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The family of cubic differential systems with two real and two complex distinct infinite singularities and invariant straight lines of the type $(3,1,1,1)$

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Abstract. In this article we consider the class $\mathbf{CSL}_7^{2r2c\infty}$ of non-degenerate real planar cubic vector fields, which possess two real and two complex distinct infinite singularities and invariant straight lines of total multiplicity 7, including the line at infinity. The classification according to the configurations of invariant lines of systems possessing invariant straight lines was given in articles published from 2014 up to 2022. We continue our investigation for the family $\mathbf{CSL}_7^{2r2c\infty}$ possessing configurations of invariant lines of type $(3,1,1,1)$ and prove that there are exactly 42 distinct configurations of this type. Moreover we construct all the orbit representatives of the systems in this class with respect to affine group of transformations and a time rescaling.

Keywords: cubic vector fields, invariant straight lines, infinite and finite singularities, multiplicity of invariant lines, configurations of invariant straight lines, multiplicity of singularity.

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1 Introduction and statement of the Main Theorem

We consider here real polynomial differential systems

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1.1}$$

where p, q are polynomials in x, y with real coefficients, i.e. p, $q \in \mathbb{R}[x,y]$. We call degree of a system (1.1) $\max(\deg(p), \deg(q))$. A *cubic* system (1.1) is of degree three. We say that a system (1.1) is non-degenerate if the polynomials $p(x,y)$ and $q(x,y)$ are co-prime, i.e. $\gcd(p,q) =$ constant.

Let

$$\mathbf{X} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$$

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be the polynomial vector field corresponding to a system (1.1).

In [17] Darboux introduced the notion of an algebraic invariant curve for differential equations on the complex plane. An algebraic curve $f(x,y)=0$ with $f(x,y)\in\mathbb{C}[x,y]$ is an invariant curve of a system of the form (1.1) where $p(x,y),q(x,y)\in\mathbb{C}[x,y]$ if and only if there exists $K[x,y]\in\mathbb{C}[x,y]$ such that

$$\mathbf{X}(f)=p(x,y)\frac{\partial f}{\partial x}+q(x,y)\frac{\partial f}{\partial y}=f(x,y)K(x,y)$$

is an identity in $\mathbb{C}[x,y]$. Since $\mathbb{R}\subset\mathbb{C}$, any system (1.1) over $\mathbb{R}$ generates a system of differential equations over $\mathbb{C}$. Using the embedding $\mathbb{C}^2\hookrightarrow\mathbb{P}_2(\mathbb{C})$, $(x,y)\mapsto[x:y:1]=[X:Y:Z]$, ($x=X/Z, y=Y/Z$ and $Z\neq 0$), we can compactify the differential equation $q(x,y)dy-p(x,y)dx=0$ to an associated differential equation over the complex projective plane. In fact the theory of Darboux in [17] is done for differential equations on the complex projective plane.

We compactify the space of all the polynomial differential systems (1.1) of degree n on $\mathbb{S}^{N-1}$ with $N=(n+1)(n+2)$ by multiplying the coefficients of each systems with $1/(\sum(a_{ij}^2+b_{ij}^2))^{1/2}$, where a_{ij} and b_{ij} are the coefficients of the polynomials $p(x,y)$ and $q(x,y)$, respectively.

Definition 1.1 ([36])**.** (1) We say that an invariant curve $\mathcal{L}: f(x,y)=0$, $f\in\mathbb{C}[x,y]$ for a polynomial system (S) of degree n has *multiplicity* m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N=(n+1)(n+2)$, such that each (S_k) has m distinct invariant curves $\mathcal{L}_{1,k}: f_{1,k}(x,y)=0,\ldots,\mathcal{L}_{m,k}: f_{m,k}(x,y)=0$ over $\mathbb{C}$, $\deg(f)=\deg(f_{i,k})=r$, converging to $\mathcal{L}$ as $k\to\infty$, in the topology of $P_{R-1}(\mathbb{C})$, with $R=(r+1)(r+2)/2$ and this does not occur for $m+1$.

(2) We say that the line at infinity $\mathcal{L}_\infty: Z=0$ of a polynomial system (S) of degree n has *multiplicity* m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N=(n+1)(n+2)$, such that each (S_k) has $m-1$ distinct invariant lines $\mathcal{L}_{1,k}: f_{1,k}(x,y)=0,\ldots,\mathcal{L}_{m,k}: f_{m-1,k}(x,y)=0$ over $\mathbb{C}$, converging to the line at infinity $\mathcal{L}_\infty$ as $k\to\infty$, in the topology of $P_2(\mathbb{C})$ and this does not occur for m.

