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KIRCHHOFF-TYPE PROBLEMS INVOLVING NONLINEARITIES SATISFYING ONLY SUBCRITICAL AND SUPERLINEAR CONDITIONS

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Abstract. In this note, we study the problem

$$-h\Big(\int_\Omega |\nabla u(x)|^2 dx\Big)\Delta u = f(u) \quad \text{in } \Omega$$
$$u\big|_{\partial\Omega} = 0.$$

As an application of a general multiplicity result, we establish the existence of at least three solutions, two of which are global minimizers of the related energy functional. The only condition assumed on f is that it be subcritical and superlinear; no condition on the behaviour of f at 0 is required.

Dedicated to the memory of Anna Aloe

1. Introduction and results

Here and in what follows, $\Omega \subset \mathbb{R}^m$ is a smooth bounded domain, with $m \geq 3$. For $q \in]0, (m+2)/(m-2)]$, we denote by $\mathcal{A}_q$ the class of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{|\xi|\to+\infty} \frac{|f(\xi)|}{|\xi|^q} < +\infty,$$
$$-\infty < \liminf_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^2} \leq \limsup_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^2} = +\infty$$

where $F(\xi) = \int_0^\xi f(t)dt$.

Given $f \in \mathcal{A}_q$ and a continuous function $h : [0, +\infty[\to \mathbb{R}$, we consider the Kirchhoff-type problem

$$-h\Big(\int_\Omega |\nabla u(x)|^2 dx\Big)\Delta u = f(u) \quad \text{in } \Omega$$
$$u\big|_{\partial\Omega} = 0.$$

A weak solution of this problem is a function $u \in H_0^1(\Omega)$ such that

$$h\Big(\int_\Omega |\nabla u(x)|^2 dx\Big) \int_\Omega \nabla u(x)\nabla v(x)\, dx = \int_\Omega f(u(x))v(x)\, dx$$

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for all $v \in H^1_0(\Omega)$.

So, the weak solutions of the problem are precisely the critical points in $H^1_0(\Omega)$ of the functional

$$u \mapsto \frac{1}{2}H\Big(\int_\Omega |\nabla u(x)|^2dx\Big) - \int_\Omega F(u(x))dx$$

where $H(t) = \int_0^t h(s)ds$.

A real-valued function g on a topological space is said to be sequentially inf-compact if, for each $r \in \mathbb{R}$, the set $g^{-1}(]-\infty, r])$ is sequentially compact.

The aim of this note is to establish the following result.

Theorem 1.1. *For each $q \in]0,(m+2)/(m-2)[$ and $f \in \mathcal{A}_q$ there exists a divergent sequence $\{a_n\}$ in $]0,+\infty[$ with the following property: for every $n \in \mathbb{N}$ and for every continuous and non-decreasing function $k : [0,+\infty[\to [0,+\infty[$, with $\lim_{t\to+\infty} K(t)/t^{(q+1)/2} = +\infty$ and $\mathrm{int}(k^{-1}(0)) = \emptyset$, there exists $b > 0$ such that the problem*

$$-\Big(a_n + bk\Big(\int_\Omega |\nabla u(x)|^2dx\Big)\Big)\Delta u = f(u) \quad \text{in } \Omega$$
$$u\big|_{\partial\Omega} = 0,$$

has at least three weak solutions, two of which are global minimizers in $H^1_0(\Omega)$ of the energy functional

$$u \mapsto \frac{a_n}{2}\int_\Omega |\nabla u(x)|^2dx + \frac{b}{2}K\Big(\int_\Omega |\nabla u(x)|^2dx\Big) - \int_\Omega F(u(x))dx$$

where $K(t) = \int_0^t k(s)ds$.

