Two nonlinear days in Urbino 2017,
*Electronic Journal of Differential Equations*, Conference 25 (2018), pp. 39–53.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# GROUND STATES OF SOME COUPLED NONLOCAL FRACTIONAL DISPERSIVE PDES

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*Dedicated to the memory of Anna Aloe*

Abstract. We show the existence of ground state solutions to the following stationary system coming from some coupled fractional dispersive equations such as: nonlinear fractional Schrödinger (NLFS) equations (for dimension $n = 1, 2, 3$) or NLFS and fractional Korteweg-de Vries equations (for $n = 1$),

$$(-\Delta)^s u + \lambda_1 u = u^3 + \beta uv, \quad u \in W^{s,2}(\mathbb{R}^n)$$
$$(-\Delta)^s v + \lambda_2 v = \frac{1}{2}v^2 + \frac{1}{2}\beta u^2, \quad v \in W^{s,2}(\mathbb{R}^n),$$

where $\lambda_j > 0$, $j = 1, 2$, $\beta \in \mathbb{R}$, $n = 1, 2, 3$, and $\frac{n}{4} < s < 1$. Precisely, we prove the existence of a positive radially symmetric ground state for any $\beta > 0$.

## 1. Introduction

In this article we study the existence of ground state solutions to the following stationary system coming from some coupled nonlocal fractional dispersive equations such as: nonlinear fractional Schrödinger (NLFS) equations (for dimension $n = 1, 2, 3$) or NLFS and fractional Korteweg-de Vries equations (FKdV) (for $n = 1$)

$$\begin{aligned}(-\Delta)^s u + \lambda_1 u &= u^3 + \beta uv, \quad u \in W^{s,2}(\mathbb{R}^n)\\ (-\Delta)^s v + \lambda_2 v &= \frac{1}{2}v^2 + \frac{1}{2}\beta u^2, \quad v \in W^{s,2}(\mathbb{R}^n),\end{aligned} \tag{1.1}$$

where $W^{s,2}(\mathbb{R}^n)$ denotes the fractional Sobolev space, $n = 1, 2, 3$. $\lambda_j > 0$, $j = 1, 2$, the coupling factor $\beta \in \mathbb{R}$, and the fraction $\frac{n}{4} < s < 1$.

The associated critical Sobolev exponent is defined by $2^*_s = \frac{2n}{n-2s}$ if $n > 2s$, and $2^*_s = \infty$ if $n \le 2s$. As a consequence, since $\frac{n}{4} < s < 1$ we have that $2^*_s > 4$.

It is well known that the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, is a nonlocal diffusive type operator. It arises in several physical phenomena like flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, in probability, American options in finance, in $\alpha$-stable Lévy processes, etc; see for instance [7, 11, 20].

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2010 *Mathematics Subject Classification.* 49J40, 35Q55, 35Q53, 35B38, 35J50.
*Key words and phrases.* Nonlinear Fractional Schrödinger equation; variational method; fractional Korteweg-de Vries equation; critical point theory; ground state.

Published September 15, 2018.

In the one-dimensional case, when $s = 1$, (1.1) comes from the following system of coupled nonlinear Schödinger (NLS) and Korteweg-de Vries (KdV) equations

$$\begin{aligned} & if_t + f_{xx} + |f|^2 f + \beta f g = 0 \\ & g_t + g_{xxx} + g g_x + \frac{1}{2}\beta(|f|^2)_x = 0, \end{aligned} \tag{1.2}$$

where $f = f(x,t) \in \mathbb{C}$ while $g = g(x,t) \in \mathbb{R}$, and $\beta \in \mathbb{R}$ is the real coupling coefficient. System (1.2) appears in phenomena of interactions between short and long dispersive waves, arising in fluid mechanics, such as the interactions of capillary - gravity water waves. Indeed, $f$ represents the short-wave, while $g$ stands for the long-wave. For more details, see for instance [2, 21, 29] and the references therein.

Looking for "traveling-wave" solutions, namely solutions to (1.2) of the form

$$(f(x,t), g(x,t)) = \big(e^{i\omega t} e^{i\frac{c}{2}x} u(x-ct), v(x-ct)\big)$$

with $u, v$ real functions, and choosing $\lambda_1 = \omega + \frac{c^2}{4}$, $\lambda_2 = c$, one finds that $u, v$ solve the problem

$$\begin{aligned} & -u'' + \lambda_1 u = u^3 + \beta u v \\ & -v'' + \lambda_2 v = \frac{1}{2} v^2 + \frac{1}{2}\beta u^2. \end{aligned} \tag{1.3}$$

This system has been studied, among others, in [2, 3, 18, 19, 23, 24]. Also, note that system (1.3) corresponds to system (1.1) when $s = 1$ and $n = 1$.

On the other hand, for $n = 2, 3$, and $s = 1$, system (1.1) corresponds to (1.3)

$$\begin{aligned} & -\Delta u + \lambda_1 u = u^3 + \beta u v \\ & -\Delta v + \lambda_2 v = \frac{1}{2} v^2 + \frac{1}{2}\beta u^2, \end{aligned} \tag{1.4}$$

for which the existence of bound and ground states have been studied in [18, 19]. We observe that system (1.4) can be seen as a stationary version of a time dependent coupled NLS system when one looks for solitary wave solutions, and $(u, v)$ are the corresponding standing waves solutions of (1.4) (see for instance [19, section 6]). It is well known that systems of NLS-NLS time-dependent equations have applications in nonlinear Optics, Hartree-Fock theory for Bose-Einstein condensates, among other physical phenomena; see for instance the earlier mathematical works [1, 4, 5, 6, 9, 33, 36, 37, 38], the more recent list (far from complete) [15, 17, 22, 35, 39] and references therein. See also a close related work; [16], in which was studied a close system of coupled NLFS equations.

Here we are interested in system (1.1), consisting of coupled NLS equations involving the so called fractional Laplacian operator (or fractional Schrödinger operator, $(-\Delta)^s + \lambda\,\mathrm{Id}$).

Note that in dimension $n = 1$, (1.1) can also be seen as a system of coupled NLFS-FKdV equations. In this case, (1.1) is the corresponding stationary system when one looks for travelling-wave solutions of the time-dependent system

$$\begin{aligned} & if_t - A_s\, f + |f|^2 f + \beta f g = 0 \\ & g_t - (A_s\, g)_x + g g_x + \frac{1}{2}\beta(|f|^2)_x = 0, \end{aligned} \tag{1.5}$$

where $A_s$ stands for the nonlocal fractional Laplacian $(-\Delta)^s$ in dimension $n = 1$.

While for $n = 1, 2, 3$, (1.1) can be seen as the stationary system when one looks for standing wave solutions of the time-dependent system of coupled NLFS equations

$$\begin{aligned} if_t - (-\Delta)^s f + |f|^2 f + \beta f g = 0 \\ ig_t - (-\Delta)^s g + \beta |f|^2 = 0. \end{aligned} \tag{1.6}$$

The main goal of this manuscript is to demonstrate that for any $\beta > 0$, problem (1.1) has a positive radially symmetric ground state $\widetilde{\mathbf{u}} = (\widetilde{u}, \widetilde{v}) \in W^{s,2}(\mathbb{R}^n) \times W^{s,2}(\mathbb{R}^n)$; see Theorems 4.1 and 4.2.

