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LOCAL MIN-ORTHOGONAL PRINCIPLE AND ITS APPLICATIONS FOR SOLVING MULTIPLE SOLUTION PROBLEMS

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In memory of Prof. John W. Neuberger

Abstract. In this article we establish a double-orthogonal principle, and a local min-orthogonal method with its step size rule, and its convergence under assumptions more general than those in its previous versions. With such a general framework, we justify mathematically the two new algorithms proposed for solving W-type problems. Numerical examples for finding multiple solutions to W-type and to mixed M-W-type problems illustrate the flexibility of this method.

1. Introduction

Consider the semilinear elliptic equation

$$-\Delta u(x) - \lambda u(x) + \kappa F(x, u(x)) = 0, \quad x \in \Omega, \tag{1.1}$$

satisfying zero Dirichlet or Neumann boundary conditions, where $\Omega \subset \mathbb{R}^n$ is an open bounded domain, λ and κ are physical parameters, and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies certain growth and regularity conditions [6]. Denoting $\frac{\partial}{\partial u} f(x, u) = F(x, u)$, its variational functional is

$$J(u) = \int_\Omega [\frac{1}{2}(|\nabla u(x)|^2 - \lambda u^2(x)) + \kappa f(x, u(x))]dx. \tag{1.2}$$

For a function $J \in \mathcal{C}^1(H, \mathbb{R})$ where H is a Hilbert space, a point $u^* \in H$ is called a critical point of J if its Frechet derivative $J'(u^*) = 0$; A critical point u^* is called a k-saddle if it is a local maximum point of J in a k-dimensional subspace and a local minimum point in the corresponding k-co-dimensional subspace. Such index k can be used to measure the instability of the critical point u^*. Thus a 0-saddle is a local minimum point of J and corresponds to a stable local equilibrium state in a physical system; while k-saddles with $k \geq 1$ correspond to unstable local equilibria or excited states. When J is $\mathcal{C}^2$ and u^* is a critical point, we denote the spectral decomposition of $J''(u^*)$ by $H = H^- \oplus H^0 \oplus H^+$ where H^-, H^0, H^+ are respectively the maximum negative, the null, and the maximum positive subspaces of $J''(u^*)$ with $\dim(H^0) < \infty$, and $MI(u^*) = \dim(H^-)$ is called the Morse index

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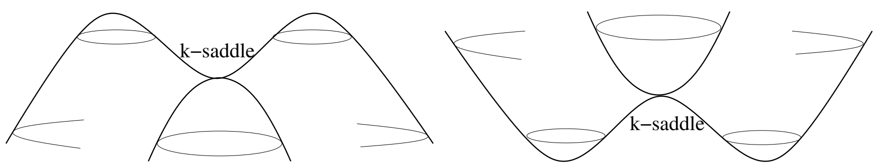
of u^*. A critical point u^* is non-degenerate if $H^0 = \{0\}$ and degenerate otherwise. A non-degenerate critical point u^* with $MI(u^*) = k$ is a k-saddle.

Let $\lambda_1 < \lambda_2 < \dots$ be the eigenvalues of $-\Delta$ with zero boundary condition, and let $v_1, v_2, \dots$ be the corresponding eigenfunctions. For simplicity, we first consider $F(x, u(x)) = |u(x)|^{p-1}u(x)$ with $p > 1$. Equation (1.1) is called focusing (M-type) if $\kappa < 0$ and defocusing (W-type) if $\kappa > 0$. It is known that when $\kappa > 0$ and $\lambda_k < \lambda < \lambda_{k+1}$, then 0 is the only index k-saddle, all nontrivial saddles have index greater than k, and for all $u \in [v_1, \dots, v_k]^\perp$, there is $t_u > 0$ such that $t_u = \arg\max_{t>0} J(tu)$; when $\kappa < 0$ and $\lambda_k < \lambda < \lambda_{k+1}$, then 0 is the only index k-saddle, all nontrivial saddles have index less than k and for all $u \in [v_1, \dots, v_k]$, there is $t_u > 0$ such that $t_u = \arg\min_{t>0} J(tu)$. These two types of problems are very different in physical nature and in mathematical structure as well, see Figure 1. In the literature, these problems have to be solved by two very different types of variational methods.



$\cap$-shape in $[v_1, \dots, v_k]$, M-shape in $[v_1, \dots, v_k]^\perp$;
$\cup$-shape in $[v_1, \dots, v_k]^\perp$, W-shape in $[v_1, \dots, v_k]$.

FIGURE 1. Typical functional profiles of M-type (left) vs. W-type (right). Because of the difference in space dimensions, they are not upside-down to each other.

The case will be much more complex if $F(x, u)$ contains both convex and concave nonlinear terms and becomes a mixed M-W-type problem, see Figure 2,

$$F(x, u) = a(x)|u(x)|^{q-1}u(x) + b(x)|u(x)|^{p-1}u(x)\,. \tag{1.3}$$

See [1, 2, 11] where $0 < q < 1 < p < 2^*$, $2^* = \frac{N+2}{N-2}$ if $N \geq 3$ or $2^* = \infty$ if $N = 1, 2$ for some given nonnegative functions $a(x)$ and $b(x)$.

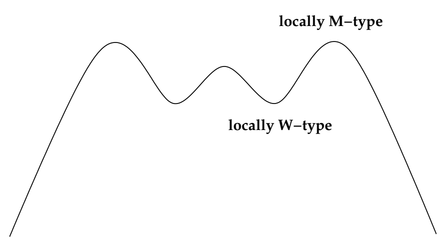


FIGURE 2. Energy profile consists of locally M and/or W parts.

To solve M-type problems for multiple solutions, a local min-max method (LMM) was developed in [4]. Theoretically it was extended to a local min-orthogonal method in [16]. But it has not been used to solve the W-type or other type problems. In this paper, we generalize the local min-orthogonal method and explore its flexibilities to solve the M-type, the W-type and even mixed M-W-type problems.

Let L be a closed subset of H. We denote $S_{L^\perp} = \{v \in H : \|v\| = 1, v \perp L\}$, and for each $v \in S_{L^\perp}$, let $[v, L] = \mathrm{Span}\{v, L\}$. Let

$$v^* = \arg \min_{v \in S_{L^\perp}} J(p(v)),$$

where for the local min-max method (LMM) [4, 2001], $p(v) = \arg\max_{u \in [v,L]} J(u)$ and for the local min-orthogonal method [16], $p(v) \in [v, L]$ such that $J'(p(v)) \perp [v, L]$. Then $u^* = p(v^*)$ is a critical point of J. Since when $p(v) \in [v, L]$ is a local maximum point of J in $[v, L]$, we have $J'(p(v)) \perp [v, L]$ and the local min-orthogonal method becomes LMM which was first developed in [4] and successfully used to solve many M-type problems.

Definition 1.1 ([16])**.** The set-valued mapping $P\colon S_{L^\perp} \to 2^H$ is called the L-orthogonal mapping if $P(v) := \{u \in [L, v] : J'(u) \perp [L, v]\}$ for each $v \in S_{L^\perp}$. A function $p\colon S_{L^\perp} \to H$ is called an L-orthogonal selection if $p(v) \in P(v)\ \forall v \in S_{L^\perp}$. If p is locally defined then p is called a local L-orthogonal selection.

