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RADIAL AND NON-RADIAL SOLUTIONS FOR A NONLINEAR SCHRÖDINGER EQUATION WITH A CONSTRAINT

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In memory of Professor Alan C. Lazer

Abstract. We study the classical nonlinear Schödinger equation with a radially symmetric potential and a constraint, for the mass subcritical case. We obtain conditions that assure the existence of non-radial solutions. Also we show symmetry breaking of the ground states, and the existence of multiple non-radial solutions under additional conditions.

1. Introduction

In this article, we consider the nonlinear Schrödinger equation under an L^2 constraint,

$$\begin{gathered} -\Delta u - Q(x)|u|^{p-2}u = \lambda u, \\ \int_{\mathbb{R}^N} |u|^2 dx = 1, \quad u \in H^1(\mathbb{R}^N) \end{gathered} \tag{1.1}$$

where $2 < p < 2 + \frac{4}{N}$ is mass subcritical. That is, we seek u and λ satisfying the above equations. We assume that the potential function is a radial, $Q(x) = Q(|x|)$. We investigate conditions which assure the existence of both radial and non-radial solutions. In particular we show that, while there always exists a radial solution, the ground state is non-radial and there can be multiple non-radial solutions.

Before stating our results we discuss some background of the problem. Solutions with a prescribed L^2-norm are referred as normalized solutions. This type of problems naturally arise from the studies of standing wave solutions of the time dependent Schrödinger equations. In mathematical physics, finding solutions with a prescribed L^2-norm is particularly relevant since this quantity is preserved along the time evolution. On this line of research we have the classical work of Cazenave-Lions [5] and Stuart [15], and some more work later works such as [1, 6, 7, 8, 9, 10, 12, 13]. We refer the reader to these and references therein for the general discussion on the existence and orbital stability of standing waves of the time dependent nonlinear Schrödinger equations.

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Solutions of this problem are critical points of the functional

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx-\frac{1}{p}\int_{\mathbb{R}^N}Q(x)|u|^pdx,\quad u\in H^1(\mathbb{R}^N) \tag{1.2}$$

under the constrained mass condition

$$\|u\|_2=1. \tag{1.3}$$

In particular, one can consider the minimization problem of J on the constraint

$$c=\inf\{J(v):v\in H^1(\mathbb{R}^N),\|v\|_2=1\}. \tag{1.4}$$

When Q is a positive constant, this goes back to the classical work of Cazenave and Lions [5] as applications of the concentration compactness principle [9, 10]. This gives rise to the existence of a minimizer solution which can be proved to be orbitally stable for the corresponding initial value problem of the time dependent nonlinear Schrödinger equation. For our purpose of notations later, we recall this result as follow.

For each positive constant d, we define

$$c_d=\inf_{u\in H^1,\|u\|_2=1}\Big(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx-\frac{d}{p}\int_{\mathbb{R}^N}|u|^pdx\Big). \tag{1.5}$$

Then it was proved in [5] that $c_d<0$ and c_d is always attained. Using the same method and being easier using the compact embedding from H^1_r into L^2, one can prove that when $Q(x)$ is a radial function, the problem is also solvable in the space of radial functions. Here we state this without giving a proof. Assume

- (A1) $Q=Q(|x|)$ is continuous and there exist positive constants $b_2\ge b_1>0$ such that $b_2\ge Q(x)\ge b_1$.

Theorem 1.1. *Assume (A1), $N\ge 2$, and $2<p<2+\frac{4}{N}$. Then*

$$c_{\rm rad}=\inf\{J(v):v\in H^1_r(\mathbb{R}^N),\|v\|_2=1\}$$

is achieved, where $H^1_r(\mathbb{R}^N)$ is the radial subspace of $H^1(\mathbb{R}^N)$.

By this theorem, the problem (1.1) has a radial solution. When we consider the minimization problem (1.4) in the full H^1 the minimization problem may or may not be solvable. Our focus in this paper is to investigate a class of potential function Q that assure the existence of minimizers, and to give conditions for the minimizers to be non-radial and for multiple non-radial solutions. We make the following assumptions.

- (A2) $\max\{q_0,q_\infty\}<q_M$ where $q_0:=Q(0)$ and $q_\infty:=\limsup_{|x|\to\infty}Q(x)$.

To study the symmetry breaking phenomenon we will magnify the conditions on the potential and study the family of problems with a small parameter $\epsilon>0$,

$$\begin{gathered}-\Delta u-Q(\epsilon x)|u|^{p-2}u=\lambda u,\\ \|u\|_2=1,\quad u\in H^1(\mathbb{R}^N).\end{gathered} \tag{1.6}$$

We write the functional depending on the parameter ϵ as

$$J_\epsilon(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx-\frac{1}{p}\int_{\mathbb{R}^N}Q(\epsilon x)|u|^pdx,\ u\in H^1(\mathbb{R}^N), \tag{1.7}$$

under the constrained mass condition

$$\|u\|_2=1. \tag{1.8}$$

In particular, one can consider the minimization problem of J_ϵ on the constraint

$$c(\epsilon) = \inf\{J_\epsilon(u) : u \in H^1(\mathbb{R}^N), \|u\|_2 = 1\}. \tag{1.9}$$