Notice that, for any $\beta \in \mathbb{R}$, (1.1) has a unique *semi-trivial* positive radially symmetric solution, that we denote by $\mathbf{v}_2 = (0, V_2)$, where $V_2(x)$ is the unique positive radially symmetric ground state of $-(\Delta)^s v + \lambda_2 v = \frac{1}{2} v^2$ in $W^{s,2}(\mathbb{R}^n)$; [27, 28]. Since we are interested in positive ground states, then we have to show that they are different from the semi-trivial solution $\mathbf{v}_2$. To do so, we will demonstrate some properties of the semi-trivial solution which will allow us to show that $\mathbf{v}_2$ is not a ground state. For example, we will show that there exists a constant $\Lambda > 0$ such that for $\beta > \Lambda$, $\mathbf{v}_2$ is a saddle point of the associated energy functional constrained on the corresponding Nehari Manifold, which actually is a natural restriction. When $\beta < \Lambda$ then $\mathbf{v}_2$ is a strict local minimum of the energy functional on the Nehari Manifold. In this case, we exclude that $\mathbf{v}_2$ is a ground state by the construction of a function in the Nehari Manifold with energy lower than the energy of $\mathbf{v}_2$. Precisely, we will demonstrate that there exists a positive radially symmetric ground state of (1.1), $\widetilde{\mathbf{u}} \neq \mathbf{v}_2$, either: $\beta > \Lambda$ (see Theorem 4.1) or $0 < \beta \leq \Lambda$ and $\lambda_2$ large enough (see Theorem 4.2).

This article is organized as follows. In Section 2 we introduce notation and preliminaries, dealing with some background on the fractional Laplacian and we give the definition of ground state. Section 3 contains some results on the method of the *natural constraint* and the main properties about the semi-trivial solution $\mathbf{v}_2$, that we will use in the proof of the main existence results stated and proved in Section 4. Finally, in Section 5 we study the existence of ground states for some systems with an arbitrary number of coupled equations.

## 2. Preliminaries and notation

The nonlocal fractional Laplacian operator $(-\Delta)^s$ in $\mathbb{R}^n$ is defined on the Schwartz class of functions $g \in \mathcal{S}$ through the Fourier transform,

$$[(-\Delta)^{\frac{\alpha}{2}} g]^\wedge(\xi) = (2\pi|\xi|)^\alpha \widehat{g}(\xi), \tag{2.1}$$

or via the Riesz potential, see for example [31, 40]. Note that $s = 1$ corresponds to the standard local Laplacian operator. See also [32, 25, 27, 28], where the fractional Schrödinger operator $((-\Delta)^s + \mathrm{Id})$ is defined and are analyzed some problems dealing with.

There is another way to define this operator. If $s = 1/2$ the square root of the Laplacian acting on a function $u$ in the whole space $\mathbb{R}^n$, can be calculated as the normal derivative on the boundary of its harmonic extension to the upper half-space $\mathbb{R}^{n+1}_+$, this is so-called Dirichlet to Neumann operator. Caffarelli-Silvestre; [14], have shown that this operator can be realized in a local way by using one more variable and the so called $s$-harmonic extension.

More precisely, given $u$ a regular function defined in $\mathbb{R}^n$ we define its $s$-harmonic extension to the upper half-space $\mathbb{R}^{n+1}_+$ by $w = \mathrm{Ext}_s(u)$, as the solution to the problem

$$
\begin{gathered}
-\operatorname{div}(y^{1-2s}\nabla w)=0 \quad \text{in } \mathbb{R}^{n+1}_{+} \\
w=u \quad \text{on } \mathbb{R}^n\times\{y=0\}.
\end{gathered} \tag{2.2}
$$

The main relevance of the $s$-harmonic extension comes from the following identity

$$
\lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y)=-\frac{1}{\kappa_s}(-\Delta)^s u(x), \tag{2.3}
$$

where $\kappa_s$ is a positive constant. The above Dirichlet-Neumann process (2.2)-(2.3) provides a formula for the fractional Laplacian, equivalent to that obtained from Fourier Transform by (2.1). In that case, the $s$-harmonic extension and the fractional Laplacian have explicit expressions in terms of the Poisson and the Riesz kernels, respectively,

$$
\begin{gathered}
w(x,y)=P_y^s * u(x)=c_{n,s}y^{2s}\int_{\mathbb{R}^n}\frac{u(z)}{(|x-z|^2+y^2)^{\frac{n+2s}{2}}}\,dz, \\
(-\Delta)^s u(x)=d_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{u(x)-u(z)}{|x-z|^{n+2s}}\,dz.
\end{gathered} \tag{2.4}
$$

The natural functional spaces are the homogeneous fractional Sobolev space $\dot H^s(\mathbb{R}^n)$ and the weighted Sobolev space $X^{2s}(\mathbb{R}^{n+1}_+)$, that can be defined as the completion of $\mathcal{C}_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ and $\mathcal{C}_0^\infty(\mathbb{R}^n)$, respectively, under the norms

$$
\begin{gathered}
\|\phi\|_{X^{2s}}^2=\kappa_s\int_{\mathbb{R}^{n+1}_+} y^{1-2s}|\nabla\phi(x,y)|^2\,dxdy, \\
\|\psi\|_{\dot H^s}^2=\int_{\mathbb{R}^n}|2\pi\xi|^{2s}|\widehat{\psi}(\xi)|^2\,d\xi=\int_{\mathbb{R}^n}|(-\Delta)^{s/2}\psi(x)|^2\,dx,
\end{gathered}
$$

where $\kappa_s$ is the constant in (2.3). Notice that, the constants in (2.4) and $\kappa_s$ satisfy the identity $s\,c_{n,s}\kappa_s=d_{n,s}$, and their explicit value can be seen in [12].

**Remark 2.1.** The $s$-harmonic extension operator defined by (2.2) is an isometry between the spaces $\dot H^s(\mathbb{R}^n)$ and $X^{2s}(\mathbb{R}^{n+1}_+)$, i.e.,

$$
\|\varphi\|_{\dot H^s}=\|E_s(\varphi)\|_{X^{2s}}\,,\quad \forall\varphi\in\dot H^s(\mathbb{R}^n). \tag{2.5}
$$

Even more, we have the following inequality for the trace $\operatorname{Tr}(w)=w(\cdot,0)$,

$$
\|\operatorname{Tr}(w)\|_{\dot H^s}\le\|w\|_{X^{2s}}\,,\quad \forall w\in X^{2s}(\mathbb{R}^{n+1}_+), \tag{2.6}
$$

see [12] for more details.