In this work we consider a particular case of invariant algebraic curves, namely the invariant straight lines of systems (1.1). A straight line over $\mathbb{C}$ is the locus $\{(x,y)\in\mathbb{C}^2 | f(x,y)=0\}$ of an equation $f(x,y)=ux+vy+w=0$ with $(u,v)\neq(0,0)$ and $(u,v,w)\in\mathbb{C}^3$. We note that by multiplying the equation by a non-zero complex number λ, the locus of the equation does not change. So that we have an injection from the lines in $\mathbb{C}^2$ to the points in $\mathbb{P}_2(\mathbb{C})\backslash\{[0:0:1]\}$. This injection induces a topology on the set of lines in $\mathbb{C}^2$ from the topology of $\mathbb{P}_2(\mathbb{C})$ and hence we can talk about a sequence of lines convergent to a line in $\mathbb{C}^2$.

For an invariant line $f(x,y)=ux+vy+w=0$ we denote $\hat{a}=(u,v,w)\in\mathbb{C}^3$ and by $[\hat{a}]=[u:v:w]$ the corresponding point in $\mathbb{P}_2(\mathbb{C})$. We say that a sequence of straight lines $f_i(x,y)=0$ converges to a straight line $f(x,y)=0$ if and only if the sequence of points $[\hat{a}_i]$ converges to $[\hat{a}]=[u:v:w]$ in the topology of $\mathbb{P}_2(\mathbb{C})$.

In view of the above definition of an invariant algebraic curve of a system (1.1), a line $f(x,y)=ux+vy+w=0$ over $\mathbb{C}$ is an invariant line if and only if it there exists $K(x,y)\in\mathbb{C}[x,y]$ which satisfies the following identity in $\mathbb{C}[x,y]$:

$$\mathbf{X}(f)=up(x,y)+vq(x,y)=(ux+vy+w)K(x,y).$$

We point out that if we have an invariant line $f(x,y)=0$ over $\mathbb{C}$ it could happen that multiplying the equation by a number $\lambda\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, the coefficients of the new equation become real, i.e. $(u\lambda, v\lambda, w\lambda) \in \mathbb{R}^3$. In this case, along with the line $f(x,y) = 0$ sitting in $\mathbb{C}^2$ we also have an associated real line, sitting in $\mathbb{R}^2$ defined by $\lambda f(x,y) = 0$.

Note that, since a system (1.1) is with real coefficients, if its associated complex system has a complex invariant straight line $ux + vy + w = 0$, then its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$ is also invariant.

A line in $\mathbb{P}_2(\mathbb{C})$ is the locus in $\mathbb{P}_2(\mathbb{C})$ of an equation $F(X,Y,Z) = uX + vY + wZ = 0$ where $(u,v,w) \in \mathbb{C}^3$ and $F(X,Y,Z) \in \mathbb{C}[X,Y,Z]$. The line $Z = 0$ in $\mathbb{P}_2(\mathbb{C})$ is called the line at infinity of the affine plane $\mathbb{C}^2$. This line is an invariant manifold of the complex differential equation on $\mathbb{P}_2(\mathbb{C})$. Clearly the lines in $\mathbb{P}_2(\mathbb{C})$ are in a one-to-one correspondence with points $[u : v : w] \in \mathbb{P}_2(\mathbb{C})$ and thus we have a topology on the set of lines in $\mathbb{P}_2(\mathbb{C})$. We can thus talk about a sequence of lines in $\mathbb{P}_2(\mathbb{C})$ convergent to a line in $\mathbb{P}_2(\mathbb{C})$.

To a line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$, $f \in \mathbb{C}[x,y]$, we associate its projective completion $F(X,Y,Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x,y) \mapsto [x : y : 1] = [X,Y,Z]$ indicated above.

We first remark that in the above definition we made an abuse of language. Indeed, we talk about complex invariant lines of real systems. However we already said that to a real system one can associate a complex systems and to a differential equation $q(x,y)dy - p(x,y)dx = 0$ corresponds a differential equation in $\mathbb{P}_2(\mathbb{C})$.

We remark that Definition 1.1 is a particular case of the definition of geometric multiplicity given in [16], and namely the "strong geometric multiplicity" with the restriction, that the corresponding perturbations are cubic systems.

The set **CS** of cubic differential systems depends on 20 parameters and for this reason people began by studying particular subclasses of **CS**. Some of these subclasses are on cubic systems having invariant straight lines.

We mention here some papers on polynomial differential systems possessing invariant straight lines. For quadratic systems see [8, 18, 31, 32, 36–40] and [41]; for cubic systems see [4, 5, 7, 9–14, 23, 25–27, 33, 34, 44] and [45]; for quartic systems see [43] and [47].

The existence of sufficiently many invariant straight lines of planar polynomial systems could be used for proving the integrability of such systems. During the past 15 years several articles were published on this theme (see for example [13, 14, 37, 39]).

According to [1, 16], for a non-degenerate polynomial differential system of degree m, the maximum number of invariant straight lines including the line at infinity and taking into account their multiplicities is $3m$. This bound is always reached (see [16]).

In particular, the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems with a finite number of infinite singularities is 9. In [25] the authors classified all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities according to their *configurations of invariant lines*. The notion of configuration of invariant lines for a polynomial differential system was first introduced in [36].