A comparison of Theorem 1.1 with known results cannot be properly done. This is due to the fact that no previous result on the problem we are dealing with guarantees the existence of at least two global minimizers of the energy functional related to it. More precisely, no such a result is known when the nonlinearity f, as in our case, does not depend on x ($x \in \Omega$). For quite special f depending necessarily on x, the only known results of that type have been obtained in [5]. But, also for what concerns the assumptions on f, Theorem 1.1 presents a novelty: it seems that, even when the energy functional in unbounded below, no existing result ensures the existence of at least three solutions of the problem assuming on f only its belonging to the class $\mathcal{A}_q$. Actually, some condition on the behaviour of f at 0 is usually assumed (see, for instance, [1, 2, 4, 8, 9, 11] and references therein).

Our proof of Theorem 1.1 is based on the use of the following new abstract multiplicity result.

Theorem 1.2. *Let X be a topological space and let $I, J : X \to \mathbb{R}$ be two sequentially lower semicontinuous functions. Assume that J is sequentially inf-compact and that, for some $c > 0$, one has*

$$\inf_{x\in J^{-1}(]c,+\infty[)} \frac{I(x)}{J(x)} = -\infty\,. \tag{1.1}$$

Then, there exists a divergent sequence $\{\lambda^*_n\}$ in $]0,+\infty[$ with the following property: for every $n \in \mathbb{N}$ and for every increasing and lower semicontinuous function $\varphi : J(X) \to \mathbb{R}$ such that $I + \mu\varphi\circ J$ is sequentially inf-compact for all $\mu > 0$, there exists

$\mu^* > 0$ such that the function $I + \lambda_n^* J + \mu^* \varphi \circ J$ has at least two global minimizers in X.

In turn, to prove Theorem 1.2, we need the two following results that we established in [6] and [7] respectively.

Theorem 1.3. *Let X be a topological space and let $\Phi, \Psi : X \to \mathbb{R}$ be two functions such that, for every $\lambda > 0$, the function $\Phi + \lambda\Psi$ is sequentially lower semicontinuos and sequentially inf-compact, and has a unique global minimizer in X. Assume also that Φ has no global minimizer. Then, for every $r \in]\inf_X \Psi, \sup_X \Psi[$, there exists $\hat{\lambda}_r > 0$ such that the unique global minimizer in X of the function $\Phi + \hat{\lambda}_r\Psi$ lies in $\Psi^{-1}(r)$.*

Theorem 1.4. *Let S be a topological space and let $P, Q : S \to \mathbb{R}$ be two functions satisfying the following conditions:*

(a) *for each $\lambda > 0$, the function $P + \lambda Q$ is sequentially lower semicontinuous and sequentially inf-compact;*
(b) *there exist $\rho \in]\inf_S Q, \sup_S Q[$ and $v_1, v_2 \in S$ such that*

$$Q(v_1) < \rho < Q(v_2), \tag{1.2}$$

$$\frac{P(v_1) - \inf_{Q^{-1}(]-\infty,\rho])} P}{\rho - Q(v_1)} < \frac{P(v_2) - \inf_{Q^{-1}(]-\infty,\rho])} P}{\rho - Q(v_2)}. \tag{1.3}$$

Under these hypotheses, there exists $\lambda^* > 0$ such that the function $P + \lambda^* Q$ has at least two global minimizers.

Proof of Theorem 1.2. Fix $\rho_0 > \inf_X J$, $x_0 \in J^{-1}(]-\infty, \rho_0[)$ and λ satisfying

$$\lambda > \frac{I(x_0) - \inf_{J^{-1}(]-\infty,\rho_0])} I}{\rho_0 - J(x_0)}.$$

Hence, one has

$$I(x_0) + \lambda J(x_0) < \lambda\rho_0 + \inf_{J^{-1}(]-\infty,\rho_0])} I. \tag{1.4}$$

Since $J^{-1}(]-\infty, \rho_0])$ is sequentially compact, by sequential lower semicontinuity, there is $\hat{x} \in J^{-1}(]-\infty, \rho_0])$ such that

$$I(\hat{x}) + \lambda J(\hat{x}) = \inf_{x \in J^{-1}(]-\infty,\rho_0])} (I(x) + \lambda J(x)). \tag{1.5}$$

