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INFINITELY MANY SMALL ENERGY SOLUTIONS FOR A FRACTIONAL KIRCHHOFF EQUATION INVOLVING SUBLINEAR NONLINEARITIES

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Abstract. This article is devoted to the study of the following fractional Kirchhoff equation

$$M\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $(-\Delta)^s$ is the fractional Laplacian, $M:\mathbb{R}_+\to\mathbb{R}_+$ is the Kirchhoff term, $V:\mathbb{R}^N\to\mathbb{R}$ is a positive continuous potential and $f(x,u)$ is only locally defined for $|u|$ small. By combining a variant of the symmetric Mountain Pass with a Moser iteration argument, we prove the existence of infinitely many weak solutions converging to zero in $L^\infty(\mathbb{R}^N)$-norm.

1. Introduction

In the previous two decades, the study of nonlocal problems driven by the fractional and nonlocal operators has received a tremendous popularity because of their intriguing structure and the great application in many different context such as optimization, finance, phase transition phenomena, population dynamics, quantum mechanics, game theory; see [14, 30] and references therein for more details.

In this article we deal with the fractional Kirchhoff equation

$$M\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)(-\Delta)^s u + V(x)u = f(x,u) \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $s\in(0,1)$, $N>2s$, $M:\mathbb{R}_+\to\mathbb{R}_+$ is a Kirchhoff function which is assumed to be continuous and satisfies the following conditions:

(A1) there exists $m_0>0$ such that $\inf_{t\in\mathbb{R}_+} M(t)\geq m_0$;
(A2) there exist $a,b,\nu>0$ such that $M(t)\leq a+bt^\nu$ for all $t\geq 0$.

A typical example of Kirchhoff function is given by

$$M(t)=a+b\gamma t^{\gamma-1}, \quad a,b\geq 0,\ a+b>0, \tag{1.2}$$

and

$$\gamma \begin{cases} \in(1,\infty) & \text{if } b>0\\ =1 & \text{if } b=0. \end{cases}$$

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When M is of the type (1.2), problem (1.1) is said to be non-degenerate when $a > 0$ and $b \geq 0$, while it is called degenerate if $a = 0$ and $b > 0$.

The potential $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function such that

(A3) $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and there exist $r > 0$ and $\alpha > N$ such that
$$\lim_{|y|\to\infty} \mathcal{L}\Big(\Big\{x \in B_r(y) : \frac{V(x)}{|x|^\alpha} \leq A\Big\}\Big) = 0 \quad \forall A > 0.$$

The fractional Laplacian $(-\Delta)^s u$ of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$ is defined by
$$\mathcal{F}((-\Delta)^s u)(k) = |k|^{2s}\mathcal{F}(u)(k), \quad k \in \mathbb{R}^N,$$
where $\mathcal{F}$ denotes the Fourier transform, that is,
$$\mathcal{F}(u)(k) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-\imath k\dot{x}} u(x)\,dx.$$
Equivalently, $(-\Delta)^s u$ can be represented by
$$(-\Delta)^s u(x) = -\frac{1}{2}C(N,s)\int_{\mathbb{R}^N} \frac{(u(x+y) + u(x-y) - 2u(x))}{|y|^{N+2s}} dy,$$
where $C(N,s)$ is a dimensional constant depending only on N and s, precisely given by
$$C(N,s) = \Big(\int_{\mathbb{R}^N} \frac{1-\cos(x_1)}{|x|^{N+2s}}\,dx\Big)^{-1}.$$

When $M(t) = a + bt$, $s = 1$ and $p = 2$, problem (1.1) is related to the stationary analogue of the Kirchhoff equation
$$\rho\, u_{tt} - \Big(\frac{p_0}{h} + \frac{E}{2L}\int_0^L |u_x|^2\,dx\Big)u_{xx} = 0, \tag{1.3}$$
which was proposed by Kirchhoff [24] in 1883 as a generalization of the well-known D'Alembert's wave equation for free vibrations of elastic strings. The Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here $u = u(x,t)$ is the transverse string displacement at the space coordinate x and time t, L is the length of the string, h is the area of the cross section, E is Young's modulus of the material, ρ is the mass density, and p_0 is the initial tension. Since we cannot review the huge bibliography on Kirchhoff equation, we only mention the early works [13], [27], [34] and some interesting results obtained in [1, 2, 17, 22, 26, 33, 40].

More recently, in [20] the authors introduced a stationary Kirchhoff model in fractional setting, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string; see Appendix in [20] for a more detailed physical description of the fractional Kirchhoff model. After that, several existence and multiplicity results for fractional Kirchhoff problems have been established by many authors [12, 10, 11, 18, 19, 29, 31, 35, 36].