Definition 1.2 ([40])**.** Consider a real planar polynomial differential system (1.1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients), including the line at infinity, of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

In [25] the authors used a weaker notion, not taking into account the multiplicities of real singularities. They detected 23 such configurations. Moreover, in [25] the necessary and sufficient conditions for the realization of each one of 23 configurations detected, are determined using invariant polynomials with respect to the action of *the group of affine transformations* $(Aff(2,\mathbb{R}))$ *and time rescaling* (i.e. $Aff(2,\mathbb{R}) \times \mathbb{R}^*$)). In [4] the author detected another class of cubic systems whose configuration of invariant lines was not detected in [25].

If two polynomial systems are equivalent under the action of the affine group and time rescaling, clearly they must have the same kinds of configurations of invariant lines. But it could happen that two distinct polynomial systems which are non-equivalent modulo the action of the affine group and time rescaling have "the same kind of configurations" of straight lines. We need to say when two configurations are considered equivalent.

Definition 1.3. Suppose we have two cubic systems $(S), (S')$ both with a finite number of singularities, finite and infinite, a finite set of invariant straight lines $\mathcal{L}_i : f_i(x,y) = 0$, $i = 1,\dots,k$, of (S) (respectively $\mathcal{L}'_i : f'_i(x,y) = 0$, $i = 1,\dots,k'$, of (S')). We say that the two configurations C, C' of invariant lines, including the line at infinity, of these systems are equivalent if there is a one-to-one correspondence ϕ between the lines of C and C' such that:

(i) ϕ sends an affine line (real or complex) to an affine line and the line at infinity to the line at infinity conserving the multiplicities of the lines and also sends an invariant line with coefficients in $\mathbb{R}$ to an invariant line with coefficients in $\mathbb{R}$;

(ii) for each line $\mathcal{L} : f(x,y) = 0$ we have a one-to-one correspondence between the real singular points on $\mathcal{L}$ and the real singular points on $\phi(\mathcal{L})$ conserving their multiplicities and their order on these lines;

(iii) we have a one-to-one correspondence ϕ_∞ between the real singular points at infinity on the (real) lines at infinity of (S) and (S') such that when we list in a counterclockwise sense the real singular points at infinity on (S) starting from a point p on the Poincaré disc, $p_1 = p,\dots,p_k$, ϕ_∞ preserves the multiplicities of the singular points and preserves or reverses the orientation;

(iv) consider the total curves

$$\mathcal{F} : \prod F_j(X,Y,Z)^{m_i} Z^m = 0, \quad \mathcal{F}' : \prod F'_j(X,Y,Z)^{m'_i} Z^{m'} = 0$$

where $F_i(X,Y,Z) = 0$ (respectively $F'_i(X,Y,Z) = 0$) are the projective completions of $\mathcal{L}_i$ (respectively $\mathcal{L}'_i$) and m_i, m'_i are the multiplicities of the curves $F_i = 0, F'_i = 0$ and m, m' are respectively the multiplicities of $Z = 0$ in the first and in the second system. Then, there is a one-to-one correspondence ψ between the real singularities of the curves $\mathcal{F}$ and $\mathcal{F}'$ conserving their multiplicities as singular points of the total curves.

Remark 1.4. In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [36]. Thus we denote by "(a,b)" the maximum number a (respectively b) of infinite (respectively finite) singularities which can be obtained by perturbation of a multiple infinite singular point.

The configurations of invariant straight lines which were detected for some families of systems (1.1), were instrumental for determining the phase portraits of those families. For example, in [37,39] it was proved that we have a total of 57 distinct configurations of invariant lines for quadratic systems with invariant lines of total multiplicity greater than or equal to 4. These 57 configurations lead to the existence of 135 topologically distinct phase portraits. In [33,34,44,45] it was proved that cubic systems with invariant lines of total parallel multiplicity six or seven (the notion of "parallel multiplicity" could be found in [45]) have 113 topologically distinct phase portraits. This was done by using the various possible configurations of invariant lines of these systems.

In what follows we define some algebraic-geometric notions which will be needed in order to describe the invariants used for distinguishing configurations of invariant lines.

Let V be an irreducible algebraic variety of dimension n over a field K.

Definition 1.5. A cycle of dimension r or r-cycle on V with coefficients in an Abelian group G is a formal sum $\Sigma_W n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in G$, and only a finite number of n_W are non-zero. The support of a cycle C is the set $\mathrm{Supp}(C) = \{W | n_W \neq 0\}$. An $(n-1)$-cycle is called a divisor $\mathcal{D}$.

Definition 1.6. We call type of a divisor $\mathcal{D}$ the set of all ordered couples (m, s_m) where m is an integer appearing as a coefficient in the divisor $\mathcal{D}$ and s_m is the number of occurrences in $\mathcal{D}$ of the coefficient m.

Clearly the notion of *type of a divisor* is an affine invariant.