Lemma 1.2 ([16])**.** *If J is $\mathcal{C}^1$, then $G = \{(u, v) : v \in S_{L^\perp}, u \in P(v)\}$ is closed.*

Since $p(v) = tv + v_L$ for some $v_L \in L$, it is clear that if p is continuous then $p(v^*)$ is a critical point of J if and only if there is a neighborhood $\mathcal{N}(v^*)$ of v^* in $L^\perp$ such that

$$J'(p(v^*)) \perp p(v) \quad \forall v \in \mathcal{N}(v^*) \cap S_{L^\perp}. \tag{1.4}$$

We call (1.4) a double-orthogonal principle for critical points. An important feature of this principle is that it involves only J' not the functional J. It implies that this principle can treat nonvariational multiple solution problems where $J'(v) = 0$ is replaced by $A(v) = 0$ for a nonlinear operator A. For example, it is applied to design algorithms to find nonvariational multiple solutions, see [5, 13]. In this paper, we focus on solving variational multiple solution problems with different mathematical structures by exploring variations of this principle. In numerical implementation, the outer-layer is an infinite-dimensional problem and the inner-layer consists of finite-dimensional sub-problems. When the outer-layer orthogonal condition in (1.4) is achieved by a minimization process. It becomes a local min-orthogonal principle for characterizing variational critical points

$$\min_{v \in S_{L^\perp}} J(p(v)). \tag{1.5}$$

This result is first announced in [4] and then published in [16]. As one of the most used mathematical frameworks in critical point theory, the Ljusternik-Schnirelman principle (LSP) [15] characterizes a critical point as a solution to the min-max problem

$$\min_{A \in \mathcal{A}} \max_{u \in A} J(u) \tag{1.6}$$

where $\mathcal{A}$ is a collection of certain *compact sets* A, e.g., a k-dimensional simplex, the min and max are all in the global sense, therefore LSP is not for algorithm implementation. Let us compare our double-orthogonal principle to LSP, one can see that the former is more general, where (1) the two global min and max in LSP are replaced by two $\perp$-conditions which can be satisfied, e.g., by local min and/or max process, and (2) compact sets in LSP are generalized to closed subspaces which can be either finite dimensional or infinite dimensional. This represents a significant advantage of our double-orthogonal principle over LSP, in particular, in algorithm implementation.

In 1960, the Nehari manifold [7]

$$\mathcal{N} = \{t_u u \neq 0 : u \in H, \|u\| = 1, J'(t_u u) \perp u\} \tag{1.7}$$

was introduced and widely applied later on in the literature to prove the existence of 1-saddle of various nonlinear problems through $\min_{u\in\mathcal{N}} J(u)$. In 2005 and after [8, 9, 12]], a generalized Nehari manifold

$$\mathcal{N}_L = \{u \in [v, L] : u \neq 0, v \in S_{L^\perp}, J'(u) \perp [v, L]\}$$

was proposed with $L = [v_1, \dots, v_k]$ where $v_1, \dots, v_k$ are the first k eigenfunctions of $-\Delta$ and applied to prove the existence of a k-saddle of semilinear elliptic PDEs through $\min_{u\in\mathcal{N}_L} J(u)$. It is also proved that such a manifold $\mathcal{N}_L$ is $\mathcal{C}^1$. It is clear that the generalized Nehari method is just a special case of our local min-orthogonal method. Actually for the double-orthogonal principle, there are many different ways to acheive the inner-layer orthogonal condition. In this paper, we first prove this principle with a weaker continuity condition, then establish two new variations to show how this principle can be modified to find multiple solutions to W-type and even mixed M-W-type problems.

2. A new local min-orthogonal method

For analysis purpose, we first extend the domain of p to $S^+_{L^\perp} = \{v \in L^\perp, \|v\| \approx 1\}$ where by $\|v\| \approx 1$ we mean $1 - \delta < \|v\| < 1 + \delta$ for a given $\delta > 0$. While in our algorithm computation and convergence analysis we will evaluate p only at v or $\frac{v+sw}{\|v+sw\|}$ where $v, w \in L^\perp, \|v\| = 1$ for small s. Thus the closed set $S_{L^\perp}$ will be enough for us to define LDLC of p.

Definition 2.1. For $J \in \mathcal{C}^1(H, \mathbb{R})$, a set-valued mapping $P : S^+_{L^\perp} \to H^2$ is called the L-orthogonal mapping if for each $v \in S^+_{L^\perp}$

$$P(v) = \{u \in [v, L] : J'(u) \perp [v, L]\}.$$

An L-orthogonal selection $p : S^+_{L^\perp} \to H$ is a mapping such that

$$p(v) \in P(v), \forall v \in S^+_{L^\perp}.$$

If p is locally defined near $v \in S^+_{L^\perp}$, then p is called a local L-orthogonal selection near v.

Definition 2.2. A local L-orthogonal selection p is said to be locally directional Lipschitz continuous (LDLC) at $v \in S_{L^\perp}$ in $w \in L^\perp$, if there is a constant ℓ_0 depending on v and w, such that for all $s > 0$ small, it holds

$$\|p(v + sw) - p(v)\| \le \ell_0 s\|w\|,$$

where the term $\|w\|$ can be removed since ℓ_0 depends on w. We say that p is LDLC at $v \in S_{L^\perp}$ if it is LDLC at $v \in S_{L^\perp}$ in each $w \in L^\perp$. We say that p is LDLC if it is LDLC at each $v \in S_{L^\perp}$.

Note that when $v \in S_{L^\perp}, w \in L^\perp$ are given, $s > 0$ is small, we have $v+sw \in S^+_{L^\perp}$; For $v(s) = \frac{v+sw}{\|v+sw\|}$, we have $\|v(s)\| = 1$ and $[v(s), L] = [v + sw, L]$ and then

$$p(v(s)) = t_s v(s) + u_L(s) = \frac{t_s}{\|v + sw\|}(v + sw) + u_L(s) = p(v + sw),$$

with $u_L(s) \in L$. It implies that if p is LDLC at $v \in S_{L^\perp}$ in $w \in L^\perp$, there is $\ell_0 > 0$ such that when $s > 0$ small, we have

$$\|p(v(s)) - p(v)\| = \|p(v+sw) - p(v)\| \le \ell_0 |s| \|w\| = O(s). \tag{2.1}$$

It is clear that if the constant ℓ_0 is independent of w, p becomes locally Lipschitz continuous at v in all directions in $L^\perp$. If the Gateaux derivative $\delta p(v;w)$ of p exists at v in $w \in L^\perp$ and

$$\lim_{s\to 0} \frac{1}{s}(p(v+sw) - p(v)) = \delta p(v;w),$$

where $\delta p(v;\alpha w) = \alpha \delta p(v;w)$ for any scalar α, but not necessarily linear in w. Denote $\ell_0 = \max\{1, \frac{2}{\|w\|}|\delta p(v;w)|\} > 0$, then there is $s_0 > 0$, such that when $s_0 > |s| > 0$, it holds

$$\|p(v+sw) - p(v)\| < \ell_0 |s| \|w\|,$$

or p is LDLC at v in w. We conclude that the Gateaux-differential of p at v in $w \in L^\perp$ implies LDLC at v in w, then the directional continuity at v in w but not the continuity at v. The directional continuity is clearly weaker than the weak-continuity since the latter implies the continuity in any finite-dimensional space.