We also note that, when $M(t) \equiv 1$, the equation (1.1) becomes the stationary fractional Schrödinger equation
$$(-\Delta)^s u + V(x)u = f(x,u) \text{ in } \mathbb{R}^N \tag{1.4}$$
derived by Laskin in [25] as a result of expanding the Feynman path integral from Brownian-like to Lévy like quantum mechanical paths. Equation (1.4) has been extensively investigated in the last years: see for instance [3, 5, 6, 7, 9, 8, 16, 21, 28, 38, 39] and the references therein.

Motivated by the above papers, in this work we focus our attention on the existence of infinitely many solutions for (1.1) without any growth conditions imposed on $f(x,u)$ at infinity with respect to u. More precisely, along this paper, we assume that $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ satisfies the following assumptions:

(A4) there exists $\delta>0$ such that $f\in C(\mathbb{R}^N\times[-\delta,\delta])$ and $f(x,-t)=-f(x,t)$ for all $x\in\mathbb{R}^N$ and $|t|\le\delta$;

(A5) there exists a ball $B_{r_0}(x_0)\subset\mathbb{R}^N$ such that
$$\liminf_{t\to 0}\Big(\inf_{x\in B_{r_0}(x_0)}\frac{F(x,t)}{t^2}\Big)>-\infty,$$
$$\limsup_{t\to 0}\Big(\inf_{x\in B_{r_0}(x_0)}\frac{F(x,t)}{t^2}\Big)=\infty,$$
where $F(x,t)=\int_0^t f(x,z)\,dz$;

(A6) there exist $\tau>0$ and $g\in C([-\tau,\tau],\mathbb{R}_+)$ such that $|f(x,t)|\le g(t)$ for all $x\in\mathbb{R}^N$ and $|t|\le\tau$.

Before stating our main result, we recall some useful facts on fractional Sobolev spaces and we fix the notation.

The fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by
$$H^s(\mathbb{R}^N)=\{u\in L^2(\mathbb{R}^N):[u]<\infty\},$$
where $[u]$ denotes the so-called Gagliardo seminorm, that is
$$[u]^2=\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy=2C(N,s)^{-1}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2dx,$$
and $H^s(\mathbb{R}^N)$ is equipped with the norm
$$\|u\|_{H^s(\mathbb{R}^N)}=\sqrt{[u]^2+|u|_2^2}.$$
We recall that $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for any $q\in[2,2_s^*]$ and compactly embedded into $L^q_{\rm loc}(\mathbb{R}^N)$ for any $q\in[1,2_s^*)$; see [14, 30].

To study our problem (1.1), we introduce the fractional space
$$\mathbb{X}_V=\big\{u\in H^s(\mathbb{R}^N):|u|_V^2:=\int_{\mathbb{R}^N}V(x)u^2\,dx<\infty\big\}$$
endowed with the norm
$$\|u\|=\sqrt{[u]^2+|u|_V^2}.$$

Definition 1.1. We say that $u\in\mathbb{X}_V$ is a weak solution to (1.1) if
$$M([u]^2)\iint_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathbb{R}^N}V(x)u\varphi\,dx$$
$$=\int_{\mathbb{R}^N}f(x,u)\varphi\,dx$$
for all $\varphi\in\mathbb{X}_V$.

It is clear that weak solutions to (1.1) can be found as critical points of the Euler-Lagrange functional
$$\mathcal{I}(u)=\frac12\mathcal{M}([u]^2)+\frac12|u|_V^2-\int_{\mathbb{R}^N}F(x,u)dx,$$

where $\mathcal{M}(t)=\int_0^t M(\tau)\,d\tau$.

Now, we state the main result of this paper.

Theorem 1.2. *Assume that* (A1)–(A6) *hold. Then there exists a sequence* $(u_n)\subset \mathbb{X}_V\cap L^\infty(\mathbb{R}^N)$ *such that* $\mathcal{I}(u_n)<0$ *and* $|u_n|_\infty\to 0$ *as* $nto\infty$*.*

We give a sketch of the proof of Theorem 1.2. Firstly we extend the nonlinear term $f(x,u)$ and we introduce a modified fractional Kirchhoff equation. By using a suitable variant of the Symmetric Mountain Pass Theorem due to Kajikiya [23], we prove that the modified problem admits infinitely many small energy solutions $(u_n)\subset\mathbb{X}_V$ such that $u_n\to 0$ in $\mathbb{X}_V$. Finally, using a Moser iteration argument [32], we prove that $u_n\to 0$ in $L^\infty(\mathbb{R}^N)$ as $nto\infty$ and that (u_n) is indeed a sequence of solutions of the original problem (1.1).

