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OPTIMAL CONDITIONS FOR THE MAXIMUM PRINCIPLE FOR SECOND-ORDER PERIODIC PROBLEMS

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Abstract. We provide alternate necessary and/or sufficient conditions on the sign-changing coefficient $p(t)$ for the maximum principle for the second-order periodic problem $u'' = p(t)u + q(t)$ to hold, i.e., for nonnegative q to yield a nonpositive periodic solution u.

1. Introduction and problem formulation

We consider the linear second-order periodic problem

$$u'' = p(t)u + q(t), \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{1.1}$$

and provide new answers to the fundamental question

For which p a nonnegative q results in a nonpositive u?

We can find several necessary and/or sufficient conditions on p in the extensive relevant literature. See for example [2, 6, 10] and references therein. Optimality and especially applicability and verifiability of these conditions are crucial for further studies of related nonlinear problems (see, e.g., [7]).

Inspired by our previous results concerning the fourth-order problems [3, 4, 5], we state alternative series of (optimal and/or verifiable) conditions on p based on the principal weighted eigenvalue of the corresponding linear operator. For details, see our main results in Section 2 and their consequences in Section 3. In Section 4 we present two examples and comparison with known conditions.

Since our text is mainly based on [6], we keep the same notation and terminology as much as possible. Throughout this article, we consider $p, q \in L_\omega$ which is the space of ω-periodic functions that are Lebesgue integrable on $(0, \omega)$. By a solution to (1.1) we mean a differentiable function u with absolutely continuous derivative in $[0, \omega]$ that satisfies (1.1) almost everywhere on $[0, \omega]$. We let C_ω denote the space of ω-periodic continuous functions with norm given by $\|u\| = \max_{t\in[0,\omega]} |u(t)|$.

Definition 1.1 (Lomtatidze [6])**.** We say that the function $p \in L_\omega$ belongs to the set $\mathcal{V}^-(\omega)$ if for any differentiable function u with absolutely continuous derivative on $[0, \omega]$, and satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a.e. } t \in [0, \omega], \quad u(0) = u(\omega),\ u'(0) = u'(\omega),$$

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the inequality $u(t) \le 0$ holds for all $t \in [0,\omega]$.

In fact (see [6]), if $p \in \mathcal{V}^-(\omega)$, then for any $q \in L_\omega$, $q \not\equiv 0$, $q(t) \ge 0$ a.e. on $[0,\omega]$, the periodic problem (1.1) possesses a unique solution u satisfying $u(t) < 0$ on $[0,\omega]$.

Let us note that $p \in \mathcal{V}^-(\omega)$ is equivalent to the statement that the *maximum principle* is satisfied by (1.1), or that the linear periodic operator $u \mapsto -u'' + p(t)u$ is (strictly) *inverse positive.* The following basic properties of $\mathcal{V}^-(\omega)$ can be found in [6]:

(i) If $p(t) \ge 0$ for a.e. $t \in [0,\omega]$ and $p \not\equiv 0$, then $p \in \mathcal{V}^-(\omega)$.
(ii) If $p_0(t) \in \mathcal{V}^-(\omega)$ and $p(t) \ge p_0(t)$ for a.e. $t \in [0,\omega]$, then $p \in \mathcal{V}^-(\omega)$ as well.
(iii) The set $\mathcal{V}^-(\omega)$ is unbounded, open and convex.
(iv) If $p \in \mathcal{V}^-(\omega)$, then $\int_0^\omega p(t)\,\mathrm{d}t > 0$.

Because of conditions (i) and (iv), it makes sense to consider only sign-changing functions p for further study.

2. Main results

Theorem 2.1. *Let $p \in L_\omega$ with $p(t) = p_1(t) - p_2(t)$, $p_i(t) \ge 0$ for a.e. $t \in [0,\omega]$, $p_i \not\equiv 0$, $i = 1,2$. Then $p \in \mathcal{V}^-(\omega)$ if and only if the principal (weighted) eigenvalue λ_0 of the problem*

$$-y'' + p_1(t)y = \lambda p_2(t)y, \quad y(0) = y(\omega),\ y'(0) = y'(\omega) \tag{2.1}$$

satisfies $\lambda_0 > 1$.

