Special Issue in honor of John W. Neuberger
Electronic Journal of Differential Equations, Special Issue 02 (2023), pp. 135–149.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

POSITIVE SOLUTIONS FOR NONLINEAR FRACTIONAL LAPLACIAN PROBLEMS

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Dedicated to the memory of Professor John W. Neuberger

Abstract. We consider a class of nonlinear fractional Laplacian problems satisfying the homogeneous Dirichlet condition on the exterior of a bounded domain. We prove the existence of a positive weak solution for classes of nonlinearities which are either sublinear or asymptotically linear at infinity. We use the method of sub-and-supersolutions to establish the results. We also provide numerical bifurcation diagrams, corresponding to the theoretical results, using the finite element method in one dimension.

1. Introduction

In this article we investigate the existence of positive solutions to reaction-diffusion problems, involving the fractional Laplacian as the diffusion operator

$$\begin{gathered}(-\Delta)^s u = \lambda f(u) \quad \text{in } \Omega; \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\,,\end{gathered} \tag{1.1}$$

with respect to the bifurcation parameter $\lambda > 0$. We will assume $\Omega \subset \mathbb{R}^N$, for $N \geq 1$, to be a bounded domain with $C^{1,1}$ boundary $\partial\Omega$ and $f : [0, +\infty) \to \mathbb{R}$ to be a Hölder continuous function. For $s \in (0, 1)$ and for a function $u : \mathbb{R}^N \to \mathbb{R}$, the fractional Laplacian is a linear operator defined pointwise for $x \in \mathbb{R}^N$ by the singular integral (see [18, pp.45],[26])

$$(-\Delta)^s u(x) := C_{N,s}\ \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\,\mathrm{d}y\,. \tag{1.2}$$

Here P.V. stands for the Cauchy principal value of the singular integral, defined as

$$\begin{gathered}\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\,\mathrm{d}y := \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\,\mathrm{d}y, \\ C_{N,s} := \frac{s2^{2s}\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(1 - s)}\end{gathered}$$

2020 *Mathematics Subject Classification.* 35J60, 35J61, 35R11.
Key words and phrases. Fractional Laplacian; sublinear; asymptotically linear; sub- and supersolution; positive weak solution.

Published March 27, 2023.

is a positive normalizing constant, with Γ the usual gamma function. For simplicity in notation, we will drop this positive constant in later sections when proving theoretical results. However, this constant is important for numerical experiments in Section 5.

Remark 1.1. There are several equivalent definitions of the fractional Laplacian in addition to the one we consider in this paper. The article [17] serves as an excellent resource on various definitions of the fractional Laplacian and their equivalence.

Recently there has been widespread interest in the study of problems involving the fractional Laplacian operator as a diffusion operator especially since the seminal paper of Caffarelli and Silvestre [10]. They showed that the fractional Laplacian operator as defined in (1.2) can be interpreted as a Dirichlet to Neumann map, effectively relating the nonlocal operator in (1.2) to a local operator. This characterization allowed them to prove several regularity results by using local techniques and provides a framework for interested researchers to further the study of the still emerging field of fractional Laplacian problems. Since then, the progress has been swift and there are already several excellent survey papers and monographs available, see [2, 5, 6, 13, 20, 23, 26, 27, 33] and references therein.

It is well known that the the existence as well as nonexistence of positive solutions of problems like (1.1), with local operators such as the classical Laplacian or p-Laplacian instead of $(-\Delta)^s$, with respect to the parameter λ, depends heavily on the behavior of the nonlinearity f near the origin as well as at infinity. For the Laplacian case ($s = 1$), see [19] for an excellent review for the case $f(0) \geq 0$, and see [9] for the case $f(0) < 0$ (*semipositone*).

In this article we establish the existence of a weak solution (to be defined) of the nonlocal problem (1.1), under suitable conditions on the nonlinearity f at infinity, using the method of sub-and supersolution. We discuss existence results of (1.1) depending only on the behavior of the nonlinearity at infinity. Our results are independent of the sign of $f(0)$.

Our first existence result reads as follows.

Theorem 1.2. *Let $f : [0, \infty) \to \mathbb{R}$ satisfy*

$$\lim_{\sigma\to+\infty} \frac{f(\sigma)}{\sigma} = 0\,, \tag{1.3}$$

$$\lim_{\sigma\to\infty} f(\sigma) = \infty\,. \tag{1.4}$$

Then, there exists $\underline{\lambda} > 0$ such that (1.1) *has a positive weak solution for $\lambda > \underline{\lambda}$.*

Examples satisfying the hypotheses of Theorem 1.2 are the reaction terms $f(\sigma) = \ln(1 + \sigma) + K$ for $K < 0$, $K = 0$, and $K > 0$, satisfying $f(0) < 0, f(0) = 0$, and $f(0) > 0$ respectively. Figure 1 gives the typical shape of the nonlinearity f satisfying (1.3) and (1.4), and the expected bifurcation diagram corresponding to these three cases. In Theorem 1.2 we establish the existence of a positive solution for each λ to the right of $\underline{\lambda}$ (indicated on each bifurcation diagram). A multi-parameter, sublinear semipositone problem was considered with pure powers in [12] to establish the existence of a positive solution. Theorem 1.2 extends their result to general semipositone nonlinearities satisfying (1.3). For the Laplacian case, the paper [22] provides a nice review of the development from the point of view of the sub- and supersolution method. The proof of Theorem 1.2 combines the ideas from [22] and [12] to construct a positive weak subsolution.

