Special Issue in honor of Alan C. Lazer
Electronic Journal of Differential Equations, Special Issue 01 (2021), pp. 91–99.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

A SEMILINEAR WAVE EQUATION WITH NON-MONOTONE NONLINEARITY AND FORCING FLAT ON CHARACTERISTICS

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Abstract. We provide sufficient conditions on the forcing term for a semilinear wave equation with non-monotone asymptotically linear nonlinearity to have a weak solution. Earlier results required the forcing term not to be flat on characteristics, now we remove those requirements. Also we provide estimates on the measure of the level sets of the forcing term, that suffice for the equation to have a weak solution.

1. Introduction

We consider the existence of weak solutions to the Dirichlet-periodic problem

$$\begin{aligned} \partial_{tt}u - \partial_{xx}u + H(u) := \Box\, u + H(u) = G(x,t), &\quad x \in (0,\pi),\ t \in \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, & \\ u(x,t) = u(x,t+2\pi), & \end{aligned} \tag{1.1}$$

with H not monotone and asymptotically linear. More precisely we assume that $H(u) = \tau u + h(u)$ with $\tau \in \mathbb{R} - \{0\}$ and

$$\lim_{|u|\to+\infty} h'(u) = 0. \tag{1.2}$$

For the sake of simplicity in the estimates, we assume that h is bounded. We also assume that $-\tau \notin \{k^2 - j^2;\ k = 1,2,\dots,\ j = 0,1,2,\dots\} := \sigma(\Box)$. The set $\sigma(\Box)$ is the spectrum of the wave operator $\Box$ subject to the boundary conditions in (1.1). The main difficulty in studying the solvability of (1.1) is the fact that 0 is an eigenvalue of infinite multiplicity. This renders useless compactness techniques extensively used in the study of related semilinear elliptic boundary value problems. If H is a monotonic function, for each $G \in L^2(\Omega) := L^2((0,\pi)\times(0,2\pi))$, equation (1.1) has a solution (see [2]). For H non-monotone it has been known from [13] and [9] that (1.1) has a solution for G in a dense subset of $L^2(\Omega)$. However the proofs in [13] and [9] do not shed light on the nature of the functions G for which (1.1) has a solution. In [8], [3] and [6] it is shown that when the forcing term G is large and *not flat in characteristics* then (1.1) has a weak solution. Here we extend such results to cases where G may be flat in characteristics and provide an estimate on the size of the subsets of characteristics on which G may be flat (constant). To date

2010 *Mathematics Subject Classification.* 35L70, 35D30.
Key words and phrases. Wave equation; flat on characteristic; non-monotone nonlinearity; infinite multiplicity eigenvalue.

Published October 6, 2021.

we do not know of Gs for which (1.1) has no solution under our hypothesis on H. However, in [4] a class of continuous G's for which the wave equation in (1.1) has no continuous solution 2π-periodic in both x and t is provided. For related results on wave equations with non-monotone nonlinearities the reader is referred to [1] and [5].

For the sake of simplicity in the notations we assume that $\tau > 0$. We denote by $\|\cdot\|_2$ the norm in L^2, and by $\mathcal{N}$ the closure of the linear subspace of $L^2(\overline{\Omega})$ generated by

$$\{\sin(kx)\cos(kt),\ \sin(kx)\sin(kt);\ k = 1, 2, \dots\}. \tag{1.3}$$

That is, $\mathcal{N}$ is the kernel of the wave operator $\Box$ subject to the boundary conditions in (1.1). We denote by $\mathcal{N}^\perp$ the orthogonal complement of $\mathcal{N}$ in $L^2(\overline{\Omega})$, and by $P_N : L^2(\overline{\Omega}) \to \mathcal{N}$, $P_{N^\perp} : L^2(\overline{\Omega}) \to \mathcal{N}^\perp$ the corresponding orthogonal projections. If $v \in \mathcal{N}$, then there exists a 2π-periodic function $p : \mathbb{R} \to \mathbb{R}$ such that

$$v(x,t) = p(t+x) - p(t-x), \quad p \in L^2([0, 2\pi]). \tag{1.4}$$

We denote by $\mathbf{H}^1$ the Sobolev space of the functions $u : (0,\pi) \times \mathbb{R} \to \mathbb{R}$ such that $u, u_x, u_t \in L^2(\overline{\Omega})$, and satisfy the boundary conditions in (1.1). The norm in $\mathbf{H}^1$ is denoted by $\|\cdot\|_{1,2}$ and $\mathbf{Y}$ denotes the subspace of functions y in $\mathbf{H}^1$, such that

$$\iint_\Omega y(t,x)v(t,x)\,dx\,dt = 0, \quad \text{for all } v \in \mathcal{N}. \tag{1.5}$$