We will also use $c_{\rm rad}(\epsilon)$ to denote the ground state energy in the radial class (which is always achieved by Theorem 1.1)

$$c_{\rm rad}(\epsilon) = \inf\{J_\epsilon(u) : u \in H_r^1(\mathbb{R}^N), \|u\|_2 = 1\}.$$

Theorem 1.2. *Assume* (A1), (A2), $N \geq 1$, *and* $2 < p < 2 + \frac{4}{N}$. *Then there exists* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$, $c(\epsilon)$ *is achieved at some* u_ϵ. *Moreover, let* u_ϵ *be a minimizer of* $c(\epsilon)$, *and* $u_\epsilon^{\rm rad}$ *be a minimizer of* $c_{\rm rad}$. *Then*

$$\lim_{\epsilon\to 0} c_{\rm rad}(\epsilon) := J_\epsilon(u_\epsilon^{\rm rad}) = c_{q_0},$$
$$\lim_{\epsilon\to 0} c_\epsilon = J_\epsilon(u_\epsilon) = c_{q_M}.$$

In particular, for ϵ *small,* u_ϵ *is non-radial.*

Next we seek more non-radial solutions of problem (1.6). Since Q is radial there exists a radial solution by Theorem 1.1. In Theorem 1.2 we prove that the ground states are non-radial. Next we show that under suitable conditions the problem has multiple non-radial solutions. These non-radial solutions appear as ground states in some other symmetric subspaces. Here is the main result and see Section 3 for more detailed descriptions of asymptotic profiles of these solutions.

Theorem 1.3. *Assume* (A1), (A2), $N = 2$, *and* $2 < p < 4$. *Let* k *be a positive integer. Suppose*

$$\frac{q_M}{\max\{q_0, q_\infty\}} > k^{\frac{p-2}{2}}.$$

Then there exists $\epsilon_k > 0$ *such that for all* $0 < \epsilon < \epsilon_k$, *problem* (1.6) *has* k *non-radial solutions.*

Being minimizers of some symmetric subspaces all these solutions from the above theorems are positive solutions. We also remark that symmetry breaking was studied in [6] for the two dimensional mass critical problem under a linear trapping potential.

The organization of the paper is as follows. In section 2 we list some preliminaries and state and prove a useful lemma on the ground state energy in terms of constant potential. Section 3 is devoted to the proof of Theorem 1.2 proving the existence and symmetry breaking of ground state solutions. In Section 4 we prove Theorem 1.3 giving multiple non-radial solutions, and discuss some possible extensions.

2. Preliminaries

We donate by $H_r^1(\mathbb{R}^N)$ the space of radially symmetric functions $u(x) = u(|x|)$ which satisfy $u(x)$, $\nabla u(x) \in L^2(\mathbb{R}^N)$. We also use notation

$$\|u\|_q = \Big(\int_{\mathbb{R}^N} |u(x)|^q\Big)^{1/q}, \quad \text{for } q\in[1,\infty) \text{ and } u\in L^q(\mathbb{R}^N), \tag{2.1}$$

$$\|u\|_{H^1} = (\|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}. \tag{2.2}$$

The following is a special case of the well-known Gagliardo Nirenberg inequality [4].

Lemma 2.1. *Let $p \geq 2$, $0 \leq \theta < 1$ be such that*

$$\frac{1}{p} = \theta(\frac{1}{2} - \frac{1}{N}) + \frac{1-\theta}{2}. \tag{2.3}$$

Then there exists constant $C = C(p)$ independent of u such that

$$\|u\|_p \leq C\|\nabla u\|_2^{\theta}\|u\|_2^{1-\theta}, \quad \forall u \in W^{1,2}(\mathbb{R}^N). \tag{2.4}$$

The following lemma 2.2 is the Concentration-Compactness principle [9, 10, 17].

Lemma 2.2. *Let μ_n be a sequence of nonnegative L^1 functions on $\mathbb{R}^N$. For $r > 0$ we define a family of concentration functions,*

$$Q_n(r) := \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} \mu_n.$$

Suppose that μ_n is a sequence of L^1-functions on $\mathbb{R}^N$,

$$\mu_n \geq 0, \quad \int_{\mathbb{R}^N} \mu_n dx = 1.$$

Then there exist a subsequence of μ_n (still denoted by μ_n) such that

$$\alpha = \lim_{m\to\infty} \lim_{n\to\infty} Q_n(m)$$

exists and one of the following three statements holds.

(i) *(vanishing) $\alpha = 0$ and for any $R > 0$ we have*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \mu_n = 0; \tag{2.5}$$

(ii) *(compactness) $\alpha = 1$ and for any $\delta > 0$ there are $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} \mu_n \geq 1 - \delta; \tag{2.6}$$

(iii) *(dichotomy) for each $\alpha \in (0, 1)$ and $\delta > 0$ there are $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that for all $r \geq R$ and $r' \geq R$*

$$\limsup_{n\to\infty} \big|\alpha - \int_{B_r(y_n)} \mu_n\big| + \big|(1-\alpha) - \int_{\mathbb{R}^N\setminus B_{r'}(y_n)} \mu_n\big| < \delta. \tag{2.7}$$

Using the concentration compactness principle, it was proved that for any $d > 0$, the c_d defined in (1.5) satisfies $c_d < 0$ and that c_d is attained. When $Q(x) = Q(|x|)$, the attained-ness of the radial case $c_{\rm rad}$ can be done in a similar and even easier by the compact embedding from $H^1_r(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for $N \geq 2$, $2 < p < \frac{2N}{N-2}$ (see [14, 4]), which gives the conclusion of Theorem 1.1. For $N = 1$, namely the case of even functions, one can easily rule out dichotomy to get compactness for the existence of a minimizer. We omit the proof here.