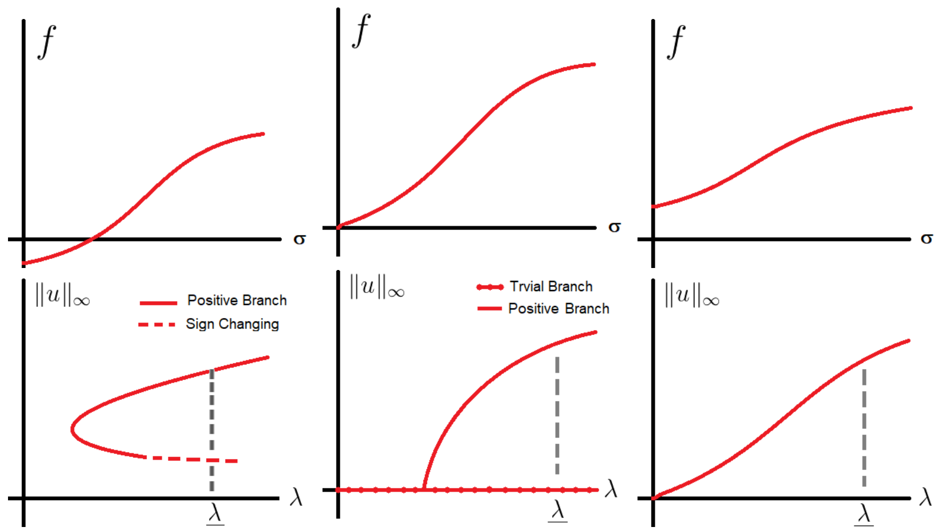

FIGURE 1. Nonlinearity and bifurcation diagrams for Theorem 1.2

Next, we consider classes of nonlinearities f that are asymptotically linear at infinity and establish the following existence result. For the purpose of stating our result, let λ_1 denote the principal eigenvalue of problem (3.5).

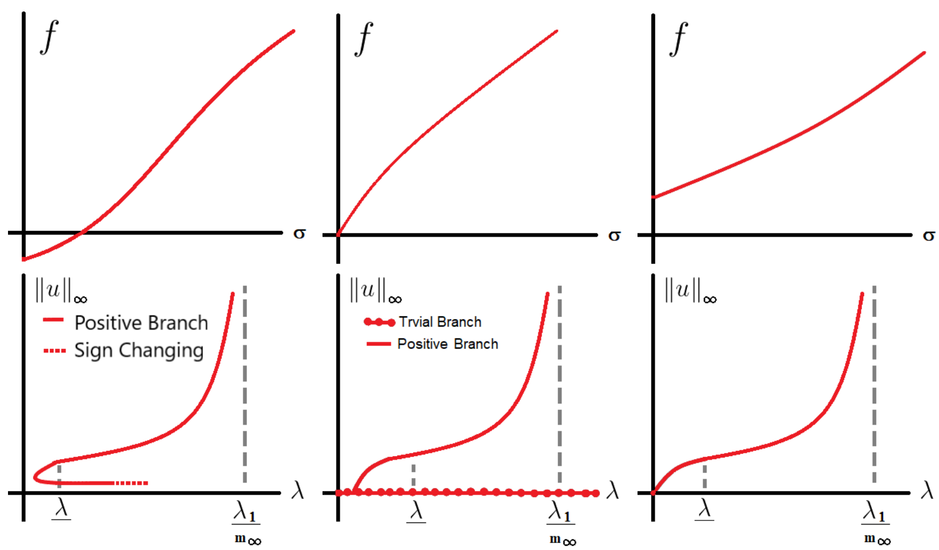

FIGURE 2. Nonlinearity and bifurcation diagrams for Theorem 1.3

Theorem 1.3. *Let $f : [0, \infty) \to \mathbb{R}$ satisfy*

$$\lim_{\sigma \to +\infty} \frac{f(\sigma)}{\sigma} = m_\infty > 0 \,. \tag{1.5}$$

Then, there exists $\underline{\lambda} > 0$ *with* $\underline{\lambda} < \frac{\lambda_1}{m_\infty}$ *such that* (1.1) *has a positive weak solution for* $\lambda \in [\underline{\lambda}, \frac{\lambda_1}{m_\infty})$.

Figure 2 gives examples of the shape of the nonlinearity f and the expected bifurcation diagram corresponding to Theorem 1.3 when f is asymptotically linear at infinity. Simple examples satisfying hypothesis of Theorem 1.3 are the reaction terms $f(\sigma) = \frac{1}{2}\sigma + 9(1+\sigma)^{\frac{1}{3}} - K$ with with $K > 10$, $K = 9$, and $K < 8$ satisfying $f(0) < 0$, $f(0) = 0$, and $f(0) > 0$ respectively. For these examples $m_\infty = 1/2$.

Using bifurcation theory, the authors in [7] discussed an existence result for the fractional Laplacian in the left neighborhood of $\frac{\lambda_1}{m_\infty}$. Our result, Theorem 1.3, extends the range of λ for existence of a positive solution farther to the left of $\frac{\lambda_1}{m_\infty}$. Existence results for such problems for the local case were discussed in [1] using bifurcation theory and in [14, 15, 16] using sub- and supersolution methods.

The rest of this article is structured as follows. In Section 2, we mention some historical development of the fractional Laplacian. Moreover, we derive a probability distribution which we use in generating and comparing random walks related to the fractional Laplacian for two choices of $s \in (0,1)$. In Section 3, we define fractional Sobolev spaces and weak solution and state a sub-and supersolution method, which we use to prove our results. In Section 4, we prove Theorems 1.2 and 1.3. In Section 5, we present one dimensional numerical experiments corresponding to our theoretical results.

2. Fractional Laplacian and random movement

In this section we demonstrate the type of random movement associated to the fractional Laplacian. We begin with the fractional heat equation and derive a probability distribution which will be used in generating random walks related to the fractional Laplacian for $N = 2$. We first recall some useful properties of the fractional Laplacian. For u from a suitable class of functions, for example the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$, one has (see [4, 11, 30])

$$\lim_{s\to 1^-} (-\Delta)^s\, u = -\Delta\, u \quad \text{and} \quad \lim_{s\to 0^+} (-\Delta)^s\, u = I\,,$$

where I is the identity operator. This connection can be seen by considering the Fourier transform of the operator (see [26, 34] for more details). Indeed, recall that the Fourier transform and the inverse Fourier transform in $\mathbb{R}^N$ are

$$\mathscr{F}[u](k) = \widehat{u}(k) := \int_{\mathbb{R}^N} u(x)e^{2\pi i\langle x,k\rangle}dx\,, \quad \mathscr{F}^{-1}[w](x) := \int_{\mathbb{R}^N} w(x)e^{-i2\pi\langle x,k\rangle}\,\mathrm{d}k\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^N$. For a smooth function $u : \mathbb{R}^N \to \mathbb{R}$, $(-\Delta)^s$ satisfies (see [13, Prop. 5.1])

$$\mathscr{F}[(-\Delta)^s u](k) = (2\pi|k|)^{2s}\mathscr{F}[u](k)\,. \tag{2.1}$$

Hence, with respect to the Fourier transform, the fractional Laplacian acts as the multiplication by the symbol (multiplier) $|k|^{2s}$.