Lemma 2.3 (Stepsize Rule). *Let p be a local L-orthogonal selection of J in $S^+_{L^\perp}$ such that p is LDLC at $v \in S_{L^\perp}$ and dis$(p(v), L) > 0$. If $d = -J'(p(v)) \neq 0$, set $w = d/C$ where $C = \max\{1, \|d\|\}$ and $v(s) = \frac{v+sw}{\|v+sw\|}$, then there is $s_0 > 0$ such that when $s_0 > s > 0$, we have a* stepsize rule

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4} t_v s \|d\|^2 / C. \tag{2.2}$$

Furthermore, if $p(v_k) \to p(v)$ as $v_k \to v$, then there exists $N > 0$ such that when $k > N$, we have a uniform stepsize rule

$$J(p(v_k)) - J(p(v)) < \frac{-t_v}{4} \|J'(p(v))\|^2 / C. \tag{2.3}$$

Proof. Denote $p(v) = t_v v + u_v, p(v(s)) = t_s v(s) + u_L(s)$ for $t_v, t_s > 0$ and $u_v, u_L(s) \in L$. Then there is $s_0 > 0$ such that when $s_0 > s > 0$, we have

$$\begin{aligned} J(p(v(s))) - J(p(v)) &= \langle J'(p(v)), p(v(s)) - p(v) \rangle + o(\|p(v(s)) - p(v)\|) \\ &= \frac{t_s s}{\|v+sw\|} \langle J'(p(v)), w \rangle + o(\|p(v+sw) - p(v)\|) \\ &= -\frac{t_s s}{\|v+sw\|} \|d\|^2 / C + o(s) \\ &< -\frac{1}{4} t_v s \|d\|^2 / C, \end{aligned} \tag{2.4}$$

where we have used the properties $J'(p(v)) \perp [v, L]$ and $\|p(v(s)) - p(v)\| = O(s)$ in (2.1). The uniform stepsize rule follows from the fact that J is $\mathcal{C}^1$, $p(v_k) \to p(v)$ leads to $J'(p(v_k)) \to J'(p(v))$ as $v_k \to v$. ☐

Theorem 2.4 (Local min-orthogonal Characterization). *Let p be an L-orthogonal selection of J in $S^+_{L^\perp}$ such that*

1. *p is LDLC at $v \in S_{L^\perp}$ and dis$(p(v), L) > 0$,*
2. *$v = \arg \operatorname{local} \min_{u \in S_{L^\perp}} J(p(u))$.*

Then $p(v)$ is a critical point of J.

Proof. If $d = -J'(p(v)) \neq 0$, set $w = d/C$ where $C = \max\{1, \|d\|\}$ and $v(s) = \frac{v+sw}{\|v+sw\|} \in S_{L^\perp}$, then by the step size rule (2.2), as $s > 0$ sufficiently small, we have

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4} t_v s \|d\|^2 / C.$$

It violates assumption (2). □

3. A numerical local min-orthogonal algorithm

The local min-orthogonal characterization of a critical point, Theorem 2.4 suggests a local min-orthogonal method with the stepsize rule (2.2) if the min is approximated by using a steepest decent method. We present the steps below:

Step 1: Given $\varepsilon > 0, \lambda > 0$ and n previously found critical points $w_1, w_2, \dots, w_n$ of J, of which w_n has the highest critical value. Set $L = \operatorname{span}\{w_1, w_2, \dots, w_n\}$. Let $v^1 \in S_{L^\perp}$ be an ascent direction at w_n. Let $t_0^0 = 1$, $v_L^0 = w_n$ and set $k = 0$.

Step 2: Using the initial guess $u = t_0^k v^k + v_L^k$, solve $t_0^k, t_1^k, \dots, t_n^k$ from

$$\langle J'(t_0 v_0^k + t_1 w_1 + \dots + t_n w_n), v^k \rangle = 0,$$
$$\langle J'(t_0 v_0^k + t_1 w_1 + \dots + t_v w_n), w^j \rangle = 0,$$

for $j = 1, \dots, n$. Denote $u^k \equiv p(v^k) = t_0^k v_0^k + v_L^k = t_0^k v_0^k + v_1^k w_1 + \cdots t_n^k w_n$.

Step 3: Compute the steepest descent vector $d^k = -J'(u^k)$.

Step 4: If $\|d^k\| \le \varepsilon$ then output $w_{n+1} = u^k$, stop; else goto Step 5.

Step 5: Set $v^k(s) = \frac{v^k + sd^k}{\|v^k + sd^k\|}$ and find

$$s^k = \max\Big\{ \frac{\lambda}{2^m} : m \in \mathbb{N}, 2^m > \|d^k\|,$$
$$J(p(v^k(\frac{\lambda}{2^m}))) - J(w^k) \le -\frac{t_0^k}{2} \|d^k\| \, \|v^k(\frac{\lambda}{2^m}) - v^k\| \Big\}.$$

Initial guess $u = t_0^k v^k(\frac{\lambda}{2^m}) + v_L^k$ is used to find $p(v^k(\frac{\lambda}{2^m}))$ in $\{L, v^k(\frac{\lambda}{2^m})\} \setminus L$ as similar in Step 2 and where t_0^k and v_L^k are found in Step 2.

Step 6: Set $v^{k+1} = v^k(s^k)$ and update $k = k + 1$ then goto Step 2.

Remark 3.1. (1) The algorithm starts with $n = 0$, $L = \{0\}$ to find w_1, then $n = 1$, $L = \{w_1\}$ to find w_2, etc. To stay away from previously found solutions contained in L, we should choose an initial guess v_0 which is at least nearly orthogonal to L.

(2) The finite-dimensional sub-problem for finding $p(v^k)$ in Step 2 or $p(v^k(\frac{\lambda}{2^m}))$ in Step 5 can be solved, for example, by Matlab subroutine fsolve, where $t_0^k > 0$ is determined by the problem structure. When $t_0^k = 0$, the algorithm fails to find a new critical point.

(3) In Step 5, the step size rule has been equivalently modified due to the inequalities

$$\frac{s\|d\|}{\|v + sd\|} \le \|v(s) - v\| \le \frac{\sqrt{2} s \|d\|}{\|v + sd\|}.$$

Also using the designated initial guess as in Step 5 is very important to continuously trace a solution branch and to keep the algorithm stable;

(4) For an initial guess u_0, when the Nehari manifold $\mathcal{N}$ defined in [7] has only one branch, we can simply find $t_0 > 0$ such that $t_0 u_0 \in \mathcal{N}$ and use $t_0 u_0$ as an initial guess; when $\mathcal{N}$ contains multiple branches, with extra information on the

branches such as the sign of J-values, we can find a proper value $t_0 > 0$ such that $t_0 u_0$ belongs to a designated branch.

4. Algorithm convergence analysis

We assume that J satisfies the Palais-Smale (PS) condition, i.e., any sequence $\{u_k\} \subset H$ with $\{J(u_k)\}$ bounded and $J'(u_k) \to 0$ has a convergent subsequence.

Let $\{w^k\} = \{p(v^k)\}$ be the sequence generated by the local min-orthogonal method with $\varepsilon = 0$.

Next we present an improved convergence result [16, 18] by modifying the proof of [18, Theorem 2.4].

Theorem 4.1. *If p is defined in $S^+_{L^\perp}$, and continuous and LDLC on $S_{L^\perp}$, $d(L, w^k) > \alpha > 0$ and $\inf_{v \in S_{L^\perp}} J(p(v)) > -\infty$, then*

(a) $s_k d_k \to 0$;
(b) *there is $\{v^{k_i}\} \subset \{v^k\}$ such that $v^{k_i} \to v^*$ with $w^* = p(v^*), J'(w^*) = 0$;*
(c) *if w^* is isolated then $v^k \to v^*$.*

Proof. By the step size rule (2.2), $\frac{\sqrt{1+(s^k\|d^k\|)^2}}{\sqrt{2}} > 1/2$ and the inequality

$$\frac{s^k\|d^k\|}{\sqrt{1+(s^k\|d_k\|)^2}} \le \|v^{k+1} - v^k\| \le \frac{\sqrt{2}s^k\|d^k\|}{\sqrt{1+(s^k\|d^k\|)^2}}, \tag{4.1}$$

we obtain

$$J(w^{k+1}) - J(w^k) \le \frac{-1}{4}|t_0^k| s^k \|d^k\|^2 C_k \le \frac{-1}{8}|t_0^k| \|J'(w^k)\| \|v^{k+1} - v^k\|.$$

Adding it up for all $k = 1, 2, \dots$ and noting $|t_0^k| > \alpha > 0$, $J(w^k) > -M > -\infty$, we obtain

$$\begin{aligned} -\infty < \lim_{k\to\infty} J(w^k) - J(w^1) &\le -\frac{1}{4}\sum_{k=1}^{\infty} |t_0^k| s^k \|d^k\|^2 C_k \\ &\le -\frac{\alpha}{4}\sum_{k=1}^{\infty} s^k\|d^k\|^2 C_k < -\frac{\alpha}{8}\sum_{k=1}^{\infty} \|J'(w^k)\| \|v^{k+1} - v^k\|. \end{aligned} \tag{4.2}$$

Thus $s^k\|d^k\|^2 C_k = s^k \|J'(w^k)\|^2 / C_k \to 0$. Then $C_k = \max\{1, \|J'(w^k)\|\}$ and $0 < s^k < \lambda$ lead to $s^k\|d^k\| \to 0$ as in (a).