The plan of the paper is the following: in Section 2 we collect some useful results which will be used along the paper. The Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminaries

From now on, we fix $s\in(0,1)$ and $N>2s$. We begin proving the following compactness result.

Lemma 2.1. *The space* $\mathbb{X}_V$ *is compactly embedded into* $L^p(\mathbb{R}^N)$ *for any* $p\in[1,2^*_s)$*.*

Proof. For any $y\in\mathbb{R}^N$ and $A>0$ we define the two sets

$$
\begin{gathered}
C_A(y)=\big\{x\in B_r(y):\frac{V(x)}{|x|^\alpha}\leq A\big\},\\
D_A(y)=\big\{x\in B_r(y):\frac{V(x)}{|x|^\alpha}> A\big\}.
\end{gathered}
$$

Let (u_n) be a bounded sequence in $\mathbb{X}_V$ and, up to a subsequence, we may assume that

$$
u_n\rightharpoonup u\quad \text{in } \mathbb{X}_V.
$$

We aim to prove that $u_n\to u$ in $L^p(\mathbb{R}^N)$ for any $p\in[1,2^*_s)$.

Firstly, we consider the case $p=1$. Set $v_n=u_n-u$. Take a sequence $(y_i)\subset\mathbb{R}^N$ such that $\mathbb{R}^N\subset\cup_{i=1}^\infty B_r(y_i)$ and each $x\in\mathbb{R}^N$ is covered by at most 2^N such balls.

Then, for all $R>2r$ we can see that

$$
\begin{aligned}
\int_{B_R^c}|v_n|\,dx&\leq\sum_{|y_i|\geq R-r}\int_{B_r(y_i)}|v_n|dx\\
&=\sum_{|y_i|\geq R-r}\Big(\int_{C_A(y_i)}|v_n|dx+\int_{D_A(y_i)}|v_n|dx\Big).
\end{aligned}\tag{2.1}
$$

Let us note that for all $|y_i|\geq R-r$ ($i\in\mathbb{N}$) we have

$$
\begin{aligned}
\int_{C_A(y_i)}|v_n|dx&\leq\Big(\int_{C_A(y_i)}|v_n|^2dx\Big)^{1/2}\mathcal{L}(C_A(y_i))^{1/2}\\
&\leq C\Big(\int_{B_r(y_i)}|(-\Delta)^{\frac{s}{2}}v_n|^2+V(x)v_n^2\,dx\Big)^{1/2}\mathcal{L}(C_A(y_i))^{1/2}.
\end{aligned}\tag{2.2}
$$

On the other hand

$$
\begin{aligned}
&\int_{D_A(y_i)} |v_n| dx \\
&= \int_{D_A(y_i)\cap\{|x|^\alpha|v_n|\leq 1\}} |v_n| dx + \int_{D_A(y_i)\cap\{|x|^\alpha|v_n|>1\}} |v_n| dx \\
&\leq \int_{D_A(y_i)\cap\{|x|^\alpha|v_n|\leq 1\}} |x|^{-\alpha}\, dx + \int_{D_A(y_i)\cap\{|x|^\alpha|v_n|>1\}} (|x|^\alpha|v_n|)|x|^{-\alpha}\, dx \\
&\leq \int_{D_A(y_i)\cap\{|x|^\alpha|v_n|\leq 1\}} |x|^{-\alpha}\, dx + \int_{D_A(y_i)\cap\{|x|^\alpha|v_n|>1\}} |x|^\alpha|v_n|^2\, dx \\
&\leq \int_{D_A(y_i)\cap\{|x|^\alpha|v_n|\leq 1\}} |x|^{-\alpha}\, dx + \int_{D_A(y_i)\cap\{|x|^\alpha|v_n|>1\}} \frac{V(x)}{A}|v_n|^2\, dx.
\end{aligned}
\tag{2.3}
$$

Putting together (2.1), (2.2) and (2.3) we obtain

$$
\begin{aligned}
\int_{B_R^c} |v_n|\, dx &\leq 2^N C\Big(\int_{B_{R-2r}^c} |(-\Delta)^{\frac{s}{2}} v_n|^2 + V(x)|v_n|^2\, dx\Big)^{1/2} \sup_{|y|\geq R-r} \mathcal{L}(C_A(y))^{1/2} \\
&\quad + 2^N \int_{B_{R-2r}^c} |x|^{-\alpha}\, dx + \frac{2^N}{A}\int_{B_{R-2r}^c} V(x)|v_n|^2\, dx \\
&\leq C \sup_{|y|\geq R-r} \mathcal{L}(C_A(y))^{1/2} + \frac{C}{(R-2r)^{\alpha-N}} + \frac{C}{A}.
\end{aligned}
$$

