} \to (-\infty, \infty)$ is a Carathéodory function, i.e., $f_i = f_i(t, u, v)$ is Lebesgue measurable in t for all $(u, v) \in (-\infty, \infty)^N \times (-\infty, \infty)^N$ and continuous in (u, v) for almost all $t \in [0, \infty)$, and $k_{ij} : (0, \infty) \to (-\infty, \infty)$ is measurable function for $1 \leq i, j \leq N$. We further assume the following conditions hold:

(H1) $\varphi_p^{-1}(\frac{1}{w_i}) \in L^1_{\rm loc}[0, \infty) \setminus L^1(0, \infty)$ and $(1 + \theta_i)k_{ij} \in L^1(0, \infty)$, where $i, j = 1, 2, \dots, N$ and
$$\theta_i(t) := \int_0^t \varphi_p^{-1}\big(\frac{1}{w_i(s)}\big) ds, \quad t \in (0, \infty);$$

(H2) $\det(K) \neq 0$, where $K = (K_{ij}) \in M_{N\times N}$ with
$$K_{ij} = \begin{cases} 1 - \int_0^\infty k_{ii}(s)ds & \text{for } i = j, \\ -\int_0^\infty k_{ij}(s)ds & \text{for } i \neq j; \end{cases}$$

(H3) for $i = 1, 2, \dots, N$, there exist nonnegative measurable functions $\alpha_i, \beta_i, \gamma$ such that
$$(1 + \theta_i)^{p-1}\alpha_i, \ \frac{\beta_i}{w_i}, \ \gamma \in L^1(0, \infty)$$
and, for almost all $t \in [0, \infty)$ and all $u, v \in (-\infty, \infty)^N$ with $u = (u_1, \dots, u_N)$ and $v = (v_1, \dots, v_N)$,
$$|f_i(t, u, v)| \leq \sum_{j=1}^N (\alpha_j(t)|u_j|^{p-1} + \beta_j(t)|v_j|^{p-1}) + \gamma(t). \tag{1.2}$$

The main tool of this paper is the following theorem, which is related to the Leray-Schauder, see, e.g., [10, p.124].

Theorem 1.1. *Let C be a convex subset of a Banach space X, and assume that $0 \in C$. Let $L : C \to C$ be a compact operator, and let*
$$\mathcal{E} = \{x \in C : x = \lambda Lx \text{ for some } \lambda \in (0, 1)\}.$$
Then either $\mathcal{E}$ is unbounded or L has a fixed point.

By using the above theorem, the existence of solutions for problem (1.1) is investigated. An example to illustrate the main result is also provided in Section 2.

2. Main result

For $i = 1, \dots, N$, X_i is the set of the functions $u_i \in C[0, \infty) \cap C^1(0, \infty)$ such that
$$\lim_{t\to 0+} (\varphi_p^{-1}(w_i)u_i')(t) \text{ and } \lim_{t\to\infty} (\varphi_p^{-1}(w_i)u_i')(t) \text{ exist.}$$
Then X_i is the Banach space with norm
$$\|u_i\|_i := \sup_{t\in[0,\infty)} \frac{|u_i(t)|}{1 + \theta_i(t)} + \sup_{t\in[0,\infty)} (\varphi_p^{-1}(w_i)|u_i'|)(t).$$
Let
$$X := \prod_{i=1}^N X_i$$

be a Banach space with norm

$$\|(u_1,\dots,u_N)\|_X := \sum_{i=1}^N \|u_i\|_i.$$

Let $Y := L^1(0,\infty)$ with norm $\|h\|_Y := \int_0^\infty |h(s)|ds$.