Let us introduce the following notation:

- $E=W^{s,2}(\mathbb{R}^n)$, denotes the fractional Sobolev space, endowed with scalar product and norm

$$
(u\mid v)_j=\int_{\mathbb{R}^n}[(-\Delta)^{s/2}u(-\Delta)^{s/2}v+\lambda_j uv]dx,\quad \|u\|_j^2=(u\mid u)_j,\quad j=1,2;
$$

- $\mathbb{E}=E\times E$; the elements in $\mathbb{E}$ will be denoted by $\mathbf{u}=(u,v)$; as a norm in $\mathbb{E}$ we will take $\|\mathbf{u}\|=\|\mathbf{u}\|_{\mathbb{E}}^2=\|u\|_1^2+\|v\|_2^2$;
- $X=X^{2s}(\mathbb{R}^{n+1}_+)$, $\mathbb{X}=X\times X$;
- for $\mathbf{u}\in\mathbb{E}$, the notation $\mathbf{u}\ge\mathbf{0}$, resp. $\mathbf{u}>\mathbf{0}$, means that $u,v\ge 0$, resp. $u,v>0$, for all $j=1,2$.

**Remark 2.2.** If we define

$$\frac{\partial w}{\partial \nu^s} = -\kappa_s \lim_{y\to 0^+} y^{1-2s}\,\frac{\partial w}{\partial y}\,,$$

we can reformulate problem (1.1) as

$$\begin{gathered}
-\operatorname{div}(y^{1-2s}\nabla w_1) = 0 \quad \text{in } \mathbb{R}^{n+1}_+\\
-\operatorname{div}(y^{1-2s}\nabla w_2) = 0 \quad \text{in } \mathbb{R}^{n+1}_+\\
\frac{\partial w_1}{\partial \nu^s} + \lambda_1 w_1 = w_1^3 + \beta w_1 w_2 \quad \text{on } \mathbb{R}\times\{y=0\}\\
\frac{\partial w_2}{\partial \nu^s} + \lambda_2 w_2 = \frac{1}{2} w_2^2 + \frac{1}{2}\beta w_1^2 \quad \text{on } \mathbb{R}\times\{y=0\},
\end{gathered}\tag{2.7}$$

with $\mathbf{w} = (w_1, w_2) \in \mathbb{X}$.

Note that if $\mathbf{w} \in \mathbb{X}$ is solution of (2.7), then $\mathrm{Tr}(\mathbf{w}(x,y)) = \mathbf{w}(x,0) \in \mathbb{E}$ is a solution of (1.1), or equivalently, if $\mathbf{u} \in \mathbb{E}$ is a solution of (1.1), then $\mathrm{Ext}_s(\mathbf{u}) \in \mathbb{X}$ is a solution of (2.7).

The introduction of this problem is only for the interested reader. As we will see along the paper, it is not necessary to make use of problem (2.7), i.e., all the results for (1.1) are going to be proved without using the $s$-harmonic extension to the upper half space, $E_s(\cdot)$.

For $\mathbf{u} = (u, v) \in \mathbb{E}$, we set

$$\begin{gathered}
I_1(u) = \frac{1}{2}\int_{\mathbb{R}^n} (|(-\Delta)^{s/2}u|^2 + \lambda_1 u^2)dx - \frac{1}{4}\int_{\mathbb{R}^n} u^4 dx,\\
I_2(v) = \frac{1}{2}\int_{\mathbb{R}^n} (|(-\Delta)^{s/2}v|^2 + \lambda_2 v^2)dx - \frac{1}{6}\int_{\mathbb{R}^n} v^3 dx,\\
\Phi(\mathbf{u}) = I_1(u) + I_2(v) - \frac{1}{2}\beta\int_{\mathbb{R}^n} u^2 v\, dx.
\end{gathered}\tag{2.8}$$

We also write

$$G_\beta(\mathbf{u}) = \frac{1}{4}\int_{\mathbb{R}^n} u^4 dx + \frac{1}{6}\int_{\mathbb{R}^n} v^3 dx + \frac{1}{2}\beta\int_{\mathbb{R}^n} u^2 v\, dx,$$

and using this notation we can rewrite the energy functional as

$$\Phi(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|^2 - G_\beta(\mathbf{u}), \quad \mathbf{u} \in \mathbb{E}.$$

We observe that $G_\beta$ makes sense because $\frac{n}{4} < s < 1 \Rightarrow 2_s^* > 4$ which implies the continuous Sobolev embedding $E \hookrightarrow L^4(\mathbb{R}^n)$. Even more, any critical point $\mathbf{u} \in \mathbb{E}$ of $\Phi$, gives rise to a solution of (1.1).

**Definition 2.3.** A non-negative critical point $\widetilde{\mathbf{u}} \in \mathbb{E}\setminus\{\mathbf{0}\}$ is called a ground state of (1.1) if its energy $\Phi(\widetilde{\mathbf{u}})$ is minimal among all the non-trivial critical points of $\Phi$.

## 3. Nehari manifold and properties of $\mathbf{v}_2$

Let us set

$$\Psi(\mathbf{u}) = (\nabla\Phi(\mathbf{u})|\mathbf{u}) = (I_1'(u)|u) + (I_2'(v)|v) - \frac{3}{2}\,\beta\int_{\mathbb{R}^n} u^2 v\, dx.$$

We define the Nehari manifold by

$$\mathcal{N} = \{\mathbf{u} \in \mathbb{E}\setminus\{\mathbf{0}\} : \Psi(\mathbf{u}) = 0\}.$$

Then, one has that

$$(\nabla\Psi(\mathbf{u}) \mid \mathbf{u}) = -\|\mathbf{u}\|^2 - \int_{\mathbb{R}^n} u^4\,dx < 0 \quad \forall \mathbf{u}\in\mathcal{N}, \tag{3.1}$$

thus $\mathcal{N}$ is a smooth manifold locally near any point $\mathbf{u}\neq\mathbf{0}$ with $\Psi(\mathbf{u})=0$. Moreover, $\Phi''(\mathbf{0}) = I_1''(0)+I_2''(0)$ is positive definite, so we infer that $\mathbf{0}$ is a strict local minimum for $\Phi$. As a consequence, $\mathbf{0}$ is an isolated point of the set $\{\Psi(\mathbf{u})=0\}$, proving that $\mathcal{N}$ is a smooth complete manifold of codimension 1, and on the other hand there exists a constant $\rho>0$ so that

$$\|\mathbf{u}\|^2 > \rho \quad \forall \mathbf{u}\in\mathcal{N}. \tag{3.2}$$

Furthermore, by (3.1) and (3.2) we can show that $\mathbf{u}\in\mathbb{E}\setminus\{\mathbf{0}\}$ is a critical point of $\Phi$ if and only if $\mathbf{u}\in\mathcal{N}$ is a critical point of $\Phi$ constrained on $\mathcal{N}$. As a consequence, we have the following result.

**Lemma 3.1.** $\mathbf{u}\in\mathbb{E}$ *is a non-trivial critical point of* $\Phi$ *if and only if* $\mathbf{u}\in\mathcal{N}$ *and is a constrained critical point of* $\Phi$ *on* $\mathcal{N}$.