Now, fixed $\varepsilon > 0$ we can take $A > 0$ sufficiently large such that

$$
\frac{C}{A} < \frac{\varepsilon}{3}.
$$

Let $R > 0$ be big enough in such way that

$$
\begin{gathered}
\frac{C}{(R-2r)^{\alpha-N}} < \frac{\varepsilon}{3}, \\
\sup_{|y|\geq R-r} \mathcal{L}(C_A(y))^{1/2} < \frac{\varepsilon}{3}.
\end{gathered}
$$

Then we can deduce that $v_n \to 0$ in $L^1(B_R^c)$. Since $v_n \to 0$ in $L^1(B_R)$ in view of the compact embedding $H^s(\mathbb{R}^N) \subset\subset L^1(B_R)$, we can infer that $v_n \to 0$ in $L^1(\mathbb{R}^N)$.

When $p \in (1, 2_s^*)$ we can use an interpolation argument and the strong convergence in $L^1(\mathbb{R}^N)$ to obtain the thesis. Indeed, taking $\theta \in (0,1)$ such that

$$
\frac{1}{p} = \theta + \frac{1-\theta}{2_s^*},
$$

we can see that

$$
|v_n|_p \leq |v_n|_1^\theta |v_n|_{2_s^*}^{1-\theta} \leq C|v_n|_1^\theta \|v_n\|^{1-\theta} \leq C|v_n|_1^\theta
$$

and recalling that $|v_n|_1 \to 0$ as $n \to \infty$ we can conclude that $|v_n|_p \to 0$ as $n \to \infty$. □

3. A variant of the Symmetric Mountain Pass Lemma

To prove the existence of infinitely many solutions to (1.1) which tend to zero, we will use a variant of the Symmetric Mountain Pass Lemma [4]. Firstly, we recall the definition of genus and some its fundamental properties; see [37] for more details.

Let E be a Banach space and A a subset of E. We say that A is symmetric if $u \in A$ implies that $-u \in A$. Let Γ be the family of closed symmetric subsets A of E such that $0 \notin A$, that is,

$$\Gamma = \{A \subset E \setminus \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to the origin}\}.$$

For $A \in \Gamma$, we define

$$\gamma(A) = \inf\{m \in \mathbb{N} : \exists \phi \in C(A, \mathbb{R}^m \setminus \{0\}), \phi(x) = -\phi(-x)\}.$$

If there is no mapping ϕ as above for any $m \in \mathbb{N}$, then $\gamma(A) = \infty$. Let us denote by Γ_k the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$. Then we have the following result.

Proposition 3.1. *Let A and B be closed symmetric subset of E which do not contain the origin. Then we have*

(i) *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$.*
(ii) *If there is an odd homeomorphism from A onto B, then $\gamma(A) = \gamma(B)$.*
(iii) *If $\gamma(B) < \infty$, then $\gamma(A \setminus B) \geq \gamma(A) - \gamma(B)$.*
(iv) *The n-dimensional sphere $\mathbb{S}^n$ has a genus of $n+1$ by the Borsuk-Ulam Theorem.*
(v) *If A is compact, then $\gamma(A) < \infty$ and there exist $\delta > 0$ and a closed and symmetric neighborhood $N_\delta(A) = \{x \in E : \|x - A\| \leq \delta\}$ of A such that $\gamma(N_\delta(A)) = \gamma(A)$.*

Also we recall the following result due to Kajikija [23].

Theorem 3.2. *Let E be an infinite-dimensional Banach space and $\mathcal{J} \in \mathcal{C}^1(E, \mathbb{R})$ be a functional satisfying the conditions below:*

(A7) *$\mathcal{J}(u)$ is even, bounded from below, $\mathcal{J}(0) = 0$ and $I(u)$ satisfies the local Palais-Smale condition; that is for some $c^* > 0$, in the case when every sequence $\{u_k\}$ in E satisfying $\mathcal{J}(u_k) \to c < c^*$ and $\mathcal{J}'(u_k) \to 0$ in E^* has a convergent subsequence;*
(A8) *For each $k \in \mathbb{N}$, there exist an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} \mathcal{J}(u) < 0$.*

Then either (i) or (ii) below holds.