Theorem 2.1 can be reformulated as follows.

Theorem 2.2. *Let $p \in L_\omega$ with $p(t) = p_1(t) - p_2(t)$, $p_i(t) \ge 0$ for a.e. $t \in [0,\omega]$, $p_i \not\equiv 0$, $i = 1,2$. Let $G_{p_1}(t,s)$ be the Green function related to the left hand side of* (2.1)*, and let $T : C_\omega \to C_\omega$ be a linear operator defined by*

$$Ty(t) := \int_0^\omega G_{p_1}(t,s)p_2(s)y(s)\,\mathrm{d}s. \tag{2.2}$$

Then $p \in \mathcal{V}^-(\omega)$ if and only if the principal characteristic value λ_0 of T (equal to $1/r(T)$ with $r(T)$ being the spectral radius of T) satisfies $\lambda_0 > 1$.

Before we prove Theorems 2.1 and 2.2, let us note that because of the nonnegativity and nontriviality of p_1, the Green function G_{p_1} exists, it is unique and continuous in both variables, hence the operator T is well defined. Moreover, it is compact. Since the eigenvalue problem (2.1) is equivalent to $y = \lambda Ty$, i.e., λ is an eigenvalue of (2.1) if and only if it is the characteristic value of T, we will prove both Theorems 2.1 and 2.2 together. Finally, let us recall that we speak about the *principal eigenvalue* (or principal characteristic value), if (at least one, in general) corresponding eigenfunction is of constant sign in $[0,\omega]$.

Proof of Theorems 2.1 and 2.2. From the assumptions on p_1, p_2, the Green function G_{p_1} satisfies $G_{p_1}(t,s) > 0$ a.e. on $[0,\omega] \times [0,\omega]$ and the spectral radius $r(T)$ of T is positive as well. Moreover, $1/r(T)$ corresponds to the principal characteristic value λ_0 of T. That is, (2.1) possesses the principal eigenvalue $\lambda_0 > 0$ with the constant-sign eigenfunction y_0 (cf., e.g., [8, Theorem 2.6 and Remark 2.1]).

First, we prove the *necessity*. Let $p = p_1 - p_2 \in \mathcal{V}^-(\omega)$, i.e., for any $q \in L_\omega$, $q(t) \geq 0$, $q \not\equiv 0$, the problem

$$-u'' + p_1(t)u = p_2(t)u - q(t), \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{2.3}$$

possesses a strictly negative solution u. Multiplying the equation in (2.3) by the eigenfunction $y_0 > 0$ and integrating over $(0, \omega)$, we easily obtain

$$\int_0^\omega (-u'' + p_1 u)y_0 = \int_0^\omega (-y_0'' + p_1 y_0)u = \lambda_0 \int_0^\omega p_2 u y_0 = \int_0^\omega p_2 u y_0 - \int_0^\omega q y_0.$$

The last equality gives $(1 - \lambda_0)\int_0^\omega p_2 u y_0 = \int_0^\omega q y_0$ and the sign properties of p_2, q, u and y_0 imply $\lambda_0 > 1$.

To prove the *sufficiency*, we use successive iterations. Let us consider $q \in L_\omega$, $q \not\equiv 0$, $q(t) \geq 0$ arbitrary but fixed, and denote

$$q^*(t) := \int_0^\omega G_{p_1}(t,s)q(s)\,\mathrm{d}s.$$

We have $q^* \in C_\omega$, $q^*(t) > 0$ for $t \in [0, \omega]$ and (2.3) is equivalent to $u = Tu - q^*$. Now, we are ready to define a sequence $\{u_n\}_{n=0}^\infty \subset C_\omega$ using the recurrence formula

$$u_{n+1} = Tu_n - q^* \quad \text{with } u_0 \equiv 0. \tag{2.4}$$

From the assumptions on p_1, p_2, the operator T is strictly monotone increasing on C_ω ordered by the cone $C_\omega^+ = \{u \in C_\omega, u(t) \geq 0 \text{ for every } t \in [0,\omega]\}$, and we obtain

$$u_0 \equiv 0 > u_1 = -q^* > u_2 > \dots > u_n > \dots.$$

Moreover,

$$u_n = Tu_{n-1} - q^* = T(Tu_{n-2} - q^*) - q^* = -\sum_{k=0}^{n-1} T^k q^*$$

and hence

$$\|u_n\| \leq \|q^*\| \sum_{k=0}^{n-1} \|T^k\| \leq \|q^*\| \sum_{k=0}^{\infty} \|T^k\|.$$