**Remark 3.2.** (i) By the previous arguments, the Nehari manifold $\mathcal{N}$ is a natural constraint of $\Phi$. Also, it is relevant to point out that working on the Nehari manifold, the functional $\Phi$ satisfies the following expression,

$$\Phi|_{\mathcal{N}}(\mathbf{u}) = \frac{1}{6}\|\mathbf{u}\|^2 + \frac{1}{12}\int_{\mathbb{R}^n} u^4 dx =: F(\mathbf{u}), \tag{3.3}$$

then using (3.2) into (3.3) we obtain

$$\Phi(\mathbf{u}) \geq \frac{1}{6}\|\mathbf{u}\|^2 > \frac{1}{6}\rho \quad \forall \mathbf{u}\in\mathcal{N}. \tag{3.4}$$

Therefore, by (3.4) the functional $\Phi$ is bounded from below on $\mathcal{N}$, as a consequence we will minimize it on the Nehari manifold. To do so, a remark about compactness is in order.

(ii) Analyzing the Palais-Smale (PS) condition, we remember that working on the radial setting, $H=E_{radial}$, the embedding of $H$ into $L^4(\mathbb{R}^n)$ is compact for $n=2,3$, but in dimension $n=1$, the embedding of $E$ or $H$ into $L^q(\mathbb{R})$ for $2<q<2_s^*$ is not compact; see [34, Remarque I.1]. However, we will analyze all the dimensional cases $n=1,2,3$, proving that for a PS sequence of $\Phi$ on $\mathcal{N}$, we can find a subsequence for which the weak limit is non-trivial and it is a solution of (1.1). This fact jointly with some properties of the Schwarz symmetrization will allow us to demonstrate the existence of positive radially symmetric ground states to (1.1). Notice that one could also try to work in the cone of non-negative radially decreasing functions, where one has the required compactness, in the one-dimensional case, thanks to Berestycki and Lions [10], but this is not our approach.

**Remark 3.3.** It is known [27, 28] that the equation

$$(-\Delta)^s v + v = v^2, \tag{3.5}$$

with $v\in E$, $v\not\equiv 0$, has a unique radially symmetric and positive solution, that we will denote by $V$. Indeed $V$ is a non-degenerate ground state of (3.5) in $H$.

Clearly, for every $\beta\in\mathbb{R}$, (1.1) already possesses a *semi-trivial* solution given by

$$\mathbf{v}_2 = (0, V_2),$$

where

$$V_2(x) = 2\lambda_2 V(\lambda_2^{1/2s}\,x) \tag{3.6}$$

is the unique positive radially symmetric solution of $(-\Delta)^s v + \lambda_2 v = \frac{1}{2}v^2$ in $E$.

To study some useful properties of $\mathbf{v}_2$, we define the corresponding Nehari manifold associated to $I_2$ in (2.8),

$$\mathcal{N}_2 = \big\{v \in E : (I_2'(v)|v) = 0\big\} = \big\{v \in E : \|v\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^n} v^3 dx = 0\big\}.$$

Let us denote $T_{\mathbf{v}_2}\mathcal{N}$ the tangent space to $\mathcal{N}$ on $\mathbf{v}_2$. Since

$$\mathbf{h} = (h_1, h_2) \in T_{\mathbf{v}_2}\mathcal{N} \Longleftrightarrow (V_2|h_2)_2 = \frac{3}{4}\int_{\mathbb{R}^n} V_2^2 h_2\, dx,$$

it follows that

$$(h_1, h_2) \in T_{\mathbf{v}_2}\mathcal{N} \Longleftrightarrow h_2 \in T_{V_2}\mathcal{N}_2. \tag{3.7}$$

**Proposition 3.4.** *There exists $\Lambda > 0$ such that:*

- (i) *if $\beta < \Lambda$, then $\mathbf{v}_2$ is a strict minimum of $\Phi$ constrained on $\mathcal{N}$,*
- (ii) *for any $\beta > \Lambda$, then $\mathbf{v}_2$ is a saddle point of $\Phi$ constrained on $\mathcal{N}$ with $\inf_{\mathcal{N}} \Phi < \Phi(\mathbf{v}_2)$.*

*Proof.* First, we observe that if $D^2\Phi_{\mathcal{N}}$ denotes the second derivative of $\Phi$ constrained on $\mathcal{N}$. Using that $\Phi'(\mathbf{v}_2) = 0$ we have that $D^2\Phi_{\mathcal{N}}(\mathbf{v}_2)[\mathbf{h}]^2 = \Phi''(\mathbf{v}_2)[\mathbf{h}]^2$ for all $\mathbf{h} \in T_{\mathbf{v}_2}\mathcal{N}$.

(i) We define

$$\Lambda = \inf_{\varphi \in E\setminus\{0\}} \frac{\|\varphi\|_1^2}{\int_{\mathbb{R}^n} V_2\varphi^2 dx}. \tag{3.8}$$

We have that for $\mathbf{h} \in T_{\mathbf{v}_2}\mathcal{N}$,

$$\Phi''(\mathbf{v}_2)[\mathbf{h}]^2 = \|h_1\|_1^2 + I_2''(V_2)[h_2]^2 - \beta\int_{\mathbb{R}^n} V_2 h_1^2 dx. \tag{3.9}$$

Let us take $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{v}_2}\mathcal{N}$, by (3.7) $h_2 \in T_{V_2}\mathcal{N}_2$, then using that $V_2$ is the minimum of $I_2$ on $\mathcal{N}_2$, there exists a constant $c > 0$ such that

$$I_2''(V_2)[h_2]^2 \geq c\|h_2\|_2^2. \tag{3.10}$$

From (3.10) jointly with (3.9), for $\beta < \Lambda$, there exists another constant $c_1 > 0$ such that,

$$\Phi''(\mathbf{v}_2)[\mathbf{h}]^2 \geq c_1(\|h_1\|_1^2 + \|h_2\|^2), \tag{3.11}$$

which proves that $\mathbf{v}_2$ is a strict local minimum of $\Phi$ on $\mathcal{N}$.

(ii) According to (3.7), $\mathbf{h} = (h_1, 0) \in T_{\mathbf{v}_2}\mathcal{N}$ for any $h_1 \in E$. We have that, for $\beta > \Lambda$, there exists $\widetilde{h} \in E$ with

$$\Lambda < \frac{\|\widetilde{h}\|_1^2}{\int_{\mathbb{R}^n} V_2\widetilde{h}^2 dx} < \beta,$$

thus, taking $\mathbf{h}_0 = (\widetilde{h}, 0) \in T_{\mathbf{v}_2}\mathcal{N}$, by (3.9) we find

$$\Phi''(\mathbf{v}_2)[\mathbf{h}_0]^2 = \|\widetilde{h}\|_1^2 - \beta\int_{\mathbb{R}^n} V_2\widetilde{h}^2 dx < 0. \tag{3.12}$$

On the other hand, by (3.7), and using again that $V_2$ is the minimum of $I_2$ on $\mathcal{N}_2$, we have that there exists $c > 0$ such that

$$I_2''(V_2)[h]^2 \geq c\|h\|_2^2,\ \forall h \in T_{V_2}\mathcal{N}_2.$$

Finally, by (3.9), $\Phi''(\mathbf{v}_2)[(0,h)]^2 = I_2''(V_2)[h]^2$ for any $h \in T_{V_2}\mathcal{N}_2$. Thus we have that there exists a constant $c > 0$ such that

$$\Phi''(\mathbf{v}_2)[\mathbf{h}_1]^2 \geq c\|\mathbf{h}_1\|^2, \quad \forall \mathbf{h}_1 = (0, h_1) \in T_{\mathbf{v}_2}\mathcal{N}.$$

□

## 4. Ground state solutions

The first result on the existence of ground states is given for the coupling parameter $\beta > \Lambda$ in the following theorem.