A function $u = y + v \in \mathbf{Y} \oplus \mathcal{N}$ is called a weak solution of (1.1) if

$$\iint_\Omega \{(y_t\hat{y}_t - y_x\hat{y}_x) - (H(u) - G)(\hat{y} + \hat{v})\}\,dx\,dt = 0, \tag{1.6}$$

for all $\hat{y} + \hat{v} \in \mathbf{Y} \oplus \mathcal{N}$.

If $\tau > 0$, $-\tau \notin \sigma(\Box)$, and $z \in L^2(\overline{\Omega})$, the equation $\Box u + \tau u = z$ subject to the boundary condition in (1.1) has only one weak solution $v + y$, which we denote $(\Box + \tau I)^{-1}(z)$. An elementary Fourier series argument shows that there exists $\kappa > 0$ such that

$$\begin{gathered} \|(\Box + \tau I)^{-1}(P_{N^\perp}(z))\|_{1,2} + \|(\Box + \tau I)^{-1}(P_{N^\perp}(z))\|_{C^{1/2}} \le \kappa\|z\|_2\,, \\ \|(\Box + \tau I)^{-1}(P_N(z))\|_2 \le \kappa\|z\|_2\,, \end{gathered} \tag{1.7}$$

where $\mathcal{C}^{1/2}$ denote the space of Hölder continuous functions with exponent $1/2$.

Throughout this paper we denote by μ the Lebesgue measure in $\mathbb{R}$. Our main result is the following theorem.

Theorem 1.1. *Let $\hat{f} \in \mathcal{N}$ with $\hat{f}(x,t) = \hat{q}(x+t) - \hat{q}(t-x)$ and $\|\hat{q}\|_2 = 1$. Let $g \in \mathcal{N}^\perp$, and $G(x,t) = Cf(x,t) + g(x,t)$ with $f \in \mathcal{N}$ and $C \in \mathbb{R}$. If*

$$\mu(\{x \in [0, 2\pi] : \hat{q}(x) = y\}) < \frac{\pi(2\tau + |h'|_\infty - \sqrt{4\tau|h'|_\infty + |h'|^2_\infty})}{|h'|_\infty} \tag{1.8}$$

for all $y \in \mathbb{R}$, then there exists $\eta > 0$ and $C_0 > 0$ such that, if $\|f - \hat{f}\|_2 < \eta$ and $|C| > C_0$ then problem (1.1) has a weak solution.

Since the smallest root of the quadratic polynomial $Q(s) = (2\pi\tau - s|h'|_\infty)^2 - 2\pi|h'|^2_\infty s$ is the right-hand side in (1.8), if

$$0 \le \alpha_1 < \frac{\pi(2\tau + |h'|_\infty - \sqrt{4\tau|h'|_\infty + |h'|^2_\infty})}{|h'|_\infty} \tag{1.9}$$

then $Q(\alpha_1) > 0$. That is,

$$2\pi|h'|_\infty^2\alpha_1 < (\pi\tau - \alpha_1|h'|_\infty)^2. \tag{1.10}$$

Also, since we are assuming H to be non-monotone, $|h'|_\infty > \tau$. Because $\varphi(s) = s - \sqrt{4\tau s - s^2}$ defines a decreasing function in $[0,\infty)$ we have $\varphi(|h'|_\infty) < \varphi(\tau) = (1-\sqrt{3})\tau$. Hence, if (1.9) holds then

$$\alpha_1|h'|_\infty < \pi(3-\sqrt{5})\pi\tau < 2\pi\tau. \tag{1.11}$$

Preliminary lemmas

In this section we state and prove some properties of the measure of level sets that play important roles in the proof of Theorem 1.1.

