Special Issue in honor of Alan C. Lazer
Electronic Journal of Differential Equations, Special Issue 01 (2021), pp. 293–300.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

EXISTENCE AND MULTIPLICITY RESULTS FOR p-q-LAPLACIAN BOUNDARY VALUE PROBLEMS

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Dedicated to the living memory of Alan C. Lazer

Abstract. We study positive solutions to the boundary value problem

$$\begin{gathered} -\Delta_p u - \Delta_q u = \lambda f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $q \in (1,p)$ and Ω is a bounded domain in $\mathbb{R}^N$, $N > 1$ with smooth boundary, λ is a positive parameter, and $f : [0,\infty) \to (0,\infty)$ is C^1, nondecreasing, and p-sublinear at infinity i.e. $\lim_{t\to\infty} f(t)/t^{p-1} = 0$. We discuss existence and multiplicity results for classes of such f. Further, when $N = 1$, we discuss an example which exhibits S-shaped bifurcation curves.

1. Introduction

In [12], the authors studied the the p-Laplacian boundary value problem

$$\begin{gathered} -\Delta_p u = \lambda f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$, $N > 1$ with smooth boundary, λ is a positive parameter, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), p > 1$ and $f : [0,\infty) \to (0,\infty)$ satisfies

(H1) f is a C^1 non-decreasing, p-sublinear function at infinity, i.e.

$$\lim_{t\to\infty} \frac{f(t)}{t^{p-1}} = 0.$$

In particular, they established the existence of a positive solution for each λ, and when there exists $0 < a < b$ such that

$$Q(a,b) := \frac{a^{p-1}/f(a)}{b^{p-1}/f(b)} \gg 1,$$

they obtained multiple positive solutions for certain range of λ. When $p = 2$, these results were discussed in [4]. Their study was motivated by the applications in chemical reaction theory (see [2]) and in combustion theory (see [11, 14]).

2010 *Mathematics Subject Classification.* 35J70, 35J55.
Key words and phrases. Multiple positive solutions; p-q Laplacian; sub-super solution.

Published January 18, 2022, 2021.

In this article, we extend this study to the p-q-Laplacian boundary value problem

$$\begin{gathered} -\Delta_p u - \Delta_q u = \lambda f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

for $q \in (1, p)$. Namely, we establish

Theorem 1.1. *Assume* (H1). *Then the following results hold:*

(1) *Equation* (1.2) *has a positive solution for each* $\lambda > 0$.
(2) *There exists positive constants* $C_0 = C_0(\Omega)$, $C_1 = C_1(p, \Omega, N)$, *and* $C_2 = C_2(\Omega)$ *such that if* $b > C_0$ *and*

$$Q(a, b) = \frac{a/(f(a))^{p-1}}{b/(f(b))^{p-1}} > \frac{C_1}{C_2},$$

for some points a *and* b, $a < b$, *then* (1.2) *has at least two positive solutions for* $\lambda \in \big(\frac{b^{p-1}}{f(b)} C_1, \frac{a^{p-1}}{f(a)} C_2\big]$.

As in [4] and [12] we use the method of sub-super solutions to establish our results. In [4], the availability of Green's function played an important role in the construction of a positive strict sub-solution that was used to establish a multiplicity result. In [12], the authors had to develop another idea to construct this special sub-solution because of the lack of a Green's function for the p-Laplacian operator for $p \neq 2$. Here we adapt and extend the ideas used in [12] to establish our results. However, unlike in [12], our multiplicity result is restricted to only two solutions. In [12], the authors used results in [1, 15] to guarantee three solutions. If the results in [1, 15] can be extended to the p-q Laplacian case, our construction of sub-super solutions will also yield at least three positive solutions for the range of λ in Theorem 1.1 part (2)

A time-dependent version of an operator such as in (1.2) often occurs in the mathematical modeling of chemical reactions and plasma physics. In recent years, a lot of attention has been given to study the boundary value problem involving p-q Laplacian, see for instance [3, 5, 10, 13] and the references therein.