Using Gelfand's formula $r(T) = \lim_{k\to\infty} \|T^k\|^{1/k}$ and the assumption $1/\lambda_0 = r(T) < 1$, we gain convergence of the series $\sum \|T^k\|$ and hence uniform boundedness of $\{u_n\}$ in C_ω. Compactness of T and monotonicity of $\{u_n\}$ yield the convergence $u_n \to u$ in C_ω with $u(t) < 0$ for all $t \in [0, \omega]$ being the solution of (2.3). Thus $p = p_1 - p_2 \in \mathcal{V}^-(\omega)$. □

The following lemma shows that the sign of $(\lambda_0 - 1)$ does not depend on the choice of p_1, p_2.

Lemma 2.3. *Let λ_0 be the principal eigenvalue of* (2.1) *with some p_1 and p_2 satisfying $p_1(t) - p_2(t) = p(t)$, $p_i(t) \geq 0$ for a.e. $t \in [0, \omega]$, $p_i \not\equiv 0$, $i = 1, 2$. Then* $\operatorname{sgn}(\lambda_0 - 1)$ *is independent on the choice of p_1, p_2.*

Proof. Let $p(t) = p_1(t) - p_2(t) = \bar{p}_1(t) - \bar{p}_2(t)$ with $p_{1,2}$, $\bar{p}_{1,2}$ all nonnegative and nontrivial. Let λ_0 be the principal eigenvalue of (2.1) with $p_{1,2}$ and let $y(t)$ be the corresponding positive eigenfunction. Similarly, let $\bar{\lambda}_0$ be the principal eigenvalue of (2.1) with $\bar{p}_{1,2}$ and $\bar{y}(t)$ be the corresponding positive eigenfunction. Multiplying (2.1) by $\bar{y}$, integrating over $(0,\omega)$ and using $\int_0^\omega y''\bar{y} = \int_0^\omega y\bar{y}''$ yields

$$\begin{aligned}&\int_0^\omega p_1(t)y(t)\bar{y}(t)\,\mathrm{d}t - \lambda_0\int_0^\omega p_2(t)y(t)\bar{y}(t)\,\mathrm{d}t\\ &= \int_0^\omega \bar{p}_1(t)y(t)\bar{y}(t)\,\mathrm{d}t - \bar{\lambda}_0\int_0^\omega \bar{p}_2(t)y(t)\bar{y}(t)\,\mathrm{d}t.\end{aligned}$$

Since $p_1 - \bar{p}_1 = p_2 - \bar{p}_2$, we obtain

$$(1-\lambda_0)\int_0^\omega p_2(t)y(t)\bar{y}(t)\,\mathrm{d}t = (1-\bar{\lambda}_0)\int_0^\omega \bar{p}_2(t)y(t)\bar{y}(t)\,\mathrm{d}t.$$

Positivity of both y, $\bar{y}$ and nonnegativity of p_2, $\bar{p}_2$ means that $\operatorname{sgn}(1-\lambda_0) = \operatorname{sgn}(1-\bar{\lambda}_0)$, which we wanted to prove. □

Remark 2.4. The natural decomposition of sign-changing p is $p_1(t) = p^+(t)$, $p_2(t) = p^-(t)$, with $p^\pm(t) = \max\{\pm p(t), 0\}$ being the positive and negative parts of p. However this choice is not convenient for computational purposes since $\min p^+(t) = 0$ (cf. Corollary 3.7 and the comment above). In further text, we will mainly use decomposition

$$p_1(t) = (p(t)-c)^+ + c, \quad p_2(t) = (p(t)-c)^- \tag{2.5}$$

with some fixed real constant $0 < c \le p_M := \operatorname{ess\,sup}_{t\in[0,\omega]} p(t)$. Notice that for $c = p_M \in \mathbb{R}$, p_1 is constant ($p_1(t) \equiv p_M$).

3. Estimates of λ_0 and consequences of main results

To find the precise value of λ_0 is not an easy task, however, for its estimates we can exploit, e.g., the following results.