We claim that

$$J(\hat{x}) < \rho_0. \tag{1.6}$$

Arguing by contradiction, assume that $J(\hat{x}) = \rho_0$. Then, in view of (1.4), we would have

$$I(x_0) + \lambda J(x_0) < I(\hat{x}) + \lambda J(\hat{x})$$

against (1.5). By (1.1), there is a sequence $\{x_n\}$ in $J^{-1}(]c, +\infty[)$ such that

$$\lim_{n\to\infty} \frac{I(x_n)}{J(x_n)} = -\infty.$$

Now, set

$$\gamma = \min\big\{0, \inf_{x \in J^{-1}(]-\infty,\rho_0])} (I(x) + \lambda J(x))\big\}$$

and fix $\hat{n} \in \mathbb{N}$ so that

$$\frac{I(x_{\hat{n}})}{J(x_{\hat{n}})} < -\lambda + \frac{\gamma}{c}.$$

We then have

$$I(x_{\hat{n}}) + \lambda J(x_{\hat{n}}) < \frac{\gamma}{c} J(x_{\hat{n}}) \leq \gamma \leq \inf_{x \in J^{-1}(]-\infty,\rho_0])} (I(x) + \lambda J(x)) . \tag{1.7}$$

In particular, this implies that

$$J(x_{\hat{n}}) > \rho_0 . \tag{1.8}$$

Put

$$\rho^*_\lambda = J(x_{\hat{n}}) .$$

At this point, we realize that it is possible to apply Theorem 1.4 taking

$$\begin{gathered} S = J^{-1}(]-\infty, \rho^*_\lambda]) , \\ P = I_{|S} + \lambda J_{|S} , \\ Q = J_{|S} . \end{gathered}$$

Indeed, (a) is satisfied since S is sequentially compact. To satisfy (b), take

$$\rho = \rho_0 , \quad v_1 = \hat{x} , \quad v_2 = x_{\hat{n}} .$$

So, with these choices, (1.2) follows from (1.6) and (1.8), while (1.3) follows from (1.5) and (1.7). Consequently, Theorem 1.4 ensures the existence of $\delta_\lambda > 0$ such that the restriction of the function $I + (\lambda + \delta_\lambda)J$ to $J^{-1}(]-\infty, \rho^*_\lambda])$ has at least two global minimizers, say w_1, w_2. Now, fix an increasing and lower semicontinuous function $\varphi : J(X) \to \mathbb{R}$ such that $I + \mu\varphi \circ J$ is sequentially inf-compact for all $\mu > 0$. We claim that, for some $\mu > 0$, the function $I + (\lambda + \delta_\lambda)J + \mu\varphi \circ J$ has at least two global minimizers in X. Arguing by contradiction, assume that, for each $\mu > 0$, there exists a unique global minimizer in X for the function $I + (\lambda + \delta_\lambda)J + \mu\varphi \circ J$ (which is clearly sequentially lower semicontinuous and sequentially inf-compact). Now, after observing that, by (1.1), the function $I + (\lambda + \delta_\lambda)J$ is unbounded below, we can apply Theorem 1.3 taking

$$\begin{gathered} \Phi = I + (\lambda + \delta_\lambda)J, \\ \Psi = \varphi \circ J . \end{gathered}$$