**Theorem 4.1.** *Assume $\beta > \Lambda$, then $\Phi$ has a positive radially symmetric ground state $\widetilde{\mathbf{u}}$, and there holds $\Phi(\widetilde{\mathbf{u}}) < \Phi(\mathbf{v}_2)$.*

*Proof.* By the Ekeland's variational principle; [26], there exists a PS sequence $\{\mathbf{u}_k\}_{k\in\mathbb{N}} \subset \mathcal{N}$, i.e.,

$$\Phi(\mathbf{u}_k) \to c_{\mathcal{N}} = \inf_{\mathcal{N}} \Phi, \tag{4.1}$$

$$\nabla_{\mathcal{N}}\Phi(\mathbf{u}_k) \to 0. \tag{4.2}$$

By (3.3) and (4.1), we find that $\{\mathbf{u}_k\}$ is a bounded sequence on $\mathbb{E}$, hence for a subsequence, we can assume that

$$\mathbf{u}_k \rightharpoonup \mathbf{u}_0 \quad \text{weakly in } \mathbb{E}, \tag{4.3}$$

$$\mathbf{u}_k \to \mathbf{u}_0 \quad \text{strongly in } \mathbb{L}^q_{\text{loc}}(\mathbb{R}) = L^q_{\text{loc}}(\mathbb{R}) \times L^q_{\text{loc}}(\mathbb{R}) \quad \forall 1 \leq q < 2^*_s, \tag{4.4}$$

and also $\mathbf{u}_k \to \mathbf{u}_0$ a. e. in $\mathbb{R}^n$. Since $\mathcal{N}$ is closed we have that $\mathbf{u}_0 \in \mathcal{N}$, even more, using that $\mathbf{0}$ is an isolated point the set $\{\Psi(\mathbf{u}) = 0\}$ we infer that $\mathbf{u}_0 \neq \mathbf{0}$. On the other hand, the constrained gradient satisfies

$$\nabla_{\mathcal{N}}\Phi(\mathbf{u}_k) = \Phi'(\mathbf{u}_k) - \eta_k\Psi'(\mathbf{u}_k) \to 0, \tag{4.5}$$

where $\eta_k$ is the corresponding Lagrange multiplier. Taking the scalar product with $\mathbf{u}_k$ in (4.5), since $\mathbf{u}_k \in \mathcal{N}$ we have that $(\Phi'(\mathbf{u}_k) \mid \mathbf{u}_k) = \Psi(\mathbf{u}_k) = 0$, then we infer that $\eta_k(\Psi'(\mathbf{u}_k) \mid \mathbf{u}_k) \to 0$; this jointly with (3.1),(3.3) and the fact that $\|\Psi'(\mathbf{u}_k)\| \leq C < \infty$ imply that $\eta_k \to 0$ and therefore $\Phi'(\mathbf{u}_k) \to 0$.

As a consequence of the discussion above, although we do not know that $\mathbf{u}_k \to \mathbf{u}_0$ in $\mathbb{E}$, we infer that $\mathbf{u}_0 \in \mathcal{N}$ is a non-trivial critical point of $\Phi$ and by Lemma 3.1 it is also a non-trivial critical point of $\Phi$ on $\mathcal{N}$.

Moreover, using that $\mathbf{u}_0 \in \mathcal{N}$ jointly with (3.3) and the Fatou's Lemma, we find

$$\Phi(\mathbf{u}_0) = F(\mathbf{u}_0) \leq \liminf_{k\to\infty} F(\mathbf{u}_k) = \liminf_{k\to\infty} \Phi(\mathbf{u}_k) = c_{\mathcal{N}}.$$

As a consequence, $\mathbf{u}_0$ is a least energy solution of (1.1). By Proposition 3.4-(ii) we know that necessarily $\Phi(\mathbf{u}_0) < \Phi(\mathbf{v}_2)$. Additionally, by the maximum principle in the fractional setting; [13], applied to the second equation in (1.1), we have that $v_0 > 0$. To show that also $u_0 > 0$, first we prove the following result.

*Claim.* We can assume without loss of generality that $u_0 \geq 0$.

To prove this, we consider $|\mathbf{u}_0| = (|u_0|, v_0)$, then we have two cases:

**Case 1.** If $|\mathbf{u}_0| \in \mathcal{N}$, by the Stroock-Varopoulos inequality [41, 42],

$$\|(-\Delta)^{s/2}(|u|)\|_{L^2} \leq \|(-\Delta)^{s/2}(u)\|_{L^2}, \tag{4.6}$$

we have, in particular, that $\||u|\|_1 \leq \|u\|_1$, then we obtain

$$\Phi(|\mathbf{u}_0|) \leq \Phi(\mathbf{u}_0) = c_{\mathcal{N}}.$$

Then, by similar arguments as in [43, Theorem 4.3], we have that $|\mathbf{u}_0|$ is a non-negative ground state.

**Case 2.** If $|\mathbf{u}_0| \notin \mathcal{N}$, we take the unique $t > 0$, $t \neq 1$ such that $t|\mathbf{u}_0| \in \mathcal{N}$, which comes from

$$\| \, |\mathbf{u}_0| \, \|^2 = t^2 \int_{\mathbb{R}^n} u_0^4 dx + t\Big(\frac{1}{2}\int_{\mathbb{R}^n} v_0^3 \, dx + \frac{3}{2}\beta \int_{\mathbb{R}^n} u_0^2 v_0 \, dx\Big). \tag{4.7}$$

Since $\mathbf{u}_0 \in \mathcal{N}$, then

$$\| \, \mathbf{u}_0 \, \|^2 = \int_{\mathbb{R}^n} u_0^4 dx + \frac{1}{2}\int_{\mathbb{R}^n} v_0^3 \, dx + \frac{3}{2}\beta \int_{\mathbb{R}^n} u_0^2 v_0 \, dx. \tag{4.8}$$

By (4.7), (4.8) and again the Stroock-Varopoulos inequality (4.6), we infer that

$$\begin{aligned} &t^2 \int_{\mathbb{R}^n} u_0^4 dx + t\Big(\frac{1}{2}\int_{\mathbb{R}^n} v_0^3 \, dx + \frac{3}{2}\beta \int_{\mathbb{R}^n} u_0^2 v_0 \, dx\Big) \\ &\le \int_{\mathbb{R}^n} u_0^4 dx + \frac{1}{2}\int_{\mathbb{R}^n} v_0^3 \, dx + \frac{3}{2}\beta \int_{\mathbb{R}^n} u_0^2 v_0 \, dx. \end{aligned} \tag{4.9}$$

Using that $t \neq 1$, as a consequence of (4.9) we deduce that $0 < t < 1$ and the inequality in (4.9) is strict. Hence, by (3.3) jointly with (4.6) and $t < 1$ we obtain

$$\begin{aligned} \Phi(t|\mathbf{u}_0|) &= t^2 \| \, |\mathbf{u}_0| \, \|^2 + t^4 \frac{1}{12}\int_{\mathbb{R}^n} u_0^4 dx \\ &< \| \, |\mathbf{u}_0| \, \|^2 + \frac{1}{12}\int_{\mathbb{R}^n} u_0^4 dx \\ &\le \Phi(\mathbf{u}_0) = c_{\mathcal{N}}. \end{aligned}$$

This is a contradiction because $t|\mathbf{u}_0| \in \mathcal{N}$. Therefore $|\mathbf{u}_0| \in \mathcal{N}$ and the claim is proved.