The rest of this article is organized as follows. In Section 2, we recall some important results that are required for the development of this article. In Section 3 we prove of Theorem 1.1. In Section 4 we provide an application of our results. Finally in Section 5, we obtain exact bifurcation diagrams for the case when $\Omega = (0, 1)$, $p = 4$ and $q = 2$, namely to the two-point boundary value problem

$$\begin{gathered} -[(u')^3]' - \mu[(u')]' = \lambda f(u);\ (0, 1) \\ u(0) = 0 = u(1) \end{gathered} \tag{1.3}$$

where $f(s) = \exp\big(\frac{\gamma s}{\gamma + s}\big)$, $\gamma > 0$, and μ is a non-negative parameter.

2. Preliminaries

In this section, we recall some results concerning a sub-super solution method for p-q-Laplacian boundary value problem. First, by a weak solution of (1.2) we mean a function $u \in W_0^{1,p}(\Omega)$ which satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi + \int_\Omega |\nabla u|^{q-2} \nabla u \cdot \nabla \phi = \lambda \int_\Omega f(u)\phi, \quad \forall \phi \in C_0^\infty(\Omega).$$

However, in this article, we in fact study $C^1(\overline{\Omega})$ solution. Next, by a sub-solution (super solution) of (1.2) we mean a function $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $v \le (\ge) 0$ on $\partial\Omega$ and satisfies

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\phi + \int_\Omega |\nabla v|^{q-2}\nabla v \cdot \nabla \phi \le (\ge) \lambda \int_\Omega f(v)\phi,$$

for all $\phi \in C_0^\infty(\Omega)$ and $\phi \ge 0$ in Ω. Then the following sub-super solution result holds.

Lemma 2.1. *Let ψ, z be sub and super solutions of* (1.2) *respectively such that $\psi \le z$ in Ω. Then* (1.2) *has a solution $u \in C^1(\overline{\Omega})$ such that $\psi \le u \le z$.*

For a proof of the above lemma see [7, Corollary 1].

3. Proof of Theorem 1.1

In this section we use sub-super solution method to prove Theorem 1.1. At first we prove the results when $\Omega = B_R$, a ball of radius R and centered at origin in $\mathbb{R}^N$. We adopt and extend the ideas presented in [12] to construct a crucial sub-solution on B_R.

Construction of two sub-solutions on B_R. Clearly $\phi_1 = 0$ is a sub-solution to the problem (1.2). Now we construct another sub-solution. For that we consider the function

$$v(r) = \begin{cases} 1, & r \le \epsilon \\ 1 - [1 - (\frac{R-r}{R-\epsilon})^\beta]^\alpha, & \epsilon \le r \le R, \end{cases}$$

where $\epsilon \in (0, R)$, $\alpha > 1$ and $\beta > 1$. Let us denote $\mu_1(r) = \frac{R-r}{R-\epsilon}$ and $\mu_2(r) = 1 - (\mu_1(r))^\beta$. Taking $\tilde{v}(r) = bv(r)$ we note that $|\tilde{v}'(r)| \le b\alpha\beta/(R-\epsilon)$. Now let ψ be a radially symmetric solution of

$$\begin{gathered} -\Delta_p\psi - \Delta_q\psi = \lambda f(\tilde{v}(|x|)) \quad \text{in } B_R, \\ \psi = 0 \quad \text{on } \partial B_R. \end{gathered} \tag{3.1}$$

Then ψ satisfies

$$\begin{gathered} -(r^{N-1}G_p(\psi'(r)))' - (r^{N-1}G_q\psi'(r))' = \lambda r^{N-1} f(\tilde{v}(r)) \\ \psi'(0) = \psi(R) = 0, \end{gathered} \tag{3.2}$$

where $G_s(t) = |t|^{s-2}t$, $s = p, q$. Integrating the above equation over $0 < r < R$ we obtain

$$-G_p(\psi'(r)) - G_q\psi'(r) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(\tilde{v}(s))\,ds. \tag{3.3}$$

Notice that $\tilde{G}(t) := G_p(t) + G_q(t)$ is a continuous, monotone function. Hence, $\tilde{G}^{-1}$ exists and is also continuous. Therefore, (3.3) yields

$$\psi'(r) = \tilde{G}^{-1}\Big(\frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(\tilde{v}(s))\,ds\Big). \tag{3.4}$$

