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POSITIVE SOLUTIONS FOR 3×3 ELLIPTIC BI-VARIATE INFINITE SEMIPOSITONE SYSTEMS WITH COMBINED NONLINEAR EFFECTS

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Abstract. We study the existence of positive solutions to 3×3 bi-variate systems of reaction diffusion equations with Dirichlet boundary conditions. In particular, we consider systems where the reaction terms approach $-\infty$ near the origin and satisfy some combined sublinear conditions at ∞. We use the method of sub-super solutions to establish our results.

1. Introduction

We study nonlinear elliptic 3×3 bi-variate systems of the form

$$
\begin{aligned}
-\Delta u_1 &= \lambda \frac{g_1(u_2, u_3)}{u_1^{\alpha_1}} \quad \text{in } \Omega, \\
-\Delta u_2 &= \lambda \frac{g_2(u_3, u_1)}{u_2^{\alpha_2}} \quad \text{in } \Omega, \\
-\Delta u_3 &= \lambda \frac{g_3(u_1, u_2)}{u_3^{\alpha_3}} \quad \text{in } \Omega, \\
u_1 &= u_2 = u_3 = 0; \quad \text{on } \partial\Omega
\end{aligned}
\tag{1.1}
$$

and

$$
\begin{aligned}
-\Delta u_1 &= \lambda \frac{g_1(u_2, u_3)}{u_2^{\alpha}} \quad \text{in } \Omega, \\
-\Delta u_2 &= \lambda \frac{g_2(u_3, u_1)}{u_3^{\alpha}} \quad \text{in } \Omega, \\
-\Delta u_3 &= \lambda \frac{g_3(u_1, u_2)}{u_1^{\alpha}} \quad \text{in } \Omega, \\
u_1 &= u_2 = u_3 = 0; \quad \text{on } \partial\Omega,
\end{aligned}
\tag{1.2}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with C^∞-boundary, $g_i \in C([0,\infty) \times [0,\infty))$, $g_i(0,0) < 0$ and α, $\alpha_i \in (0,1)$, for $i = 1,2,3$.

Here, if $\alpha = \alpha_i = 0$, for $i = 1,2,3$, the reaction terms are negative but finite. Such problems are referred to as semipositone problems. (see [1, 3, 4, 6, 7, 8, 10,

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11]). It is well documented in the literature that the study of positive solutions to such semipositone problems are mathematically very challenging. Since the test functions for positive subsolutions must come from positive functions ψ such that $-\Delta\psi < 0$ near $\partial\Omega$ while $-\Delta\psi > 0$ in a large part of the interior of Ω (see [5, 14]). In this paper, we study the more challenging semipositone problem where the nonlinearities approach $-\infty$ at the origin. Here we not only need to produce subsolutions such that $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$ but also they must satisfy $lim_{x\to\partial\Omega}(-\Delta\psi) = -\infty$. We refer to such problems as infinite semipositone systems. We will seek positive solutions in $[C^1(\Omega) \cap C(\bar{\Omega})]^3$.

To state our results precisely we introduce the following hypotheses:

(H1) There exist $\sigma > 0$ and $A > 0$ such that $\overline{\alpha} - \underline{\alpha} < \sigma < \overline{\alpha}$ and $g_i(s,t) > As^\sigma$ for $s \gg 1, t \gg 1$, for $i = 1,2,3$ where $\overline{\alpha} = \max\{\alpha_1,\alpha_2,\alpha_3\}$ and $\underline{\alpha} = \min\{\alpha_1,\alpha_2,\alpha_3\}$.

(H2)
$$\lim_{s\to\infty} \frac{g_1(s, Mg_3(s,s))}{s^{1+\alpha_1}} = 0, \quad \forall M > 0.$$

(H3)
$$\lim_{s\to\infty} \frac{g_2(Mg_3(s,s), s)}{s^{1+\alpha_2}} = 0, \quad \forall M > 0.$$

(H4) There exist $\sigma > 0$ and $A > 0$ such that $0 < \sigma < \alpha$ and $g_i(s,t) > As^\sigma$ for $s \gg 1$, $t \gg 1$, for $i = 1,2,3$.

(H5)
$$\lim_{s\to\infty} \frac{\tilde{g}_1(s, M\tilde{g}_3(s,s))}{s} = 0, \quad \forall M > 0.$$

(H6)
$$\lim_{s\to\infty} \frac{\tilde{g}_2(M\tilde{g}_3(s,s), s)}{s} = 0, \quad \forall M > 0,$$

where $\tilde{g}_i(s,t) = g_i(s,t)/s^\alpha$.