(i) *There exists a sequence $\{u_k\}$ such that $\mathcal{J}'(u_k) = 0$, $\mathcal{J}(u_k) < 0$ and $\{u_k\}$ converges to zero.*
(ii) *There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $\mathcal{J}'(u_k) = 0$, $\mathcal{J}(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $\mathcal{J}'(v_k) = 0$, $\mathcal{J}(v_k) < 0$, $\lim_{k\to\infty} \mathcal{J}(v_k) = 0$ and $\{v_k\}$ converges to a non-zero limit.*

4. Proof of Theorem 1.2

Take $\ell > 0$ be such that $\ell < \frac{1}{2}\min\{\delta, \tau\}$, where δ and τ are as in (A4) and (A6). Let us define a function $h \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$\begin{gathered}
0 \leq h(t) \leq 1, \\
h(-t) = h(t) \quad \text{for all } t \in \mathbb{R}, \\
h(t) = 1 \quad \text{if } |t| \leq \ell, \\
h(t) = 0 \quad \text{if } |t| \geq 2\ell, \\
h \text{ is decreasing in } [\ell, 2\ell].
\end{gathered}$$

Set $\tilde{f}(x,t) = f(x,t)h(t)$ and we denote

$$\tilde{F}(x,t) = \int_0^t \tilde{f}(x,z)dz \quad \text{for } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Let us consider the equation

$$M\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)(-\Delta)^s u + V(x)u = \tilde{f}(x,u) \text{ in } \mathbb{R}^N, \tag{4.1}$$

and we observe that if u is a weak solution to (4.1) such that $|u|_\infty \leq \ell$ then u is also a solution to (1.1). For this reason we look for critical points of the functional $\mathcal{J} : \mathbb{X}_V \to \mathbb{R}$ defined as

$$\mathcal{J}(u) = \frac{1}{2}\mathcal{M}([u]^2) + \frac{1}{2}|u|_V^2 - \int_{\mathbb{R}^N} \tilde{F}(x,u)\,dx.$$

In view of (A6) we can see that there exists $C_0 > 0$ such that

$$\begin{gathered} |\tilde{f}(x,t)| \leq C_0 \quad \text{for any } (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ |\tilde{F}(x,t)| \leq C_0|t| \quad \text{for any } (x,t) \in \mathbb{R}^N \times \mathbb{R}. \end{gathered} \tag{4.2}$$

Therefore, using Lemma 2.1, we can deduce that $\mathcal{J}$ is well-defined on $\mathbb{X}_V$. Moreover, it is easy to verify that $\mathcal{J} \in C^1(\mathbb{X}_V, \mathbb{R})$. We also note that for all $u, \varphi \in \mathbb{X}_V$,

$$\begin{aligned} \langle \mathcal{J}'(u), \varphi\rangle = M([u]^2) &\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &+ \int_{\mathbb{R}^N} V(x)u\varphi\,dx - \int_{\mathbb{R}^N} \tilde{f}(x,u)\varphi\,dx. \end{aligned}$$

Next we show that $\mathcal{J}$ satisfies the assumptions of Theorem 3.2.

Lemma 4.1. *$\mathcal{J}$ is bounded from below and satisfies the (PS) condition on $\mathbb{X}_V$.*

Proof. By using (A1), (4.2) and Lemma 2.1 we can see that for all $u \in \mathbb{X}_V$,

$$\mathcal{J}(u) \geq \frac{\min\{m_0, 1\}}{2}\|u\|^2 - C_0 \int_{\mathbb{R}^N} |u|dx \geq C_1\|u\|^2 - C_2\|u\| \tag{4.3}$$

which implies that $\mathcal{J}$ is bounded from below.

Let (u_n) be a (PS) sequence at the level $c \in \mathbb{R}$, that is

$$\mathcal{J}(u_n) \to c \quad \text{and} \quad \mathcal{J}'(u_n) \to 0.$$

Then we can use (4.3) to show that

$$c + o(1) = \mathcal{J}(u_n) \geq C_1\|u_n\|^2 - C_2\|u_n\|,$$

which gives the boundedness of (u_n) in $\mathbb{X}_V$. Then, in view of Lemma 2.1, we may assume that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } \mathbb{X}_V, \\ u_n \to u \quad \text{in } L^p(\mathbb{R}^N) \text{ for all } p \in [1, 2_s^*). \end{gathered}$$

In particular, from (A4), (4.2) and the Dominated Convergence Theorem, we can infer that

$$\int_{\mathbb{R}^N} \tilde{f}(x,u_n)u_n\,dx = \int_{\mathbb{R}^N} \tilde{f}(x,u)u\,dx + o(1) = \int_{\mathbb{R}^N} \tilde{f}(x,u_n)u\,dx.$$