These notions (see [21]) which occur frequently in algebraic geometry, were used for classification purposes of planar quadratic differential systems by Pal and Schlomiuk [29], [35] and by Llibre and Schlomiuk in [24]. They are also helpful here as we indicate below.

We apply the preceding notions to planar polynomial differential systems (1.1). We denote by $\mathbf{PSL}_{n,\mathfrak{L}}$ the class of all non-degenerate planar polynomial differential systems of degree n with a finite number of infinite singularities and possessing invariant lines, including the line at infinity, of total multiplicity $\mathfrak{L}$.

We define here below an important divisor which is used in this work and which we call *the parallelism divisor*. Consider a system in $(S) \in \mathbf{PSL}_{n,\mathfrak{L}}$. Let $p_1, p_2, \ldots, p_s$ be the set of all the real singular points at infinity of (S). Let j_k, $k \in \{1, \ldots, s\}$ be the total multiplicity of all invariant affine lines which cut the line at infinity at p_k. Let i_k, $k \in \{1, \ldots, s\}$ be the maximum number of distinct invariant affine lines which can appear from the line at infinity in a perturbation of (S) in the class $\mathbf{PSL}_{n,\mathfrak{L}}$ and which cut the line at infinity at p_k.

Definition 1.7. We call parallelism divisor on $Z = 0$ with coefficients in $\mathbb{Z}^2$ the divisor $D_L(S; Z)$ defined as follows:

$$D_L(S; Z) = \sum_{k=1}^{s} \begin{pmatrix} i_k \\ j_k \end{pmatrix} p_k.$$

Observation 1.8. In this definition we spell out the affine part j_k (the finite parallelism index) as well as the infinite part expressed by i_k (the infinite parallelism index). We could form another divisor on the line at infinity, namely $\sum_{k=1}^{s} (i_k + j_k) p_k$ whose coefficients are the total parallelism indices.

Definition 1.9. We define the parallelism type of the configuration (or simply type of the configuration) of invariant lines occurring for a cubic polynomial system (S), the sequence of non-zero numbers, $\tau_k = i_k + j_k$, $k \in \{1, \ldots, s\}$ attached to $D_L(S; Z)$, listed according to descending magnitudes:

$$\mathfrak{T} = (\tau_1, \tau_2, \ldots, \tau_l), \quad 1 \leq l \leq s.$$

Clearly $\mathfrak{T}$ is an affine invariant of systems in the class $\mathbf{PSL}_{n,\mathfrak{L}}$ and of their configurations of invariant lines.

Notation 1.10. As already used in the Abstract $\mathbf{CSL}_7^{2r2c\infty}$ is meant to be the class of non-degenerate cubic systems with invariant lines of total multiplicity seven which have two real and two complex distinct singularities at infinity.

As we have two real and two complex infinite singularities and the total multiplicity of the invariant lines (including the line at infinity) must be 7, then the cubic systems in $\mathbf{CSL}_7^{2r2c\infty}$ could only have one of the following four possible types of configurations of invariant lines:

$$(i)\ \mathfrak{T}=(3,3);\quad (ii)\ \mathfrak{T}=(3,1,1,1);\quad (iii)\ \mathfrak{T}=(2,2,2);\quad (iv)\ \mathfrak{T}=(2,2,1,1). \tag{1.2}$$

Remark 1.11. We remark that the cubic systems in $\mathbf{CSL}_7^{2r2c\infty}$ possessing the configurations of invariant lines of the type $\mathfrak{T}=(3,3)$ were already investigated in [6], where the existence of 14 distinct configurations *Config. 7.1a* $-$*Config. 7.14a* of this type are determined.

In this article we classify the subfamily of cubic systems in $\mathbf{CSL}_7^{2r2c\infty}$, possessing configurations of invariant line of the type $(3,1,1,1)$, according to the relation of equivalence of configurations. We denote this subfamily by $\mathbf{CSL}_{(3,1,1,1)}^{2r2c\infty}$.

Our main result is the following one.

Main Theorem.

(A) *A non-degenerate cubic system* (1.1) *belongs to the class* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$ *if and only if* $\mathcal{D}_1<0$, $\mathcal{V}_4=\mathcal{U}_2=0$ *and one of the following set of conditions holds:*