For $i = 1,2,\dots,N$, we define $P_i : Y \to X_i$ by, for $h_i \in Y$ and $t \in [0,\infty)$,

$$P_i(h_i)(t) := \int_0^t \varphi_p^{-1}\Big(\frac{1}{w_i(s)}\Big(\varphi_p(l_i^\infty) + \int_s^\infty h_i(\tau)d\tau\Big)\Big)ds.$$

For $h = (h_1,\dots,h_N) \in Y^N$, we define $F_i : Y^N \to (-\infty,\infty)$ by, for $i = 1,2,\dots,N$,

$$F_i(h) := \sum_{j=1}^N \int_0^\infty k_{ij}(s)P_j(h_j)(s)ds.$$

From (H2), $K^{-1} := (a_{ij}) \in M_{N\times N}$ exists and let

$$A_i(h) := \sum_{j=1}^N a_{ij}F_j(h) \quad \text{for } h \in Y^N \text{ and } i = 1,2,\dots,N,$$

i.e.,

$$(A_1(h),\dots,A_N(h))^T = K^{-1}(F_1(h),\dots,F_N(h))^T.$$

Then

$$F_i(h) = -\sum_{j=1}^N A_j(h)\int_0^\infty k_{ij}(s)ds + A_i(h). \tag{2.1}$$

Define $T : Y^N \to X$ by, for $h \in Y^N$,

$$T(h)(t) := (T_1(h)(t), T_2(h)(t),\dots,T_N(h)(t)) \quad \text{for all } t \in [0,\infty).$$

Here, for $i = 1,2,\dots,N$, $T_i : Y^N \to X_i$ is defined by, for $h = (h_1,\dots,h_N) \in Y^N$,

$$T_i(h)(t) := A_i(h) + P_i(h_i)(t) \quad \text{for all } t \in [0,\infty).$$

For $h = (h_1,\dots,h_N) \in Y^N$, consider the problem

$$\begin{gathered}(w_i\varphi_p(u_i'))'(t) + h_i(t) = 0, \quad \text{a.e. } t \in (0,\infty),\\ u_i(0) = \sum_{j=1}^N \int_0^\infty k_{ij}(s)u_j(s)ds, \quad \lim_{t\to\infty}(\varphi_p^{-1}(w_i)u_i')(t) = l_i^\infty, \quad 1 \le i \le N.\end{gathered} \tag{2.2}$$

Then we have the following lemma.

Lemma 2.1. *Assume that* (H1), (H2) *hold. For each* $h = (h_1,\dots,h_N) \in Y^N$, (2.2) *has a unique solution* $u = T(h)$ *in* X.

Proof. Let $u = (u_1,\dots,u_N)$ be a solution of (2.2) with a fixed $h = (h_1,\dots,h_N) \in Y^N$. Then, for $j = 1,2,\dots,N$,

$$u_j(t) = u_j(0) + P_j(h_j)(t),\ t \in [0,\infty).$$

Thus, for $1 \le i,j \le N$,

$$\int_0^\infty k_{ij}(s)u_j(s)ds = u_j(0)\int_0^\infty k_{ij}(s)ds + \int_0^\infty k_{ij}(s)P_j(h_j)(s)ds,$$

which implies

$$u_i(0)=\sum_{j=1}^N u_j(0)\int_0^\infty k_{ij}(s)ds+F_i(h),\quad 1\le i\le N.$$

Thus,

$$K(u_1(0),\dots,u_N(0))^T=(F_1(h),\dots,F_N(h))^T,$$

where K is the matrix in the assumption (H2). By (H2), $u_i(0)=A_i(h)$ for $1\le i\le N$, and thus $u=T(h)$. In a similar manner, one can show that $T(h)$ is a solution to (2.2) for each $h\in Y^N$ by (2.1). The proof is complete. □

Lemma 2.2. *Assume that* (H1)–(H3) *hold. For* $h=(h_1,\dots,h_N)\in Y^N$,

$$\|T_j(h)\|_j\le\sum_{i=1}^N C_{j,i}(|l_i^\infty|^{p-1}+\|h_i\|_Y)^{\frac{1}{p-1}}$$

where $1\le j,\ i\le N$ *and*

$$C_{j,i}:=\begin{cases}\sum_{z=1}^N|a_{iz}|\|k_{zi}\theta_i\|_Y+2 & \text{if } i=j,\\ \sum_{z=1}^N|a_{jz}|\|k_{zi}\theta_i\|_Y & \text{if } i\ne j.\end{cases}$$