Next to prove (b), there are totally two cases, either (1) $\|d^k\| > \eta > 0$ for $k = 1, 2, \dots$ for some $\frac{1}{2} > \eta > 0$, or (2) there is a subsequence $\{d^{k_i}\} \subset \{d^k\}$ such that $d^{k_i} \to 0$.

In case (1), $\eta < \|d^k\| \le 1$, then (4.2) becomes

$$\begin{aligned} -\infty < \lim_{k\to\infty} J(w^k) - J(w^1) &\le -\frac{\alpha}{8}\sum_{k=1}^{\infty} \|J'(w^k)\| \|v^{k+1} - v^k\| \\ &\le -\frac{\alpha\eta}{8}\sum_{k=1}^{\infty} \|v^{k+1} - v^k\|, \end{aligned} \tag{4.3}$$

i.e., $\{v^k\} \subset S_{L^\perp}$ is a Cauchy sequence in the Hilbert space H. Thus there exists $v^* \in S_{L^\perp}$ such that $v^k \to v^*$ as $k \to \infty$. Since p is continuous and J is $\mathcal{C}^1$, we have $w^* = p(v^*) = \lim_{k\to\infty} p(v^k)$ and $\|J'(w^*)\| \ge \eta$. Hence w^* is not a critical point. Then $s^k d^k \to 0$ and $\|d^k\| > \eta > 0$ imply $s^k \to 0$, which contradicts the uniform step size rule, Lemma 2.3. Thus case (2) must hold, i.e., there is a subsequence

$\{d^{k_i}\} \subset \{d^k\}$ such that $d^{k_i} \to 0$. Since $\{J(w^{k_i})\}$ is bounded, by the PS condition, there is a subsequence denoted by $w^{k_i} = p(v^{k_i}) = t_0^{k_i} v^{k_i} + w_L^{k_i}$ again for some $w_L^{k_i} \in L$, such that $w^{k_i} \to w^* = w_\perp^* + w_L^*$ with $w_\perp^* \in L^\perp, w_L^* \in L$. By the condition $|t_0^k| > \alpha > 0$, we must have $t_0^{k_i} v^{k_i} \to w_\perp^*, w_L^{k_i} \to w_L^*$ and $v^{k_i} \to v^*$ for some $v^* \in S_{L^\perp}$. Thus $t_0^{k_i} \to t_0^*$ for some $|t_0^*| \geq \alpha > 0$. The continuity of p then leads to $w^* = p(v^*)$. We have proved that p is a homeomorphism and $J'(w^*) = \lim_{i\to\infty} J'(w^{k_i}) = \lim_{i\to\infty} d^{k_i} = 0$. So (b) is proved.

Finally to prove (c) $v^k \to v^*$ if $w^* = p(v^*)$ is an isolated saddle, since p is a homeomorphism, this means that there is $\delta > 0$ such that there is no point $v' \in S_{L^\perp}$ satisfying $\|v^* - v'\| < \delta$ and $J'(p(v')) = 0$. Let $I \subset N = \{1, 2, \dots\}$. We call $\sum_{i\in I} \|v^{i+1} - v^i\|$ the total distance traveled by the subsequence $\{v^i\}_{i\in I}$. For any $\eta > 0$, let $i \in I \subset N$ denote the whole index set in N with $\|d^i\| > \eta$. Then (4.2) leads to

$$\begin{aligned} -\infty < \lim_{i\to\infty} J(w^i) - J(w^1) &\leq -\frac{\alpha}{8} \sum_{i\in I} \|J'(w^i)\| \|v^{i+1} - v^i\| \\ &< -\frac{\alpha\eta}{8} \sum_{i\in I} \|v^{i+1} - v^i\|, \end{aligned} \tag{4.4}$$

i.e., the total distance traveled by $\{v^i\}_{i\in I}$ is finite. Note that by (a), we have

$$\|v^{k+1} - v^k\| \leq \frac{\sqrt{2} s^k \|d^k\|}{\sqrt{1 + (s^k \|d^k\|)^2}} \to 0. \tag{4.5}$$

Suppose there is $\delta_3 > 0$ such that *there are infinitely many points* v in $\{v^k\}$ with $\|v - v^*\| > \delta_3$. By the inequality (4.5), for any $0 < \delta_1 < \delta_2 < \delta_3$, there is $K > 0$ such that when $k > K$, $\|v^{k+1} - v^k\| < \frac{1}{2}(\delta_2 - \delta_1)$. Since v^* is a limit point of $\{v^k\}$, there are infinitely many points $v \in \{v^k\}$ such that $0 < \|v - v^*\| < \delta_1$ and there are also infinitely many points $\{v^{k_i}\} \subset \{v^k\}$ such that $0 < \delta_1 < \|v^{k_i} - v^*\| < \delta_2$, i.e., the sequence $\{v^k\}$ enters the three ring regions centered at v^* and defined by $0 < \delta_1 < \delta_2 < \delta_3$ infinitely many times. Thus the total distance traveled by such $\{v^{k_i}\}$ has to be infinite. However by (4.4), for any $\eta > 0$, the total distance traveled by all the points $v^{k_i'} \in \{v^k\}$ with $\|J'(p(v^{k_i'}))\| > \eta$ is finite. Thus there must be infinitely many points $\{v^{k_{i'}}\} \subset \{v^{k_i}\}$ such that $J'(p(v^{k_{i'}})) \to 0$. By the PS condition, there is a subsequence, denoted by $\{w^{k_{i'}}\} = \{p(v^{k_{i'}})\}$ again, such that $w^{k_{i'}} \to w' = t'v' + w_L'$ for some $t' > 0, v' \in S_{L^\perp}, w_L' \in L$ with $J'(w') = 0$ and $\delta_1 \leq \|v^* - v'\| \leq \delta_2$. Since $0 < \delta_1 < \delta_2$ can be any numbers less than δ_3, this contradicts to the assumption that $w^* = p(v^*)$ is an isolated critical point. Thus for any $\delta_3 > 0$, there can be *at most a finite number* of points v in $\{v^k\}$ with $\|v^* - v\| > \delta_3$, i.e., $v^k \to v^*$. (c) is proved. This completes the proof of the theorem. □

5. Two new variations to the min-orthogonal method

The local min-orthogonal method is first introduced in [16] as a theoretical extension of LMM for solving M-type multiple solution problems. Next we show that this method turns out to be quite flexible. It can be actually modified to solve W-type or even mixed M-W type problems for multiple solutions. First we observe closely the profile of a typical W-type functional J in Figure 1, we found that (1) 0 is the only k-saddle and all other critical points must have saddle index less than k;

(2) For any nontrivial solution u^*, it must be a local minimum point of J along its direction with $J(u^*) < J(0) = 0$. Based on the variational structures we observed, we propose two new variations of the min-orthogonal method:

(A1) A min-min-max algorithm. (This method was proposed by Dr. Zhi-Qiang Wang in 2012 in a discussion.)

$$\min_{v\in S_{L^\perp}} \min_{r>0} \max_{u\in[v,L],\|u\|=r} J(u).$$

(A2) A min-max-min algorithm.

$$\min_{v\in S_{L^\perp}} \max_{u\in[v,L],\|u\|\approx 1} \min_{t>0} J(tu).$$

Note that those two algorithms start with $L = \{0\}$. In this case, the max-operation in the algorithms is void and the algorithms find a local minimum point u_0^* of J. Then by setting $L = \{u_0^*\}$, the algorithms find a 1-saddle,...,etc. For each $v \in S_{L^\perp}$, we denote, in the min-min-max algorithm (A1),

$$p(v) = \arg\min_{r>0} \max_{u\in[v,L],\|u\|=r} J(u) \tag{5.1}$$

and in the min-max-min algorithm (A2),

$$p(v) = \arg \max_{u\in[v,L],\|u\|\approx 1} \min_{t>0} J(tu), \tag{5.2}$$

then the two algorithms can be expressed as $\min_{v\in S_{L^\perp}} J(p(v))$. Thus to establish their mathematical justifications by the local min-orthogonal principle, we only have to show $J'(p(v))\perp[v,L]$.