We establish the following results.

Theorem 1.1. *Assume* (H1)–(H3) *hold and $g_i(s,t)$ is nondecreasing in both variables for $i = 1,2,3$. Then system* (1.1) *has a positive solution for $\lambda \gg 1$.*

Theorem 1.2. *Assume* (H4)–(H6) *hold and $g_i(s,t)/s^\alpha$ is nondecreasing in both variables for $i = 1,2,3$. Then system* (1.2) *has a positive solution for $\lambda \gg 1$.*

We use the method of sub-super solutions to establish our results. Consider the system

$$\begin{gathered} -\Delta u_1 = \lambda h_1(u_1,u_2,u_3) \quad \text{in } \Omega \\ -\Delta u_2 = \lambda h_2(u_1,u_2,u_3) \quad \text{in } \Omega \\ -\Delta u_3 = \lambda h_3(u_1,u_2,u_3) \quad \text{in } \Omega \\ u_1 = u_2 = u_3 = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

We define (ψ_1,ψ_2,ψ_3) to be a subsolution of (1.3) if $\psi_i \in C^1(\Omega)\cap C(\bar{\Omega})$ and

$$\begin{gathered} -\Delta\psi_i \le \lambda h_i(\psi_1,\psi_2,\psi_3) \quad \text{in } \Omega \\ \psi_i > 0 \quad \text{in } \Omega \\ \psi_i = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

for $i = 1, 2, 3$, and (Z_1, Z_2, Z_3) to be a supersolution of (1.3) if $Z_i \in C^1(\Omega) \cap C(\bar{\Omega})$ and

$$\begin{gathered} -\Delta Z_i \geq \lambda h_i(Z_1, Z_2, Z_3) \quad \text{in } \Omega \\ Z_i > 0 \quad \text{in } \Omega \\ Z_i = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

for $i = 1, 2, 3$. For systems (1.1) and (1.2), if there exist subsolutions (ψ_1, ψ_2, ψ_3) and supersolutions (Z_1, Z_2, Z_3) such that $(\psi_1, \psi_2, \psi_3) \leq (Z_1, Z_2, Z_3)$ on $\bar{\Omega}$, then these systems have at least one solution $(u_1, u_2, u_3) \in [C^1(\Omega) \cap C(\bar{\Omega})]^3$ satisfying $(\psi_1, \psi_2, \psi_3) \leq (u_1, u_2, u_3) \leq (Z_1, Z_2, Z_3)$ on $\bar{\Omega}$. This follows by the natural extension of the result in [9] for scalar equations to systems (1.1) and (1.2) under the assumptions that $g_i(s,t)$'s are nondecreasing and $\frac{g_i(s,t)}{s^\alpha}$'s are nondecreasing in both variables, respectively.

In [13], the authors study such singular systems in the case $n = 2$. (See also [15] for a study in the case $n = 1$.) Here we extend this study to 3×3 bi-variate systems (1.1) and (1.2). The main difference in these new systems is that our nonlinearities depend on two variables instead of one variable, and this is more challenging in constructing both sub and super solutions. We will prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3. In Section 4, we will consider the natural extension of our results to p-Laplacian systems.

2. Proof of main results

Theorem 1.1. Let $\phi > 0$ such that $\|\phi\|_\infty = 1$ be the eigenfunction corresponding to the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary condition, i.e. ϕ satisfies

$$\begin{gathered} -\Delta\phi = \lambda_1\phi, \quad \text{in } \Omega \\ \phi = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

For $\gamma \in \big(\frac{1}{1+\underline{\alpha}}, \frac{1}{1+(\overline{\alpha}-\sigma)}\big)$, let $\psi_i = \lambda^\gamma \phi^{\frac{2}{1+\alpha_i}}$. Then

$$-\Delta\psi_i = \big(\lambda^\gamma \frac{2}{1+\alpha_i}\big)\phi^{\frac{-2\alpha_i}{1+\alpha_i}}\big[\lambda_1\phi^2 - \big(\frac{1-\alpha_i}{1+\alpha_i}\big)|\nabla\phi|^2\big].$$