Once we can assume without loss of generality that $u_0 \ge 0$, by the maximum principle applied to the first equation in (1.1) we find $u_0 > 0$ proving that indeed $\mathbf{u}_0$ is a positive ground state.

To complete the proof, we have to show that the ground state is indeed radially symmetric.

If $\mathbf{u}_0$ is not radially symmetric, we set $\widetilde{\mathbf{u}} = \mathbf{u}_0^\star = (u_0^\star, v_0^\star)$, where $u_0^\star$, $v_0^\star$ denote the Schwarz symmetric functions associated to $u_0, v_0$ respectively. By the properties of the Schwarz symmetrization; see for instance [30] for the fractional setting and [8] for the classical one, there hold

$$\|\mathbf{u}^\star\|^2 \le \|\mathbf{u}\|^2, \quad G_\beta(\mathbf{u}^\star) \ge G_\beta(\mathbf{u}). \tag{4.10}$$

Furthermore, there exists a unique $t_\star > 0$ such that $t_\star \, \widetilde{\mathbf{u}} \in \mathcal{N}$. If $t_\star = 1$, by (4.10) we have $\Phi(\widetilde{\mathbf{u}}) \le \Phi(\mathbf{u}_0) = c_{\mathcal{N}}$ with $\widetilde{\mathbf{u}} \in \mathcal{N}$ thus $\widetilde{\mathbf{u}}$ is a positive radially symmetric ground state of (1.1).

On the contrary, i.e., if $t_\star \neq 1$, as in (4.7), $t_\star$ comes from

$$\|\widetilde{\mathbf{u}}\|^2 = t_\star^2 \int_{\mathbb{R}^n} (u_0^\star)^4 dx + t_\star\Big(\frac{1}{2}\int_{\mathbb{R}^n} (v_0^\star)^3 dx + \frac{3}{2}\beta \int_{\mathbb{R}^n} (u_0^\star)^2 v_0^\star \, dx\Big). \tag{4.11}$$

Because $\mathbf{u}_0 \in \mathcal{N}$, (4.10), (4.11), that $\mathbf{u}_0 > \mathbf{0}$ and that $t_\star > 0$, we find

$$\begin{aligned} &\int_{\mathbb{R}^n} u_0^4 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} v_0^3 \, dx + \frac{3}{2} \beta \int_{\mathbb{R}^n} u_0^2 v_0 \, dx \\ &\geq t_\star^2 \int_{\mathbb{R}^n} u_0^4 dx + t_\star \Big( \frac{1}{2} \int_{\mathbb{R}^n} v_0^3 \, dx + \frac{3}{2} \beta \int_{\mathbb{R}^n} u_0^2 v_0 \, dx \Big). \end{aligned} \tag{4.12}$$

Thus, using that $0 < t_\star \neq 1$ in (4.12), we obtain $0 < t_\star < 1$, this and (4.10) show that

$$\Phi(t_\star \widetilde{\mathbf{u}}) = \frac{1}{6} t_\star^2 \|\mathbf{u}^\star\|^2 + \frac{1}{12} t_\star^4 \int_{\mathbb{R}^n} (u_0^\star)^4 \, dx < \frac{1}{6} \|\mathbf{u}_0\|^2 + \frac{1}{12} \int_{\mathbb{R}^n} u_0^4 \, dx = \Phi(\mathbf{u}_0) = c_{\mathcal{N}}, \tag{4.13}$$

with $t_\star \widetilde{\mathbf{u}} \in \mathcal{N}$ which is a contradiction with (4.13), proving that $t_\star = 1$ and as above, the proof is complete. □

The second result about existence of ground states cover the range $0 < \beta \leq \Lambda$, provided $\lambda_2$ is large enough.

**Theorem 4.2.** *There exists $\Lambda_2 > 0$ such that if $\lambda_2 > \Lambda_2$, System* (1.1) *has a radially symmetric ground state $\widetilde{\mathbf{u}} > \mathbf{0}$ for every $0 < \beta \leq \Lambda$.*

*Proof.* Arguing as in the proof of Theorem 4.1, we prove that there exists a radially symmetric ground state $\widetilde{\mathbf{u}} \geq \mathbf{0}$. Moreover, in Theorem 4.1 for $\beta > \Lambda$ we proved that $\widetilde{\mathbf{u}} > \mathbf{0}$. Now we need to show that for $0 < \beta \leq \Lambda$ indeed $\widetilde{\mathbf{u}} > \mathbf{0}$ which follows by the maximum principle provided $\widetilde{\mathbf{u}} \neq \mathbf{v}_2$. Taking into account Proposition 3.4-(i), $\mathbf{v}_2$ is a strict local minimum of $\Phi$ on $\mathcal{N}$, and this does not guarantee that $\mathbf{u}_0 \not\equiv \mathbf{v}_2$. Following [19], the idea consists on the construction of a function $\mathbf{u}_0 = (u_0, v_0) \in \mathcal{N}$ with $\Phi(\mathbf{u}_0) < \Phi(\mathbf{v}_2)$. To do so, since $\mathbf{v}_2 = (0, V_2)$ is a local minimum of $\Phi$ on $\mathcal{N}$ provided $0 < \beta < \Lambda$, we cannot find $\mathbf{u}_0$ in a neighborhood of $\mathbf{v}_2$ on $\mathcal{N}$. Thus, we define $\mathbf{u}_0 = t(V_2, V_2)$ where $t > 0$ is the unique value such that $\mathbf{u}_0 \in \mathcal{N}$.

Now, we show that

$$\mathbf{u}_0 = t(V_2, V_2) \in \mathcal{N} \quad \text{with } \Phi(\mathbf{u}_0) < \Phi(\mathbf{v}_2),$$

for $\lambda_2$ large enough.