Next we mention some of the historical development and demonstrate the type of random movement associated to the fractional Laplacian. The following derivation, for $s = \frac{1}{3}$ and $N = 2$, was carried out in [24] in 1955 without the use of the modern notation $(-\Delta)^s$, see [32] for more details. Consider the initial value problem

$$\frac{\partial u}{\partial t} + (-\Delta)^s u = 0 \text{ in } \mathbb{R}^2 \times [0,\infty) \quad \text{with } u(x,0) = \delta_x\,, \tag{2.2}$$

where δ_x is the Dirac-delta function. By computing the Fourier transform of (2.2), we obtain

$$\frac{\partial \widehat{u}}{\partial t} = -(2\pi|k|)^{2s}\widehat{u}(k,t) \quad \text{with } \widehat{u}(k,0) = \widehat{\delta_x} = e^{i\langle k,0\rangle} = 1\,, \tag{2.3}$$

where $\widehat{f}$ denotes the Fourier transform of f. Then, (2.3) defines an ordinary differential equation in the t variable with solution

$$\widehat{u}(k,t) = e^{-(2\pi|k|)^{2s}t}\,.$$

Using the inverse Fourier transform, we obtain the solution of (2.2)

$$u(x,t) = \int_{\mathbb{R}^2} e^{-i2\pi\langle k,x\rangle} e^{-(2\pi|k|)^{2s}t}\,\mathrm{d}k\,, \tag{2.4}$$

which is a Lévy distribution with stretching factor t. To simplify the expression (2.4), we note that $b(k) := e^{-(2\pi|k|)^{2s}}$ is a radial function. The inverse Fourier transform of the radial function b is given by (see [13, Thm. 4.4])

$$\mathscr{F}^{-1}[b](x) = 2\pi\int_0^\infty b(\rho)J_0(2\pi|x|\rho)\,\mathrm{d}\rho\,,$$

where J_0 is the Bessel function of order zero. Then, setting the stretching factor $t = 1$, (2.4) has the form

$$u(x,1) = \int_{\mathbb{R}^2} e^{-i2\pi\langle k,x\rangle} e^{-(2\pi|k|)^{2s}}\,\mathrm{d}k = 2\pi\int_{\mathbb{R}^2} e^{-(2\pi\rho)^{2s}} J_0(2\pi|x|\rho)\rho\,\mathrm{d}\rho\,. \tag{2.5}$$

Using polar coordinates $x = (r\cos(\theta'), r\sin(\theta'))$, (2.5) becomes

$$u(r,1) = 2\pi\int_0^\infty e^{-(2\pi\rho)^{2s}} J_0(2\pi\rho r)\rho\,\mathrm{d}\rho\,. \tag{2.6}$$

The expression (2.6) defines a Lévy distribution which is a type of heavy-tailed probability distribution, that is, a probability distribution whose tail is not bounded above by an exponential distribution. Now we are ready to describe how to generate a random walk using (2.6). Given a sequence of vectors $\{(x_k, y_k)\}$ we define

$$(x_{k+1}, y_{k+1}) := (x_k, y_k) + r_k(\cos(\theta_k), \sin(\theta_k))$$

At each step choose the direction $\theta_k \in [0, 2\pi)$ with a uniform distribution, then we choose the length of jump $r_k \in (0,\infty)$ by numerical integration and sampling from the distribution given by (2.6). See Figure 3 for two realizations of random walks with $s = 3/8$ and $s = 7/8$.

We observe that the random movement for $s = 7/8 \approx 1$ resembles Brownian movement compared to $s = 3/8$ which has occasional long jumps, a characteristic associated with the fractional Laplacian. For more on Lévy distributions and their connection to the fractional Laplacian, see [20].

Remark 2.1. The derivation of (2.6) above gives a probability distribution only when $s \in (0,1)$. Moreover, the fractional Laplacian operator defined by (1.2) is the operator satisfying (2.1) only when $s \in (0,1)$, see [31]. However, when $s > 1$, the operator satisfying (2.1) is a hypersingular integral, see [29].

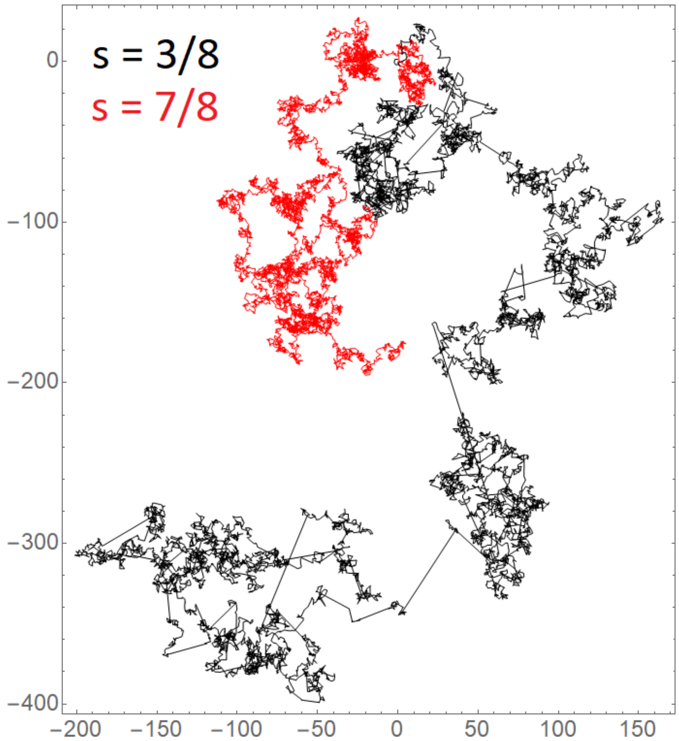

Figure 3. Comparison of Random Walks With $s = 3/8$ and $7/8$

3. Preliminaries

In this section we define function spaces, weak solution, and weak sub-and supersolution. We also state a sub-and supersolution result. Finally, we discuss two auxiliary problems whose positive solutions are used in the construction of positive weak sub-and supersolutions.