Lemma 1.2. *Let (X,B,m) be a measure space. If $q \in L^1(X)$ and $m(X) < +\infty$ then there exists $y \in \mathbb{R}$ such that*

$$m(\{x \in X : q(x) = y\}) = \max\big\{m(\{x \in X : q(x) = z\}); z \in \mathbb{R}\big\} := \alpha(q). \tag{1.12}$$

Proof. If $m(\{x \in X : q(x) = z\}) = 0$ for all $z \in \mathbb{R}$ then $\alpha(q) = 0$ and we can take y to be any real number.

If there exists $\hat{z} \in \mathbb{R}$ such that $m(\{x \in X : q(x) = \hat{z}\}) > 0$ then $\{z; m(\{x \in X : q(x) = z\}) \geq m(\{x \in X : q(x) = \hat{z}\})\}$ is finite, say $\{z_1,\dots,z_n\}$. Therefore, there exists $j \in \{1,\dots,n\}$ such that $m(\{x \in X : q(x) = z_j\}) \geq m(\{x \in X : q(x) = z_i\})$ for $i = 1,\dots,n$. Taking $y = z_j$ the lemma is proven. □

Lemma 1.3. *Let $\hat{q} \in L^2([0,2\pi])$ with $\|\hat{q}\|_2 = 1$, and $\alpha(\hat{q})$ as in Lemma 1.2. If $\alpha(\hat{q}) < \alpha_1$ then there exists $\delta > 0$ such that if $\|q - \hat{q}\|_2 < \delta$, then $\alpha(q) < \alpha_1$ for all $y \in \mathbb{R}$.*

Proof. Suppose there are sequences $\{q_j\}_j$ in $L^2(0,2\pi)$ such that $\lim_{j\to\infty}\|q_j - \hat{q}\|_2 = 0$, $\{\delta_j\}_j$ in $(0,\infty)$ such that $\lim_{j\to+\infty}\delta_j = 0$, and $\{y_j\}_j$ in $\mathbb{R}$ such that

$$\mu(\{x \in [0,2\pi] : |q_j(x) - y_j| < \delta_j\}) \geq \alpha_1. \tag{1.13}$$

Since $\{q_j\}_j$ is bounded in $L^1(0,2\pi)$, $\alpha_1 > 0$, and $\lim_{j\to+\infty}\delta_j = 0$, $\{y_j\}_j$ is bounded. By passing to a subsequence we may assume that $\{y_j\}_j$ converges. Let $\hat{y} = \lim_{j\to+\infty} y_j$. By Egoroff's theorem, see [10], there exists $E \subset [0,2\pi]$ such that $\mu(E) < (\alpha_1 - \alpha(\hat{q}))/4$ such that $\{q_j\}_j$ converges uniformly to $\hat{q}$ in $[0,2\pi] - E$. Since

$$\cap_{n=1}^\infty \{x \in [0,2\pi] : |\hat{q}(x) - \hat{y}| < 1/n\} = \{x \in [0,2\pi] : \hat{q}(x) = \hat{y}\}, \tag{1.14}$$

there exists $\eta > 0$ such that

$$\mu(\{x \in [0,2\pi] : |\hat{q}(x) - \hat{y}| < \eta\}) < \frac{\alpha + \alpha_1}{2}. \tag{1.15}$$

Let J be such that, for $j \geq J$, $|q_j(x) - \hat{q}(x)| < \eta/4$ for $x \in [0,2\pi] - E$ and $\delta_j < \eta/4$. Hence, for $j \geq J$,

$$\begin{aligned}
&\mu(\{x \in [0,2\pi] : |q_j(x) - \hat{y}| < \delta_j\}) \\
&\leq \mu(\{x \in [0,2\pi] : |q_j(x) - \hat{y}| < \frac{\eta}{2}\}) \\
&\leq \mu(\{x \in [0,2\pi] : x \in E \text{ and } |q_j(x) - \hat{y}| < \eta\}) \\
&\quad + \mu(\{x \in [0,2\pi] : x \in [0,2\pi] - E \text{ and } |\hat{q}(x) - \hat{y}| < \eta\}) \\
&< \frac{\alpha_1 - \alpha}{8} + \frac{\alpha + \alpha_1}{2} < \alpha_1,
\end{aligned} \tag{1.16}$$

which contradicts (1.13). The proof is complete. $\square$

Proof of Theorem 1.1

Let $f(x,t) = q(x+t) - q(t-x)$. An elementary Fourier series argument and Parseval's identity prove that $\sqrt{2\pi}\|q-\hat{q}\|_2 = \|f-\hat{f}\|_2$. Hence taking δ as in Lemma 1.3, for $\|f-\hat{f}\|_2 < \sqrt{2\pi}\delta := \eta$ we have