We finish this section with a simple but useful result relating the minimum values of c_d in terms of d.

Proposition 2.3. *When the potential function in* (1.5) *is constant, it holds*

$$c_d = d^{\frac{4}{4-N(p-2)}} c_1. \tag{2.8}$$

Proof. From now on for any function u and $\lambda > 0$ we use the notation $u^\lambda(x) = \lambda^{N/2}u(\lambda x)$. Note that $\|u^\lambda\|_2 = \|u\|_2$ for any $\lambda > 0$. Choose u_1 such that $c_1 = J(u_1)$. Using $u_1^\lambda(x)$ as a testing function we have

$$c_d \le \lambda^2\Big(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} \frac{\lambda^{\frac{N(p-2)}{2}}}{\lambda^2} d|u_1|^p dx\Big).$$

Choosing λ_0 such that

$$\frac{\lambda_0^{\frac{N(p-2)}{2}}}{\lambda_0^2} d = 1,$$

i.e., $\lambda_0 = d^{\frac{4}{4-N(p-2)}}$, we obtain

$$c_d \le \lambda_0^2 J(u_1) = \lambda_0^2 c_1 = d^{\frac{4}{4-N(p-2)}} c_1. \tag{2.9}$$

In a similar way,

$$c_1 \le \lambda_1^2\Big(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_d|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} \frac{\lambda_1^{\frac{N(p-2)}{2}}}{\lambda_1^2} |u_d|^p dx\Big)$$

where $c_d = J(u_d)$, and

$$u_d^{\lambda_1} = \lambda_1^{N/2} u_d(\lambda_1 x).$$

Therefore when we set:

$$d = \frac{\lambda_1^{\frac{N(p-2)}{2}}}{\lambda_1^2} \tag{2.10}$$

and obtain

$$c_1 \le \lambda_0^2 J(u_1) = \lambda_0^2 c_1 = d^{-\frac{4}{4-N(p-2)}} c_d. \tag{2.11}$$

Then the result follows from (2.9) and (2.11). □

3. Existence and symmetry breaking of the ground states

In this section, we study the existence and symmetry property of the ground state solutions under our conditions (A1) and (A2). We will show that for ϵ small, the ground states are not radially symmetric.

Proposition 3.1.

$$\lim_{\epsilon\to 0} c(\epsilon) = c_{q_M}. \tag{3.1}$$

Proof. Let u_M be a minimizer of c_{q_M}. By the definition of c_{q_M}, we have

$$\begin{aligned} c_{q_M} &= \inf_{\|u\|_2=1} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p} q_M \int_{\mathbb{R}^N} |u|^p dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_M|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} q_M |u_M|^p dx. \end{aligned}$$

Let x_0 be such that $Q(x_0) = q_M$. We set $v_n(x) = u_M(x - \frac{x_0}{\epsilon_n})$, where $n \to \infty, \epsilon_n \to 0$. Hence we have $\|v_n\|_2 = \|u_M\|_2 = 1$. Then

$$\begin{aligned} c(\epsilon_n) &= \inf_{\|u\|_2=1} \Big(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} Q(\epsilon_n x)|u|^p dx\Big) \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} Q(\epsilon_n x)|v_n|^p dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{1}{p} q_M \int_{\mathbb{R}^N} |v_n|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} (q_M - Q(\epsilon_n x))|v_n|^p dx \end{aligned}$$

$$= c_{q_M} + \frac{1}{p}\int_{\mathbb{R}^N}(q_M - Q(\epsilon_n x))|v_n|^p dx.$$

So we turn the complicated problem into proving the claim

$$\frac{1}{p}\int_{\mathbb{R}^N}(q_M - Q(\epsilon_n x))|v_n|^p dx \to 0.$$

By a change of variable we have

$$\int_{\mathbb{R}^N}(q_M - Q(\epsilon_n x))|v_n|^p dx = \int_{\mathbb{R}^N}(q_M - Q(\epsilon_n x + x_0))|u_M|^p dx.$$

By the dominated convergence theorem we have the desired estimate. Hence we obtain

$$\limsup_{\epsilon\to 0} c(\epsilon) \le c_{q_M}. \tag{3.2}$$

Next by the definition of q_M we have $Q(\epsilon_n x) \le q_M$, and by the energy functional form, we have

$$c(\epsilon) \ge c_{q_M}. \tag{3.3}$$

Consequently, $\lim_{\epsilon\to 0} c(\epsilon) = c_{q_M}$. □

Lemma 3.2. *For a fixed $\epsilon > 0$, if $c(\epsilon) < c_{q_\infty}$, then $c(\epsilon)$ is attained.*