Proof. Let $h=(h_1,\dots,h_N)\in Y^N$ be given. Then, for $1\le j,z\le N$,

$$\begin{aligned}|a_{jz}F_z(h)|&\le|a_{jz}|\sum_{i=1}^N\int_0^\infty|k_{zi}(s)P_i(h_i)(s)|ds\\&\le\sum_{i=1}^N|a_{jz}|\|k_{zi}\theta_i\|_Y(|l_i^\infty|^{p-1}+\|h_i\|_Y)^{\frac{1}{p-1}}\end{aligned}$$

which implies

$$\begin{aligned}\|A_j(h)\|_j&=\sup_{t\in[0,\infty)}\frac{|A_j(h)|}{1+\theta_j(t)}\le\sum_{z=1}^N|a_{jz}F_z(h)|\\&\le\sum_{i=1}^N\sum_{z=1}^N|a_{jz}|\|k_{zi}\theta_i\|_Y(|l_i^\infty|^{p-1}+\|h_i\|_Y)^{\frac{1}{p-1}},\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\|P_j(h_j)\|_j&=\sup_{t\in[0,\infty)}\frac{|P_j(h_j)(t)|}{1+\theta_j(t)}+\sup_{t\in[0,\infty)}(\varphi_p^{-1}(w_j)|(P_j(h_j))'|)(t)\\&\le 2(|l_j^\infty|^{p-1}+\|h_j\|_Y)^{\frac{1}{p-1}}.\end{aligned}\tag{2.4}$$

By (2.3) and (2.4),

$$\|T_j(h)\|_j\le\|A_j(h)\|_j+\|P_j(h_j)\|_j\le\sum_{i=1}^N C_{j,i}(|l_i^\infty|^{p-1}+\|h_i\|_Y)^{\frac{1}{p-1}}$$

for $1\le j\le N$, and thus the proof is complete. □

We define the Nemytskii operators $N_{f_i}:X\to Y$ by, for $u=(u_1,\dots,u_N)\in X$,

$$N_{f_i}(u)(t):=f_i(t,u(t),u'(t))\quad\text{for almost all } t\in(0,\infty),$$

and define $L:X\to X$ by

$$L(u)=(L_1(u),\dots,L_N(u)):=T(N_{f_1}(u),\dots,N_{f_N}(u))\quad\text{for } u\in X.$$

Then L is well defined, and by Lemma 2.1, problem (1.1) has a solution u in X if and only if L has a fixed point u in X.

To show the compactness of the operator L, we use the following compactness criterion:

Theorem 2.3 ([4]). *Let Z be the space of all bounded continuous real-valued functions on $[0,\infty)$ and $S \subset Z$. Then S is relatively compact in Z if the following conditions hold:*

(i) *S is bounded in Z;*
(ii) *S is equicontinuous on any compact interval of $[0,\infty)$;*
(iii) *S is equiconvergent at ∞, that is, given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $|\phi(t) - \phi(\infty)| < \epsilon$ for all $t > T$ and all $\phi \in S$.*

Lemma 2.4. *Assume that* (H1)–(H3) *hold. Then the operator $L : X \to X$ is completely continuous.*

Proof. We only prove that $L_1 : X \to X_1$ is compact, since the compactness of $L_i : X \to X_i$, $2 \le i \le N$, can be proved in a similar manner, and consequently $L : X \to X$ is compact. Recall that $L_1(u) = T_1(N_{f_1}(u), \dots, N_{f_N}(u))$ for $u \in X$.