Next, we claim that $\tilde{v}(r) \le \psi(r)$, for $0 \le r \le R$. If this claim is true, then ψ is a sub-solution as f is nondecreasing. Now, since $\psi(R) = v(R) = 0$, it is sufficient to show that $\psi'(R) \le v'(R)$ for all $0 \le r \le R$. Observe that $\psi'(r) = 0$ for $0 \le r \le R$ and $\tilde{v}'(r) = 0$ for $0 \le r \le \epsilon$. Where as for $r \ge \epsilon$ we have

$$\int_0^r s^{N-1} f(\tilde{v}(s))\,ds \ge \int_0^\epsilon s^{N-1} f(\tilde{v}(s))\,ds \ge f(b)\frac{\epsilon^N}{N}.$$

It follows from (3.4) that

$$-\psi'(r) \geq \tilde{G}^{-1}\Big(f(b)\frac{\lambda\epsilon^N}{NR^{N-1}}\Big).$$

Recall that $|\tilde{v}'(r)| \leq \frac{b\alpha\beta}{(R-\epsilon)}$. Thus, $\psi'(r) \leq v'(r)$ if $\tilde{G}^{-1}(f(b)\frac{\lambda\epsilon^N}{NR^{N-1}}) \geq \frac{b\alpha\beta}{R-\epsilon}$ i.e. if

$$f(b)\frac{\lambda\epsilon^N}{NR^{N-1}} \geq \tilde{G}(\frac{b\alpha\beta}{R-\epsilon}). \tag{3.5}$$

Note that (3.5) will be satisfied if

$$f(b)\frac{\lambda\epsilon^N}{NR^{N-1}} \geq \max\big\{2, 2G_p(\frac{b\alpha\beta}{R-\epsilon})\big\},$$

i.e. if

$$\lambda \geq \frac{2NR^{N-1}}{f(b)\epsilon^N}\max\big\{1, (\frac{b\alpha\beta}{R-\epsilon})^{p-1}\big\}. \tag{3.6}$$

Let $b > R$. Then we can choose $\alpha \approx 1$, $\beta \approx 1$ so that $(\frac{b\alpha\beta}{R-\epsilon}) > 1$ and (3.6) will be satisfied if

$$\lambda \geq \frac{b^{p-1}}{f(b)}C_1(\alpha\beta)^{p-1}, \tag{3.7}$$

where $C_1 = \inf_\epsilon \frac{2NR^{N-1}}{\epsilon^N(R-\epsilon)^{p-1}}$. In fact, this infimum is achieved at $\epsilon = \epsilon_0 = \frac{NR}{N+p-1}$, which will be our choice of ϵ. Assume that

$$\lambda > \frac{b^{p-1}}{f(b)}C_1\,. \tag{3.8}$$

Then clearly we can fix $\alpha(>1) \approx 1$, $\beta(>1) \approx 1$ so that (3.7) holds. Hence, for these choice of α, β, ϵ, ψ will be a sub-solution, when (3.8) is satisfied. Further, since $\psi \geq \tilde{v}$, $\|\psi\|_\infty \geq b$.

Construction of super solutions on B_R. Let $\sigma(r) = (1-(\frac{r}{R})^{p'})/p'$ on B_R, where $\frac{1}{p}+\frac{1}{p'} = 1$. Notice that $0 \leq \sigma \leq 1$. Also that for $0 \leq r \leq R$,

$$\begin{gathered} \sigma'(r) = -\frac{r^{p'-1}}{R^{p'}}, \\ -(r^{N-1}G_s(\sigma'(r)))' = -(r^{N-1}|\sigma'(r)|^{s-2}\sigma'(r))' = \Big(\frac{r^{N-1}r^{(p'-1)(s-1)}}{R^{p'(s-1)}}\Big)' \geq 0, \end{gathered} \tag{3.9}$$

In particular, $-(r^{N-1}G_p(\sigma'(r)))' = \frac{Nr^{N-1}}{R^p}$. Now let $\xi_a = N^{\frac{1}{1-p}}a\sigma$, where a as in the assumption of Theorem 1.1. Then, since f is nondecreasing,

$$-(r^{N-1}G_p(\xi_a'(r)))' - (r^{N-1}G_q(\xi_a'(r)))' \geq \frac{r^{N-1}a^{p-1}}{R^p} \geq \lambda r^{N-1}f(\xi_a(r))$$

if $\lambda \leq \frac{a^{p-1}}{f(a)R^p}$. Thus, $\widetilde{\xi_a} := \xi_a(|x|)$ satisfies

$$-\Delta_p\widetilde{\xi_a} - \Delta_q\widetilde{\xi_a} \geq \frac{a^{p-1}}{R^p} \geq \lambda f(\widetilde{\xi_a}) \quad \text{if } \lambda \leq \frac{a^{p-1}}{f(a)R^p}. \tag{3.10}$$