Notice that $t > 0$ comes from $\Psi(\mathbf{u}_0) = 0$, i.e.,

$$t^2 \|(V_2, V_2)\|^2 - t^4 \int_{\mathbb{R}^n} V_2^4 \, dx - \frac{1}{2} t^3 (1 + 3\beta) \int_{\mathbb{R}^n} V_2^3 \, dx = 0. \tag{4.14}$$

We also have

$$\|(V_2, V_2)\|^2 = 2\|V_2\|_2^2 + (\lambda_1 - \lambda_2) \int_{\mathbb{R}^n} V_2^2 \, dx. \tag{4.15}$$

Moreover, since $V_2 \in \mathcal{N}_2$, we have

$$\|V_2\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^n} V_2^3 \, dx = 0. \tag{4.16}$$

Substituting (4.15) and (4.16) in (4.14) it follows

$$t^2 \Big( \int_{\mathbb{R}^n} V_2^3 \, dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}^n} V_2^2 \, dx \Big) \quad - t^4 \int_{\mathbb{R}^n} V_2^4 \, dx - \frac{1}{2} t^3 (1 + 3\beta) \int_{\mathbb{R}^n} V_2^3 \, dx = 0. \tag{4.17}$$

Hence, applying the scaling (3.6) yields

$$\int_{\mathbb{R}^n} V_2^r \, dx = 2^r \lambda_2^{r - \frac{n}{2s}} \int_{\mathbb{R}^n} V^r \, dx. \tag{4.18}$$

Subsequently, substituting (4.18) for $r = 2, 3, 4$ into (4.17) and dividing in (4.17) by $2^3\lambda_2^{3-\frac{n}{2s}}t^2$ we have

$$\int_{\mathbb{R}^n} V^3\,dx+\frac{\lambda_1-\lambda_2}{2\lambda_2}\int_{\mathbb{R}^n} V^2\,dx-2\lambda_2 t^2\int_{\mathbb{R}^n} V^4\,dx-\frac{1}{2}t(1+3\beta)\int_{\mathbb{R}^n} V^3\,dx=0. \quad (4.19)$$

Moreover, by (3.3), (4.15) and (4.16) we find respectively the expressions

$$\Phi(\mathbf{u}_0)=\frac{1}{6}t^2\Big(\int_{\mathbb{R}^n} V_2^3\,dx+(\lambda_1-\lambda_2)\int_{\mathbb{R}^n} V_2^2\,dx\Big)+\frac{1}{12}t^4\int_{\mathbb{R}^n} V_2^4\,dx, \quad (4.20)$$

$$\Phi(\mathbf{v}_2)=I_2(V_2)=\frac{1}{2}\|V_2\|_2^2-\frac{1}{6}\int_{\mathbb{R}^n} V_2^3=\frac{1}{12}\int_{\mathbb{R}^n} V_2^3. \quad (4.21)$$

By (4.20), (4.21) we have $\Phi(\mathbf{u}_0) < \Phi(\mathbf{v}_2)$ is equivalent to

$$\begin{aligned}&\frac{1}{6}t^2\Big(\int_{\mathbb{R}^n} V_2^3\,dx+(\lambda_1-\lambda_2)\int_{\mathbb{R}^n} V_2^2\,dx\Big)\\&+\frac{1}{12}t^4\int_{\mathbb{R}^n} V_2^4\,dx-\frac{1}{12}\int_{\mathbb{R}^n} V_2^3\,dx<0,\end{aligned} \quad (4.22)$$

and then, applying again (4.18) and multiplying (4.22) by $6\lambda_2^{\frac{n}{2s}-3}$, we actually have

$$t^2\Big(\int_{\mathbb{R}^n} V^3\,dx+\frac{\lambda_1-\lambda_2}{\lambda_2}\int_{\mathbb{R}^n} V^2\,dx\Big)+\frac{1}{2}t^4\lambda_2\int_{\mathbb{R}^n} V^4\,dx-\frac{1}{2}\int_{\mathbb{R}^n} V^3\,dx<0. \quad (4.23)$$

For $\lambda_2$ large enough we find that (4.19) will provide us with (4.23). Therefore, there exists a positive constant $\Lambda_2$ such that for $\lambda_2 > \Lambda_2$ inequality (4.23) holds, proving that

$$\Phi(\widetilde{\mathbf{u}}) \le \Phi(\mathbf{u}_0) < \Phi(\mathbf{v}_2).$$

Finally, this shows that $\widetilde{\mathbf{u}} \neq \mathbf{v}_2$ and we finish. □

## 5. Systems with more than 2 equations

In this last subsection, we deal with some extended systems of (1.1) to more than two equations. We start with the study of the following system coming from NLFS-2FKdV equations if $n = 1$ or 3NLFS equations if $n = 1, 2, 3$,

$$\begin{aligned}(-\Delta)^s u+\lambda_0 u&=u^3+\beta_1 uv_1+\beta_2 uv_2,\\(-\Delta)^s v_1+\lambda_1 v_1&=\frac{1}{2}v_1^2+\frac{1}{2}\beta_1 u^2,\\(-\Delta)^s v_2+\lambda_2 v_2&=\frac{1}{2}v_2^2+\frac{1}{2}\beta_2 u^2,\end{aligned} \quad (5.1)$$

where $u, v_1, v_2 \in E$. This system can be seen as a perturbation of (1.1) when $|\beta_1|$ or $|\beta_2|$ is small.

We use similar notation as in previous sections with natural meaning, for example, $\mathbb{E} = E \times E \times E$, $\mathbf{0} = (0, 0, 0)$,

$$\Phi(\mathbf{u})=\frac{1}{2}\|\mathbf{u}\|^2-\frac{1}{4}\int_{\mathbb{R}^n} u^4\,dx-\frac{1}{6}\int_{\mathbb{R}^n}(v_1^3+v_2^3)\,dx-\frac{1}{2}\int_{\mathbb{R}^n} u^2(\beta_1 v_1+\beta_2 v_2)\,dx \quad (5.2)$$

$$\mathcal{N}=\{\mathbf{u}\in\mathbb{E}\setminus\{\mathbf{0}\}:(\Phi'(\mathbf{u})|\mathbf{u})=0\}, \quad (5.3)$$

Let $U^*, V_j^*$ be the unique positive radially symmetric solutions of $(-\Delta)^s u+\lambda_0 u = u^3$, $(-\Delta)^s v + \lambda_j v = \frac{1}{2}v^2$ in $E$ respectively, $j = 1, 2$; see [27, 28].

**Remark 5.1.** The unique non-negative semi-trivial solutions of (5.1) are given by $\mathbf{v}_1^* = (0, V_1^*, 0)$, $\mathbf{v}_2^* = (0, 0, V_2^*)$ and $\mathbf{v}_{12}^* = (0, V_1^*, V_2^*)$.

As in Section 4, the first result about existence of ground states is the following theorem.

**Theorem 5.2.** *Assume $\beta_j > \Lambda_j$ for $j = 1, 2$, then* (5.1) *has a positive radially symmetric ground state $\widetilde{\mathbf{u}}$.*

*Proof.* We define

$$\Lambda_j = \inf_{\varphi \in E \setminus \{0\}} \frac{\|\varphi\|_0^2}{\int_{\mathbb{R}^n} V_j^* \varphi^2 dx} \quad j = 1, 2. \tag{5.4}$$

where $\| \cdot \|_0$ is the norm in $E$ with $\lambda_0$.