Lemma 3.1 (Webb and Lan [8])**.** *Let λ_0 be the principal eigenvalue of* (2.1) *and p_1, p_2, and G_{p_1} be as in Theorem 2.2. Then $m \le \lambda_0 \le M$, where*

$$m = \Big(\sup_{0\le t\le\omega}\int_0^\omega G_{p_1}(t,s)p_2(s)\,\mathrm{d}s\Big)^{-1},$$

$$M = \inf_{0\le a<b\le\omega}\Big(\inf_{a\le t\le b}\int_a^b G_{p_1}(t,s)p_2(s)\,\mathrm{d}s\Big)^{-1}.$$

Lemma 3.2 (Bo Yang [9])**.** *In addition to the assumptions of Lemma 3.1, let θ_0, σ_0 be a priori bounds for the constant-sign eigenfunction y_0 corresponding to λ_0, i.e., $\sigma_0 \le y_0/\|y_0\| \le \theta_0$. Let us define sequences $\{\sigma_n\}$ and $\{\theta_n\}$ by*

$$\theta_{n+1}(t) = \int_0^\omega G_{p_1}(t,s)p_2(s)\theta_n(s)\,\mathrm{d}s, \quad \sigma_{n+1}(t) = \int_0^\omega G_{p_1}(t,s)p_2(s)\sigma_n(s)\,\mathrm{d}s,$$

and the values

$$m_n := \Big(\sup_{0\le t\le\omega}\theta_n(t)\Big)^{-1/n} \quad \textit{and} \quad M_n := \Big(\sup_{0\le t\le\omega}\sigma_n(t)\Big)^{-1/n}.$$

Then for each $n\in\mathbb{N}$, we have $m_n \le \lambda_0 \le M_n$.

Remark 3.3. Let us note that the original result in [8] is formulated for much more general kernels and weight functions, and G_{p_1}, p_2 meet all the required assumptions. On the other hand, result in [9] is stated for the particular $(n-1,1)$ conjugate boundary value problem but it is valid for a wider class of operators including our case.

Remark 3.4. Obviously, we can choose $\theta_0 \equiv 1$ and then m_1 from Lemma 3.2 coincides with m given by Lemma 3.1. Moreover, if both p_1, p_2 are positive constant functions, then $m = \lambda_0 = p_1/p_2$ with $y_0 \equiv 1$ being the corresponding constant-sign eigenfunction. To find the lower bound σ_0, we can proceed similarly as in [9]. In particular, for any y satisfying $-y'' + p_1(t)y \geq 0$ and $y(t) > 0$ for all $t \in [0, \omega]$, we can write

$$\begin{aligned}\|y(t)\| = \max_{t\in[0,\omega]} y(t) &= \max_{t\in[0,\omega]} \int_0^\omega G_{p_1}(t,s)\left(-y''(s) + p_1(s)y(s)\right) \mathrm{d}s \\ &\leq \max_{(t,s)} G_{p_1}(t,s) \int_0^1 \left(-y''(s) + p_1(s)y(s)\right) \mathrm{d}s.\end{aligned}$$

Similarly,

$$y(t) \geq \min_{(t,s)} G_{p_1}(t,s) \int_0^1 \left(-y''(s) + p_1(s)y(s)\right) \mathrm{d}s \geq \frac{\min_{(t,s)} G_{p_1}(t,s)}{\max_{(t,s)} G_{p_1}(t,s)} \|y\|.$$

Hence, we can take $\sigma_0 = \min_{(t,s)} G_{p_1}(t,s)/\max_{(t,s)} G_{p_1}(t,s) > 0$.

Now, we are ready to formulate several corollaries of our main results that provide verifiable necessary and sufficient conditions for $p \in \mathcal{V}^-(\omega)$. The first one is a direct consequence of Theorem 2.2 and Lemma 3.1.