Proof. We fix an $\epsilon > 0$ such that $c(\epsilon) < c_{q_\infty}$. Let u_n be a minimizing sequence of $c(\epsilon)$. Because $c(\epsilon) < 0$, there is no vanishing for u_n. We claim that there is no dichotomy for u_n. Otherwise, by Brezis-Lieb lemma, we have $\alpha = \int_{\mathbb{R}^N}|u|^2dx, 1-\alpha = \int_{\mathbb{R}^N}|u_n - u|^2dx + o(1)$, and $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Since $1-\alpha \in (0,1)$, it follows that

$$\alpha^{p/2} < \alpha, (1-\alpha)^{p/2} < (1-\alpha).$$

Hence we obtain

$$\begin{aligned}
c(\epsilon) &\ge \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2dx - \frac{1}{p}\int_{\mathbb{R}^N}Q(\epsilon x)|u_n|^p dx + o(1)\\
&\ge (1-\alpha)c(\epsilon) + \alpha c(\epsilon) + o(1)\\
&\quad + [(1-\alpha) - (1-\alpha)^{p/2}]\int_{\mathbb{R}^N}Q(\epsilon x)\Big|\frac{(u_n-u)}{\|u_n-u\|_2}\Big|^p dx\\
&\quad + [\alpha - \alpha^{p/2}]\int_{\mathbb{R}^N}Q(\epsilon x)\Big|\frac{u}{\|u\|_2}\Big|^p dx.
\end{aligned}$$

Sending $n\to\infty$ we obtain $c(\epsilon) > c(\epsilon)$, a contradiction.

With vanishing and dichotomy ruled out, we have compactness for u_n from Lemma 2.2 (ii), i.e., up to a subsequence there exist (x_n) such that for any $\delta > 0$ there is $R > 0$, it holds

$$\liminf_{n\to\infty}\int_{B_R(x_n)} u_n^2 dx \ge 1-\delta.$$

Let $\eta(t)$ be a cut-off function such that $\eta(t) = 1$ for $|t| \le 1$ and $\eta(t) = 0$ for $|t| \ge 2$. Define $\eta_R(t) = \eta(t/R)$. Define $v_n = \eta_R(|x-x_n|)u_n(x)/\|\eta_R(|x-x_n|)u_n(x)\|_2$. Then we have

$$c(\epsilon) = \inf_{\|u\|_2=1} J_\epsilon(u) \ge \frac{1-\delta}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2dx - \frac{1}{p}\int_{B_{2R}(x_n)}Q(\epsilon x)|v_n|^p dx + O(\delta) + o(1/R).$$

Since $q_\infty < q_M$ for n large we have for $\gamma > 0$, $Q(\epsilon x) \le q_\infty + \gamma < q_M$ for all $x \in B_{2R}(x_n)$. Thus for n large we have

$$\begin{aligned} c(\epsilon) &\ge \frac{1-\delta}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{1}{p}\int_{B_{2R}(x_n)} (q_\infty+\gamma)|v_n|^p dx + O(\delta) + o(1/R) \\ &\ge c_{q_\infty+\gamma} + o(1/R) + O(\epsilon) + O(\delta). \end{aligned}$$

Sending $\delta \to 0$ and $R \to \infty$ we obtain $\lim_{\epsilon\to 0} c(\epsilon) \ge c_{q_\infty+\gamma}$. Since $\gamma > 0$ is arbitrary, we obtain $\lim_{\epsilon\to 0} c(\epsilon) \ge c_{q_\infty}$ which is a contradiction. □

Next we prove the limiting behavior of the ground state energy in the radially symmetric class.

Proposition 3.3. $\lim_{\epsilon\to 0} c_{\rm rad}(\epsilon) = \lim_{\epsilon\to 0} J_\epsilon(u_\epsilon^{\rm rad}) = c_{q_0}$.

Proof. Let u a radial minimizer of c_{q_0}. Using this as a testing function, we easily have $\lim_{\epsilon\to 0} J_\epsilon(u) = c_{q_0}$ which implies $\limsup_{\epsilon\to 0} c(\epsilon) \le c_{q_0}$.

Now for any sequence $\epsilon_n \to 0$ we write $u_n = u_{\epsilon_n}^{\rm rad}$. By Lions Lemma [17], there is no vanishing for this sequence.

Because u_ns are radial functions, it is easy to rule out dichotomy of this sequence. Also because of the radial symmetry [14], for compactness we have for any $\delta > 0$ there is $R > 0$ such that $\liminf_{n\to\infty} \int_{B_R(0)} u_n^2 dx \ge 1-\delta$. This implies $u_n \to u$ in $L^s(\mathbb{R}^2)$ for any $2 < s < \infty$ [14]. Then we have

$$c_{q_0} \le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} q_0 |u|^p dx \le \liminf_{n\to\infty} J_{\epsilon_n}(u_n).$$

Thus we obtain the desired estimate. □

Then Theorem 1.3 follows from Theorem 1.2, Propositions 3.1, 3.2 and 3.3.