3.1. **Function spaces and solutions.** For a fixed $s \in (0,1)$, let

$$H^s(\mathbb{R}^N) := \{w \in L^2(\mathbb{R}^N) : \|w\|_{H^s(\mathbb{R}^N)} < +\infty\},$$

where $\|w\|_{H^s(\mathbb{R}^N)} := \big(\|w\|^2_{L^2(\mathbb{R}^N)} + [w]^2_{H^s(\mathbb{R}^N)}\big)^{1/2}$ and

$$[w]_{H^s(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/2}$$

is the *Gagliardo seminorm* of w. Then, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is a Hilbert space with respect to the inner product

$$\langle v, w\rangle_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} vw\,\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v(x)-v(y)][w(x)-w(y)]}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

Further, the fractional Sobolev space $H^s_0(\Omega) := \{w \in H^s(\mathbb{R}^N) : w \equiv 0 \text{ a.e. } \mathbb{R}^N \setminus \Omega\}$ is also a Hilbert space with respect to the inner product

$$\langle v, w\rangle_{H^s_0(\Omega)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v(x)-v(y)]\,[w(x)-w(y)]}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

See [11, 23] for more on these spaces. We will use the following equivalence of the inner product $\langle\cdot,\cdot\rangle_{H_0^s(\Omega)}$ and the fractional Laplacian defined by (1.2).

Definition 3.1. We say that a function $u \in H_0^s(\Omega)$ is a *weak solution* of (1.1) if for all $\phi \in H_0^s(\Omega)$, it satisfies the integral identity

$$\mathcal{E}(u,\phi) = \lambda \int_\Omega f(u)\phi(x)\,\mathrm{d}x\,,$$

where $\mathcal{E}(u,\phi) := \langle u,\phi\rangle_{H_0^s(\Omega)}$.

Definition 3.2. A function $\overline{u} \in H^s(\mathbb{R}^N)$ is called a *weak supersolution* of (1.1) if, for all $\phi \in H_0^s(\Omega)$ with $0 \le \phi$ in Ω, the following inequalities hold:

$$\mathcal{E}(\overline{u},\phi) \ge \lambda \int_\Omega f(\overline{u}(x))\,\phi(x)\,\mathrm{d}x \tag{3.1}$$

$$\overline{u} \ge 0 \quad \text{a.e. in } \mathbb{R}^N \setminus \Omega\,. \tag{3.2}$$

A function $\underline{u} \in H^s(\mathbb{R}^N)$ is called a *weak subsolution* of (1.1) if the inequalities are reversed in (3.1) and (3.2).

Finally, we state the sub-and supersolution result which we employ to prove Theorems 1.2 and 1.3.

Proposition 3.3 ([8]). *Suppose f is a Hölder continuous function. Let $\underline{u}$ and $\overline{u} \in H^s(\mathbb{R}^N) \cap L^\infty(\Omega)$ be a weak subsolution and weak supersolution, respectively, of* (1.1) *satisfying $\underline{u} \le \overline{u}$ a.e. in Ω. Then, there exists a weak solution $u \in H_0^s(\Omega)$ to* (1.1) *satisfying $\underline{u} \le u \le \overline{u}$ a.e. in Ω.*

3.2. **Auxiliary problems.** Here we introduce the problems whose positive solutions are used in the construction of sub-and supersolutions. First, Consider the following fractional linear problem

$$\begin{aligned} &(-\Delta)^s e = 1 \quad \text{in } \Omega;\\ &e = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\,. \end{aligned} \tag{3.3}$$

There exists a unique weak solution $e \in H_0^s(\Omega)$ of (3.3) such that $e > 0$ a.e. in Ω, see [21, Thm. 12] for $N \ge 2$, and for $N = 1$ the explicit formula of the solution is given in [28, eqn. (1.4)]. Moreover, it follows from [27, Lem. 7.3] and [28, Thm. 1.2] that there exist $c_1, c_2 > 0$ such that

$$c_1\delta^s(x) \le e(x) \le c_2\delta^s(x) \quad \text{a.e. in } \Omega, \tag{3.4}$$

where $\delta(x)$ is the distance function to the boundary $\partial\Omega$. Solutions of (3.3) can have at most $C^s(\overline{\Omega})$ regularity in the bounded domain. Indeed, in the unit ball, the explicit solution of (3.3) is given by a positive constant multiple of $e := (1-|x|^2)^s$, see [28].

Next, consider the fractional Laplacian eigenvalue problem

$$\begin{aligned} &(-\Delta)^s \varphi = \lambda\varphi \quad \text{in } \Omega;\\ &\varphi = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\,. \end{aligned} \tag{3.5}$$

It is known that (3.5) has a simple eigenvalue $\lambda_1 > 0$ and a corresponding positive eigenfunction $\varphi_1 \in H_0^s(\Omega)$, see [23, Prop 3.1 & Cor. 4.8]. Moreover, it follows from [27, Lem. 7.3] and [28, Thm. 1.2] that there exist $d_1\,, d_2 > 0$ such that

$$d_1\delta^s(x) \le \varphi_1(x) \le d_2\delta^s(x) \quad \text{a.e. in } \Omega\,. \tag{3.6}$$

The following estimate involving the positive eigenfunction φ_1, established in [12], is crucial in the construction of a positive weak subsolution.

Proposition 3.4 ([12]). *Let $\varphi_1 > 0$ be the eigenfunction corresponding to the principle eigenvalue λ_1 of the eigenvalue problem* (3.5)*. Then, there exists $\gamma > 0$ such that $\gamma < h(x) < +\infty$ for all $x \in \Omega$, where*

$$h(x) := \int_{\mathbb{R}^N} \frac{[\varphi_1(x) - \varphi_1(y)]^2}{|x-y|^{N+2s}} \, \mathrm{d}y \,,$$

4. Proofs of Theorems 1.2 and T1.3

Here we prove Theorem 1.2 and Theorem 1.3 by employing Proposition 3.3. We construct an ordered pair of positive weak sub-and supersolutions.