Observe that the function $\varphi \circ J$ is unbounded above. Indeed, if not, the sequential inf-compactness of $\varphi \circ J + I$ jointly with the sequential lower semicontinuity of I would contradict (1.1). Moreover, since $J(x_0) < \rho^*_\lambda$, we have

$$\inf_X \varphi \circ J \leq \varphi(J(x_0)) < \varphi(\rho^*_\lambda) .$$

Then, Theorem 1.3 ensures the existence of $\hat{\mu} > 0$ such that the unique global minimizer in X of the function $I + (\lambda + \delta_\lambda)J + \hat{\mu}\varphi \circ J$, say $\hat{w}$, lies in $(\varphi \circ J)^{-1}(\varphi(\rho^*_\lambda))$. Since φ is increasing, we have

$$J^{-1}(]-\infty, \rho^*_\lambda]) = (\varphi \circ J)^{-1}(]-\infty, \varphi(\rho^*_\lambda)])$$

and hence, for $i = 1, 2$, we have

$$\begin{aligned} &\inf_{x \in X} (I(x) + (\lambda + \delta_\lambda)J(x) + \hat{\mu}\varphi(J(x))) \\ &\leq I(w_i) + (\lambda + \delta_\lambda)J(w_i) + \hat{\mu}\varphi(J(w_i)) \\ &\leq I(\hat{w}) + (\lambda + \delta_\lambda)J(\hat{w}) + \hat{\mu}\varphi(J(\hat{w})) \\ &= \inf_{x \in X} (I(x) + (\lambda + \delta_\lambda)J(x) + \hat{\mu}\varphi(J(x))) . \end{aligned}$$

That is to say, w_1 and w_2 would be two global minimizers in X of the function $I + (\lambda + \delta_\lambda)J + \hat{\mu}\varphi \circ J$, a contradiction. Therefore, it remains proved that there exists $\mu^* > 0$ such that the function $I + (\lambda + \delta_\lambda)J + \mu^* \varphi \circ J$ has at least two global minimizers in X. Finally, observe that the set

$$A := \big\{ \lambda + \delta_\lambda : \lambda > \frac{I(x_0) - \inf_{J^{-1}(]-\infty,\rho_0])} I}{\rho_0 - J(x_0)} \big\}$$

is unbounded above. So, for what we have seen above, any divergent sequence $\{\lambda_n^*\}$ in A satisfies the thesis. $\square$

Proof of Theorem 1.1. Fix $q \in]0,(m+2)/(m-2)[$ and $f \in \mathcal{A}_q$. We are going to apply Theorem 1.2 taking $X = H_0^1(\Omega)$, endowed with the weak topology, and $I, J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) = -\int_\Omega F(u(x))dx\,,$$
$$J(u) = \frac{1}{2}\|u\|^2\,,$$

where

$$\|u\|^2 = \int_\Omega |\nabla u(x)|^2 dx\,.$$

Clearly, J is weakly inf-compact and I (since f has a subcritical growth) is sequentially weakly continuous. Now, fix a measurable set $C \subset \Omega$, of positive measure, and a function $w \in H_0^1(\Omega)$ such that $w(x) = 1$ for all $x \in C$. Since $f \in \mathcal{A}_q$, there exist a sequence $\{\xi_n\}$ in $\mathbb{R}$, with $\lim_{n\to\infty} |\xi_n| = +\infty$, and a constant $\alpha > 0$ such that

$$-\alpha(\xi^2 + 1) \le F(\xi)$$

for all $\xi \in \mathbb{R}$ and

$$\lim_{n\to+\infty} \frac{F(\xi_n)}{\xi_n^2} = +\infty\,.$$

Thus, we have

$$\begin{aligned}
\frac{\int_\Omega F(\xi_n w(x))dx}{\int_\Omega |\nabla \xi_n w(x)|^2 dx} &= \frac{\operatorname{meas}(C)F(\xi_n) + \int_{\Omega\setminus C} F(\xi_n w(x)dx}{\xi_n^2 \int_\Omega |\nabla w(x)|^2 dx} \\
&\ge \frac{\operatorname{meas}(C)F(\xi_n)}{\xi_n^2 \int_\Omega |\nabla w(x)|^2 dx} - \alpha \frac{\int_\Omega |w(x)|^2 dx + \frac{\operatorname{meas}(\Omega)}{\xi_n^2}}{\int_\Omega |\nabla w(x)|^2 dx}
\end{aligned}$$