4. Multiple non-radial solutions

To construct multiple non-radial solutions we use the group invariance property of the problem. First, for each integer $k \ge 2$ we define the group $G = G_k$ as follows

$$G = \Big\{ g, g^2, \dots, g^k = Id : g = \begin{pmatrix} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix} \Big\}. \tag{4.1}$$

We will work in the G-invariant subspace of functions $H_G^1(\mathbb{R}^2)$. We define the working space as the following

$$\Gamma^G = \{ u \in H_G^1(\mathbb{R}^2) : \|u\|_2 = 1 \} \tag{4.2}$$

and we have the ground state energy in the G-invariant subspace

$$c(\epsilon, k) = \inf_{u\in\Gamma^G} J_\epsilon(u). \tag{4.3}$$

Lemma 4.1.

$$\limsup_{\epsilon\to 0} c(\epsilon,k) \le c_{k^{\frac{2-p}{2}} q_M} \tag{4.4}$$

where as in (1.5),

$$c_{k^{\frac{2-p}{2}} q_M} = \inf_{u\in H^1, \|u\|_2=1} \Big(\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{k^{\frac{2-p}{2}} q_M}{p}\int_{\mathbb{R}^2} |u|^p dx \Big). \tag{4.5}$$

Proof. First of all, let x_0 be such that $Q(x_0) = q_M$ and consider

$$u_\epsilon^i(x) = u(x - g_i x_\epsilon), \quad i = 1, 2, \dots, k \tag{4.6}$$

where u is the solution that corresponds to the ground state energy $c_{k^{\frac{2-p}{2}} q_M}$, and $x_\epsilon = \frac{x_0}{\epsilon}$. Then we can obtain $\| \sum_{i=1}^k u_\epsilon^i \|_2^2 = \sum_{i=1}^k \|u_\epsilon^i\|_2^2 + o(1) = k + o(1)$, where $o(1) \to 0$ as $\epsilon \to 0$. Note that $\sum_{i=1}^k u_\epsilon^i / \| \sum_{i=1}^k u_\epsilon^i \|_2 \in \Gamma^G$. Thus we have

$$\begin{aligned}
c(\epsilon, k) &\le J_\epsilon\Big(\sum_{i=1}^k u_i / \| \sum_{i=1}^k u_\epsilon^i \|_2\Big) \\
&= \frac{1}{2(k+o(1))} \int_{\mathbb{R}^2} |\nabla(\sum_{i=1}^k u_i)|^2 dx - \frac{1}{p(k+o(1))^{\frac{p}{2}}} \int_{\mathbb{R}^2} Q(\epsilon x) |\sum_{i=1}^k u_i|^p dx \\
&= \frac{1}{2}\frac{1}{k} \sum_{i=1}^k \int_{\mathbb{R}^2} |\nabla u_i|^2 dx - \frac{k^{-\frac{p}{2}}}{p} \sum_{i=1}^k \int_{\mathbb{R}^2} Q(\epsilon x) |u_i|^p dx + o(1) \\
&= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{k^{\frac{2-p}{2}}}{p} \int_{\mathbb{R}^2} q_M |u|^p dx + o(1).
\end{aligned}$$

This gives

$$\limsup_{\epsilon \to 0} c(\epsilon, k) \le c_{k^{\frac{2-p}{2}} q_M}. \qquad \square$$

Lemma 4.2. *For fixed k, when $c(\epsilon, k) < c_{q_\infty}$, $c(\epsilon, k)$ is achieved.*

Proof. Assume that (u_n) is minimizing sequence, i.e. $J_\epsilon(u_n) \to c(\epsilon, k) < c_{q_\infty}$. Then the minimizing sequence (u_n) is bounded in H^1, and we may assume $u_n \rightharpoonup u$ in H^1 and $u_n \to u$ a.e. in $\mathbb{R}^2$.

We claim that $u \neq 0$. If not, we assume $u = 0$. Since $\limsup_{|x| \to \infty} Q(x) = q_\infty < q_M$ for any $\gamma > 0$, there exists $R > 0$ such that $Q(\epsilon x) \le q_\infty + \gamma < q_M$ for all $x \in B_R^c(0)$. Then we introduce a cut-off function

$$\eta(t) = \begin{cases} 1, & \text{if } |t| \ge 2, \\ 0, & \text{if } |t| \le 1. \end{cases} \tag{4.7}$$

We let $v_n = u_n \cdot \eta(\frac{|x|}{R})$. Since $u = 0$, when $n \to \infty$, we have $\|v_n - u_n\|_{H^1} \to 0$ and $\|v_n\|_2 \to 1, n \to \infty$. Then we normalize it

$$\widetilde{v}_n = \frac{v_n}{\|v_n\|_2} \in \Gamma^G. \tag{4.8}$$

Then we have

$$c(\epsilon, k) = J_\epsilon(u_n) = J_\epsilon(\widetilde{v}_n) + o(1) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \widetilde{v}_n|^2 dx - \frac{1}{p} \int_{B_R^c(0)} Q(\epsilon x) |\widetilde{v}_n|^p dx + o(1).$$

Hence we can obtain

$$c(\epsilon, k) \ge \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \widetilde{v}_n|^2 dx - \frac{1}{p} \int_{\mathbb{R}^2} (q_\infty + \gamma) |\widetilde{v}_n|^p dx + o(1).$$

Sending $n \to \infty$, we have $c(\epsilon, k) \ge c_{q_\infty + \gamma}$. Since $\gamma > 0$ is arbitrary we obtain $c(\epsilon, k) \ge c_{q_\infty}$, a contradiction. Similar to the proof of Lemma 3.2 we can use the Lemma 2.2 to prove that $\|u\|_2 = 1$. Then we can obtain u_n is converges strongly to u in L^p. Using the weak lower continuity of norm and the definition of $c(\epsilon, k)$, we can obtain $J_\epsilon(u) = c(\epsilon, k)$, which implies that $c(\epsilon, k)$ can be achieved. $\square$

From the previous two lemmas we see for fixed k, there exists $\epsilon_k > 0$ such that for all $\epsilon < \epsilon_k$, $c(\epsilon, k)$ is attained. We will examine the asymptotic behavior of $c(\epsilon, k)$ as $\epsilon \to 0$. Then we will be able to distinguish between these ground state energies.