As in Proposition 3.4-(ii), using that $\beta_j > \Lambda_j$, $j = 1, 2$, one can show that both $\mathbf{v}_1^*$, $\mathbf{v}_2^*$ are saddle points of the energy functional $\Phi$ (defined by (5.2)) constrained on the Nehari manifold $\mathcal{N}$ (defined by (5.2)). Then

$$c_{\mathcal{N}} = \inf_{\mathcal{N}} \Phi < \min\{\Phi(\mathbf{v}_1^*), \Phi(\mathbf{v}_2^*)\} < \Phi(\mathbf{v}_{12}^*) = \Phi(\mathbf{v}_1^*) + \Phi(\mathbf{v}_2^*). \tag{5.5}$$

By the Ekeland's variational principle, there exists a PS sequence $\{\mathbf{u}_k\}_{k \in \mathbb{N}} \subset \mathcal{N}$, i.e.,

$$\Phi(\mathbf{u}_k) \to c_{\mathcal{N}}, \quad \nabla_{\mathcal{N}} \Phi(\mathbf{u}_k) \to 0. \tag{5.6}$$

The lack of compactness can be circumvent arguing in a similar way as in the proof of Theorem 4.1, proving that for a subsequence, $\mathbf{u}_k \rightharpoonup \widetilde{\mathbf{u}}$ weakly in $\mathbb{E}$ with $\widetilde{\mathbf{u}} \gneqq \mathbf{0}$, $\widetilde{\mathbf{u}} \in \mathcal{N}$ a critical point of $\Phi$ satisfying $\Phi(\widetilde{\mathbf{u}}) = c_{\mathcal{N}}$, then $\widetilde{\mathbf{u}}$ is a non-negative ground state.

To prove the positivity of $\widetilde{\mathbf{u}}$, if one supposes that the first component $u^* \equiv 0$, since the only non-negative solutions of (5.1) are the semi-trivial solutions defined in Remark 5.1, we obtain a contradiction with (5.5). Furthermore, if the second or third component vanish then $\widetilde{\mathbf{u}}$ must be $\mathbf{0}$, and this is not possible because $\Phi|_{\mathcal{N}}$ is bounded bellow by a positive constant like in (3.4), then $\mathbf{0}$ is an isolated point of the set $\{\mathbf{u} \in \mathbb{E} : \Psi(\mathbf{u}) = (\Phi'(\mathbf{u})|\mathbf{u}) = 0\}$, proving that $\mathcal{N}$ is a complete manifold, as in the previous sections. Then, the maximum principle shows that $\widetilde{\mathbf{u}} > \mathbf{0}$. Finally, to show that we have a radially symmetric ground state, we argue as in the proof of Theorem 4.1. □

Furthermore, following the ideas in the proof of Theorem 4.2 we have the following result.

**Theorem 5.3.** *Assume that $\beta_1, \beta_2 > 0$ (but not necessarily $\beta_j > \Lambda_j$ as in Theorem 5.2). Then there exists a positive radially symmetric ground state $\widetilde{\mathbf{u}}$ provided $\lambda_1, \lambda_2$ are sufficiently large.*

*Proof.* The proof follows the same ideas as the one of Theorem 4.2 with appropriate changes. For example, to prove the positivity, one has to show that there exists $\mathbf{u}_0 \in \mathcal{N}$ with $\Phi(\mathbf{u}_0) < \min\{\Phi(\mathbf{v}_1^*), \Phi(\mathbf{v}_2^*)\}$, that holds true provided $\lambda_1, \lambda_2$ are large enough. We omit details here. □

Plainly we can extend these results to systems with an arbitrary number of equations $N > 3$ as follows,

$$
\begin{aligned}
(-\Delta)^s u + \lambda_0 u &= u^3 + \sum_{k=1}^{N-1} \beta_k \, u v_k \\
(-\Delta)^s v_j + \lambda_j v_j &= \frac{1}{2} v_j^2 + \frac{1}{2}\beta_j u^2; \quad j = 1, \cdots, N-1.
\end{aligned}
\tag{5.7}
$$

Arguing as in Theorems 5.2 and 5.3 we can show the next result.

**Theorem 5.4.** *There exists a positive radially symmetric ground state of* (5.7) *if*

- *either*

$$
\beta_k > \Lambda_k = \inf_{\varphi \in E\setminus\{0\}} \frac{\|\varphi\|_0^2}{\int_{\mathbb{R}^n} V_k^* \varphi^2 dx}; \quad k = 1, \cdots N-1,
$$

*where $V_k^*$ denotes the unique positive radial solution of $\Delta v + \lambda_k v = \frac{1}{2}v^2$ in $E$; $k = 1, \cdots, N-1$,*

- *or $\beta_j > 0$ are arbitrary and $\lambda_j$ are large enough; $j = 1, \dots, N-1$.*

**Remark 5.5.** As was commented in [19] for the local setting, here in the nonlocal fractional framework, another natural extension of (1.3) to more than two equations different from (5.1) is the following system coming from 2NLFS-FKdV equations if $n = 1$ or 3NLFS equations if $n = 1, 2, 3$,

$$
\begin{aligned}
(-\Delta)^s u_1 + \lambda_1 u_1 &= u_1^3 + \beta_{12} u_1 u_2^2 + \beta_{13} u_1 v \\
(-\Delta)^s u_2 + \lambda_2 u_2 &= u_2^3 + \frac{1}{2}\beta_{12} u_1^2 u_2 + \beta_{23} u_2 v. \\
(-\Delta)^s v + \lambda v &= \frac{1}{2} v^2 + \frac{1}{2}\beta_{13} u_1^2 + \frac{1}{2}\beta_{23} u_2^2.
\end{aligned}
\tag{5.8}
$$

We denote $U_j$ the unique positive radially symmetric solution of $(-\Delta)^s u + \lambda_j u = u^3$ in $E$; $j = 1, 2$; and $V$ the corresponding positive radially symmetric solution to $(-\Delta)^s v + \lambda v = \frac{1}{2}v^2$ in $E$.

Note that the non-negative radially symmetric semi-trivial solution $(0, 0, V)$ is a strict local minimum of the associated energy functional constrained on the corresponding Nehari manifold provided

$$
\beta_{j3} < \Lambda_j = \inf_{\varphi \in E\setminus\{0\}} \frac{\|\varphi\|_{\lambda_j}^2}{\int_{\mathbb{R}^n} V \varphi^2 dx} \quad j = 1, 2.
$$

While if either $\beta_{13} > \Lambda_1$ or $\beta_{23} > \Lambda_2$ then $(0, 0, V)$ is a saddle point of $\Phi$ on $\mathcal{N}$.

There also exist semi-trivial solutions coming from the solutions studied in Section 4, with the first component or the second one $\equiv 0$. This fact makes different the analysis of (5.8) with respect to the previous studied systems (5.1) and (5.7). To finish, one could study more general extended systems of (5.1), (5.8) with $N = m + \ell$; coming from $m$-NLFS and $\ell$-FKdV coupled equations with $m, \ell \geq 2$ in the one dimensional case, or $N$-NLFS equations if $n = 1, 2, 3$. Indeed, the existence of positive ground states it is still unknown in the local setting ($s = 1$) for this last kind of systems, including (5.8) with $s = 1$.

**Acknowledgments.** The author is partially supported by the Ministry of Economy and Competitiveness of Spain and FEDER under Research Project MTM2016-80618-P, and by the INdAM - GNAMPA Project 2017 "Teoria e modelli non locali".

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