In both cases, we construct a positive weak subsolution as a multiple of φ_1^2, where $0 < \varphi_1 \in H_0^s(\Omega)$ is the eigenfunction corresponding to the principle eigenvalue λ_1 of the eigenvalue problem (3.5). It follows from [12] that φ_1^2 satisfies

$$\begin{aligned}
(-\Delta)^s \varphi_1^2(x) &= \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\varphi_1^2(x) - \varphi_1^2(y)}{|x-y|^{N+2s}} \, \mathrm{d}y \\
&= \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{[\varphi_1(x) + \varphi_1(y)][\varphi_1(x) - \varphi_1(y)]}{|x-y|^{N+2s}} \, \mathrm{d}y \\
&= 2\varphi_1(x) \, \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\varphi_1(x) - \varphi_1(y)}{|x-y|^{N+2s}} \, \mathrm{d}y - \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{[\varphi_1(x) - \varphi_1(y)]^2}{|x-y|^{N+2s}} \, \mathrm{d}y \\
&= 2\varphi_1(x)(-\Delta)^s \varphi_1(x) - \mathrm{P.V.} \, h(x) \,.
\end{aligned}$$

Moreover, by Proposition 3.4, $\mathrm{P.V.}\ h(x) = h(x)$ in Ω. Hence

$$(-\Delta)^s \varphi_1^2(x) = 2\lambda_1 \varphi_1^2(x) - h(x) \quad \text{in } \Omega \,,$$

and for all $\phi \in H_0^s(\Omega)$, it holds ([12, Lem. 3.1])

$$\mathcal{E}(\varphi_1^2, \phi) = \int_\Omega \{2\lambda_1 \varphi_1^2(x) - h(x)\} \phi(x) \, \mathrm{d}x \,.$$

Without loss of generality, assume $\|\varphi_1\|_\infty = 1$. Then, it follows from Proposition 3.4, using $\varphi_1 = 0$ in $\mathbb{R}^N \setminus \Omega$, that there exist $\eta, \beta, \mu > 0$ such that

$$\beta < h(x) - 2\lambda_1 \varphi_1^2(x) \quad \text{in } \Omega_\eta \,, \tag{4.1}$$

$$\mu \le \varphi_1 \le 1 \quad \text{in } \Omega \setminus \overline{\Omega}_\eta \,, \tag{4.2}$$

where $\Omega_\eta := \{x \in \Omega : \delta(x) < \eta\}$.

Proof of Theorem 1.2. We first construct a positive subsolution. Since f is continuous on $[0, +\infty)$ and satisfies (1.4), there exists $b_0 > 0$ such that

$$f(\sigma) \ge -b_0 \quad \text{for all } \sigma \ge 0. \tag{4.3}$$

For $\lambda > 0$, let $\underline{u} := \frac{b_0 \lambda}{\beta} \varphi_1^2 \in H_0^s(\Omega)$. Then, for every $\phi \in H_0^s(\Omega)$, it holds

$$\mathcal{E}(\overline{u}, \phi) = \frac{b_0 \lambda}{\beta} \mathcal{E}(\varphi_1^2, \phi) = \frac{b_0 \lambda}{\beta} \int_\Omega \{2\lambda_1 \varphi_1^2(x) - h(x)\} \phi(x) \, \mathrm{d}x \,.$$

Thus, $\underline{u}$ is a weak subsolution of (1.1) if

$$\frac{b_0 \lambda}{\beta} \int_\Omega \{2\lambda_1 \varphi_1^2(x) - h(x)\} \phi(x) \, \mathrm{d}x \le \lambda \int_\Omega f\Big(\frac{b_0 \lambda}{\beta} \varphi_1^2(x)\Big) \phi(x) \, \mathrm{d}x \tag{4.4}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \le \phi$ in Ω. We split the analysis into two cases: $x \in \Omega_\eta$ and $x \in \Omega \setminus \overline{\Omega}_\eta$. If $x \in \Omega_\eta$, by (4.1) and (4.3), it holds

$$\begin{aligned}
\frac{b_0\lambda}{\beta}\int_{\Omega_\eta}\{2\lambda_1\varphi_1^2(x) - h(x)\}\phi(x)\,\mathrm{d}x &= -\frac{b_0\lambda}{\beta}\int_{\Omega_\eta}\{h(x) - 2\lambda_1\varphi_1^2(x)\}\phi(x)\,\mathrm{d}x \\
&< \lambda\int_{\Omega_\eta} -b_0\phi(x)\,\mathrm{d}x \\
&\le \lambda\int_{\Omega_\eta} f\Big(\frac{b_0\lambda}{\beta}\varphi_1^2(x)\Big)\phi(x)\,\mathrm{d}x \\
&= \lambda\int_{\Omega_\eta} f(\underline{u}(x))\phi(x)\,\mathrm{d}x
\end{aligned} \tag{4.5}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \le \phi$ in Ω. Now let $x \in \Omega\setminus\overline{\Omega}_\eta$. Then, using (4.2), it follows from the hypothesis (1.4) that for $\lambda \gg 1$, we have

$$\frac{2b_0\lambda_1}{\beta} \le f\Big(\frac{b_0\lambda}{\beta}\varphi_1^2\Big)\,.$$

Therefore, using that $h(x) > \gamma > 0$, it follows that

$$\begin{aligned}
\frac{b_0\lambda}{\beta}\int_{\Omega\setminus\overline{\Omega}_\eta}\{2\lambda_1\varphi_1^2(x) - h(x)\}\phi(x)\,\mathrm{d}x &\le \frac{2b_0\lambda\lambda_1}{\beta}\int_{\Omega\setminus\overline{\Omega}_\eta}\varphi_1^2(x)\phi(x)\,\mathrm{d}x \\
&\le \lambda\int_{\Omega\setminus\overline{\Omega}_\eta} f\Big(\frac{b_0\lambda}{\beta}\varphi_1^2(x)\Big)\phi(x)\,\mathrm{d}x \\
&= \lambda\int_{\Omega\setminus\overline{\Omega}_\eta} f(\underline{u}(x))\phi(x)\,\mathrm{d}x
\end{aligned} \tag{4.6}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \le \phi$ in Ω. Combining (4.5) and (4.6), it follows that (4.4) holds. Therefore, $\underline{u} = \frac{b_0\lambda}{\beta}\varphi_1^2$ is a positive weak subsolution of (1.1) for λ sufficiently large.