Lemma 4.3. *Under the conditions of Theorem 1.3, we have*

$$\lim_{\epsilon\to 0} c(\epsilon, k) = c_{k^{\frac{2-p}{2}} q_M}.$$

Proof. We just need to consider the reverse inequality

$$\liminf_{\epsilon\to 0} c(\epsilon, k) \geq c_{k^{\frac{2-p}{2}} q_M}. \tag{4.9}$$

Let $\epsilon_n \to 0$. By the last two lemmas, we assume that $u_n \in \Gamma^G$ is such that $J_{\epsilon_n}(u_n) = c(\epsilon_n, k)$. From Lemma 4.1, we have

$$\limsup_{\epsilon\to 0} c(\epsilon, k) \leq c_{k^{\frac{2-p}{2}} q_M} < 0. \tag{4.10}$$

From this and Lemma 2.1, (u_n) is bounded in H^1. Consequently, we may assume that $u_n \rightharpoonup u$ in H^1, and $u_n \to u$ a.e. in $\mathbb{R}^2$.

We claim that $u = 0$. If not, assume $u \neq 0$. Let $v_n = u_n - u$. By the Brezis-Lieb lemma [2, 17], we can assume $\|u\|_2^2 = \alpha, \|v_n\|_2^2 = 1 - \alpha + o(1)$.

However, if $\alpha = 1$, we can obtain $c(\epsilon_n, k) \to c_{q_0}, n \to \infty$. It contradicts Lemma 4.1, so we have $\alpha \in (0, 1)$. Since $\alpha \in (0, 1)$, we have

$$(1-\alpha) > (1-\alpha)^{p/2}, \alpha > \alpha^{p/2}. \tag{4.11}$$

Hence

$$\begin{aligned}
&c(\epsilon_n, k)\\
&= J_{\epsilon_n}(u_n) = J_{\epsilon_n}(u) + J_{\epsilon_n}(v_n) + o(1)\\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)|u|^p dx + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 dx\\
&\quad - \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)|v_n|^p dx + o(1)\\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^2} q_0|u|^p dx + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 dx\\
&\quad - \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)|v_n|^p dx + o(1)\\
&= \|u\|_2^2\Big(\frac{1}{2}\int_{\mathbb{R}^2} \big|\frac{\nabla u}{\|u\|_2}\big|^2 dx - \frac{q_0}{p}\int_{\mathbb{R}^2} \big|\frac{u}{\|u\|_2}\big|^p dx\Big) + \frac{q_0}{p}\|u\|_2^2 \int_{\mathbb{R}^2} \big|\frac{u}{\|u\|_2}\big|^p dx\\
&\quad - \frac{1}{p}\int_{\mathbb{R}^2} q_0\big|\frac{u}{\|u\|_2}\big|^p dx (\|u\|_2^2)^{p/2}\\
&\quad + \|v_n\|_2^2\Big(\frac{1}{2}\int_{\mathbb{R}^2} \big|\frac{\nabla v_n}{\|v_n\|_2}\big|^2 dx - \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)\big|\frac{v_n}{\|v_n\|_2}\big|^p dx\Big)\\
&\quad + \frac{1}{p}\|v_n\|_2^2 \int_{\mathbb{R}^2} Q(\epsilon_n x)\big|\frac{v_n}{\|v_n\|_2}\big|^p dx - \frac{1}{p}(\|v_n\|_2^2)^{p/2}\int_{\mathbb{R}^2} Q(\epsilon_n x)\big|\frac{v_n}{\|v_n\|_2}\big|^p dx + o(1)\\
&\geq \alpha c_{q_0} + \frac{\alpha - \alpha^{p/2}}{p}\int_{\mathbb{R}^2} q_0\big|\frac{u}{\|u\|_2}\big|^p dx + (1-\alpha)c(\epsilon, k)\\
&\quad + \frac{1}{p}[(1-\alpha) - (1-\alpha)^{p/2}]\int_{\mathbb{R}^2} Q(\epsilon_n x)\big|\frac{v_n}{\|v_n\|_2}\big|^p dx + o(1).
\end{aligned}$$

We assume $A = \lim_{n\to\infty} c(\epsilon_n, k)$. Then

$$\begin{aligned}
A &\geq \alpha c_{q_0} + (1-\alpha)A + \frac{\alpha - \alpha^{p/2}}{p}\int_{\mathbb{R}^2} q_0\Big|\frac{u}{\|u\|_2}\Big|^p dx \\
&\quad + \frac{1}{p}[(1-\alpha) - (1-\alpha)^{p/2}]\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{v_n}{\|v_n\|_2}\Big|^p dx \\
&\geq \alpha \limsup_{n\to\infty} c(\epsilon_n, k) + (1-\alpha)A + \frac{\alpha - \alpha^{p/2}}{p}\int_{\mathbb{R}^2} q_0\Big|\frac{u}{\|u\|_2}\Big|^p dx \\
&\quad + \frac{1}{p}[(1-\alpha) - (1-\alpha)^{p/2}]\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{v_n}{\|v_n\|_2}\Big|^p dx \\
&\geq A + \frac{\alpha - \alpha^{p/2}}{p}\int_{\mathbb{R}^2} q_0\Big|\frac{u}{\|u\|_2}\Big|^p dx \\
&\quad + \frac{1}{p}[(1-\alpha) - (1-\alpha)^{p/2}]\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{v_n}{\|v_n\|_2}\Big|^p dx
\end{aligned}$$

which is a contradiction; therefore $u = 0$.