Corollary 3.5. *Let $p \in L_\omega$ be decomposed to $p(t) = p_1(t) - p_2(t)$ with $p_i(t) \geq 0$ for a.e. $t \in [0, \omega]$, $p_i \not\equiv 0$, $i = 1, 2$. If*

$$\sup_{0\leq t\leq\omega} \int_0^\omega G_{p_1}(t,s)p_2(s)\,\mathrm{d}s < 1, \tag{3.1}$$

then $p \in \mathcal{V}^-(\omega)$. If for some $[a,b] \subset [0,\omega]$ we have

$$\inf_{a\leq t\leq b} \int_a^b G_{p_1}(t,s)p_2(s)\,\mathrm{d}s > 1, \tag{3.2}$$

then $p \notin \mathcal{V}^-(\omega)$.

Similarly, Lemma 3.2 provides the following conditions.

Corollary 3.6. *Let $p \in L_\omega$ be decomposed to $p(t) = p_1(t) - p_2(t)$ with $p_i(t) \geq 0$ for a.e. $t \in [0, \omega]$, $p_i \not\equiv 0$, $i = 1, 2$, and let θ_n, σ_n be defined as in Lemma 3.2. If there exists $n \in \mathbb{N}$ such that*

$$\sup_{0\leq t\leq\omega} \theta_n(t) < 1, \tag{3.3}$$

then $p \in \mathcal{V}^-(\omega)$. Also, if there exists $n \in \mathbb{N}$ such that

$$\sup_{0\leq t\leq\omega} \sigma_n(t) > 1, \tag{3.4}$$

then $p \notin \mathcal{V}^-(\omega)$.

If p_1 is bounded away from zero, i.e., if there exists $c > 0$ such that $p_1(t) \geq c$ for a.e. $t \in [0, \omega]$, then from the comparison principle, $G_{p_1}(t,s) \leq G_c(t,s)$ for all

$(t,s) \in [0,\omega] \times [0,\omega]$. Moreover, G_c with constant positive c can be given explicitly, namely (see, e.g., [1])

$$G_c(t,s) = \begin{cases} \frac{\cosh \sqrt{c}(t-s-\frac{\omega}{2})}{2\sqrt{c}\sinh \sqrt{c}\frac{\omega}{2}}, & 0 \le s \le t \le \omega, \\ \frac{\cosh \sqrt{c}(t-s+\frac{\omega}{2})}{2\sqrt{c}\sinh \sqrt{c}\frac{\omega}{2}}, & 0 \le t \le s \le \omega. \end{cases} \tag{3.5}$$

Hence, we can state stricter but easier to apply sufficient condition. In particular, taking $p_1(t) = (p(t)-c)^+ + c$, $p_2(t) = (p(t)-c)^-$ with $c > 0$, Corollary 3.5 directly implies the following assertion.

Corollary 3.7. *Let $p \in L_\omega$ and G_c be given by* (3.5). *If there exists $c > 0$ such that*

$$\sup_{0\le t\le\omega} \int_0^\omega G_c(t,s)(p(s)-c)^- \,\mathrm{d}s < 1, \tag{3.6}$$

then $p \in \mathcal{V}^-(\omega)$.

Finally, using that $\max G_c(t,s) = G_c(t,t) = \left(2\sqrt{c}\tanh \sqrt{c}\frac{\omega}{2}\right)^{-1}$ and $G_c(t,s)$ is non-constant, we can obtain the simplest sufficient condition that fits exactly Theorem 11.4 in [6].

Corollary 3.8. *Let $p \in L_\omega$. If there exists $c > 0$ such that*

$$\int_0^\omega (p(s)-c)^-(s)\,\mathrm{d}s \le 2\sqrt{c}\tanh\sqrt{c}\frac{\omega}{2}, \tag{3.7}$$

then $p \in \mathcal{V}^-(\omega)$.

Remark 3.9. From the opposite point of view, if p is bounded from above, taking $p_1(t) \equiv p_M = \operatorname{ess\,sup}_{t\in[0,\omega]} p(t)$, $p_2(t) = p_M - p(t)$ and G_{p_M} given by (3.5) with $c = p_M$, the latter statement in Corollary 3.5 directly implies that if for some $[a,b] \subset [0,\omega]$

$$\inf_{a\le t\le b} \int_a^b G_{p_M}(t,s)(p_M - p(s))\,\mathrm{d}s > 1, \tag{3.8}$$

then $p \notin \mathcal{V}^-(\omega)$. Since $\int_0^\omega G_{p_M}(t,s)\,\mathrm{d}s = 1/p_M$, condition (3.8) with $[a,b] = [0,\omega]$ reads as follows