$$\alpha(q) < \alpha_1. \tag{1.17}$$

From [13] and [9], there exist sequences $\{\phi_n\}, \{u_n\} \subset L^2(\Omega)$ with $u_n = z_n + w_n \in \mathcal{N} \oplus \mathbf{Y}$, and $\lim_{n\to+\infty} \|\phi_n\|_2 = 0$ such that

$$\square\, w_n + \tau(z_n + w_n) + h(z_n + w_n) = Cf(x,t) + g(x,t) + \phi_n(x,t) \tag{1.18}$$

in the weak sense. Projecting onto $\mathcal{N}^\perp$ and $\mathcal{N}$ one sees that (1.18) is equivalent to

$$(\square + \tau I)w_n = g + P_{N^\perp}(\phi_n - h(z_n + w_n)), \tag{1.19}$$

$$\tau z_n + P_N(h(z_n + w_n)) = Cf + P_N(\phi_n). \tag{1.20}$$

In turn, (1.19) and (1.20) are equivalent to

$$w_n = (\square + \tau I)^{-1}(g) + (\square + \tau I)^{-1} P_{N^\perp}(\phi_n - h(z_n + w_n)), \tag{1.21}$$

$$z_n = \frac{C}{\tau} f + \frac{1}{\tau} P_N(\phi_n - h(z_n + w_n)) \equiv \frac{C}{\tau} f + \frac{1}{\tau} v_n. \tag{1.22}$$

Since $\lim_{n\to\infty} \|\phi_n\|_2 = 0$ in L^2 and h is bounded, there exist k_1 such that $\|\phi_n - h(z_n + w_n)\|_2 \le k_1$. Hence, (1.21) and (1.22) imply

$$\|w_n\|_{1,2} \le \kappa k_1 \quad \text{and} \quad \|v_n\|_2 \le k_1 + \|h(u_n)\|_2. \tag{1.23}$$

Since $v_n, P_N(\phi_n) \in \mathcal{N}$, there exist 2π-periodic functions $p_n, \gamma_n : \mathbb{R} \to \mathbb{R}$, with $p_n, \gamma_n \in L^2([0,2\pi])$ such that

$$v_n(x,t) = p_n(t+x) - p_n(t-x), \quad P_N(\phi_n)(x,t) = \gamma_n(t+x) - \gamma_n(t-x), \tag{1.24}$$

for all $n \in \mathbb{Z}_+$, $x \in [0,\pi], t \in \mathbb{R}$.

By (1.20) and [3, Lemma 5.2], we have

$$2\pi p_n(r) = 2\pi\gamma_n(r) - I_n(r) + \Gamma_n(r), \quad \text{a.e. in } [0,2\pi], \tag{1.25}$$

with

$$\begin{aligned}
I_n(r) &= \int_0^\pi h\Big(w_n(x, r-x) + \frac{1}{\tau}(p_n(r) - p_n(r-2x) + Cf(x, r-x))\Big)dx \\
&= \frac{1}{2}\int_{r-2\pi}^{r} h\Big(\tilde{w}_n(r,y) + \frac{1}{\tau}(q_n(r) - q_n(y))\Big)dy, \\
\Gamma_n(r) &= \int_0^\pi h(w_n(x, r+x) + \frac{1}{\tau}(p_n(r+2x) - p_n(r) + Cf(x, r+x)))dx \\
&= \frac{1}{2}\int_{r}^{r+2\pi} h(\hat{w}_n(r,y) + \frac{1}{\tau}(q_n(y) - q_n(r)))dy,
\end{aligned}$$

where $q_n(s) = p_n(s) + Cq(s)$, $\tilde{w}_n(r,y) = w_n(\frac{r-y}{2}, \frac{r+y}{2})$, and $\hat{w}_n(r,y) = w_n(\frac{y-r}{2}, \frac{r+y}{2})$.