Let Σ be bounded in X, i.e., there exists $R_1 > 0$ such that $\|u\|_X \le R_1$ for all $u = (u_1, \dots, u_N) \in \Sigma$. Let $Z = (C[0,\infty) \cap L^\infty(0,\infty)) \times (C[0,\infty) \cap L^\infty(0,\infty))$ with norm $\|(u,v)\|_Z = \|u\|_{L^\infty(0,\infty)} + \|v\|_{L^\infty(0,\infty)}$, and

$$S = \Big\{ \Big(\frac{L_1(u)}{1+\theta_1}, \varphi_p^{-1}(w_1)(L_1(u))' \Big) \in Z : u \in \Sigma \Big\}.$$

Set, for almost all $t \in [0,\infty)$,

$$h_\Sigma(t) := \sum_{j=1}^N \Big((1+\theta_j(t))^{p-1}\alpha_j(t) + \frac{\beta_j(t)}{w_j(t)} \Big) R_1^{p-1} + \gamma(t).$$

By (H3), $h_\Sigma \in Y$ and, for each $1 \le i \le N$,

$$|N_{f_i}(u)(t)| \le h_\Sigma(t) \tag{2.5}$$

for almost all $t \in [0,\infty)$ and all $u \in \Sigma$. Indeed, for $u \in \Sigma$ and for almost all $t \in [0,\infty)$, by (H3),

$$\begin{aligned} |N_{f_i}(u)(t)| &\le \sum_{j=1}^N (\alpha_j(t)|u_j(t)|^{p-1} + \beta_j(t)|u_j'(t)|^{p-1}) + \gamma(t) \\ &\le \sum_{j=1}^N \left((1+\theta_j(t))^{p-1}\alpha_j(t) + \frac{\beta_j(t)}{w_j(t)} \right) \|u\|_X^{p-1} + \gamma(t) = h_\Sigma(t). \end{aligned}$$

Thus $N_{f_i}(\Sigma)$ is bounded in Y for all $1 \le i \le N$. It follows from Lemma 2.2 that S is bounded in Z.

Let $R > 0$ be fixed and $t_1, t_2 \in [0,R]$ with $t_1 < t_2$, for $u \in \Sigma$,

$$\begin{aligned} &\Big| \frac{T_1(N_{f_1}(u), \dots, N_{f_N}(u))(t_1)}{1+\theta_1(t_1)} - \frac{T_1(N_{f_1}(u), \dots, N_{f_N}(u))(t_2)}{1+\theta_1(t_2)} \Big| \\ &\le |A_1(N_{f_1}(u), \dots, N_{f_N}(u))| \Big| \frac{1}{1+\theta_1(t_1)} - \frac{1}{1+\theta_1(t_2)} \Big| \\ &\quad + \Big| \frac{P_1(N_{f_1}(u))(t_1)}{1+\theta_1(t_1)} - \frac{P_1(N_{f_1}(u))(t_2)}{1+\theta_1(t_2)} \Big| \end{aligned}$$

$$
\begin{aligned}
&\le |A_1(N_{f_1}(u),\dots,N_{f_N}(u))|(\theta_1(t_2)-\theta_1(t_1))\\
&\quad+\Big(\frac{1}{1+\theta_1(t_1)}-\frac{1}{1+\theta_1(t_2)}\Big)|P_1(N_{f_1}(u))(t_1)|\\
&\quad+\frac{1}{1+\theta_1(t_2)}|P_1(N_{f_1}(u))(t_2)-P_1(N_{f_1}(u))(t_1)|\\
&\le |A_1(N_{f_1}(u),\dots,N_{f_N}(u))|(\theta_1(t_2)-\theta_1(t_1))+|P_1(N_{f_1}(u))(R)|(\theta_1(t_2)-\theta_1(t_1))\\
&\quad+|P_1(N_{f_1}(u))(t_2)-P_1(N_{f_1}(u))(t_1)|\\
&\le \Big[\sup_{u\in\Sigma}\{|A_1(N_{f_1}(u),\dots,N_{f_N}(u))|\}\\
&\quad+(\theta_1(R)+1)(|l_1^\infty|^{p-1}+\|h_\Sigma\|_Y)^{\frac{1}{p-1}}\Big](\theta_1(t_2)-\theta_1(t_1))\\
&\quad+|P_1(N_{f_1}(u))(t_2)-P_1(N_{f_1}(u))(t_1)|
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|(\varphi_p^{-1}(w_1)(T_1(N_{f_1}(u),\dots,N_{f_N}(u)))')(t_1)\\
&-(\varphi_p^{-1}(w_1)(T_1(N_{f_1}(u),\dots,N_{f_N}(u)))')(t_2)\big|\\
&=\Big|\varphi_p^{-1}\Big(\varphi_p(l_1^\infty)+\int_{t_1}^\infty N_{f_1}(u)(s)ds\Big)\\
&\quad-\varphi_p^{-1}\Big(\varphi_p(l_1^\infty)+\int_{t_2}^\infty N_{f_1}(u)(s)ds\Big)\Big|,
\end{aligned}
$$