Next, we show there exists $M_\lambda > 0$ such that $\overline{u} := Me$ is a weak supersolution of (1.1) for all $M \ge M_\lambda$, where $0 < e \in H_0^s(\Omega)$ is the weak solution of (3.3). We observe that while f is not assumed to be nondecreasing, $\overline{f}(t) := \max_{\sigma\in[0,t]} f(\sigma)$ is nondecreasing. Moreover, $f(t) \le \overline{f}(t)$ for all $t \ge 0$, and $\overline{f}$ satisfies the sublinear condition at infinity

$$\lim_{t\to+\infty}\frac{\overline{f}(t)}{t} = 0\,. \tag{4.7}$$

Therefore, since $\overline{f}$ satisfies (4.7), there exists $M_\lambda > 0$ sufficiently large such that for all $M \ge M_\lambda$,

$$\frac{\overline{f}(M\|e\|_\infty)}{M\|e\|_\infty} \le \frac{1}{\lambda\|e\|_\infty} \quad \text{or equivalently} \quad \lambda\overline{f}(M\|e\|_\infty) \le M\,.$$

Therefore, with $M \ge M_\lambda$, we obtain $\overline{u} = Me \in H_0^s(\Omega)$ satisfies

$$\begin{aligned}
\mathcal{E}(\overline{u},\phi) &= M\mathcal{E}(e,\phi) \\
&= M\int_\Omega \phi(x)\,\mathrm{d}x \\
&\ge \lambda\int_\Omega \overline{f}(M\|e\|_\infty)\phi(x)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\geq \lambda \int_\Omega \overline{f}(Me(x))\phi(x)\,\mathrm{d}x \\
&\geq \lambda \int_\Omega f(Me(x))\phi(x)\,\mathrm{d}x \\
&= \lambda \int_\Omega f(\overline{u})\phi(x)\,\mathrm{d}x\,,
\end{aligned}
$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leq \phi$ in Ω. Hence, $\overline{u} := Me$ is a weak supersolution of (1.1) for $M \geq M_\lambda$.

Finally, using the right estimate in (3.6), the left estimate in (3.4), and taking M larger, if necessary, we obtain

$$\underline{u} = \frac{b_0\lambda}{\beta}\varphi_1^2 \leq Me = \overline{u} \text{ a.e. in } \Omega\,.$$

Hence, by Proposition 3.3, (1.1) has a positive weak solution u such that $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω for λ sufficiently large. This completes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.3. As in the proof of Theorem 1.2, we will show that a suitable positive constant multiple of φ_1^2 serves as a positive weak subsolution of (1.1). Since f is continuous on $[0,\infty)$ and satisfies (1.5), there exist $\sigma_0, b_1 > 0$ and $m_1 > 2m_\infty$ such that

$$f(\sigma) \geq m_1\sigma - b_1 \quad \text{for all } 0 \leq \sigma \leq \sigma_0. \tag{4.8}$$

Assume $\lambda \in [\underline{\lambda}, \frac{\lambda_1}{m_\infty})$, where $\underline{\lambda} := \frac{2\lambda_1}{m_1}$. Note that since $m_1 > 2m_\infty$, one has $\underline{\lambda} < \frac{\lambda_1}{m_\infty}$. Let $\underline{u} := k_0\varphi_1^2$, where k_0 satisfies

$$k_0 > \frac{\lambda_1 b_1}{m_\infty \gamma}\,, \tag{4.9}$$

with γ as in Proposition 3.4. Then, for all $\phi \in H_0^s(\Omega)$, it holds

$$\mathcal{E}(\underline{u},\phi) = k_0\mathcal{E}(\varphi_1^2,\phi) = \int_\Omega \{2\lambda_1 k_0\varphi_1^2(x) - k_0h(x)\}\phi(x)\,\mathrm{d}x\,.$$

Then, setting $\sigma_0 := k_0$, it follows from (4.8) that $\underline{u} = k_0\varphi_1^2$ is a weak subsolution if we have

$$\int_\Omega \{2\lambda_1\, k_0\, \varphi_1^2(x) - k_0\, h(x)\}\phi(x)\,\mathrm{d}x \leq \lambda \int_\Omega \{m_1k_0\varphi_1^2(x) - b_1\}\phi(x)\,\mathrm{d}x \tag{4.10}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leq \phi$ in Ω. If $\lambda \geq \frac{2\lambda_1}{m_1}$, then

$$2\lambda_1 k_0\varphi_1^2(x) \leq \lambda m_1 k_0 \varphi_1^2(x) \quad \text{for a.e. } x \in \Omega\,. \tag{4.11}$$

On the other hand, for $\lambda < \frac{\lambda_1}{m_\infty}$, it follows from the choice of k_0 in (4.9) that

$$\lambda b_1 < \frac{\lambda_1 b_1}{m_\infty} < k_0\gamma < k_0 h(x) \quad \text{for a.e. } x \in \Omega\,. \tag{4.12}$$

Then, it follows from (4.11) and (4.12) that inequality (4.10) holds for $\lambda \in [\frac{2\lambda_1}{m_1}, \frac{\lambda_1}{m_\infty})$. Hence $\underline{u} = k_0\varphi_1^2$ is a positive weak subsolution for $\lambda \in [\frac{2\lambda_1}{m_1}, \frac{\lambda_1}{m_\infty})$.