Next we prove that the sequence $\{p_n\}$, defined in (1.25), converges in $L^2(0,2\pi)$. By (1.10) and (1.11), there exists $\epsilon_1 > 0$ such that

$$\frac{2\pi|h'|_\infty^2\alpha + 2\pi^2\epsilon_1^2}{\tau^2} < (\pi - \frac{|h'|_\infty\alpha + 2\pi\epsilon_1}{\tau})^2 \quad \text{and} \quad |h'|_\infty\alpha + \pi\epsilon_1 < 2\pi\tau. \tag{1.26}$$

Let $\epsilon \in (0, \epsilon_1)$. By (1.7), $\{w_n\}$ is bounded in the Holder space $C^{1/2}$. Hence it is bounded and equicontinuous. This and the Arzela-Ascoli theorem imply that $\{w_n\}$ has a uniformly convergent subsequence. Thus, without loss of generality, we may assume that $\{w_n\}$ converges uniformly on Ω. Hence there exists N_1 such that if $n, m \geq N_1$ then

$$|w_n(x,t) - w_m(x,t)| < \epsilon \quad \text{for all } (x,t) \in \Omega. \tag{1.27}$$

Since $\{\gamma_n\}_n$ converges to zero in $L^2(0, 2\pi)$, by Egoroff's theorem there exists $D \subset [0, 2\pi]$ be such that $\mu(D) < \epsilon$ and $\{\gamma_n\}_n$ converges uniformly to zero in $[0, 2\pi] - D$. Hence, there exists $N_2 \geq N_1$ such that if $n \geq N_2$ then

$$\|\gamma_n\|_2 < \epsilon \text{ and } |\gamma_n(s)| < \epsilon \quad \text{for almost all } s \in [0, 2\pi] - D. \tag{1.28}$$

Hence,

$$\int_D (|I_n(r) - I_m(r)| + |\Gamma_n(r) - \Gamma_m(r)|) dr \leq 4\pi |h|_\infty \epsilon \tag{1.29}$$

for all $n, m \geq N_2$. For $r \in [0, 2\pi] - D$, we have

$$\begin{aligned} |I_n(r) - I_m(r)| \leq \frac{1}{2} \int_{D_r} \Big| & h\Big(\tilde{w}_n(r,y) + \frac{1}{\tau}(q_n(r) - q_n(y))\Big) \\ & - h\Big(\tilde{w}_m(r,y) + \frac{1}{\tau}(q_m(r) - q_m(y))\Big)\Big| \, dy \\ & + \frac{1}{2} \int_{E_r} \Big| h\Big(\tilde{w}_n(r,y) + \frac{1}{\tau}(q_n(r) - q_n(y))\Big) \\ & - h\Big(\tilde{w}_m(r,y) + \frac{1}{\tau}(q_m(r) - q_m(y))\Big)\Big| \, dy, \end{aligned} \tag{1.30}$$

where $D_r = \{s \in [r - 2\pi, r] : s \in D \text{ or } s + 2\pi \in D\}$ and $E_r = [r - 2\pi, r] - D_r$. Since $\mu(D_r) = \mu(D) < \epsilon$, applying the mean value theorem we have

$$\begin{aligned} & |I_n(r) - I_m(r)| \\ & \leq |h|_\infty \epsilon + \frac{1}{2} \int_{E_r} \Big| h\Big(\tilde{w}_n(r,y) + \frac{1}{\tau}(q_n(r) - q_n(y))\Big) \\ & \quad - h\Big(\tilde{w}_m(r,y) + \frac{1}{\tau}(q_m(r) - q_m(y))\Big)\Big| \, dy \\ & \leq |h|_\infty \epsilon + \frac{1}{2} \int_{E_r} \Big| \frac{(q_n - q_m)(r) - (q_n - q_m)(y)}{\tau} h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y)) \Big| \, dy \\ & \quad + \frac{1}{2} \int_{E_r} \Big| (\tilde{w}_n - \tilde{w}_m)(r,y) h'\Big(\frac{1}{\tau}(q_n(r) - q_n(y) + \bar{\zeta}_{nm}(r,y))\Big) \Big| dy, \end{aligned} \tag{1.31}$$

where $\zeta_{nm}(r,y)$ is in the bounded interval defined by $(q_n(r) - q_n(y))/\tau$ and $(q_m(r) - q_m(y))/\tau$, and $\bar{\zeta}_{nm}(r,y)$ is in the bounded interval defined by $w_n((r-y)/2, (r+y)/2)$ and $w_m((r-y)/2, (r+y)/2)$. From (1.27), for $n, m \geq N_2$, we have

$$\int_{r-2\pi}^{r} \Big| (\tilde{w}_n - \tilde{w}_m)(r,y) h'\Big(\frac{1}{\tau}(q_n(r) - q_n(y)) + \bar{\zeta}_{nm}(r,y)\Big) \Big| dy < 2\pi\epsilon |h'|_\infty. \tag{1.32}$$