which yield that S is equicontinuous on any finite subinterval of $[0,\infty)$ by (H1) and (2.5).

For $u\in\Sigma$, by L'Hospital's rule,

$$
\lim_{t\to\infty}\frac{T_1(N_{f_1}(u),\dots,N_{f_N}(u))(t)}{1+\theta_1(t)}=\lim_{t\to\infty}\varphi_p^{-1}\Big(\varphi_p(l_1^\infty)+\int_t^\infty N_{f_1}(u)(\tau)d\tau\Big)
$$

and

$$
\begin{aligned}
&\lim_{t\to\infty}(\varphi_p^{-1}(w_1)(T_1(N_{f_1}(u),\dots,N_{f_N}(u)))')(t)\\
&=\lim_{t\to\infty}\varphi_p^{-1}\Big(\varphi_p(l_1^\infty)+\int_t^\infty N_{f_1}(u)(\tau)d\tau\Big).
\end{aligned}
$$

It follows from (2.5) that, as $t\to\infty$,

$$
\frac{T_1(N_{f_1}(u),\dots,N_{f_N}(u))(t)}{1+\theta_1(t)}\to l_1^\infty
$$

and

$$
(\varphi_p^{-1}(w_1)(T_1(N_{f_1}(u),\dots,N_{f_N}(u)))')(t)\to l_1^\infty
$$

uniformly on Σ. Then, S is equiconvergent at ∞. Thus, in view of Theorem 2.3, S is relatively compact in Z, i.e., $L_1:X\to X_1$ is compact. $\square$

Now we give the main result in this paper.

Theorem 2.5. *Assume that* (H1)–(H3) *hold. Then problem* (1.1) *has at least one solution $u=(u_1,\dots,u_N)$ in X provided that, for each $1\le i\le N$,*

$$
\sum_{j=1}^N \kappa_p D_{j,i}^{p-1}\|(1+\theta_j)^{p-1}\alpha_j\|_Y+\|\frac{\beta_i}{w_i}\|_Y<\frac{1}{N} \tag{2.6}
$$

holds. Here $\kappa_p = \max\{1, 2^{(N-1)(p-2)}\}$*, and for* $1 \le j, i \le N$*,*

$$D_{j,i} := \begin{cases} C_{j,i} - 1 & \text{if } j = i, \\ C_{j,i} & \text{if } j \neq i. \end{cases}$$

Proof. Let $u = (u_1, \dots, u_N) \in X$ satisfying $u = \lambda L(u)$ for some $\lambda \in (0,1)$. Then

$$u_i = \lambda T_i(N_{f_1}(u), \dots, N_{f_N}(u)).$$

It is well known that, for $q > 0$ and for any $a_m \in (-\infty, \infty)$ for $1 \le m \le N$,

$$\Big| \sum_{m=1}^N a_m \Big|^q \le \max\{1, 2^{(N-1)(q-1)}\} \sum_{m=1}^N |a_m|^q. \tag{2.7}$$