5.1. **Justification of the min-min-max algorithm.** In the min-min-max algorithm (A1), when L is k-dimensional and $v \in S_{L^\perp}$ is given, choose an orthonormal basis in $[v,L]$. For each $u \in [v,L]$, let $(r,\theta) = (r,\theta_1,\dots,\theta_k)$ be the k+1- dimensional spherical coordinates of u in $[v,L]$. Denote $\chi : (r,\theta) \to u = \chi(r,\theta)$ the coordinate transformation from the spherical coordinates to $[v,L]$ with respect to the orthonormal basis. We denote $u(r,\theta)$ the spherical coordinates of u. Since $p(v) \in [v,L]$, to prove $J'(p(v))\perp[v,L]$, we consider

$$J^S(r,\theta) = J(\chi(r,\theta)) = J(u).$$

Denote, for each $r > 0$,

$$\theta(r) = \arg\max_{\theta} J^S(r,\theta) \tag{5.3}$$

and $u = p(v)$ in the spherical coordinates

$$u(r_v,\theta(r_v)) = \arg\min_{r>0} J^S(r,\theta(r)). \tag{5.4}$$

Assume $\theta(r)$ is continuous, we show that

$$dJ^S(r_v,\theta(r_v)) = (J_r^S(r_v,\theta(r_v)), J_\theta^S(r_v,\theta(r_v))) = (0,0,\dots,0).$$

By (5.3) we have

$$J_\theta^S(r,\theta(r)) = (J_{\theta_1}^S(r,\theta(r)),\dots,J_{\theta_k}^S(r,\theta(r))) = (0,\dots,0)$$

or

$$dJ^S(r,\theta(r)) = (J_r^S(r,\theta(r)), J_\theta^S(r,\theta(r))) = (J_r^S(r,\theta(r)),0,\dots,0).$$

Suppose $J_r^S(r_v,\theta(r_v)) \neq 0$. Denote

$$r_v(s) = r_v - sJ_r^S(r_v,\theta(r_v)).$$

Let $(r_v(s), \theta(s))$ be defined in (5.3) where $\theta(s) = \theta_v(r_v(s))$. For each fixed $s > 0$, we define

$$g(\lambda) = J^S(r_v - \lambda s J_r^S(r_v, \theta(r_v)), \theta(s)).$$

By the mean-value theorem, there is $0 < \lambda_s < 1$ such that

$$g'(\lambda_s) = g(1) - g(0) = J^S(r_v(s), \theta(s)) - J^S(r_v, \theta(s)) \geq J^S(r_v(s), \theta(s)) - J^S(r_v, \theta(r_v))$$

since by (5.3), $J^S(r_v, \theta(r_v)) \geq J^S(r_v, \theta(s))$ when $s > 0$ is small. On the other hand

$$g'(\lambda_s) = \langle J_r^S(r_v - \lambda_s s J_r^S(r_v, \theta(r_v)), \theta(s)), -s J_r^S(r_v, \theta(r_v))\rangle \leq -\frac{s}{4}\|J_r^S(r_v, \theta(r_v))\|^2$$

since $r_v - \lambda_s s J_r^S(r_v, \theta(r_v)) \to r_v$ implies $\theta(s) \to \theta(r_v)$ as $s \to 0$. Then we have

$$J^S(r_v(s), \theta(s)) - J^S(r_v, \theta(r_v)) < -\frac{s}{4}\|J_r^S(r_v, \theta(r_v))\|^2$$

when $s > 0$ is small. This contradicts (5.4) if $J_r^S(r_v, \theta(r_v)) \neq 0$. Thus we have shown $dJ^S(r_v, \theta(r_v)) = (J_r^S(r_v, \theta(r_v)), J_\theta^S(r_v, \theta(r_v))) = (0, 0, \dots, 0)$.

To show $J'(p(v)) \perp [v, L]$, we use the same notation as before. For $u = p(v)$, we have $u = \chi(r_v, \theta(r_v))$. We note that $\chi_r'(r_v, \theta(r_v))$ is the ray-direction of the point u and $\chi_\theta'(r_v, \theta(r_v))$ forms a basis for the tangent space of the k+1-dimensional sphere at the point u. By the chain rule, we have

$$\begin{aligned} 0 = \frac{\partial}{\partial r} J^S(r_v, \theta(r_v)) = \frac{\partial}{\partial r} J(\chi(r, \theta))|_{(r,\theta)=(r_v,\theta(r_v))} = J'(p(v))\chi_r'(r_v, \theta(r_v)), \\ (0, \dots, 0) = \frac{\partial}{\partial \theta} J^S(r_v, \theta(r_v)) = \frac{\partial}{\partial \theta} J(\chi(r, \theta))|_{(r,\theta)=(r_v,\theta(r_v))} \\ = J'(p(v))\chi_\theta'(r_v, \theta(r_v)), \end{aligned}$$

i.e., in the space $[v, L]$, $J'(p(v))$ is orthogonal to the vector (ray) $u = p(v)$ and the tangent space of the sphere at $u = p(v)$. Consequently $J'(p(v)) \perp [v, L]$.

We have theoretically verified that the min-min-max algorithm (A1) fits into the local min-orthogonal principle framework by introducing a k+1-dimensional spherical coordinates. In numerical implementation, this is not necessary, we may conveniently work with the original coordinates system while matching the above analysis.

Also numerically the two mins can be put into one to form a two-level algorithm. A similar method has been proposed independently in [14] in the framework of LSP which actually uses global min and max in its mathematical formulation while our local min-orthogonal principle uses local min and max in its mathematical formulation as showed in the previous sections. Since some numerical examples on W-type problems are already carried out in [14], we will not go further on this algorithm.

5.2. **Justification of the min-max-min algorithm.** In the min-max-min algorithm (A2), for each $v \in S_{L^\perp}$, $u \in [v, L]$, $\|u\| \approx 1$, denote

$$\bar{p}(u) = t_u u \quad \text{where } t_u = \arg\min_{t>0} J(tu) \tag{5.5}$$

and

$$p(v) = \bar{p}(u_v) = t_v u_v \quad \text{where } u_v = \arg\max_{u \in [v,L], \|u\| \approx 1} J(\bar{p}(u)).$$

Assume $\bar{p}$ is continuous and suppose the projection w of $J'(p(v))$ onto the subspace $[v,L]$ satisfies $w = J'(p(v))_{[v,L]} \neq 0$. For $s > 0$ small, we let

$$u_v(s) = \frac{u_v + sw}{\|u_v + sw\|} \in S_{[v,L]} \tag{5.6}$$

and $\bar{p}(u_v(s)) = t_s u_v(s) = t_s \frac{u_v+sw}{\|u_v+sw\|}$ for some $t_s > 0$. Thus $\bar{p}(u_v(s)) \to \bar{p}(u_v)$ as $s \to 0$. For each $s > 0$, we define

$$g(\lambda) = J(\frac{t_s u_v}{\|u_v + sw\|} + \lambda \frac{t_s sw}{\|u_v + sw\|}).$$