Let us assume that $[u_n] \to t_0 \geq 0$. Taking into account (A1), $\langle \mathcal{J}'(u_n), u_n\rangle = o(1)$ and $\langle \mathcal{J}'(u_n), u\rangle = o(1)$ we can see that

$$M(t_0^2)[u_n]^2 + |u_n|_V^2 = \int_{\mathbb{R}^N} \tilde{f}(x,u)u dx + o(1) = M(t_0^2)[u]^2 + |u|_V^2,$$

from which we can easily deduce the thesis. □

Lemma 4.2. *For each $k \in \mathbb{N}$, there exists a closed symmetric subset $A_k \subset X$ such that $0 \in A_k$, the genus $\gamma(A_k) \geq k$ and $\sup_{u\in A_k} \mathcal{J}(u) < 0$.*

Proof. For any $r > 0$ we define

$$D(r) = \{x \in \mathbb{R}^N : 0 \leq x_i \leq r \text{ for all } i = 1, \dots, N\}$$

By using (A5) we take ℓ sufficiently small such that there exists a positive constant C and two sequences of positive numbers $\delta_n \to 0$ and $M_n \to \infty$ as $n \to \infty$ such that

$$F(x,u) \geq -Cu^2 \quad \forall x \in D(r_0), |u| \leq 2\ell \tag{4.4}$$

$$F(x,\delta_n) \geq M_n\delta_n^2 \quad \forall x \in D(r_0), n \in N. \tag{4.5}$$

Fix $k \in \mathbb{N}$ and we construct $A_k \in \Gamma_k$ satisfying (A8). Let $p \in \mathbb{N}$ be the smallest integer that satisfies $p^N \geq k$. We divide $D(r_0)$ equally into p^N small cubes by planes parallel to each face of $D(r_0)$. Denote them by D_i with $1 \leq i \leq p^N$. We use D_i with$1 \leq i \leq k$ only. Set $a = r_0/p$. Then the edge of D_i has the length of a.We make a cube E_i in D_i such that E_i has the same center as that of D_i, the faces of E_i and D_i are parallel and the edge of E_i has the length of $a/2$.

Let us introduce a function $\phi \in C(\mathbb{R},\mathbb{R})$ such that

$$\phi(t) = \begin{cases} 0 & \text{for } t \in (-\infty, 0] \cap [a,\infty), \\ \frac{4t}{a} & \text{for } t \in [0, \frac{a}{4}], \\ 1 & \text{for } t \in [\frac{a}{4}, \frac{3}{4}a], \\ -\frac{4t}{a} + 4 & \text{for } t \in [\frac{3}{4}a, a]. \end{cases}$$

Then $\phi \in H^s(\mathbb{R})$. Let us define

$$\psi(x) = \phi(x_1)\cdot\phi(x_2)\dots\phi(x_N) = \prod_{i=1}^N \phi(x_i),$$

and we note that $\operatorname{supp}(\psi) = [0,a]^N$. We define $\psi_i(x)$ by a parallel translation $\psi(x - y_i)$ with a suitable $y_i \in \mathbb{R}^N$ such that the support of ψ_i coincides with D_i. Then

$$0 \leq \psi_i \leq 1 \text{ in } \mathbb{R}^N, \quad \psi_i = 1 \text{ on } E_i,$$
$$\operatorname{supp}(\psi_i) = D_i, \text{ and } \operatorname{supp}(\psi_i) \cap \operatorname{supp}(\psi_j) = \emptyset\ textforalli \neq j.$$

Finally, let us define

$$V_k = \{t \in \mathbb{R}^k : \max_{1\leq i\leq k} |t_i| = 1\},$$

$$W_k = \{\sum_{i=1}^k t_i\psi_i : t \in V_k\}.$$

Since V_k is the surface of the k-dimensional cube, it is homeomorphic to the sphere $\mathbb{S}^{k-1}$ by an odd mapping. Hence $\gamma(V_k) = \gamma(\mathbb{S}^{k-1}) = k$. Moreover, $\gamma(W_k) = \gamma(V_k) = k$ because the mapping $(t_1, \dots, t_k) \mapsto \sum t_i\psi_i$ is odd and homeomorphic.