- (A_1) *If* $\mathcal{D}_7\neq 0$, $\mathcal{D}_8\neq 0$, $\chi_1=0$, $\mathcal{D}_6\neq 0$ *then* $\chi_3=\chi_6=0$.
- (A_2) *If* $\mathcal{D}_7\neq 0$, $\mathcal{D}_8\neq 0$, $\chi_1=0$, $\mathcal{D}_6=0$ *then* $\chi_2=\chi_3=0$.
- (A_3) *If* $\mathcal{D}_7\neq 0$, $\mathcal{D}_8\neq 0$, $\chi_1\neq 0$, $\mathcal{D}_4\neq 0$ *then* $\chi_7=\chi_8=\chi_9=\chi_{10}$ *and either* $\mathcal{D}_5\neq 0$, $\chi_{11}=0$ *or* $\mathcal{D}_5=\chi_{12}=0$.
- (A_4) *If* $\mathcal{D}_7\neq 0$, $\mathcal{D}_8\neq 0$, $\chi_1\neq 0$, $\mathcal{D}_4=0$ *then* $\chi_4=\chi_5=\chi_7=\chi_9=\chi_{13}=\chi_{14}=0$.
- (A_5) *If* $\mathcal{D}_7\neq 0$, $\mathcal{D}_8=0$, $\mathcal{D}_6\neq 0$, $\mathcal{D}_4\neq 0$ *then* $\chi_1=\chi_3=\chi_6=0$.
- (A_6) *If* $\mathcal{D}_7\neq 0$, $\mathcal{D}_8=0$, $\mathcal{D}_6\neq 0$, $\mathcal{D}_4=0$ *then* $\chi_1=\chi_3=\chi_8=\chi_{16}=0$, $\chi_{15}\neq 0$.
- (A_7) *If* $\mathcal{D}_7\neq 0$, $\mathcal{D}_8=0$, $\mathcal{D}_6=0$ *then* $\chi_1=\chi_2=\chi_4=\chi_6=\chi_{17}=0$, $\chi_{11}\neq 0$, $\zeta_4\leq 0$.
- (A_8) *If* $\mathcal{D}_7=0$, $\widetilde{\chi}_1\neq 0$ *then* $\chi_1=\chi_2=\chi_3=0$.
- (A_9) *If* $\mathcal{D}_7=0$, $\widetilde{\chi}_1=0$, $\widetilde{\chi}_2\neq 0$ *then* $\chi_1=\chi_3=\chi_6=0$.

If $\mathcal{D}_7=\widetilde{\chi}_1=\widetilde{\chi}_2=0$ *then a cubic system* (1.1) *could not belong to the class* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$.

(B) *Assume that a non-degenerate cubic system* (1.1) *belongs to the class* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$, *i.e. one of the sets of conditions provided by statement* (A) *holds. Then this system possesses one of the configurations Config. 7.1b – Config. 7.42b, presented in Figure 1.1. Moreover the necessary and sufficient conditions for the realization of each one the mentioned configurations are given in Diagrams from Figures 1.2, 1.3 and 1.4, correspondingly.*

(C) *In Figure 1.1 are given all the configurations that could occur for systems in the class* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$. *We prove that all these configurations are realizable within* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$ *(see the examples given in the proof of the statement* (A)*) and that these 42 configurations are distinct. This proof is done in Subsection 3.3 using geometric invariants and it is presented in the corresponding diagram from Figures 3.1.*

Notation 1.12. We give here the directions for reading the pictures representing the configurations. An invariant line with multiplicity $k>1$ will appear in a configuration in bold face and will have next to it the number k. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. The multiplicities of the real singular points of the system located on the invariant lines, will be indicated next to the singular points.