By the mean-value theorem, there is $0 < \lambda_s < 1$ such that

$$g'(\lambda_s) = g(1) - g(0) = J(\bar{p}(u_v(s))) - J(\frac{t_s u_v}{\|u_v + sw\|}) \leq J(\bar{p}(u_v(s))) - J(\bar{p}(u_v)),$$

since $t_s \to t_v\|u_v\|$ as $s \to 0$ and when $s > 0$ is small, we have $J(\frac{t_s u_v}{\|u_v+sw\|}) \geq J(\bar{p}(u_v))$ by (5.5). On the other hand when $s > 0$ is small, we have $t_s \to t_v\|u_v\|$ and then

$$g'(\lambda_s) = \langle J'(\frac{t_s u_v}{\|u_v + sw\|} + \lambda_s \frac{t_s sw}{\|u_v + sw\|}), \frac{t_s sw}{\|u_v + sw\|}\rangle > \frac{1}{4} t_s s \|w\|^2,$$

where the last inequality is due to the facts that as $s \to 0$, $J'(\frac{t_s u_v}{\|u_v+sw\|} + \lambda_s \frac{t_s sw}{\|u_v+sw\|}) \to J'(\bar{p}(u_v))$ and $\langle J'(\bar{p}(u_v)), w\rangle = \|w\|^2 > 0$. Consequently as $s > 0$ small, we have

$$J(\bar{p}(u_v(s))) - J(\bar{p}(u_v) = J(\bar{p}(u_v(s))) - J(p(v) > \frac{1}{4} t_s s \|w\|^2 > 0, \tag{5.7}$$

which violates (5.2) unless $w = J'(p(v))_{[v,L]} = 0$ or $J'(p(v)) \perp [v,L]$.

Thus we have proved that the min-max-min algorithm (A2) fits into the local min-orthogonal principle framework. On the other hand, denote $F(t,u) = \langle J'(tu), u\rangle = 0$, when

$$F'_t = \langle J''(t_u u)u, u\rangle \neq 0 \ (> 0 \text{ for W-type}), \tag{5.8}$$

by the implicit function theorem, $t'(u) = \bar{p}'(u)$ is actually locally $\mathcal{C}^1$ at u. For many problems in application, $\bar{p}(u) = t_u u$ has an explicit expression, then the algorithm becomes to solve $\min_{v \in S_{L^\perp}} \max_{u \in [v,L], \|u\| \approx 1} J(\bar{p}(u))$, a two-level local min-max algorithm.

6. Numerical examples

Since the M-type problems have been successfully solved before, in this section, we present numerical multiple solutions to a W-type problem by the local min-max-min algorithm (A2) and to a mixed M-W-type problem by the local min-orthogonal algorithm.

6.1. Numerical examples by the mini-max-min method. A W-type problem.

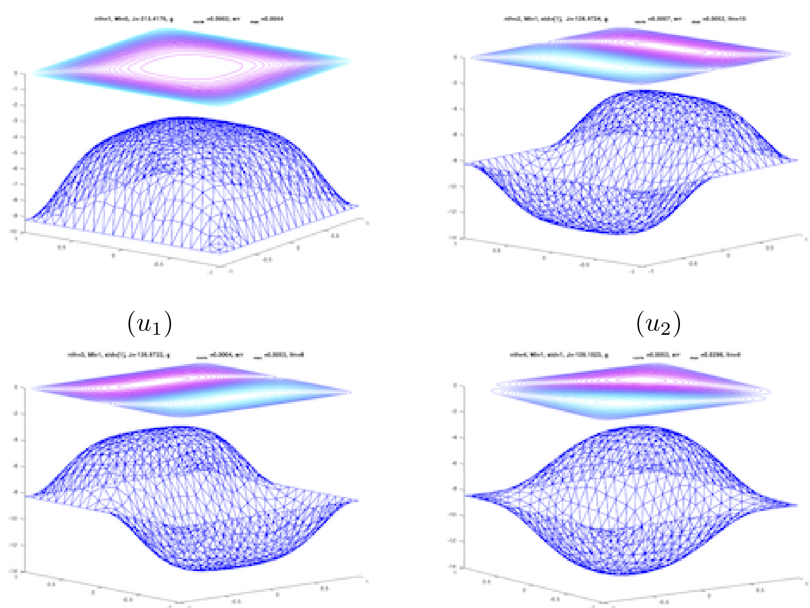
In (1.1), we set $\kappa = 1, F(x,u(x)) = |u(x)|^{p-1}u(x)$ with $H = H_0^1(\Omega), \Omega = (0,1)^2 \subset \mathbb{R}^2$ and $p = 3$.

The operator $-\Delta$ subject to zero Dirichlet boundary condition. has eigenvalues $\mu_i = 4.9348,\ 12.3370,\ 12.3370,\ 19.7392,\ 24.6740,\ 24.6740,\ 32.0762,\ 32.0762,\ \dots$. It is known that when $\mu_n < \lambda < \mu_{n+1}$, the problem has at least n pairs of solutions.

The local min-max-min algorithm (A2) has numerical solutions u_1–u_{10} shown in Figs. 3-4, and their numerical data documented in Table 1.

TABLE 1. W-type solutions

nth	figure	MI	support	$J(\cdot)$	$\|J'(\cdot)\|$	err_{max}	N_{it}
1	(u_1)	0	NA	-313.4176	0.0002	0.0044	8
2	(u_2)	1	1	-126.9724	0.0007	0.0052	10
3	(u_3)	1	1	-126.9722	0.0004	0.0053	8
4	(u_4)	1	1	-109.1923	0.0052	0.0296	9
5	(u_5)	1	1	-109.1914	0.006	0.0719	6
6	(u_6)	2	[1,2]	-31.7034	0.001	0.0075	7
7	(u_7)	3	[1,2,6]	-5.1074	0.0029	0.0457	5
8	(u_8)	3	[1,2,6]	-5.1073	0.0014	0.0304	6
9	(u_9)	3	[1,2,4]	-4.2985	0.001	0.0215	4
10	(u_{10})	3	[1,2,4]	-4.2607	0.0011	0.0709	5



(u_1) (u_2)

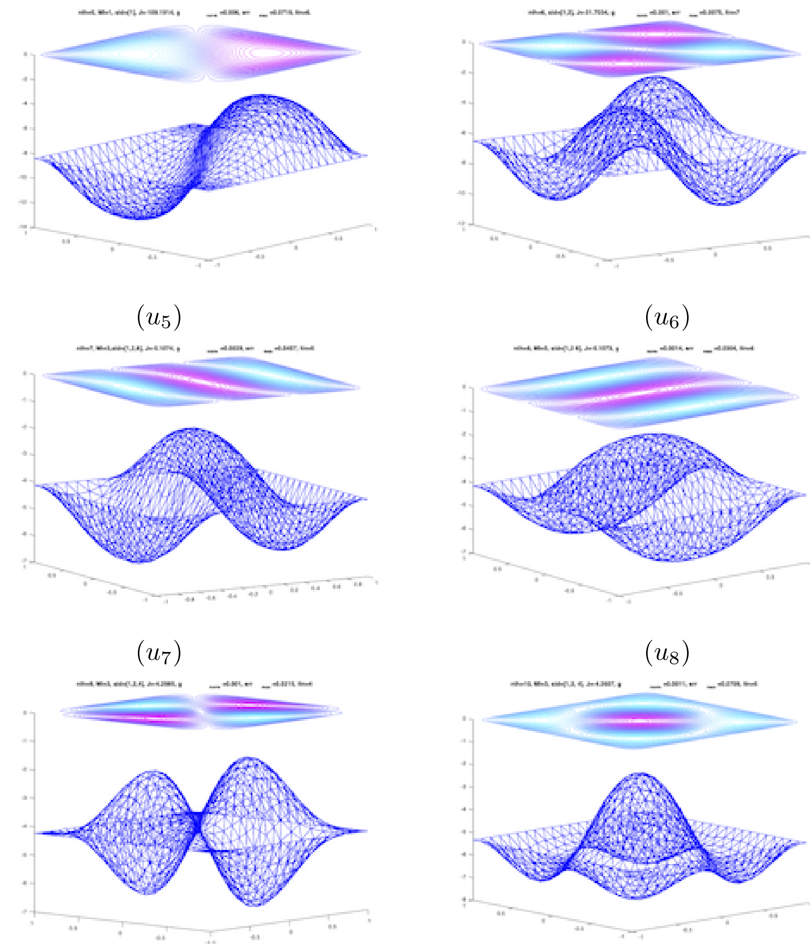
(u_3) (u_4)

FIGURE 3. Critical points u_1–u_4 when $\lambda = 28$

As stated in the last paragraph of Section 5, for this problem, $\bar{p}(u)$ has an explicit expression, the local min-max method can actually be used to find multiple solutions. Much more numerical details can be found in [3].