From the definition of q_n we have

$$\begin{aligned}
&\int_{E_r} \Big|((q_n - q_m)(r) - (q_n - q_m)(y))h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))\Big|\,dy \\
&\le |(p_n - p_m)(r)| \int_{E_r} \Big|h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))\Big|\,dy \\
&\quad + \int_{E_r} \Big|(p_n - p_m)(y)h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))\Big|\,dy.
\end{aligned} \tag{1.33}$$

By (1.2), there exists $b > 0$ such that $|h'(s)| < \epsilon$ for any $|s| \ge b$. For $C > 0$ and $r \in [0, 2\pi] - D$, we define the set

$$\begin{aligned}
A(r, C) &= \big\{y \in E_r : \big|\frac{C}{\tau}(q(r) - q(y))\big| < b\big\} \\
&= \big\{y \in E_r : |q(r) - q(y)| < \frac{\tau b}{|C|}\big\}.
\end{aligned}$$

Let δ be as in Lemma 1.3. By Lemma 1.3 and (1.17) for $|C| > \tau b/\delta := C_0$ we have

$$\mu(A(r, C)) < \alpha_1 \quad \text{for all } r \in [0, 2\pi]. \tag{1.34}$$

If $y \notin A(r, C)$ and $C \ge C_0$, then $\frac{C}{\tau}|q(r) - q(y)| \ge b$. Hence

$$|h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))| < \epsilon. \tag{1.35}$$

Also, for all $r \in [0, 2\pi] - D$ and $C \ge C_0$,

$$\begin{aligned}
&\Big|\int_{r-2\pi}^{r} h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))\,dy\Big| \\
&= \Big|\int_{A(r,C)} h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))\,dy \\
&\quad + \int_{A^c(r,C)} h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))dy\Big| \\
&< (|h'|_\infty \alpha_1 + 2\pi\epsilon).
\end{aligned} \tag{1.36}$$

By the Cauchy-Schwartz inequality

$$\begin{aligned}
&\int_{r-2\pi}^{r} |h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y)) \cdot (p_n(y) - p_m(y))|\,dy \\
&\le \Big(\int_{r-2\pi}^{r} [h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))]^2\,dy\Big)^{1/2} \Big(\int_0^{2\pi} [p_n(y) - p_m(y)]^2 dy\Big)^{1/2} \\
&\le \|p_n - p_m\|_2 \Big(\int_{r-2\pi}^{r} [h'(w_m(r,y) + \zeta_{nm}(r,y))]^2 dy\Big)^{1/2}.
\end{aligned}$$

Reasoning as in (1.36) we have

$$\begin{aligned}
&\int_{r-2\pi}^{r} [h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))]^2 dy \\
&= \int_{A(r,C)} [h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y)]^2 dy \\
&\quad + \int_{A^c(r,C)} [h'(\tilde{w}_m(r,y) + \zeta_{nm}(r,y))]^2 dy \\
&< |h'|_\infty^2 \alpha_1 + \pi\epsilon^2 := K + \pi 2\epsilon^2.
\end{aligned} \tag{1.37}$$

From (1.31), (1.36), and (1.37), for $r \in [0, 2\pi] - D$ and $n, m \geq N_2$, we have

$$
\begin{aligned}
&|I_n(r) - I_m(r)| \\
&< |h'|_\infty \epsilon + \frac{(K + 2\pi\epsilon^2)^{1/2}}{2\tau} \|p_n - p_m\|_2 + \frac{|h'|_\infty \alpha_1 + 2\pi\epsilon}{2\tau} |p_n(r) - p_m(r)|,
\end{aligned} \tag{1.38}
$$