$$\sup_{0\le t\le\omega} \int_0^\omega G_{p_M}(t,s)p(s)\,\mathrm{d}s < 0\,.$$

Similarly, the latter statement in Corollary 3.6 implies that if

$$\sup_{0\le t\le\omega} \int_0^\omega G_{p_M}(t,s)(p_M - p(s))\sigma_0(t)\,\mathrm{d}s > 1, \tag{3.9}$$

then $p \notin \mathcal{V}^-(\omega)$. Since $\min G_{p_M}(t,s) = G_{p_M}(\frac{\omega}{2},0) = \left(2\sqrt{p_M}\sinh\sqrt{p_M}\frac{\omega}{2}\right)^{-1}$, it follows (3.9) with the choice $\sigma_0 = \min G_{p_M} / \max G_{p_M} = \cosh^{-1}\sqrt{p_M}\frac{\omega}{2}$ (cf. Remark 3.4) reads as follows

$$\inf_{0\le t\le\omega} \int_0^\omega G_{p_M}(t,s)p(s)\,\mathrm{d}s < 1 - \cosh\sqrt{p_M}\frac{\omega}{2}.$$

From the profile of G_{p_M}, neither of these conditions provide better information than the already known implication $\int_0^\omega p(t)\,\mathrm{d}t \le 0 \ \Rightarrow\ p \notin \mathcal{V}^-(\omega)$.

4. Examples and comparison of results

In this section, we consider $\omega = 2\pi$ and two 2π-periodic test functions:

$$p(t) = \alpha^2 + \beta\cos t \quad \text{and} \quad p(t) = \begin{cases} \alpha^2 + \beta & \text{for } t \in (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi) \\ \alpha^2 - \beta & \text{for } t \in (\frac{\pi}{2}, \frac{3\pi}{2}). \end{cases}$$

and their various decompositions. The first function was used as an example in [6]. The latter (step) function has intentionally the same parameters and nodal domains. Its advantage is that we are able to find directly the corresponding principal eigenvalue.

Using Wolfram Mathematica, we illustrate our results in Sections 2 and 3 for these functions in the $\alpha\beta$-plane, and compare them with sufficient conditions stated in [6, Theorems 11.1, 11.3, 11.4, 11.5].

Example 4.1. Let $\omega = 2\pi$ and

$$p(t) = \alpha^2 + \beta\cos t \tag{4.1}$$

with $\alpha, \beta \in \mathbb{R}^+$. For this function, the best sufficient condition in [6] is provided by Theorem 11.1 therein. That's why we take it as the reference set for our comparison. Let us also recall that [6, Theorem 11.4] coincides with our Corollary 3.8 and [6, Theorem 11.5] gives the sufficient condition $\alpha \geq \beta$ for $p \in \mathcal{V}^-(\omega)$.

We start with the decomposition

$$p_1(t) \equiv p_M = \alpha^2 + \beta, \quad p_2(t) = p_M - p(t) = \beta(1 - \cos t).$$

For this choice, condition (3.1) of Corollary 3.5 coincides with condition (3.6) of Corollary 3.7 and with (3.3) for $n = 1$ of Corollary 3.6. These are visualized in $\alpha\beta$-plane in Figure 1 left (orange area) and we see that they do not provide better results than [6, Theorem 11.1] (blue area). Figure 1 right (orange area) illustrates (3.3) for $n = 3$, i.e. the third iteration of the lower estimate of λ_0. Here we can see a considerable improvement.

Next we use decomposition (2.5) with $c = \alpha^2$, i.e., c is the mean value of $p(t)$ and

$$p_1(t) = \alpha^2 + \beta(\cos t)^+, \quad p_2(t) = \beta(\cos t)^-.$$

In this case, we are not able to determine G_{p_1} and have to be content with the weaker condition (3.6) of Corollary 3.7 involving G_{α^2}. The obtained region in $\alpha\beta$-plane for which $p(t) \in \mathcal{V}^-(2\pi)$ is visualized in Figure 2 in green color. Picture on the left illustrates comparison with result of [6, Theorem 11.1] (blue area), picture on the right illustrates comparison with our third iteration (3.3) for the previous choice $c = p_M$ (orange area). In general, we can observe that smaller c improves the sufficient condition for larger values of α, β, but spoils the result close to the origin. Moreover, for greater c, the higher iterations in (3.3) are easier to compute.