and so

$$\liminf_{\|u\|\to+\infty} \frac{I(u)}{J(u)} = -\infty\,.$$

Therefore, the assumptions of Theorem 1.2 are satisfied. Let $\{\lambda_n^*\}$ be a divergent sequence with the property expressed in Theorem 1.2. Fix $n \in \mathbb{N}$ and a continuous and non-decreasing function $k : [0,+\infty[\to [0,+\infty[$, with $\lim_{t\to+\infty} \frac{K(t)}{t^{(q+1)/2}} = +\infty$ and $\operatorname{int}(k^{-1}(0)) = \emptyset$. Let $\varphi : [0,+\infty[\to [0,+\infty[$ be defined by

$$\varphi(t) = \frac{1}{2}K(2t)$$

for all $t \ge 0$. Clearly, the function φ is increasing (and continuous). Moreover, due to the Sobolev imbedding, there is a constant $\beta > 0$ such that

$$I(u) \ge -\beta(1 + \|u\|^{q+1})$$

for all $u \in X$ and so, for each $\mu > 0$, we have

$$\begin{aligned} I(u) + \mu\varphi(J(u)) &\geq -\beta(1 + \|u\|^{q+1}) + \frac{\mu}{2} K(\|u\|^2) \\ &= \|u\|^{q+1}\Big(-\beta\big(1 + \frac{1}{\|u\|^{q+1}}\big) + \frac{\mu}{2} \frac{K(\|u\|^2)}{\|u\|^{q+1}}\Big) \end{aligned} \tag{1.9}$$

for all $u \in X$. Since

$$\lim_{\|u\| \to +\infty} \frac{K(\|u\|^2)}{\|u\|^{q+1}} = +\infty \,,$$

from (1.9) we infer that the functional $I + \mu\varphi \circ J$ is sequentially weakly inf-compact. As a consequence, there exists $\mu^* > 0$ such that the functional $I + \lambda_n^* J + \mu^* \varphi \circ J$ has at least two global minimizers in X which, therefore, are weak solutions of the problem we are dealing with. Now, observe that the function $t \to t(\lambda_n^* + \mu^* k(t^2))$ is increasing in $[0, +\infty[$ and its range is $[0, +\infty[$. Denote by ψ its inverse. Let $T : X \to X$ be the operator defined by

$$T(v) = \begin{cases} \frac{\psi(\|v\|)}{\|v\|} v & \text{if } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$$

Since ψ is continuous and $\psi(0) = 0$, the operator T is continuous in X. For each $u \in X \setminus \{0\}$, we have

$$\begin{aligned} T((\lambda_n^* + \mu^* k(\|u\|^2))u) &= \frac{\psi((\lambda_n^* + \mu^* k(\|u\|^2))\|u\|)}{(\lambda_n^* + \mu^* k(\|u\|^2))\|u\|} (\lambda_n^* + \mu^* k(\|u\|^2))u \\ &= \frac{\|u\|}{(\lambda_n^* + \mu^* k(\|u\|^2))\|u\|} (\lambda_n^* + \mu^* k(\|u\|^2))u = u \,. \end{aligned}$$

In other words, T is a continuous inverse of the derivative of the functional $\lambda_n^* J + \mu^* \varphi \circ J$. Then, since the derivative of I is compact, the functional $I + \lambda_n^* J + \mu^* \varphi \circ J$ satisfies the Palais-Smale condition [10, Example 38.25] and hence the existence of a third critical point of the same functional is assured by [3, Corollary 1]. The proof is complete. □

We conclude by formulating two open problems.

Problem 1. In Theorem 1.1, can the role of the sequence $\{a_n\}$ be assumed by a suitable unbounded interval?

Problem 2. Does Theorem 1.1 hold for $q = (m + 2)/(m - 2)$?

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