Because $\limsup_{\epsilon\to 0} c(\epsilon, k) < 0$, there is no vanishing for u_n. By Lemma 2.2, exists $|x_n| \to \infty, \alpha_1 > 0$, for $\forall \delta > 0, \exists R > 0$, we have for $i = 1, 2, \dots, k$,

$$\liminf_{n\to\infty} \int_{B_R(g_i x_n)} |u_n(x)|^2 dx \geq \alpha_1 - \delta. \tag{4.12}$$

Then we define the sequences v_n, w_n via a smooth non-increasing cut-off function ξ,

$$v_n(x) = \sum_{i=1}^{k} \xi\big(\frac{|x - g_i x_n|}{R}\big)u_n(x), \quad w_n(x) = \sum_{i=1}^{k}\big[1 - \xi\big(\frac{|x - g_i x_n|}{R}\big)\big]u_n(x) \tag{4.13}$$

where $x_n = x_0/\epsilon_n$ and

$$\xi(t) = \begin{cases} 1, & \text{if } |t| \leq 1, \\ 0, & \text{if } |t| \geq 2. \end{cases} \tag{4.14}$$

Obviously,

$$u_n = v_n + w_n. \tag{4.15}$$

Then we have $\|v_n\|_2^2 \to k\alpha_1$, $\|w_n\|_2^2 \to 1 - k\alpha_1$ due to Lemma 2.2. Then we need to prove that $k\alpha_1 = 1$.

If not, then $1 - k\alpha_1 > 0$, hence

$$k\alpha_1 \in (0, 1), (1 - k\alpha_1) > (1 - k\alpha_1)^{p/2}, k\alpha > (k\alpha_1)^{p/2}. \tag{4.16}$$

Therefore

$$\begin{aligned}
&c(\epsilon_n, k) \\
&= J_{\epsilon_n}(u_n) \geq J_{\epsilon_n}(v_n) + J_{\epsilon_n}(w_n) + O(\delta) + O(\frac{1}{R}) \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 dx - \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)|v_n|^p dx + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla w_n|^2 dx \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)|w_n|^p dx + O(\delta) + O(\frac{1}{R}) \\
&= \|v_n\|_2^2\Big(\frac{1}{2}\int_{\mathbb{R}^2}\Big|\frac{\nabla v_n}{\|v_n\|_2}\Big|^2 dx - \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{v_n}{\|v_n\|_2}\Big|^p dx\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \|v_n\|_2^2 \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{v_n}{\|v_n\|_2}\Big|^p dx - (\|v_n\|_2^2)^{p/2}\frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{v_n}{\|v_n\|_2}\Big|^p dx \\
&+ \|w_n\|_2^2\Big(\frac{1}{2}\int_{\mathbb{R}^2}\Big|\frac{\nabla w_n}{\|w_n\|_2}\Big|^2 dx - \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{w_n}{\|w_n\|_2}\Big|^p dx\Big) \\
&+ \|w_n\|_2^2 \frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{w_n}{\|w_n\|_2}\Big|^p dx \\
&- (\|w_n\|_2^2)^{p/2}\frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{w_n}{\|w_n\|_2}\Big|^p dx + O(\delta) + O(\frac{1}{R}) \\
\geq{}& k\alpha_1 c(\epsilon_n, k) + (1-k\alpha_1)c(\epsilon_n,k) \\
&+ [k\alpha_1 - (k\alpha_1)^{p/2}]\frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{v_n}{\|v_n\|_2}\Big|^p dx \\
&+ [(1-k\alpha_1) - (1-k\alpha_1)^{p/2}]\frac{1}{p}\int_{\mathbb{R}^2} Q(\epsilon_n x)\Big|\frac{w_n}{\|w_n\|_2}\Big|^p dx + O(\delta) + O(\frac{1}{R}).
\end{aligned}
$$

Sending $\delta \to 0$ (and therefore $R \to \infty$) we obtain a contradiction. Consequently, $k\alpha_1 = 1$.