Since W_k is compact we can find a constant $C_k > 0$ such that

$$\|u\| \le C_k \quad \forall u \in W_k. \tag{4.6}$$

For any $t \in (0, \ell)$ and $u = \sum_{i=1}^k t_i\psi_i \in W_k$ we can use the definition of $\tilde{F}(x, u)$ and the condition (A2) to see that

$$\begin{aligned} \mathcal{J}(tu) &\le \frac{at^2}{2}\|u\|^2 + \frac{bt^{2(\nu+1)}}{\nu+1}\|u\|^{2(\nu+1)} - \sum_{i=1}^k \int_{D_i} \tilde{F}(x, tt_i\psi_i)\,dx \\ &= \frac{at^2}{2}\|u\|^2 + \frac{bt^{2(\nu+1)}}{\nu+1}\|u\|^{2(\nu+1)} - \sum_{i=1}^k \int_{D_i} F(x, tt_i\psi_i)\,dx. \end{aligned} \tag{4.7}$$

In view of the definition of V_k, there is $j \in [1, k]$ such that $|t_j| = 1$ and $|t_i| \le 1$ for other i. Hence

$$\begin{aligned} \sum_{i=1}^k \int_{D_i} F(x, tt_i\psi_i)\,dx &= \int_{E_j} F(x, tt_i\psi_i)\,dx + \int_{D_j\setminus E_j} F(x, tt_i\psi_i)\,dx \\ &\quad + \sum_{i\neq j} \int_{D_i} F(x, tt_i\psi_i)\,dx. \end{aligned} \tag{4.8}$$

Now, by using (4.4), we note that

$$\int_{D_j\setminus E_j} F(x, tt_i\psi_i)\,dx + \sum_{i\neq j} \int_{D_i} F(x, tt_i\psi_i)\,dx \ge -Cr_0^N t^2. \tag{4.9}$$

On the other hand, taking into account (A4), $|t_j| = 1$ and $\psi_j = 1$ on E_j we obtain

$$\int_{E_j} F(x, tt_i\psi_i)\,dx = \int_{E_j} F(x, t)\,dx. \tag{4.10}$$

Then, putting together (4.7), (4.8), (4.9) and (4.10), for each $\delta_n \in (0, \ell)$, we obtain

$$\mathcal{J}(\delta_n u) \le \delta_n^2\big[\frac{a}{2}C_k^2 + \frac{b\delta_n^{2\nu}}{\nu+1}C_k^{2(\nu+1)} + Cr_0^N - \Big(\frac{a}{2}\Big)^N M_n\big] \quad \forall u \in W_k. \tag{4.11}$$

Since $\delta_n \to 0$ and $M_n \to \infty$ as $n \to \infty$ we can see that (4.11) yields $\mathcal{J}(\delta_m u) < 0$ for all $u \in W_k$, for some $m \in \mathbb{N}$. Then, setting $A_k = \{\delta_m u : u \in W_k\}$, we can deduce that $\gamma(A_k) = \gamma(W_k) = k$ and $\sup_{u\in A_k} \mathcal{J}(u) < 0$. $\square$

Now, we are ready to give the proof of the main result of this paper.

Proof of Theorem 1.2. In view of (A4) and the definition of $\tilde{F}$ we know that $\mathcal{J}$ is even and $\mathcal{J}(0) = 0$. Taking into account Lemma 4.1, Lemma 4.2 and Theorem 3.2 we can deduce that there exists a sequence $(u_n) \subset \mathbb{X}_V$ such that $\mathcal{J}(u_n) < 0$ and $u_n \to 0$ in $\mathbb{X}_V$ as $n \to \infty$.

Therefore, if we prove that $|u_n|_\infty \le \ell$, it follows that $\mathcal{J}(u_n) = \mathcal{I}(u_n) < 0$ and (u_n) is the desired sequence. In order to achieve our aim, we use a Moser iteration argument [32].

For all $\beta \geq 0$ and $L > 0$ we denote by $v_{L,n} = u_n u_{L,n}^{2\beta}$ and $w_{L,n} = u_n u_{L,n}^{\beta}$, where $u_{L,n} = \min\{|u_n|, L\}$. Taking $v_{L,n}$ as test function in (4.1) we can see that

$$\begin{aligned} &M([u_n])^2 \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(v_{L,n}(x)-v_{L,n}(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &+ \int_{\mathbb{R}^N} V(x)u_n^2 u_{L,n}^{2\beta}\,dx \\ &= \int_{\mathbb{R}^N} \tilde{f}(x,u_n)u_n u_{L,n}^{2\beta}. \end{aligned} \tag{4.12}$$

By using (4.2) and $0 \leq u_{L,n} \leq |u_n|$ we have

$$|\int_{\mathbb{R}^N} \tilde{f}(x,u_n)u_n u_{L,n}^{2\beta}\,dx| \leq C_0 \int_{\mathbb{R}^N} |u_n|^{2\beta+1} = C_0|u_n|_{2\beta+1}^{2\beta+1}. \tag{4.13}$$