Let $\delta > 0$, $m > 0$ and $\mu > 0$ be such that

$$\big(\frac{1-\alpha_i}{1+\alpha_i}\big)|\nabla\phi|^2 - \lambda_1\phi^2 \geq m, \quad \text{in } \overline{\Omega}_\delta, \quad \text{for } i = 1, 2, 3,$$

and $\phi \geq \mu > 0$ in $\Omega \setminus \overline{\Omega}_\delta$, where $\overline{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. This is possible since $|\nabla\phi| \neq 0$ on $\partial\Omega$. Hence even though $g_i(0,0) < 0$, for $\lambda \gg 1$, in $\overline{\Omega}_\delta$,

$$\big(\lambda^\gamma \frac{2}{1+\alpha_i}\big)\big[\lambda_1\phi^2 - \big(\frac{1-\alpha_i}{1+\alpha_i}\big)|\nabla\phi|^2\big] \leq \lambda\frac{g_i(0,0)}{(\lambda^\gamma)^{\alpha_i}},$$

since $1 - \gamma - \alpha_i\gamma < 0$. Therefore,

$$-\Delta\psi_i \leq \lambda\frac{g_i(0,0)}{(\lambda^\gamma\phi^{\frac{2}{1+\alpha_i}})^{\alpha_i}} \leq \lambda\frac{g_i(\psi_{i+1}, \psi_{i+2})}{\psi_i^{\alpha_i}} \quad \text{in } \overline{\Omega}_\delta \tag{2.1}$$

for $\lambda \gg 1$.

Next, in $\Omega \setminus \overline{\Omega}_\delta$, since $\phi \geq \mu > 0$, from (H1), we know that for $\lambda \gg 1$,

$$g_i(\lambda^\gamma\phi^{\frac{2}{1+\alpha_{i+1}}}, \lambda^\gamma\phi^{\frac{2}{1+\alpha_{i+2}}}) \geq A(\lambda^\gamma\phi^{\frac{2}{1+\alpha_{i+1}}})^\sigma.$$

Also, since $0 < \mu \leq \phi < 1$ and $1 + (\sigma - \alpha_i)\gamma - \gamma > 0$, for $\lambda \gg 1$,

$$\big(\lambda^{\gamma}\frac{2}{1+\alpha_i}\big)\lambda_1\phi^2 \leq \lambda\frac{A(\lambda^{\gamma}\phi^{\frac{2}{1+\alpha_{i+1}}})^{\sigma}}{\lambda^{\gamma\alpha_i}}.$$

Then in $\Omega \setminus \overline{\Omega}_\delta$, for $\lambda \gg 1$,

$$\begin{aligned} -\Delta\psi_i &\leq \big(\lambda^{\gamma}\frac{2}{1+\alpha_i}\big)\lambda_1\phi^{\frac{-2\alpha_i}{1+\alpha_i}+2} \\ &\leq \lambda\frac{g_i(\lambda^{\gamma}\phi^{\frac{2}{1+\alpha_{i+1}}}, \lambda^{\gamma}\phi^{\frac{2}{1+\alpha_{i+2}}})}{(\lambda^{\gamma}\phi^{\frac{2}{1+\alpha_i}})^{\alpha_i}} \\ &= \lambda\frac{g_i(\psi_{i+1}, \psi_{i+2})}{\psi_i^{\alpha_i}}. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we see that for $\lambda \gg 1$,

$$-\Delta\psi_i \leq \lambda\frac{g_i(\psi_{i+1}, \psi_{i+2})}{\psi_i^{\alpha_i}} \quad \text{in } \Omega.$$

Thus (ψ_1, ψ_2, ψ_3) is a positive subsolution of (1.1).

Now, we construct a supersolution $(Z_1, Z_2, Z_3) \geq (\psi_1, \psi_2, \psi_3)$. From [12], we know that $w_i \in C^1(\Omega) \cap C(\overline{\Omega})$ exists such that

$$\begin{gathered} -\Delta w_i = \frac{1}{w_i^{\alpha_i}}, \quad \text{in } \Omega, \\ w_i = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

and satisfying $w_i \geq \varepsilon e$ for some $\varepsilon > 0$. Here e is a positive solution of $-\Delta e = 1$ in Ω and $e = 0$ on $\partial\Omega$ which satisfies $e \in C_0^1(\overline{\Omega})$ and $\frac{\partial e}{\partial\nu} < 0$ on $\partial\Omega$, where ν is the outward normal vector on $\partial\Omega$. Let $\omega = \max\{\|w_1\|, \|w_2\|, \|w_3\|\}$, and

$$(Z_1, Z_2, Z_3) = (m(\lambda)w_1, m(\lambda)w_2, g_3(m(\lambda)\|w_1\|, m(\lambda)\|w_2\|)w_3).$$