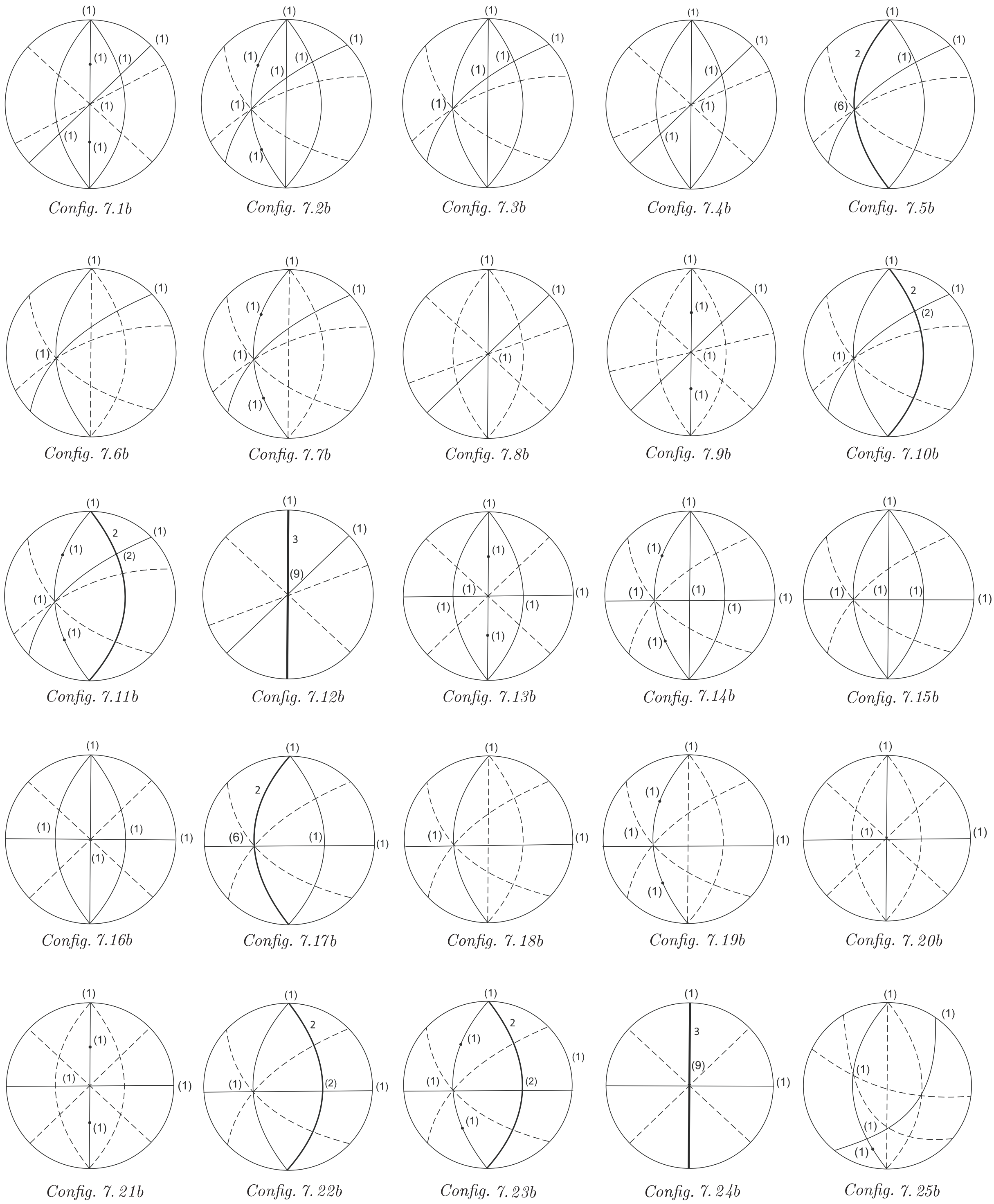

Figure 1.1: The configurations of invariant lines for cubic systems in the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$

Since a configuration of invariant lines of a system (1.1) could contain simultaneously real and complex invariant lines, there appears the problem of indicating these lines simultaneously on a picture in the Poincaré disc in order to capture and see schematically this phenomenon. So in order to fix the positions of real lines with respect to the complex ones in