Repeating the arguments leading to (1.38) for $|\Gamma_n(r) - \Gamma_m(r)|$ it is seen that the estimate on the right of (1.38) also holds for $|\Gamma_n - \Gamma_m|$. This and (1.25) give

$$
\begin{aligned}
\pi|p_n(r) - p_m(r)| < \pi|\gamma_n(r) - \gamma_m(r)| &+ \frac{(K + 2\pi\epsilon^2)^{1/2}}{2\tau} \|p_n - p_m\|_2 \\
&+ \frac{|h'|_\infty \alpha_1 + 2\pi\epsilon}{\tau} |p_n(r) - p_m(r)| + |h'|_\infty \epsilon,
\end{aligned} \tag{1.39}
$$

for $r \in [0, 2\pi] - D$. Letting $C_1 = \pi - \frac{|h'|_\infty \alpha_1}{2\tau} > 0$ we have

$$
\big(C_1 - \frac{\epsilon\pi}{\tau}\big)|p_n(r) - p_m(r)| < (\pi + |h'|_\infty)\epsilon + \frac{(K + 2\pi\epsilon^2)^{1/2}}{2\tau} \|p_n - p_m\|_2 \tag{1.40}
$$

for all $r \in [0, 2\pi] - D$ and all $n, m \geq N_2$. Let $M > 0$ be such that

$$
\begin{aligned}
&2\pi(\pi + |h'|_\infty)^2 \epsilon_1 + 4\pi(\pi + |h'|_\infty) \frac{(K + 2\pi\epsilon_1^2)^{1/2}}{\tau} \|p_n - p_m\|_2 \\
&+ C_1^2(8|h|_\infty^2 \pi^2 + 2\epsilon_1) \leq M,
\end{aligned} \tag{1.41}
$$

for all $n, m \geq N_2$. Squaring in (1.40) and integrating with respect to r we obtain

$$
\begin{aligned}
&(C_1 - \frac{\epsilon\pi}{\tau})^2 \int_0^{2\pi} |p_n(r) - p_m(r)|^2 dr \\
&< 2\pi(\pi + |h'|_\infty)^2 \epsilon^2 + 2\pi(\pi + |h'|_\infty)\epsilon \frac{(K + \pi\epsilon^2)^{1/2}}{\tau} \|p_n - p_m\|_2 \\
&\quad + \frac{\pi(K + 2\pi\epsilon^2)}{2\tau^2} \|p_n - p_m\|_2^2 + C_1^2 \int_D |p_n - p_m|^2 dr.
\end{aligned} \tag{1.42}
$$

Also, by (1.25), (1.28), and (1.29), we have

$$
\begin{aligned}
\int_D |p_n - p_m|^2 dr &\leq 2 \int_D |(p_n - \gamma_n) - (p_m - \gamma_m)|^2 dx + 2 \int_D |\gamma_m - \gamma_n|^2 dr \\
&= 2 \int_D |(I_m - \Gamma_m - I_n + \Gamma_n)/(2\pi)|^2 dr + 2\epsilon^2 \\
&\leq 8|h|_\infty^2 \epsilon + 2\epsilon^2.
\end{aligned} \tag{1.43}
$$

By (1.26), taking

$$
\rho = \big(C_1 - \frac{\epsilon_1\pi}{\tau}\big)^2 - \frac{\pi K + \pi^2\epsilon_1^2}{2\tau^2} > 0, \tag{1.44}
$$

we have

$$
\|p_n - p_m\|_2^2 \leq \frac{M}{\rho}\epsilon. \tag{1.45}
$$

Since $\epsilon > 0$ may be chosen arbitrarily small, $\{p_n\}_n$ is a Cauchy sequence in $L^2([0, 2\pi])$. Hence $\{v_n\}_n$ and $\{z_n\}_n$ are Cauchy sequences in $L^2(\Omega)$.

Let $w \in \mathbf{Y}$ and $z \in \mathcal{N}$ be such that $\lim_{n\to+\infty} w_n = w$ and $\lim_{n\to z_n} z_n = z$ in $L_2(\Omega)$. Because of (1.21), (1.22), and $\lim_{n\to+\infty} \phi_n = 0$ in $L^2(\Omega)$, we have

$$\begin{gathered} w = (\Box + \tau I)^{-1}(g) + (\Box + \tau I)^{-1} P_{N^\perp}(h(z+w)), \\ z = \frac{C}{\tau} f + \frac{1}{\tau} P_N(h(z+w)), \end{gathered} \tag{1.46}$$

which proves that $z + w$ is a weak solution to (1.1). The proof of the theorem is complete.

Acknowledgements. The authors are grateful to the anonymous referees for their careful reading of the original manuscript and for their helpful suggestions. The authors are also grateful to editor Julio G. Dix for obtaining referee reports and accepting this article.

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