Then, from (H2), we can choose $m(\lambda) \gg 1$ such that

$$\frac{g_1(m(\lambda)w, g_3(m(\lambda)w, m(\lambda)w)w)}{(m(\lambda))^{1+\alpha_1}} \leq \frac{1}{\lambda}.$$

Then

$$\begin{aligned} -\Delta Z_1 = \frac{m(\lambda)}{w_1^{\alpha_1}} &\geq \lambda\frac{g_1(m(\lambda)w, g_3(m(\lambda)w, m(\lambda)w)w)}{(m(\lambda)w_1)^{\alpha_1}} \\ &\geq \lambda\frac{g_1(m(\lambda)w_2, g_3(m(\lambda)\|w_1\|, m(\lambda)\|w_2\|)w_3)}{(m(\lambda)w_1)^{\alpha_1}} \\ &= \lambda\frac{g_1(Z_2, Z_3)}{Z_1^{\alpha_1}}. \end{aligned}$$

From (H3), choose $m(\lambda) \gg 1$ such that

$$\frac{g_2(g_3(m(\lambda)w, m(\lambda)w)w, m(\lambda)w)}{(m(\lambda))^{1+\alpha_2}} \leq \frac{1}{\lambda}.$$

Then

$$\begin{aligned} -\Delta Z_2 = \frac{m(\lambda)}{w_2^{\alpha_2}} &\geq \lambda\frac{g_2(g_3(m(\lambda)w, m(\lambda)w)w, m(\lambda)w)}{(m(\lambda)w_2)^{\alpha_2}} \\ &\geq \lambda\frac{g_2(g_3(m(\lambda)\|w_1\|, m(\lambda)\|w_2\|)w_3, m(\lambda)w_1)}{(m(\lambda)w_2)^{\alpha_2}} \end{aligned}$$

$$= \lambda \frac{g_2(Z_3, Z_1)}{Z_2^{\alpha_2}}.$$

From (H1), choose $m(\lambda) \gg 1$ such that

$$\frac{\lambda}{(g_3(m(\lambda)\|w_1\|, m(\lambda)\|w_2\|))^{\alpha_3}} < 1.$$

Then

$$\begin{aligned}
-\Delta Z_3 &= \frac{g_3(m(\lambda)\|w_1\|, m(\lambda)\|w_2\|)}{w_3^{\alpha_3}} \\
&\geq \lambda \frac{g_3(m(\lambda)w_1, m(\lambda)w_2)}{(g_3(m(\lambda)\|w_1\|, m(\lambda)\|w_2\|))^{\alpha_3} w_3^{\alpha_3}} \\
&= \lambda \frac{g_3(Z_1, Z_2)}{Z_3^{\alpha_3}}.
\end{aligned}$$

Thus (Z_1, Z_2, Z_3) is a supersolution of (1.1). Further, $m(\lambda)$ can be chosen large enough so that $(Z_1, Z_2, Z_3) \geq (\psi_1, \psi_2, \psi_3)$ in $\overline{\Omega}$. Therefore, problem (1.1) has a positive solution $(u_1, u_2, u_3) \in [(\psi_1, \psi_2, \psi_3), (Z_1, Z_2, Z_3)]$. $\square$

Proof of Theorem 1.2. Let $\psi = \lambda^\gamma \phi^{\frac{2}{1+\alpha}}$, $\gamma \in \big(\frac{1}{1+\alpha}, \frac{1}{1+(\alpha-\sigma)}\big)$, and ϕ as before. Then by arguments similar to that in the proof of Theorem 1.1, we can show that (ψ, ψ, ψ) is a subsolution. Now, we construct a supersolution $(Z_1, Z_2, Z_3) \geq (\psi, \psi, \psi)$. From (H5), (H6), we can choose $m(\lambda) \gg 1$ such that

$$\frac{\tilde{g}_1(m(\lambda)\|e\|, \lambda\tilde{g}_3(m(\lambda)\|e\|, m(\lambda)\|e\|)\|e\|)}{m(\lambda)} \leq \frac{1}{\lambda}, \tag{2.3}$$

$$\frac{\tilde{g}_2(\lambda\tilde{g}_3(m(\lambda)\|e\|, m(\lambda)\|e\|)\|e\|, m(\lambda)\|e\|)}{m(\lambda)} \leq \frac{1}{\lambda}, \tag{2.4}$$

where e is as described before in the proof of Theorem 1.1. Let

$$(Z_1, Z_2, Z_3) := (m(\lambda)e, m(\lambda)e, \lambda\tilde{g}_3(m(\lambda)\|e\|, m(\lambda)\|e\|)e).$$