By the same arguments as in the proof of Lemma 2.2, for $1 \le j \le N$,

$$\sup_{t\in[0,\infty)} \frac{|T_j(N_{f_1}(u), \dots, N_{f_N}(u))(t)|}{1+\theta_j(t)} \le \sum_{m=1}^N D_{j,m}(|l_m^\infty|^{p-1} + \|N_{f_m}(u)\|_Y)^{\frac{1}{p-1}}$$

and

$$\sup_{t\in[0,\infty)} \big|(\varphi_p^{-1}(w_j)(T_j(N_{f_1}(u), \dots, N_{f_N}(u)))')(t)\big| \le (|l_j^\infty|^{p-1} + \|N_{f_j}(u)\|_Y)^{\frac{1}{p-1}}.$$

Then, using the assumption (H3) and the inequality (2.7) with $q = p-1$ and $a_m = D_{j,m}(|l_m^\infty|^{p-1} + \|N_{f_m}(u)\|_Y)^{\frac{1}{p-1}}$, for almost all $t \in (0,\infty)$ and for all $1 \le i \le N$,

$$\begin{aligned}
&|N_{f_i}(u)(t)| \\
&\le \sum_{j=1}^N (\alpha_j(t)|u_j(t)|^{p-1} + \beta_j(t)|u_j'(t)|^{p-1}) + \gamma(t) \\
&= \sum_{j=1}^N \Big((1+\theta_j(t))^{p-1}\alpha_j(t)\Big(\frac{|u_j(t)|}{1+\theta_j(t)}\Big)^{p-1} + \frac{\beta_j(t)}{w_j(t)}|(\varphi_p^{-1}(w_j)u_j')(t)|^{p-1}\Big) + \gamma(t) \\
&\le \sum_{j=1}^N \Big[(1+\theta_j(t))^{p-1}\alpha_j(t)\Big(\frac{|T_j(N_{f_1}(u), \dots, N_{f_N}(u))(t)|}{1+\theta_j(t)}\Big)^{p-1}\Big] \\
&\quad + \sum_{j=1}^N \Big[\frac{\beta_j(t)}{w_j(t)}\, \big|(\varphi_p^{-1}(w_j)(T_j(N_{f_1}(u), \dots, N_{f_N}(u)))')(t)\big|^{p-1}\Big] + \gamma(t) \\
&\le \sum_{j=1}^N \Big[(1+\theta_j(t))^{p-1}\alpha_j(t)\Big(\sum_{m=1}^N D_{j,m}(|l_m^\infty|^{p-1} + \|N_{f_m}(u)\|_Y)^{\frac{1}{p-1}}\Big)^{p-1}\Big] \\
&\quad + \sum_{j=1}^N \Big[\frac{\beta_j(t)}{w_j(t)}(|l_j^\infty|^{p-1} + \|N_{f_j}(u)\|_Y)\Big] + \gamma(t) \\
&\le \sum_{m=1}^N \sum_{j=1}^N \kappa_p D_{j,m}^{p-1}(1+\theta_j(t))^{p-1}\alpha_j(t)(|l_m^\infty|^{p-1} + \|N_{f_m}(u)\|_Y) \\
&\quad + \sum_{m=1}^N \frac{\beta_m(t)}{w_m(t)}(|l_m^\infty|^{p-1} + \|N_{f_m}(u)\|_Y) + \gamma(t)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{N}\Big[\sum_{j=1}^{N}\kappa_p D_{j,m}^{p-1}(1+\theta_j(t))^{p-1}\alpha_j(t)+\frac{\beta_m(t)}{w_m(t)}\Big]\|N_{f_m}(u)\|_Y\\
&\quad+\sum_{m=1}^{N}\Big[\sum_{j=1}^{N}\kappa_p D_{j,m}^{p-1}(1+\theta_j(t))^{p-1}\alpha_j(t)+\frac{\beta_m(t)}{w_m(t)}\Big]|l_m^\infty|^{p-1}+\gamma(t).
\end{aligned}
$$