6.2. **Numerical examples by the min-orthogonal method. A mixed M-W-type problem.** Consider numerically solutions to the problem mixed with concave and convex nonlinearities [1, 2, 11]

$$-\Delta u + \lambda(x)u - a|u(x)|^{q-1}u(x) - b|u(x)|^{p-1}u(x) = 0, \tag{6.1}$$



(u_5) (u_6)

(u_7) (u_8)

(u_9) (u_{10})

FIGURE 4. Critical points u_5–u_{10} when $\lambda = 28$

where $u \in H = H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^N$ is open bounded, $0 < q < 1 < p < 2^*$, $2^* = \frac{N+2}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N \leq 2$, a and b are nonnegative functions on Ω. Its energy functional is

$$J(u) = \int_\Omega \big[\frac{1}{2}|\nabla u(x)|^2 + \frac{1}{2}\lambda(x)u^2(x) - \frac{a(x)}{q+1}|u(x)|^{q+1} - \frac{b(x)}{p+1}|u(x)|^{p+1}\big]dx. \quad (6.2)$$

We note that its function profile shows a mixed locally M-type and locally W-type feature, see Figure 2, and its Nehari manifold $\mathcal{N}$ defined in (1.7) consists of two branches, one is a local M-type where J-values are positive and the other is a local W-type where J-values are negative. Usually two different variational methods, e.g., the local min-max method for the locally M-type branch and the min-min-max algorithm (A1) for the locally W-type branch have to be used separately. We use the min-orthogonal method described in Section 3, since the $\perp$-operation does not differentiate and can treat both types. The difficulty is how the algorithm search will consistently stay on the desired branch. To resolve this difficulty, in our numerical computation we add a constraint $J > 0$ or $J < 0$ to the $\perp$-operation.

Case 1. We set $\lambda(x) = 0$, $a(x) = 1.4$, $b(x) = 1$, $p = 4$, $q = 0.05$, $r = 0$, and $\Omega = (-1, 1)^2$.

Locally W-type saddles with $J < 0$. The algorithm finds the solutions in Figure 5 and generates the numerical data in Table 2.

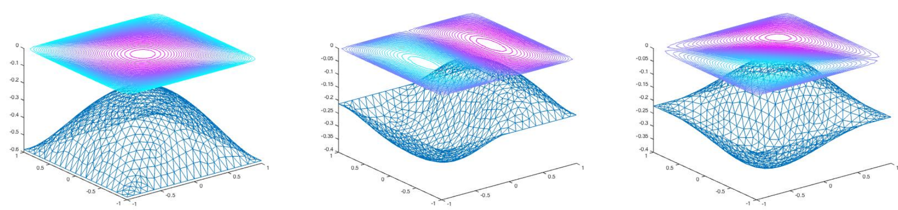
TABLE 2. Numerical data for W-type saddles

	$\|d\|$	ϵ	$\|J'(\cdot)\|_\infty$	$J(\cdot)$	N_{It}
A Local Min	0.0007	1e-3	0.0031	-0.4314	9
1-saddle 1	0.0008	1e-3	1.7330	-0.1589	8
1-saddle 2	0.0064	7e-3	1.9031	-0.1445	6
2-saddle 1	0.0009	1e-3	1.6276	-0.0930	7
2-saddle 2	0.0008	1e-3	1.6924	-0.0671	8
3-saddle	0.0007	1e-3	1.7164	-0.0756	8

$J = -0.4314$, $\|u\|_\infty = 0.3898$ at $(-0.0005, -0.0032)$

$J = -0.1589$, $\|u\|_\infty = 0.1428$ at $(0.5003, 0.0031)$

$J = -0.1445$, $\|u\|_\infty = 0.1497$ at $(-0.3853, -0.3897)$

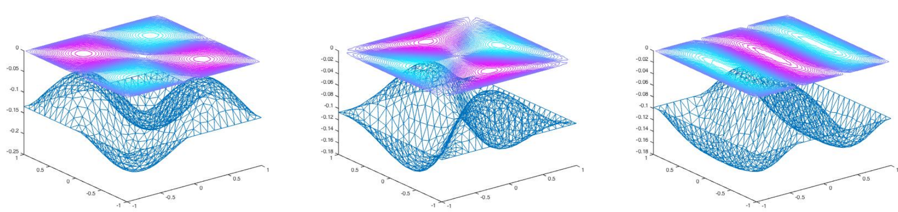


(a) A Local Min (b) 1-saddle 1 (c) 1-saddle 2

$J = -0.0930$, $\|u\|_\infty = 0.0901$ at $(-0.5012, 0.4955)$, sid $= a, b$

$J = -0.0671$, $\|u\|_\infty = 0.0714$ at $(0.0018, -0.6103)$, sid$= a, d$

$J = -0.0756$, $\|u\|_\infty = 0.0662$ at $(0.6662, 0.0038)$, sid $= a, b, d$



(d) 2-saddle 1 (e) 2-saddle 2 (f) 3-saddle

FIGURE 5. Locally W-type saddles with with $J < 0$

Locally M-type saddles with $J > 0$ The algorithm finds the solutions in Figure 6 and generates the numerical data in Table 3.

Case 2. To investigate possible bifurcation phenomenon, we set $\lambda(x) = 0$, $a(x) = 1.4$, $b(x) = 1$, $p = 4$, $q = 0.05$, $r = 4$, and $\Omega = (-1, 1)^2$.

TABLE 3. Numerical data for locally M-type saddles

	$\|d\|$	ϵ	$\|J'(\cdot)\|_\infty$	$J(\cdot)$	N_{It}
1-saddle	0.0008	1e-3	0.0281	2.1680	18
2-saddle 1	0.0009	1e-3	1.7601	18.0417	45
2-saddle 2	0.0009	1e-3	0.0630	19.5637	31
2-saddle 3	0.0009	1e-3	0.0556	19.5638	31
3-saddle	0.0029	3e-3	1.7163	56.6547	44
4-saddle 1	0.0007	1e-3	0.9094	53.2727	36
4-saddle 2	0.0036	4e-3	1.5387	65.1722	60

Locally M-type saddles with $J > 0$ The algorithm finds the solutions in Figure 7 and generates the numerical data in Table 4.