Then by (2.3)

$$\begin{aligned}
-\Delta Z_1 = m(\lambda) &\geq \lambda\tilde{g}_1(m(\lambda)\|e\|, \lambda\tilde{g}_3(m(\lambda)\|e\|, m(\lambda)\|e\|)\|e\|) \\
&\geq \lambda \frac{\tilde{g}_1(m(\lambda)e_2, \lambda\tilde{g}_3(m(\lambda)\|e\|, m(\lambda)\|e\|)e)}{(m(\lambda)e)^\alpha} \\
&= \lambda \frac{g_1(Z_2, Z_3)}{Z_2^\alpha},
\end{aligned}$$

and by (3.2)

$$\begin{aligned}
-\Delta Z_2 = m(\lambda) &\geq \lambda\tilde{g}_2(\lambda\tilde{g}_3(m(\lambda)\|e\|, m(\lambda)\|e\|)e, m(\lambda)\|e\|) \\
&\geq \lambda \frac{g_2(\lambda\tilde{g}_3(m(\lambda)\|e\|, m(\lambda)\|e\|)e, m(\lambda)e)}{(\lambda\tilde{g}_3((m(\lambda)\|e\|, m(\lambda)\|e\|)e)^\alpha} \\
&= \lambda \frac{g_2(Z_3, Z_1)}{Z_3^\alpha},
\end{aligned}$$

and

$$\begin{aligned}
-\Delta Z_3 &= \lambda\tilde{g}_3(m(\lambda)\|e\|, m(\lambda)\|e\|) \\
&\geq \lambda\tilde{g}_3(m(\lambda)e, m(\lambda)e)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda \frac{g_3(m(\lambda)e, m(\lambda)e)}{(m(\lambda)e)^{\alpha}} \\
&= \lambda \frac{g_3(Z_1, Z_2)}{Z_1^{\alpha}}.
\end{aligned}
$$

Thus (Z_1, Z_2, Z_3) is a supersolution of (1.2). Further, $m(\lambda)$ can be chosen large enough so that $(Z_1, Z_2, Z_3) \geq (\psi_1, \psi_2, \psi_3)$ in $\overline{\Omega}$. Therefore, problem (1.2) has a positive solution $(u_1, u_2, u_3) \in [(\psi_1, \psi_2, \psi_3), (Z_1, Z_2, Z_3)]$. $\square$

3. p-Laplacian systems

In this section, we discuss the extensions of our main results to the following two p-Laplacian systems:

$$
\begin{gathered}
-\Delta_p u_1 = \lambda \frac{g_1(u_2, u_3)}{u_1^{\alpha_1}}, \quad \text{in } \Omega, \\
-\Delta_p u_2 = \lambda \frac{g_2(u_3, u_1)}{u_2^{\alpha_2}}, \quad \text{in } \Omega, \\
-\Delta_p u_3 = \lambda \frac{g_3(u_1, u_2)}{u_3^{\alpha_3}}, \quad \text{in } \Omega, \\
u_1 = u_2 = u_3 = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.1}
$$

and

$$
\begin{gathered}
-\Delta_p u_1 = \lambda \frac{g_1(u_2, u_3)}{u_2^{\alpha}}, \quad \text{in } \Omega, \\
-\Delta_p u_2 = \lambda \frac{g_2(u_3, u_1)}{u_3^{\alpha}}, \quad \text{in } \Omega, \\
-\Delta_p u_3 = \lambda \frac{g_3(u_1, u_2)}{u_1^{\alpha}}, \quad \text{in } \Omega, \\
u_1 = u_2 = u_3 = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.2}
$$

Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, Ω is a bounded domain in $\mathbb{R}^N$ with C^{∞}-boundary, $g_i \in C([0,\infty) \times [0,\infty))$, $g_i(0,0) < 0$ and α, $\alpha_i \in (0,1)$, for $i = 1,2,3$.