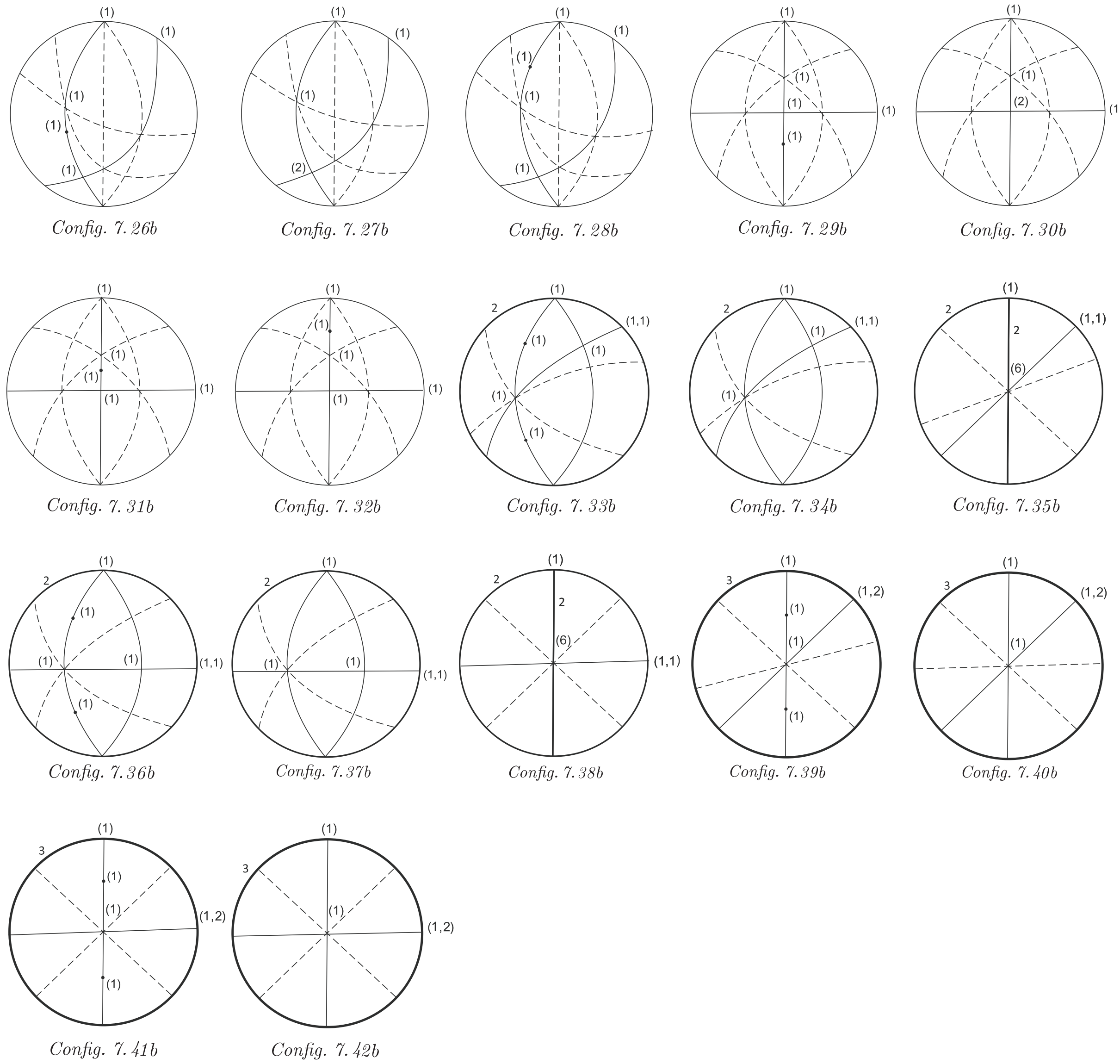

Figure 1.1 (*cont.*): The configurations of invariant lines for cubic systems in the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$

a coherent way, we present here the following mode of representing complex invariant lines of systems (1.1) along with the real invariant ones on the Poincaré disc.

Convention. Assume that a system (1.1) possesses an invariant line with complex coefficients such that we cannot multiply all its coefficients by a non-zero complex number and obtain real coefficients. Then clearly the corresponding conjugate line is also invariant for this system having the same property. Suppose that such invariant lines are:

$$L : Ax + By + C = 0, \qquad \bar{L} : \bar{A}x + \bar{B}y + \bar{C} = 0, \quad A, B, C \in \mathbb{C}, \quad (A, B) \neq (0, 0).$$

These lines are affine lines in $\mathbb{C}^2$ ($\cong \mathbb{R}^4$) and hence planes in $\mathbb{R}^4$.

Without loss of generality, due to the change $x \leftrightarrow y$ we may assume $B \neq 0$ and then the lines become:

$$y = (a \pm bi)x + (c \pm di) = (ax + c) \pm i(bx + d), \quad (a, b, c, d) \in \mathbb{R}^4, \ b^2 + d^2 \neq 0. \tag{1.3}$$