Thus, for each $1\le i\le N$,

$$
\begin{aligned}
&\|N_{f_i}(u)\|_Y\\
&\le \sum_{m=1}^{N}\Big(\sum_{j=1}^{N}\kappa_p D_{j,m}^{p-1}\|(1+\theta_j)^{p-1}\alpha_j\|_Y+\|\frac{\beta_m}{w_m}\|_Y\Big)\|N_{f_m}(u)\|_Y\\
&\quad+\sum_{m=1}^{N}\Big(\sum_{j=1}^{N}\kappa_p D_{j,m}^{p-1}\|(1+\theta_j)^{p-1}\alpha_j\|_Y+\|\frac{\beta_m}{w_m}\|_Y\Big)|l_m^\infty|^{p-1}+\|\gamma\|_Y.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\sum_{i=1}^{N}\|N_{f_i}(u)\|_Y\\
&\le N\Big[\sum_{m=1}^{N}\Big(\sum_{j=1}^{N}\kappa_p D_{j,m}^{p-1}\|(1+\theta_j)^{p-1}\alpha_j\|_Y+\|\frac{\beta_m}{w_m}\|_Y\Big)\|N_{f_m}(u)\|_Y\Big]\\
&\quad+N\Big[\sum_{m=1}^{N}\Big(\sum_{j=1}^{N}\kappa_p D_{j,m}^{p-1}\|(1+\theta_j)^{p-1}\alpha_j\|_Y+\|\frac{\beta_m}{w_m}\|_Y\Big)|l_m^\infty|^{p-1}+\|\gamma\|_Y\Big]\\
&\le \sum_{i=1}^{N}N\Big(\sum_{j=1}^{N}\kappa_p D_{j,i}^{p-1}\|(1+\theta_j)^{p-1}\alpha_j\|_Y+\|\frac{\beta_i}{w_i}\|_Y\Big)\|N_{f_i}(u)\|_Y\\
&\quad+N\Big[\sum_{i=1}^{N}\Big(\sum_{j=1}^{N}\kappa_p D_{j,i}^{p-1}\|(1+\theta_j)^{p-1}\alpha_j\|_Y+\|\frac{\beta_i}{w_i}\|_Y\Big)|l_i^\infty|^{p-1}+\|\gamma\|_Y\Big],
\end{aligned}
$$

and it follows from (2.6) that there exists a constant $C>0$ such that, for all $1\le i\le N$,

$$\|N_{f_i}(u)\|_Y\le C.$$

By Lemma 2.2, there exists $R>0$ such that $\|u\|_X<R$ for all u satisfying $u=\lambda L(u)$ for some $\lambda\in(0,1)$. Thus problem (1.1) has at least one solution u in X in view of Theorem 1.1. □

Finally, we give an example to illustrate the main result.

Example 2.6. Consider the system

$$
\begin{gathered}
(|u_1'(t)|u_1'(t))'+f_1(t,u_1(t),u_2(t),u_1'(t),u_2'(t))=0,\quad t\in(0,\infty),\\
(|u_2'(t)|u_2'(t))'+f_2(t,u_1(t),u_2(t),u_1'(t),u_2'(t))=0,\quad t\in(0,\infty),\\
u_1(0)=\int_0^\infty\big(-e^{-s}u_1(s)+2e^{-s}u_2(s)\big)\,ds,\quad \lim_{t\to\infty}u_1'(t)=l_1^\infty,\\
u_2(0)=\int_0^\infty\big(e^{-s}u_1(s)+e^{-2s}u_2(s)\big)\,ds,\quad \lim_{t\to\infty}u_2'(t)=l_2^\infty,
\end{gathered}
\tag{2.8}
$$