To state our results for these p-Laplacian systems, we introduce the following hypotheses:

(H7)
$$
\lim_{s\to\infty} \frac{g_1(s, M(g_3(s,s))^{\frac{1}{p-1}})}{s^{p-1+\alpha_1}} = 0, \quad \forall M > 0.
$$

(H8)
$$
\lim_{s\to\infty} \frac{g_2(M(g_3(s,s))^{\frac{1}{p-1}}, s)}{s^{p-1+\alpha_2}} = 0, \quad \forall M > 0.
$$

(H9)
$$
\lim_{s\to\infty} \frac{\tilde{g}_1(s, M(\tilde{g}_3(s,s))^{\frac{1}{p-1}})}{s^{p-1}} = 0, \quad \forall M > 0.
$$

(H10)
$$
\lim_{s\to\infty} \frac{\tilde{g}_2(M(\tilde{g}_3(s,s))^{\frac{1}{p-1}}, s)}{s^{p-1}} = 0, \quad \forall M > 0,
$$

where $\tilde{g}_i(s,t) = g_i(s,t)/s^{\alpha}$.

Theorem 3.1. *Assume (A) $p \geq 3$ or (B) $p < 3$ and $\alpha_i < \frac{p}{3}$. Let* (H1), (H7), (H8) *hold and $g_i(s,t)$ be nondecreasing in both variables for $i = 1, 2, 3$. Then system* (3.1) *has a positive solution for $\lambda \gg 1$.*

Theorem 3.2. *Assume* (H4), (H9), (H10) *hold and $g_i(s,t)/s^\alpha$ is nondecreasing in both variables for $i = 1, 2, 3$. Then system* (3.2) *has a positive solution for $\lambda \gg 1$.*

Here we prove these results again by the method of sub-super solutions. As described in [13], the method of sub-super solutions holds for systems (3.1) and (3.2) with the assumptions that $g_i(s,t)$'s are nondecreasing and the functions $g_i(s,t)/s^\alpha$ are nondecreasing in both variables. First, by an argument similar to the proof of Theorem 1.1, we can show that if $\psi_i := \lambda^\gamma \phi_p^{\frac{p}{p-1+\alpha_i}}$, for

$$\gamma \in (\frac{1}{p-1+\underline{\alpha}}, \frac{1}{p-1+(\overline{\alpha}-\sigma)}),$$

then (ψ_1, ψ_2, ψ_3) is subsolution of (3.1) for $\lambda \gg 1$. Here $\phi_p > 0$ such that $\|\phi_p\|_\infty = 1$ is the eigenfunction corresponding to the first eigenvalue of the operator $-\Delta_p$ with Dirichlet boundary condition, i.e. ϕ_p satisfies:

$$\begin{gathered} -\Delta_p \phi_p = \lambda_1 \phi_p^{p-1}, \quad \text{in } \Omega \\ \phi_p = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

Also, by [2], for (A) $p \geq n$, or (B) $p < n$ and $\alpha_i < \frac{p}{n}$, the problem

$$\begin{gathered} -\Delta_p w_i = \frac{1}{w_i^{\alpha_i}}, \quad \text{in } \Omega \\ w_i = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

has a solution $w_i \in C^1(\Omega) \times C(\overline{\Omega})$ such that $w_i \geq \varepsilon e_p$, where $-\Delta_p e_p = 1$ in Ω, $e_p = 0$ on $\partial\Omega$. Let $(Z_1, Z_2, Z_3) := (m(\lambda)w_1, m(\lambda)w_2, g_3(m(\lambda)\|w_1\|, m(\lambda)\|w_2\|)w_3)$. Then for $m(\lambda) \gg 1$, (Z_1, Z_2, Z_3) is a supersolution of (3.1) and $(Z_1, Z_2, Z_3) \geq (\psi_1, \psi_2, \psi_3)$, by an argument similar to that in the proof of Theorem 1.1. Hence Theorem 3.1 holds.

Next, to establish theorem 3.2, let $\psi := \lambda^\gamma \phi_p^{\frac{p}{p-1+\alpha}}$, for $\gamma \in (\frac{1}{p-1+\alpha}, \frac{1}{p-1+(\alpha-\sigma)})$, and

$$(Z_1, Z_2, Z_3) := (m(\lambda)e_p, m(\lambda)e_p, \lambda^{\frac{1}{p-1}} \tilde{g}_3(m(\lambda)\|e_p\|, m(\lambda)\|e_p\|)e_p).$$

Then by an argument similar to that in the proof of Theorem 1.2, (ψ, ψ, ψ) is a subsolution of (3.2) for $\lambda \gg 1$ and for $m(\lambda) \gg 1$, (Z_1, Z_2, Z_3) is a supersolution of (3.2) with $(Z_1, Z_2, Z_3) \geq (\psi, \psi, \psi)$. Hence Theorem 3.2 holds.

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