Hence, $\widetilde{\xi_a}$ is a super solution when $\lambda \leq \frac{a^{p-1}}{f(a)R^p}$. Next for a given $\lambda > 0$, let $\widetilde{\xi_\lambda} = N^{\frac{1}{1-p}}M(\lambda)\sigma(|x|)$, where $M(\lambda) >> 1$ so that $\frac{[m(\lambda)]^{p-1}}{f(M(\lambda))} \geq \lambda R^p$. Then, again using f is nondecreasing we have

$$-\Delta_p\widetilde{\xi_\lambda} - \Delta_q\widetilde{\xi_\lambda} \geq \frac{M(\lambda)^{p-1}}{R^p} \geq \lambda f(\widetilde{\xi_\lambda}), \tag{3.11}$$

hence $\widetilde{\xi_\lambda}$ is a super solution on B_R.

Comparison of Sub-Sup solutions on B_R**.** For any $\lambda > 0$, $\psi_1 \equiv 0$ is a strict sub-solution (as $f(0) > 0$) and $z_2 = \widetilde{\xi_\lambda} = N^{\frac{1}{1-p}} M(\lambda)\sigma(|x|)$ with $M(\lambda) >> 1$ is a super solution. Hence, by Lemma 2.1, (1.2) has a positive solution for each $\lambda > 0$. Next let $\lambda \in (\frac{b^{p-1}}{f(b)}C_1, \frac{a^{p-1}}{f(a)}C_2]$, where $C_2 = \frac{1}{R^p}$ and $0 < a < b$ such that $b > R = C_0(\Omega)$ and $Q(a,b) \geq \frac{C_1}{C_2}$. For such λ, $\psi_1 \equiv 0$ is a strict sub-solution, $\psi_2 = \psi$ (ψ as in (3.1)) is a sub-solution, $z_1 = \widetilde{\xi_a} = N^{\frac{1}{1-p}} a\sigma(|x|)$ is a super solution (see (3.10)) and $z_2 = \widetilde{\xi_\lambda} = N^{\frac{1}{1-p}} M(\lambda)\sigma(|x|)$ is a super solution. Hence, by Lemma 2.1, (1.2) has two solutions u_1, u_2 for such $\lambda > 0$, where $u_1 \in [\psi_1, z_1]$ and $u_2 \in [\psi_2, z_2]$. Note that, u_1 and u_2 are distinct since $\|z_1\|_\infty \leq a$, $\|\psi_2\|_\infty \geq b$ and $a < b$.

Now we proceed to prove our result for any open, bounded subset Ω of $\mathbb{R}^N$.

Proof of Theorem 1.1. Note that $\tilde{\psi}_1 = 0$ still remains a sub-solution on Ω to (1.2) on Ω. Now let B_R be the largest inscribed ball inside Ω and we define

$$\tilde{\psi}_2(x) = \begin{cases} \psi(x), & \text{if } x \in B_R \\ 0, & \text{if } x \in \overline{\Omega} \setminus B_R \end{cases}$$

where ψ is as in (3.1). Clearly, $\tilde{\psi}_2 \in W_0^{1,p}(\Omega)$ and when $\lambda > C_1 b^{p-1}/f(b)$ we have

$$-\Delta_p \tilde{\psi}_2(x) = -\Delta_p \psi_2(x) \leq \lambda f(\psi_2(x)) = \lambda f(\psi_2(x)) \quad \text{on } B_R.$$

Also $-\Delta_p \tilde{\psi}_2(x) = 0 < \lambda f(0) = \lambda f(\psi_2(x))$ in $\Omega \setminus B_R$. Hence, $\tilde{\psi}_2$ is a strict sub-solution when $\lambda > C_1 b^{p-1}/f(b)$. Also $\|\tilde{\phi}_2\|_\infty \geq b$. Next, we consider a ball $B_{\overline{R}}$ containing Ω. By taking $z_1 = \widetilde{\xi_a} = N^{\frac{1}{1-p}} a\sigma(|x|)$ and $z_2 = \widetilde{\xi_\lambda} = N^{\frac{1}{1-p}} M(\lambda)\sigma(|x|)$ as earlier (but now in ball $B_{\overline{R}}$) and taking their restrictions on Ω, it is easy to see that z_1 is a strict super solution (1.2) on Ω if $\lambda \leq \frac{a^{p-1}}{R^p f(a)}$, while $z_2 = \widetilde{\xi_\lambda}$ with $M(\lambda) >> 1$ is a super solution to (1.2) on Ω for any $\lambda > 0$. Noticing again $\|z_1\|_\infty \leq a$ and using Lemma 2.1, the proof of Theorem 1.1 follows in the general region Ω. $\square$