Now, we can argue as in [7] to infer that

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(v_{L,n}(x)-v_{L,n}(y))}{|x-y|^{N+2s}}\,dx\,dy \geq S_*(\beta+1)^{-2}|w_{L,n}|_{2_s^*}^2. \tag{4.14}$$

Indeed, to deduce the above estimate, we note that $\gamma(t) = tt_L^{2\beta}$ is an increasing function, so we have

$$(a-b)(\gamma(a)-\gamma(b)) \geq 0 \quad \text{for any } a,b \in \mathbb{R}.$$

Let us consider

$$\mathcal{E}(t) = \frac{|t|^2}{2} \quad \text{and} \quad \Gamma(t) = \int_0^t (\gamma'(\tau))^{1/2} d\tau.$$

Then, by applying Jensen inequality we obtain for all $a,b \in \mathbb{R}$ such that $a > b$,

$$\begin{aligned} \mathcal{E}'(a-b)(\gamma(a)-\gamma(b)) &= (a-b)(\gamma(a)-\gamma(b)) \\ &= (a-b)\int_b^a \gamma'(t)dt \\ &= (a-b)\int_b^a (\Gamma'(t))^2 dt \\ &\geq \Big(\int_b^a (\Gamma'(t))dt\Big)^2 \\ &= (\Gamma(b)-\Gamma(a))^2. \end{aligned}$$

The same argument works when $a \leq b$. Therefore

$$\mathcal{E}'(a-b)(\gamma(a)-\gamma(b)) \geq |\Gamma(a)-\Gamma(b)|^2 \text{ for any } a,b \in \mathbb{R}. \tag{4.15}$$

By using (4.15), we can see that

$$|\Gamma(u_n)(x)-\Gamma(u_n)(y)|^2 \leq (u_n(x)-u_n(y))((u_n u_{L,n}^{2\beta})(x)-(u_n u_{L,n}^{2\beta})(y)). \tag{4.16}$$

Since

$$\Gamma(u_n) \geq \frac{1}{(\beta+1)} u_n u_{L,n}^{\beta-1},$$

and invoking the following Sobolev inequality

$$S_*|u|_{2_s^*}^2 \leq [u]^2 \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$$

where

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \{u \in L^{2_s^*}(\mathbb{R}^N) : [u] < \infty\},$$

we have

$$[\Gamma(u_n)]^2 \geq S_* |\Gamma(u_n)|^2_{2^*_s} \geq \Big(\frac{1}{\beta+1}\Big)^2 S_* |u_n u_{L,n}^{\beta-1}|^2_{2^*_s} \tag{4.17}$$

which shows that (4.14) holds.

Putting together (A1), (A3), (4.12), (4.13) and (4.14) we obtain

$$m_0 S_* (\beta+1)^{-2} |w_{L,n}|^2_{2^*_s} \leq C_0 |u_n|^{2\beta+1}_{2\beta+1},$$

and taking the limit as $L \to \infty$ we deduce that

$$|u_n|_{2^*_s(\beta+1)} \leq [C(\beta+1)]^{\frac{1}{\beta+1}} |u_n|^{\frac{2\beta+1}{2(\beta+1)}}_{2\beta+1}. \tag{4.18}$$

Set $\beta_0 = \frac{2^*_s-1}{2}$ and we define β_k for $k \geq 1$ such that $2\beta_{k+1}+1 = 2^*_s(\beta_k+1)$. Then, iterating the formula (4.18) we can see that

$$|u_n|_{2\beta_{k+1}+1} \leq e^{r_k} |u_n|^{\sigma_k}_{2^*_s} \tag{4.19}$$

where

$$r_k = \sum_{i=0}^{k} \frac{\log(C(\beta_i+1))}{\beta_i+1}, \quad \sigma_k = \prod_{i=0}^{k} \frac{2\beta_i+1}{2(\beta_i+1)}.$$

We note that (r_k) and (σ_k) are two convergent subsequences and

$$r := \lim_{k\to\infty} r_k > 0,$$
$$\sigma := \lim_{k\to\infty} \sigma_k \in (0,1).$$

Taking the limit as $k \to \infty$ in (4.19) and using Lemma 2.1 we can infer that

$$|u_n|_\infty \leq e^r |u_n|^\sigma_{2^*_s} \leq C\|u_n\|^\sigma.$$

Recalling that $\|u_n\| \to 0$ as $n \to \infty$ we can infer that $|u_n|_\infty \to 0$ as $n \to \infty$. As a consequence, we can find $n_0 \in \mathbb{N}$ such that $|u_n|_\infty \leq \ell$ for all $n \geq n_0$. This completes the proof. □

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