Next, we construct a supersolution for $\lambda < \frac{\lambda_1}{m_\infty}$. Let $\epsilon > 0$ be such that $\lambda_1 > \lambda(m_\infty + \epsilon)$. Since f is continuous on $[0,+\infty)$ and satisfies (1.4), there exists $L > 0$ such that

$$f(\sigma) \leq (m_\infty + \epsilon)\sigma + L \quad \text{for all } \sigma \geq 0. \tag{4.13}$$

Since e and φ_1 satisfy the estimates (3.4) and (3.6), respectively, there exists $c > 0$ such that $e \leq c\varphi_1$ in Ω. Let $\overline{u} := M\varphi_1 + \lambda Le$, where $M \geq M_\lambda := \frac{\lambda^2 cL(m_\infty+\epsilon)}{\lambda_1-\lambda(m_\infty+\epsilon)}$. Then

$$\mathcal{E}(\overline{u},\phi) = M\mathcal{E}(\varphi_1,\phi) + \lambda L\mathcal{E}(e,\phi) = \int_\Omega [M\lambda_1\varphi_1(x) + \lambda L]\phi(x)\,\mathrm{d}x$$

for all $\phi \in H_0^s(\Omega)$. Therefore, by using (4.13) and the choices of M and c, it follows that

$$\begin{aligned}
\lambda\int_\Omega f(\overline{u}(x))\phi(x)\,\mathrm{d}x &\leq \lambda\int_\Omega [L + (m_\infty+\epsilon)\overline{u}(x)]\phi(x)\,\mathrm{d}x \\
&= \lambda\int_\Omega \left[L + (m_\infty+\epsilon)(M\varphi_1(x) + \lambda Le(x))\right]\phi(x)\,\mathrm{d}x \\
&\leq \lambda\int_\Omega \left[L + (m_\infty+\epsilon)(M\varphi_1(x) + \lambda Lc\varphi_1(x))\right]\phi(x)\,\mathrm{d}x \\
&= \int_\Omega \left[\lambda L + M\lambda(m_\infty+\epsilon) + \lambda^2 Lc(m_\infty+\epsilon)\right]\varphi_1(x)\phi(x)\,\mathrm{d}x \\
&\leq \int_\Omega [\lambda L + M\lambda_1\varphi_1(x)]\phi(x)\,\mathrm{d}x
\end{aligned}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leq \phi$ in Ω. Hence, $\overline{u}$ is a weak supersolution for $\lambda < \frac{\lambda_1}{m_\infty}$. Finally, using the right estimate in (3.6), the left estimate in (3.4), and taking M larger, if necessary, we obtain

$$\underline{u} = k_0\varphi_1^2 \leq M\varphi_1 + Le = \overline{u} \quad \text{a.e. in } \Omega\,.$$

Hence, by Proposition 3.3, (1.1) has a positive weak solution u such that $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω for $\lambda \in [\underline{\lambda}, \frac{\lambda_1}{m_\infty})$. This completes the proof. □

5. Numerical experiments

Here we investigate positive numerical weak solutions of the nonlinear fractional Laplacian problems

$$\begin{gathered}
(-\Delta)^s u = \lambda f(u) \quad \text{in } \Omega; \\
u = 0 \quad \text{in } \mathbb{R}\setminus\Omega\,,
\end{gathered} \tag{5.1}$$

for $s \in (0,1)$ with a bounded open set $\Omega = (0,1) \subset \mathbb{R}$ ($N = 1$) and Hölder continuous nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfying the hypotheses of Theorem 1.2 or Theorem 1.3.

We use the finite element method (FEM) developed for linear fractional Laplacian problems of in [3] to construct positive numerical solutions u of problem (5.1). Moreover, using the branch following technique of [25], we construct bifurcation diagrams $\|u\|_\infty$ vs. λ. We also present profiles of positive solutions obtained for various choices of $s \in (0,1)$.

As in [3], we use the weak formulation of (5.1) to seek solution $u \in H_0^s(\Omega)$ such that

$$\frac{C_{N,s}}{2}\mathcal{E}(u,\phi) = \lambda\int_\Omega f(u)\phi dx \text{ for all } \phi \in H_0^s(\Omega)\,.$$

We first describe the approximation method when $\Omega := (0,1) \subset \mathbb{R}$. Fix a uniform partition $0 = x_0 < x_1 < x_2 \ldots < x_{n+1} = 1$ of $[0,1]$ with step size $h = x_i - x_{i-1}$

for $i = 1, \dots, n+1$. Let V_h be an n-dimensional subset of $H_0^s(0,1)$ spanned by $\{\phi_1, \dots, \phi_n\}$, where

$$\phi_i(x) := \begin{cases} 1 - |x - x_i|/h & \text{if } x \in [x_{i-1}, x_{i+1}]\,, \\ 0 & \text{if } x \in \mathbb{R} \setminus [x_{i-1}, x_{i+1}] \end{cases}$$

for $i = 1, \dots, n$. The finite element approximation $u_h \in V_h$ for a weak solution $u \in H_0^s(0,1)$ of (5.1) is expressed as

$$u_h(x) := \sum_{i=1}^{n} u_i \phi_i(x)\,,$$

where $u_i \in \mathbb{R}$ are unknowns and u_h satisfies system of n equations

$$\frac{C_{1,s}}{2}\mathcal{E}(u_h, \phi) = \lambda \int_0^1 f\left(x, u_h(x)\right) \phi_j(x)\,\mathrm{d}x \tag{5.2}$$

for all $j = 1, \dots, n$. To implement the finite element scheme, we express (5.2) in matrix notation.

For a column vector $\mathbf{u} := [u_1, \dots, u_n]^T$, the left hand side of (5.2) can be expressed as $\mathcal{A}\mathbf{u}$, where $\mathcal{A}$ is the $n \times n$ stiffness matrix corresponding to the left hand side of (5.2) derived in [3]. Then, defining the column vector $\mathbf{F}$ by

$$\mathbf{F}(\mathbf{u}) := h[f(x_1, u_1), f(x_2, u_2), \dots, f(x_n, u_n)]^T\,,$$

we rewrite (5.2) as a matrix equation

$$\mathcal{A}\mathbf{u} = \lambda \mathbf{F}(\mathbf{u})\,. \tag{5.3}$$

We solve the system (5.3) for a given nonlinearity f and $\lambda > 0$ with Newton's method, provided a suitable initial guess for the iteration. A multiple of the solution of the linear problem $(-\Delta)^s e = 1$ in $(0,1)$ with $u = 0$ in $\mathbb{R}\setminus(0,1)$ is a good candidate for an initial guess in many cases. The pseudo-code for constructing numerical solutions and numerical bifurcation diagrams can be found in [8, p. 15-16].