By Lemma 2.2, there exists a sequence (x_n) satisfying $|x_n| \to \infty$ such that for any $\delta > 0$ there exists $R > 0$, we have, for $i = 1, \dots, k$,

$$
\liminf_{n\to\infty} \int_{B_R(g_i x_n)} |u_n(x)|^2 dx \geq \frac{1}{k} - \delta. \tag{4.17}
$$

We let

$$
v_n(x) = \sum_{i=1}^{k} \xi(\frac{|x - g_i x_n|}{R}) u_n(x). \tag{4.18}
$$

Then we have

$$
\lim_{n\to\infty} \int_{\mathbb{R}^2} |v_n|^2 dx = 1. \tag{4.19}
$$

Now we have

$$
\begin{aligned}
&c(\epsilon_n, k) \\
&= J_{\epsilon_n}(u_n) \\
&\geq J_{\epsilon_n}(v_n) + O(\delta) + O(\frac{1}{R}) \\
&= \frac{k}{2}\int_{B_{2R}(x_n)} |\nabla u_n|^2 dx - \frac{k}{p}\int_{B_{2R}(x_n)} Q(\epsilon_n x)|u_n|^p dx + O(\delta) + O(\frac{1}{R}) \\
&\geq k\cdot\frac{1}{k}\Big(\frac{1}{2}\int_{B_{2R}(x_n)} \frac{|\nabla u_n|^2}{\frac{1}{k}} dx - \frac{1}{p}\alpha_1^{\frac{p-2}{2}} q_M \int_{B_{2R}(x_n)} \frac{|u_n|^p}{\frac{1}{k}^{p/2}} dx\Big) + O(\delta) + O(\frac{1}{R}) \\
&= \frac{1}{2}\int_{B_{2R}(x_n)} \Big|\frac{\nabla u_n}{\sqrt{1/k}}\Big|^2 dx - \frac{1}{p} k^{\frac{2-p}{2}} q_M \int_{B_{2R}(x_n)} \Big|\frac{u_n}{\sqrt{1/k}}\Big|^p dx + O(\delta) + O(\frac{1}{R}) \\
&= \frac{1}{2}\int_{B_{2R}(x_n)} \Big|\frac{\nabla u_n}{\|u_n\|_2}\Big|^2 dx - \frac{1}{p} k^{\frac{2-p}{2}} q_M \int_{B_{2R}(x_n)} \Big|\frac{u_n}{\|u_n\|_2}\Big|^p dx + O(\delta) + O(\frac{1}{R}) \\
&\geq c_{k^{\frac{2-p}{2}} q_M} + O(\delta) + O(\frac{1}{R}).
\end{aligned}
$$

Sending $n \to \infty$ we have

$$
\liminf_{\epsilon\to 0} c(\epsilon, k) \geq c_{k^{\frac{2-p}{2}} q_M} + O(\delta) + O(\frac{1}{R}).
$$

Then sending $\delta \to 0$ (and therefore $R \to \infty$) we obtain the result. $\square$

Proof of Theorem 1.3. By using the Proposition 2.3, we obtain

$$c_d = d^{-\frac{2}{\frac{N(p-2)}{2}-2}} c_1. \tag{4.20}$$

Hence by Lemma 4.1, we have

$$\begin{aligned}
\limsup_{\epsilon \to 0} c(\epsilon, k) &\le c_{k^{\frac{2-p}{2}} q_M} \\
&= (k^{\frac{2-p}{2}} q_M)^{-\frac{2}{\frac{N(p-2)}{2}-2}} c_1 \\
&= c_1 (q_M)^{\frac{4}{2-N(p-2)}} k^{-\frac{2(p-2)}{2-N(p-2)}} \\
&< \min\left\{ c_{q_0} = c_1(q_0)^{\frac{4}{4-N(p-2)}}, c_{q_\infty} = c_1(q_\infty)^{\frac{4}{4-N(p-2)}} \right\}.
\end{aligned}$$

Then, for fixed k, exists ϵ_k, when $\epsilon < \epsilon_k$, we have $c(\epsilon, k) < \min\{c_{q_0}, c_{q_\infty}\}$. Therefore, by Lemma 4.2, for $\epsilon < \epsilon_k$, $c(\epsilon, i)$ is achieved at some u_i for $i = 1, \dots, k$. Using Lemma 4.3 we have for $i = 1, \dots, k$

$$\lim_{\epsilon \to 0} c(\epsilon, i) = c_{i^{\frac{2-p}{2}} q_M}, \tag{4.21}$$

which implies by Lemma 2.3 that these u_i for $i = 1, \dots, k$ are mutually different non-radial solutions. $\square$

Remark 4.4. From the proof of Lemma 4.3 we can see that these solutions are multi-bump type solutions. In particular, for $i = 1, 2, \dots, k$, solution u_i behaves like a normalized sum of translations of a minimizer of $c_{i^{\frac{2-p}{2}} q_M}$ at a symmetric orbit of the group action.

Remark 4.5. The result is still true for $N \ge 4$. We may use the group $G = G_k \times O(N-2)$, and consider the subspace of G-invariant functions. These functions are radially symmetric with respect to the last $N-2$ variables. Since $N - 2 \ge 2$, when we do concentration compactness analysis the concentration points stay around the two dimensional subspace of the first two variables. Therefore our arguments go through with little modifications. We omit the proof here. It would be interesting to see whether the same phenomena is still valid for the case of $N = 3$, though our minimization arguments seem to break down here.

For problems similar to (1.1) but without a constraint, we refer [3, 11, 16] for references on results of similar natures in particular [3, 16] where as a small parameter tends to zero there are more and more non-radial solutions. However for our problem (1.6) we do not know whether this would be the case, i.e., as $\epsilon \to 0$ the number of non-radial solutions tends to infinity.

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