where $l_1^\infty, l_2^\infty \in (-\infty, \infty)$. Corresponding to the problem (1.1), $p = 3$, $N = 2$, $k_{11}(t) = -e^{-t}$, $k_{12}(t) = 2e^{-t}$, $k_{21}(t) = e^{-t}$, $k_{22}(t) = e^{-2t}$, $w_1(t) = 1$, and $w_2(t) = 1$. Then $\theta_1(t) = t$, $\theta_2(t) = t$ and

$$K = \begin{pmatrix} 2 & -2 \\ -1 & 1/2 \end{pmatrix},$$

and thus (H1) and (H2) hold. Then

$$K^{-1} = \begin{pmatrix} -1/2 & -2 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix} = \begin{pmatrix} 7/2 & 3/2 \\ 3 & 7/2 \end{pmatrix}.$$

Let

$$f_1(t,u,v,y,z) = \alpha_1(t)u\sin(tv) + \alpha_2(t)v^2\big(\frac{y^2}{1+y^2}\big) + \beta_1(t)y^2 + \beta_2(t)z\cos(yz) + \gamma_1(t)$$

and

$$f_2(t,u,v,y,z) = \alpha_1(t)u^2 + \alpha_2(t)v + \beta_1(t)y^2\big(\frac{y^2+z^2}{1+y^2+z^2}\big) + \beta_2(t)z^2 + \gamma_2(t),$$

where $\alpha_1(t) = 10^{-2}e^{-t}(1+t)^{-2}$, $\alpha_2(t) = 10^{-2}e^{-2t}(1+t)^{-2}$,

$$\beta_1(t) = \begin{cases} 10^{-2}t^{-\frac{1}{2}}, & t \in (0,1) \\ 10^{-2}t^{-2}, & t \in [1,\infty) \end{cases}, \quad \beta_2(t) = \begin{cases} 10^{-1}t^{-\frac{1}{5}}, & t \in (0,1) \\ 10^{-1}t^{-5}, & t \in [1,\infty) \end{cases},$$

and γ_1, γ_2 are any functions in Y. Then

$$|f_1(t,u,v,y,z)| \le \alpha_1(t)u^2 + \alpha_2(t)v^2 + \beta_1(t)y^2 + \beta_2(t)z^2 + \alpha_1(t) + \beta_2(t) + |\gamma_1(t)|$$

and

$$|f_2(t,u,v,y,z)| \le \alpha_1(t)u^2 + \alpha_2(t)v^2 + \beta_1(t)y^2 + \beta_2(t)z^2 + \alpha_2(t) + |\gamma_2(t)|.$$

Taking $\gamma(t) = \alpha_1(t) + \alpha_2(t) + \beta_2(t) + |\gamma_1(t)| + |\gamma_2(t)|$, then (H3) holds, and

$$\|(1+\theta_1)^2\alpha_1\|_Y = \frac{1}{100}, \quad \|(1+\theta_2)^2\alpha_2\|_Y = \frac{1}{200}, \quad \|\frac{\beta_1}{w_1}\|_Y = \frac{3}{100}, \quad \|\frac{\beta_2}{w_2}\|_Y = \frac{3}{20}.$$

By direct calculation, $\kappa_3 = 2$,

$$2D_{1,1}^2\|(1+\theta_1)^2\alpha_1\|_Y + 2D_{2,1}^2\|(1+\theta_2)^2\alpha_2\|_Y + \|\frac{\beta_1}{w_1}\|_Y < \frac{1}{2},$$
$$2D_{1,2}^2\|(1+\theta_1)^2\alpha_1\|_Y + 2D_{2,2}^2\|(1+\theta_2)^2\alpha_2\|_Y + \|\frac{\beta_2}{w_2}\|_Y < \frac{1}{2}.$$

Consequently, (2.6) holds for $i = 1, 2$. By Theorem 2.5, the system (2.8) has at least one solution for any $l_1^\infty, l_2^\infty \in (-\infty, \infty)$.

Acknowledgements. The authors thank the anonymous referee for his/her valuable comments on this article. The third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A1011225)

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