5.1. **Numerical experiments corresponding to Theorem 1.2.** Consider $f(\sigma) = \ln(1+\sigma) + K$ for $\sigma \geq 0$ satisfying the hypotheses of Theorem 1.2. In Figure 4 (A), (B), and (C) we give a plot of the nonlinearity satisfying the cases $f(0) < 0$, $f(0) = 0$, and $f(0) > 0$ respectively. The bifurcation diagrams corresponding to the nonlinearities in Figure 4 (A), (B), and (C) are given in Figure 4 (D), (E), and (F), respectively for the specific choices of $s \in (0,1)$ indicated. We note that these bifurcation diagrams can be numerically constructed for any $s \in (0,1)$, and that they will be qualitatively similar to those shown in Figure 4 (D), (E), and (F). The inset of Figure 4 (D), (E), and (F) shows the numerical positive solution(s) for the $\lambda = 30$. Since f is sublinear at infinity, Theorem 1.2 guarantees a positive solution for λ sufficiently large. However, the solution set $\mathcal{S}$ is not necessarily monotone with respect to λ. In particular, Figure 4 (D), for the case $f(0) < 0$, there is a range of λ, depending on s, for which two positive solutions exist. For each case, the numerical experiment suggests that positive solution for λ sufficiently large must be unique. To the best of our knowledge, this has not been investigated theoretically.

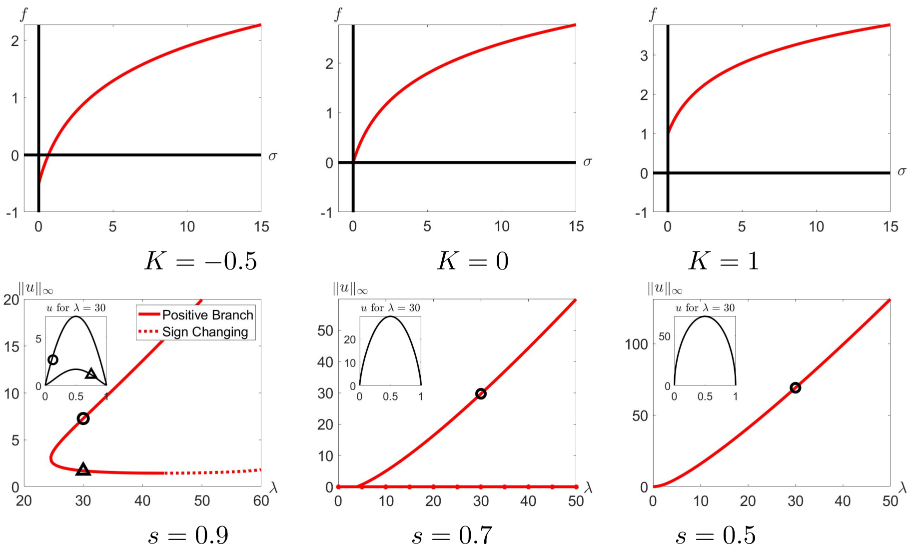

FIGURE 4. Nonlinearity $f(\sigma) = \ln(1+\sigma) + K$ and bifurcation diagrams for Theorem 1.2

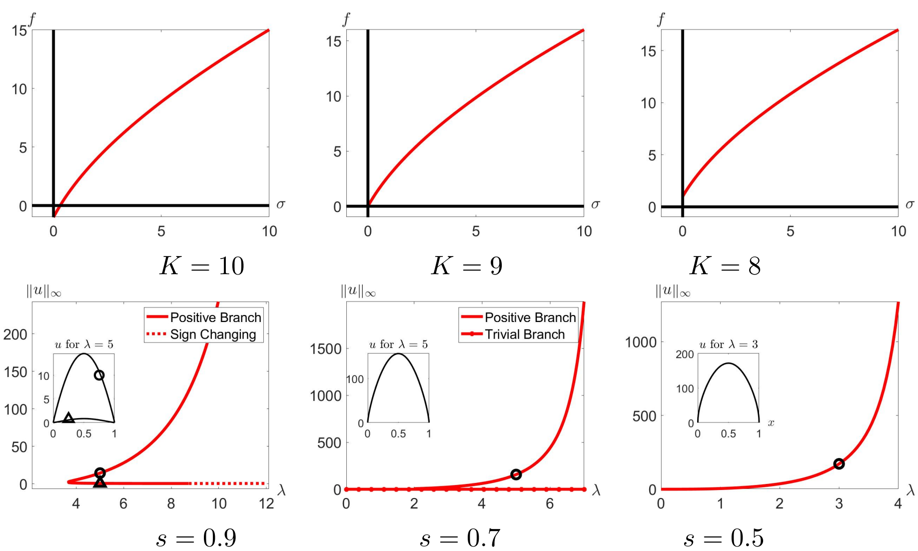

FIGURE 5. Nonlinearity $f(\sigma) = 0.5\sigma + 9\sqrt[3]{1+\sigma} - K$ and bifurcation diagrams for Theorem 1.3

5.2. **Numerical experiments corresponding to Theorem 1.3.** Consider $f(\sigma) = 0.5\sigma + 9\sqrt[3]{1+\sigma} + K$ for $\sigma \geq 0$ satisfying the hypothesis of Theorem 1.3. In Figure 5 (A), (B), and (C) we give a plot of the nonlinearity satisfying the cases $f(0) < 0$, $f(0) = 0$, and $f(0) > 0$ respectively. The bifurcation diagrams corresponding to the nonlinearities in Figure 5 (A), (B), and (C) are given in Figure 5 (D), (E), and (F), respectively for the specific choices of $s \in (0,1)$ indicated. The inset of Figure 5 (D), (E), and (F) shows the numerical positive solution(s) for the choice of λ given. Since f is asymptotically linear at infinity, Theorem 1.3 guarantees the existence of a range of λ for which there exists a positive solution. We see that depending on the behavior of the nonlinearity at zero, there may exists a positive numerical solution on an interval bounded away from zero and to the left of $\frac{\lambda_1}{m_\infty}$. Because of the dependence of λ_1 on s, we can see the interval for which there exists a positive numerical solution shifts to the left as $s \in (0,1)$ decreases. For each case, the numerical experiments suggests that the positive solution for λ sufficiently close to $\frac{\lambda_1}{m_\infty}$ must be unique.

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