Remark 3.1. Under assumption (H1), if $f(s)/s^{q-1}$ is strictly decreasing for $s > 0$, then (1.2) has a unique positive solution [8, Theorem 2.2].

4. Application in combustion theory

For $1 < q < p$, we consider

$$\begin{gathered} -\Delta_p u - \Delta_q u = \lambda \exp\Big(\frac{\gamma u}{\gamma + u}\Big) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.1}$$

The reaction term $f(s) = \exp\big(\frac{\gamma s}{\gamma+s}\big)$, $\gamma > 0$ occurs in the theory of combustion and it has been discussed in [4] (Laplacian case), and in [12] (p-Laplacian case). In [4], the authors obtained that $\gamma > 4$ is a necessary condition for multiple positive solutions for the Laplacian case; while in [12] the authors obtained the same for $\gamma > 4(p-1)$ in the p-Laplacian case. Here we present analogous result for p-q Laplacian. Towards this we first notice that $\tilde{f}(u) := \frac{f(u)}{u^{q-1}}$ is decreasing if $\gamma \leq 4(q-1)$. Thus, Remark 3.1 ensures that $\gamma > 4(q-1)$ is a necessary condition for (4.1) to have multiple solutions. Further, taking $a = 1$ and $b = \gamma$, we have

$$Q(a,b) := \frac{a^{p-1}/f(a)}{b^{p-1}/f(b)} = \gamma^{(1-p)} \exp\big(\frac{\gamma}{2} - \frac{\gamma}{\gamma+1}\big).$$

Thus, for any C_0, C_1 and C_2, we can choose γ large enough such that $b > C_0$ $Q(1,\gamma) > \frac{C_1}{C_2}$ and hence, by Theorem 1.1, (4.1) admits at least two solutions at least for certain range of λ.

5. Bifurcation diagram for positive solutions to (5.1)

Here we study the two-point boundary value problem

$$\begin{gathered} -[(u')^3]' - \mu[(u')]' = \lambda f(u);\ (0,1) \\ u(0) = 0 = u(1) \end{gathered} \tag{5.1}$$

where $f(s) = \exp\left(\frac{\gamma s}{\gamma+s}\right); \gamma > 0$, and μ is a non-negative parameter. We will provide the exact bifurcation diagram via a quadrature method and Mathematica computations. We will also study how this bifurcation curve evolves when γ and μ vary.

Here we use the quadrature method described in [6] which was obtained by extending the method initially introduced in [9]. First we note that since (5.1) is autonomous, any positive solution u must be symmetric about $x = 1/2$, increasing on $(0, 1/2)$, and decreasing on $(1/2, 1)$. Assume u is a positive solution of (5.1) and let $u(1/2) = \rho$.

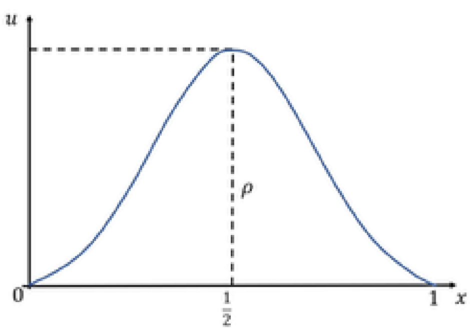


Figure 1. Shape of a positive solution to (5.1)

Multiplying (5.1) by u' and integrating we obtain

$$-\frac{3}{4}[(u')^4]' - \frac{\mu}{2}[(u')^2]' = \lambda(F(u))' \quad \text{in} \quad (0,1),$$

where $F(s) = \int_0^s f(z)dz$. Further integrating we obtain

$$3[u'(x)]^4 + 2\mu[u'(x)]^2 = 4\lambda[F(\rho) - F(u(x))] \quad \text{in } [0, \frac{1}{2}],$$

and hence

$$u'(x) = \frac{\sqrt{[\mu^2 + 12\lambda(F(\rho) - F(u(x)))]^{\frac{1}{2}} - \mu}}{\sqrt{3}} \quad \text{in } [0, \frac{1}{2}]. \tag{5.2}$$