Example 4.2. Let $\omega = 2\pi$ and

$$p(t) = \begin{cases} \alpha^2 + \beta & \text{for } t \in (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi) \\ \alpha^2 - \beta & \text{for } t \in (\frac{\pi}{2}, \frac{3\pi}{2}). \end{cases} \tag{4.2}$$

As we mentioned above, for this step function p, we are able to find directly the corresponding principal eigenvalue λ_0. In particular, choosing, e.g.,

$$p_1(t) \equiv p_M = \alpha^2 + \beta, \quad p_2(t) = p_M - p(t) = \begin{cases} 2\beta & \text{for } t \in (\frac{\pi}{2}, \frac{3\pi}{2}), \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

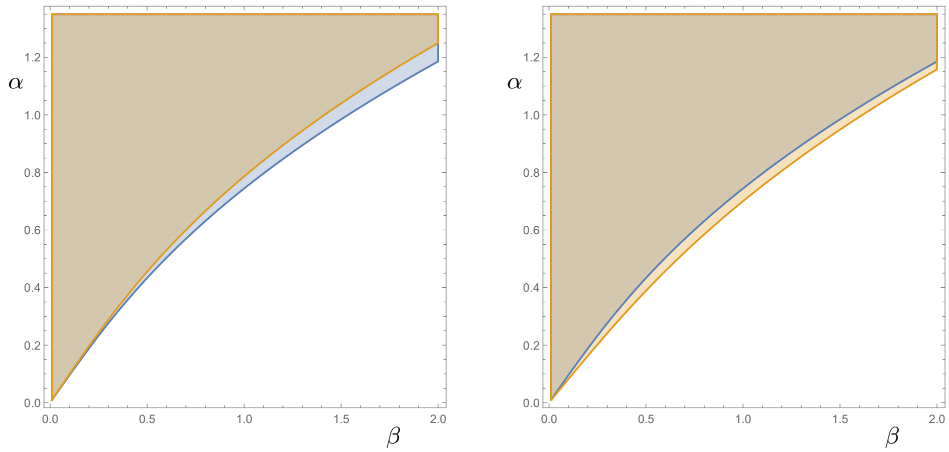


FIGURE 1. Values of α, β for which $p(t) = \alpha^2 + \beta \cos t \in \mathcal{V}^-(2\pi)$. Blue areas correspond to the condition given by [6, Theorem 11.1]. Orange areas correspond to the decomposition with $p_1(t) \equiv p_M$ and the condition (3.3) of Corollary 3.6 for $n = 1$ (left) and $n = 3$ (right).

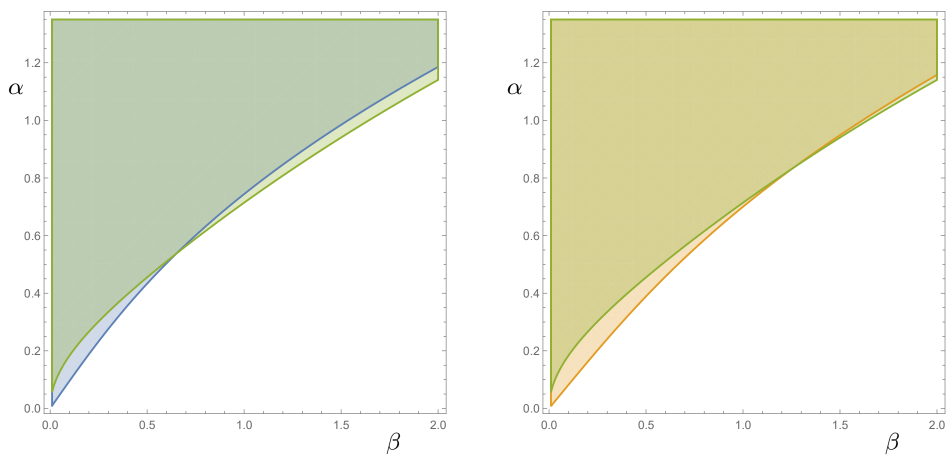


FIGURE 2. Values of α, β for which $p(t) = \alpha^2 + \beta \cos t \in \mathcal{V}^-(2\pi)$. Green areas correspond to the decomposition (2.5) with $c = \alpha^2$ and the condition (3.6) of Corollary 3.7. Blue area on the left corresponds to the condition given by [6, Theorem 11.1]. Orange area on the right is the same as in Figure 1 (right).