TABLE 4. Numerical data for locally M-type saddles

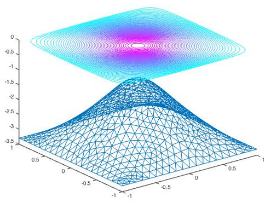
	$\|d\|$	ϵ	$\|J'(\cdot)\|_\infty$	$J(\cdot)$	N_{It}
1-saddle	0.0009	1e-3	0.0916	17.6073	98
2-saddle 1	0.0009	1e-3	1.6902	38.7233	98
2-saddle 2	0.0009	1e-3	1.6926	39.8931	93
2-saddle 3	0.0010	1e-3	1.6629	39.8933	93
3-saddle	0.0009	1e-3	0.9003	56.6546	171
4-saddle 1	0.0010	1e-3	1.7205	82.5416	154

From Figure 7 (a), a symmetry breaking phenomenon can be clearly observed. To further explore such a bifurcation process, we use the same initial guess u_0, the eigenfunction of $-\Delta$ corresponding to the first eigenvalue μ_1. Thus u_0 is symmetric about the x-axis, the y-axis and the lines $x = y$ and $x = -y$. The min-orthogonal method is applied with $L = \{0\}$ for the first N_{MO} times and then it is followed by a Newton method to speedup local convergence. Since the Newton method is symmetry invariant and insensitive to numerical errors [10], we obtain the following three solutions with different symmetries as in Figure 8, a typical symmetry-breaking phenomenon due to a bifurcation process along the parameter r.

Since in this case, the locally W-type saddles did not show any bifurcation phenomenon and their solution profiles are similar to those in Figure 5, they are omitted here. It is interesting to compare the three solutions in Figure 8 with the three solutions in [18, Figure 9], where $a = 0, r = 2$ and $N_{MO} = 4, 14, 20$. They have exactly the same symmetries in order. Since $a = 0$, the latter is an M-type problem. The differences in N_{MO} values imply that the latter is much easier to bifurcate to an asymmetric solution. This may indicate the influence by the concave term $-a|u(x)|^{q-1}u(x)$ in the problem.

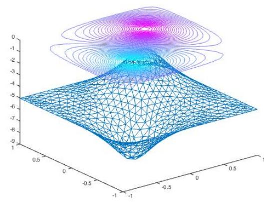
Final remarks. From a double-orthogonal principle, which does not have to have a variational structure, to a local min-orthogonal principle, which needs only a general variational structure, to its numerical algorithm, we established its solution characterization, step size rule and convergence. Then we used the local min-orthogonal principle as a general mathematical framework to justify two algorithms, a min-max-min algorithm and a min-min-max algorithm, used to solve W-type problems for multiple solutions. In the final section, to illustrate the flexibilities of

$J = 2.1680$,
$\|u\|_\infty = 2.1990$ at
$(-0.0005, -0.0032)$



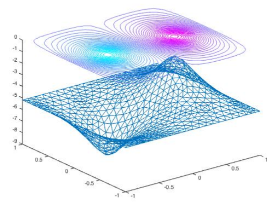
(a) 1-saddle

$J = 18.0417$,
$\|u\|_\infty = 3.3582$ at
$(0.3972, 0.3990)$



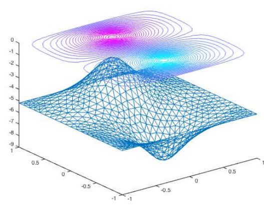
(b) 2-saddle 1

$J = 19.5637$,
$\|u\|_\infty = 3.4559$ at
$(0.5003, 0.0031)$



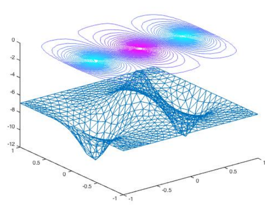
(c) 2-saddle 2

$J = 19.5638$,
$\|u\|_\infty = 3.4559$ at
$(-0.0014, -0.5010)$



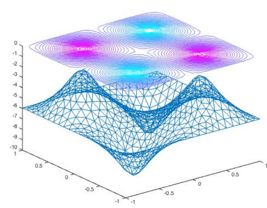
(d) 2-saddle 3

$J = 56.6547$,
$\|u\|_\infty = 4.6177$ at
$(-0.0005, -0.0032)$, sid
$= a, c$



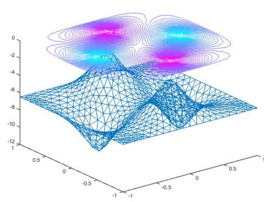
(e) 3-saddle

$J = 53.2727$,
$\|u\|_\infty = 3.9597$ at
$(0.4978, 0.5000)$, sid
$= a, c, d$



(f) 4-saddle 1

$J = 65.1722$,
$\|u\|_\infty = 4.3352$ at
$(-0.6056, 0.0019)$, sid
$= a, b, c$



4-saddle 2

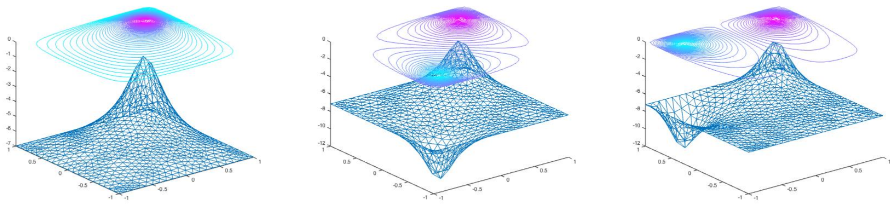
FIGURE 6. Locally M-type saddles with $J > 0$

the method developed in the previous sections, we present some numerical examples for solving W-type problems and, in particular, for mixed M-W-type problems where both convex and concave nonlinearities are present, for multiple solutions. As a trade-off for the generalization of the method, so far we can establish only a one side bound (inequality) estimate for the Morse index of a solution found by the local min-orthogonal method [3], not as an equality established by the local min-max method [17].

$J = 17.6073$, $\|u\|_\infty = 4.6509$ at $(0.6375, 0.6408)$

$J = 38.7233$, $\|u\|_\infty = 4.7529$ at $(-0.6686, -0.6668)$

$J = 39.8931$, $\|u\|_\infty = 4.8060$ at $(-0.6855, 0.6760)$

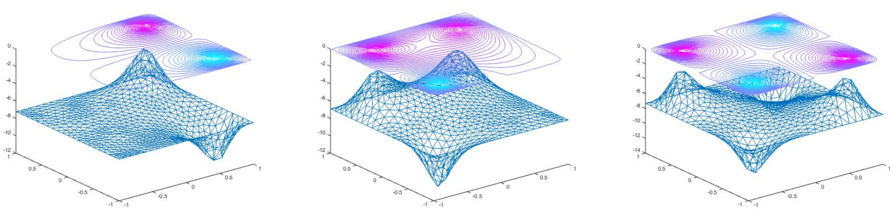


(a) 1-saddle (b) 2-saddle 1 2-saddle 2

$J = 39.8933$, $\|u\|_\infty = 4.8022$ at $(0.6838, -0.6831)$

$J = 56.6546$, $\|u\|_\infty = 4.9724$ at $(-0.7159, -0.7175)$, sid $= a, c$

$J = 82.5416$, $\|u\|_\infty = 4.8665$ at $(0.6910, -0.6946)$, sid $= a, c, d$



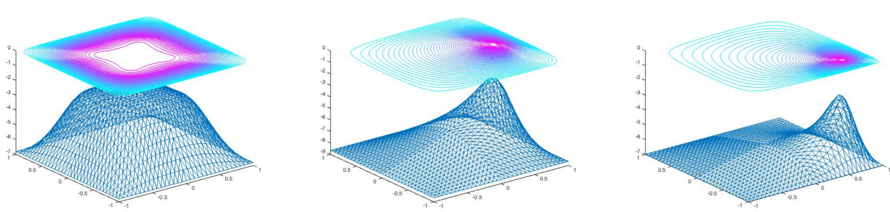
(d) 2-saddle 3 (e) 3-saddle (f) 4-saddle

FIGURE 7. Locally M-type saddles with $J > 0$

$J = 46.1140$, $\|u\|_\infty = 4.5370$ at $(0.0208, -0.0104)$

$J = 29.4731$, $\|u\|_\infty = 5.7561$ at $(0.6510, -0.0052)$

$J = 17.6390$, $\|u\|_\infty = 4.6441$ at $(0.6458, -0.6354)$



(a)$N_{MO} = 15$ $N_{MO} = 90$ $N_{MO} = 110$

FIGURE 8. Locally M-type positive solutions with Newton's method

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