Integrating (5.2), we obtain

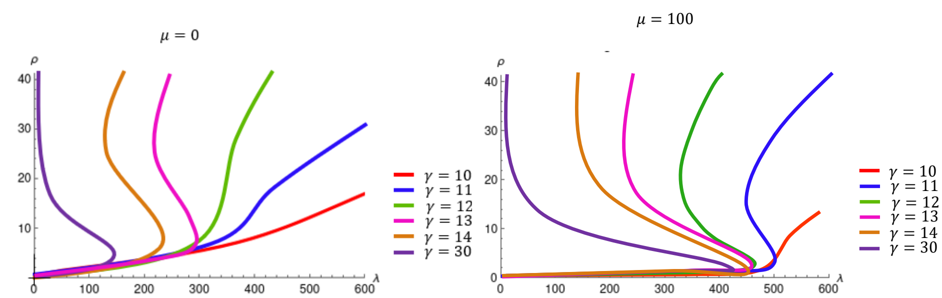
$$\int_0^{u(x)} \frac{ds}{\sqrt{[\mu^2 + 12\lambda(F(\rho) - F(s))]^{\frac{1}{2}} - \mu}} = \frac{x}{\sqrt{3}} \quad \text{in } [0, \frac{1}{2}), \tag{5.3}$$

and setting $x \to \left(\frac{1}{2}\right)^-$, we obtain

$$G(\lambda,\rho)=\int_0^{\rho}\frac{ds}{\sqrt{[\mu^2+12\lambda(F(\rho)-F(s))]^{\frac{1}{2}}-\mu}}=\frac{1}{2\sqrt{3}}. \tag{5.4}$$

Inversely, if λ, ρ are such that (5.4) is satisfied, $u(x)$ is defined via (5.3) for $x \in [0, \frac{1}{2}), u(1/2) = \rho$, and $u(x) = u(1-x)$ for $x \in (1/2, 1]$, it follows that u will be a positive solution of (5.1). Hence (5.4) determines the bifurcation diagram of positive solutions for (5.1). Now, for $f(s) = \exp\left(\frac{\gamma s}{\gamma+s}\right)$, we use Mathematica computations to obtain the bifurcation diagram using (5.4).

Observations. For a given $\mu \geq 0$, there exists $\gamma_0(\mu)$ such that for $\gamma < \gamma_0(\mu)$, we obtain a unique solution of (5.1) for all $\lambda > 0$, and for $\gamma > \gamma_0(\mu)$ the bifurcation curve is $S-$shaped with multiplicity in the region (λ_1, λ_2). Further, $\gamma_0(\mu)$ decreases in μ and λ_1 decreases in γ (see Figure 2). Furthermore, strength of the multiplicity range (i.e. the length $(\lambda_2 - \lambda_1)$) increases in γ (See Figure 3).



$\mu = 0$ $\mu = 100$

FIGURE 2. Bifurcation diagrams of (5.1) for different values of γ for given μ.

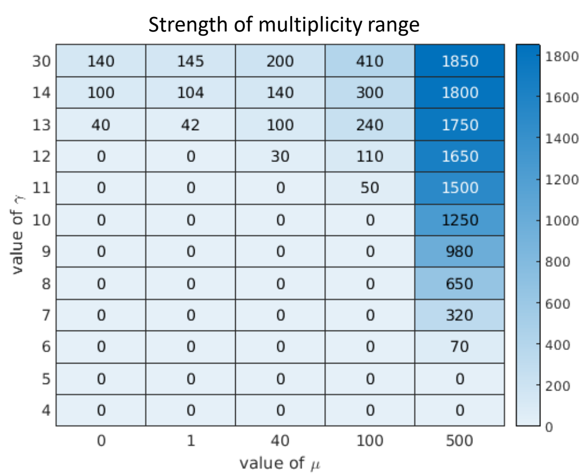


FIGURE 3. Heat map showing the strength of multiplicity range w.r.t γ and μ.

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