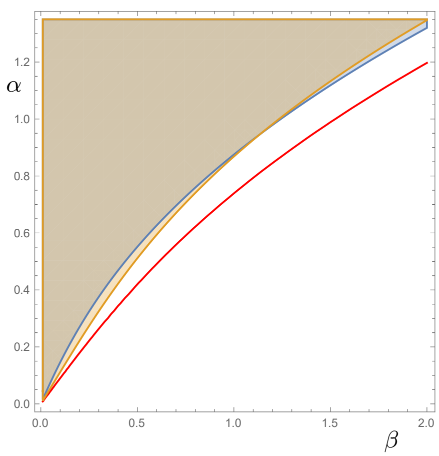
we easily find out that λ_0 corresponds to the first positive root of the equation

$$\sqrt{\alpha^2 + \beta} \tanh \sqrt{\alpha^2 + \beta} \frac{\pi}{2} = \sqrt{2\beta x - \alpha^2 - \beta} \tan \sqrt{2\beta x - \alpha^2 - \beta} \frac{\pi}{2}. \tag{4.4}$$

The corresponding normalized constant-sign eigenfuction takes the form

$$y(t) = \begin{cases} A\cosh\sqrt{\alpha^2+\beta}\, t & \text{for } t \in (0, \frac{\pi}{2}), \\ \cos\sqrt{2\beta\lambda_0 - \alpha^2 - \beta}(t-\pi) & \text{for } t \in (\frac{\pi}{2}, \frac{3\pi}{2}), \\ A\cosh\sqrt{\alpha^2+\beta}(2\pi - t) & \text{for } t \in (\frac{3\pi}{2}, 2\pi), \end{cases}$$

with $A = \cos\sqrt{2\beta\lambda_0 - \alpha^2 - \beta}\,\frac{\pi}{2} / \cosh\sqrt{\alpha^2+\beta}\,\frac{\pi}{2}$. The curve $\lambda_0 = 1$ is plotted in red in Figure 3. Hence, according to Theorems 2.1 and/or 2.2, we have $p \in \mathcal{V}^-(2\pi)$ above this curve, and $p \notin \mathcal{V}^-(2\pi)$ below this curve. For comparison, we visualize also the approximate condition (3.3) of Corollary 3.6 for the same decomposition (4.3), see the orange areas in Figure 3 ($n = 1$ on the left, $n = 3$ on the right). Similarly as in Example 4.1, the blue regions in Figure 3 correspond to the best result of [6], which is in this case provided by Theorem 11.3 therein. Again, we can observe a considerable improvement for higher iterations.



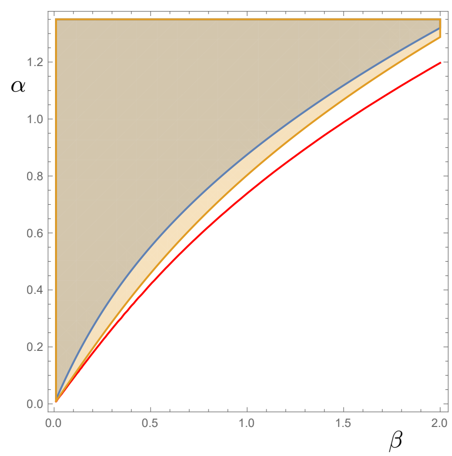


Figure 3. Values of α, β for which p given by (4.2) belongs to $\mathcal{V}^-(2\pi)$. Blue areas correspond to the condition given by [6, Theorem 11.3]. Orange areas correspond to the decomposition with $p_1(t) \equiv p_M$ and the condition (3.3) of Corollary 3.6 for $n = 1$ (left) and $n = 3$ (right). Red curve depicts the precise border $\lambda_0 = 1$ of $\mathcal{V}^-(2\pi)$.

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