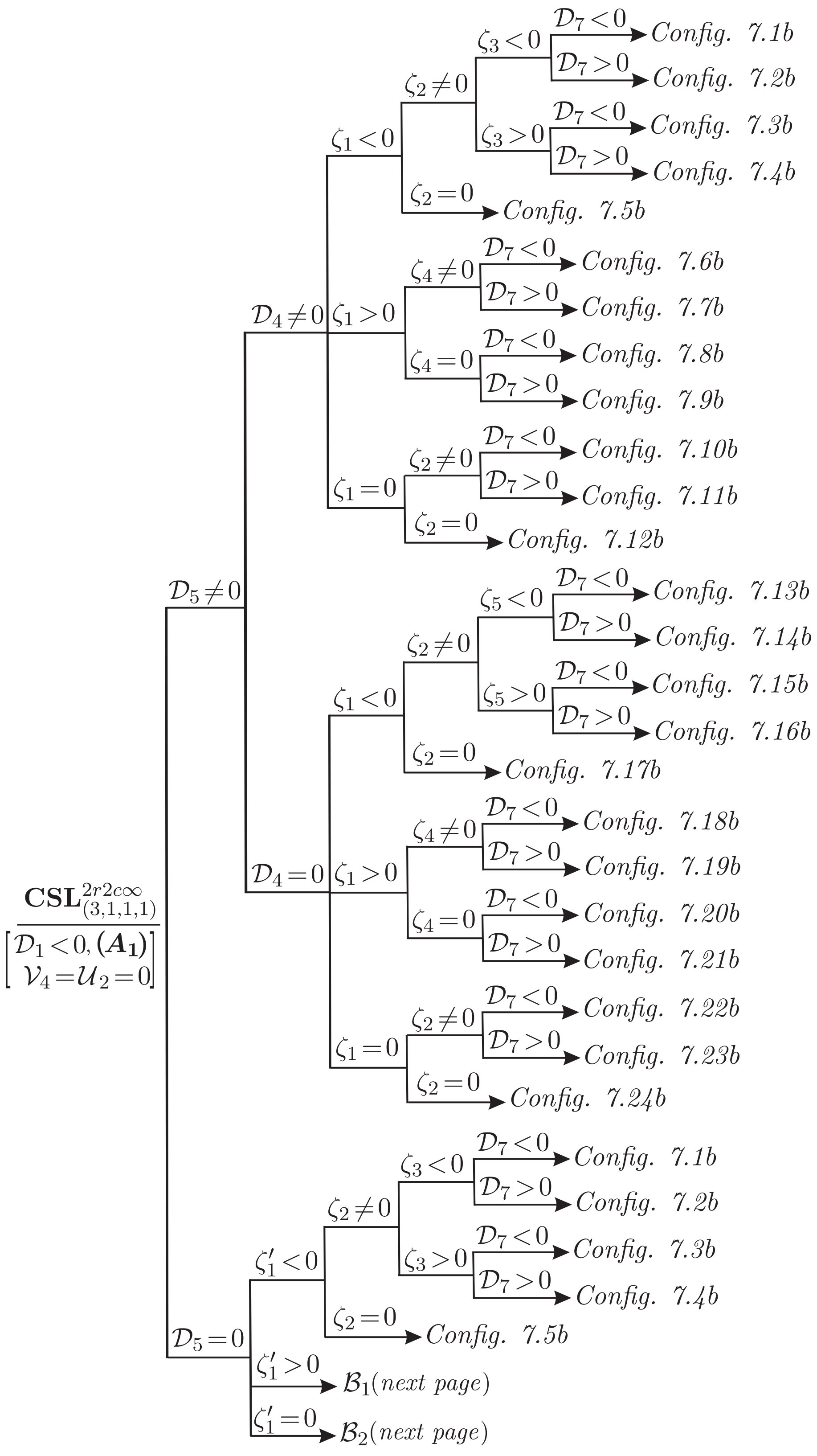

Figure 1.2: Diagram of the configurations for the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$: statement $(\boldsymbol{A_1})$

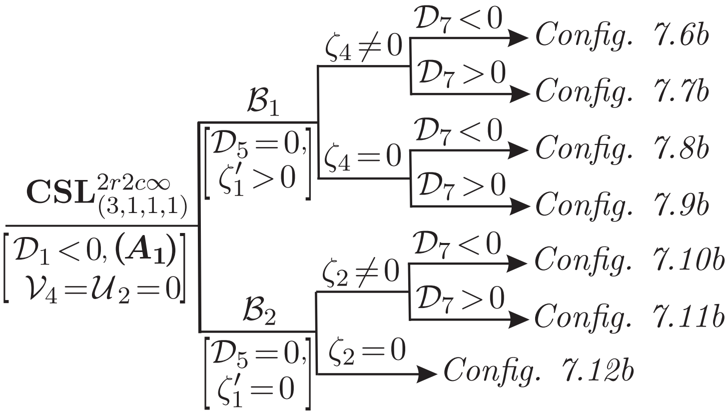

Figure 1.2 (*cont.*): Diagram of the configurations for the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$: statement $(\boldsymbol{A_1})$

We associate to the lines (1.3) over $\mathbb{C}$ the two lines with real coefficients: the real line $l = \mathcal{R}(L,\bar{L})$: $y = ax + c$ as well as its complexification Cl defined by the same equation but letting x, y run over the complex plane. The real line l can be drawn on the Poincaré disk. Consider now the two cases $b \neq 0$ and $b = 0$.

Case $b \neq 0$. In this case the two lines (1.3) intersect at the real point $M_0 = (-d/b, -(ad - bc)/b) \in \mathbb{R}^2$ that also lies on the real line $l \subset Cl$. Being at the intersection of the two complex invariant lines (1.3) , the real point M_0 is a singular point for systems (1.1). To signal the presence of the complex lines (1.3) we make the convention to represent them on the Poincaré disk as two dashed lines both passing through M_0. Thus the real line l will appear inside two of the four curvilinear triangles described by the dashed lines and parts of the circle at infinity. We denote this domain by $\mathcal{D}$.

Suppose now that the system S has a real invariant line l' also passing through M_0 and consider its complexification $L' = Cl'$ that is also an invariant line.

We assume that our system is included in a family of systems possessing the invariant lines (1.3) and the line L′. If the parameters b and d tend simultaneously to zero, then it is clear that the two complex lines tend to the complexification of the real line $y = ax + c$. Then clearly this line is an invariant line that is a multiple line of multiplicity two or three. We now distinguish two subcases: $l' = l$ or $l' \neq l$.

Subcase $l' = l$. In this case two complex invariant lines (1.3) coalesced with the invariant line L' and hence this is a triple line. In this case we will draw the real line l' inside the domain $\mathcal{D}$.

Subcase $l' \neq l$. In this case if both b and d tend to zero then the lines (1.3) will tend to a double line, the complexification of the real line $y = ax + c$. In this case we draw the line l' outside $\mathcal{D}$.

Case $b = 0$. In this case the lines (1.3) intersect at infinity at the real point $[1 : a : 0]$. The real line $l : y = ax + c$ passes also through this point. We draw by dashed lines these two complex lines placing inside the domain delimited by them and denoted by $\mathcal{D}'$ the real line l. Suppose the line L' passes through the same point at infinity $[1 : a : 0]$. We make the following convention:

If $l' = l$ then we will draw l' inside the domain $\mathcal{D}'$. If $l' \neq l$ then we will draw l' outside the domain $\mathcal{D}'$.

The work is organized as follows. In Section 2 we give some preliminary results needed

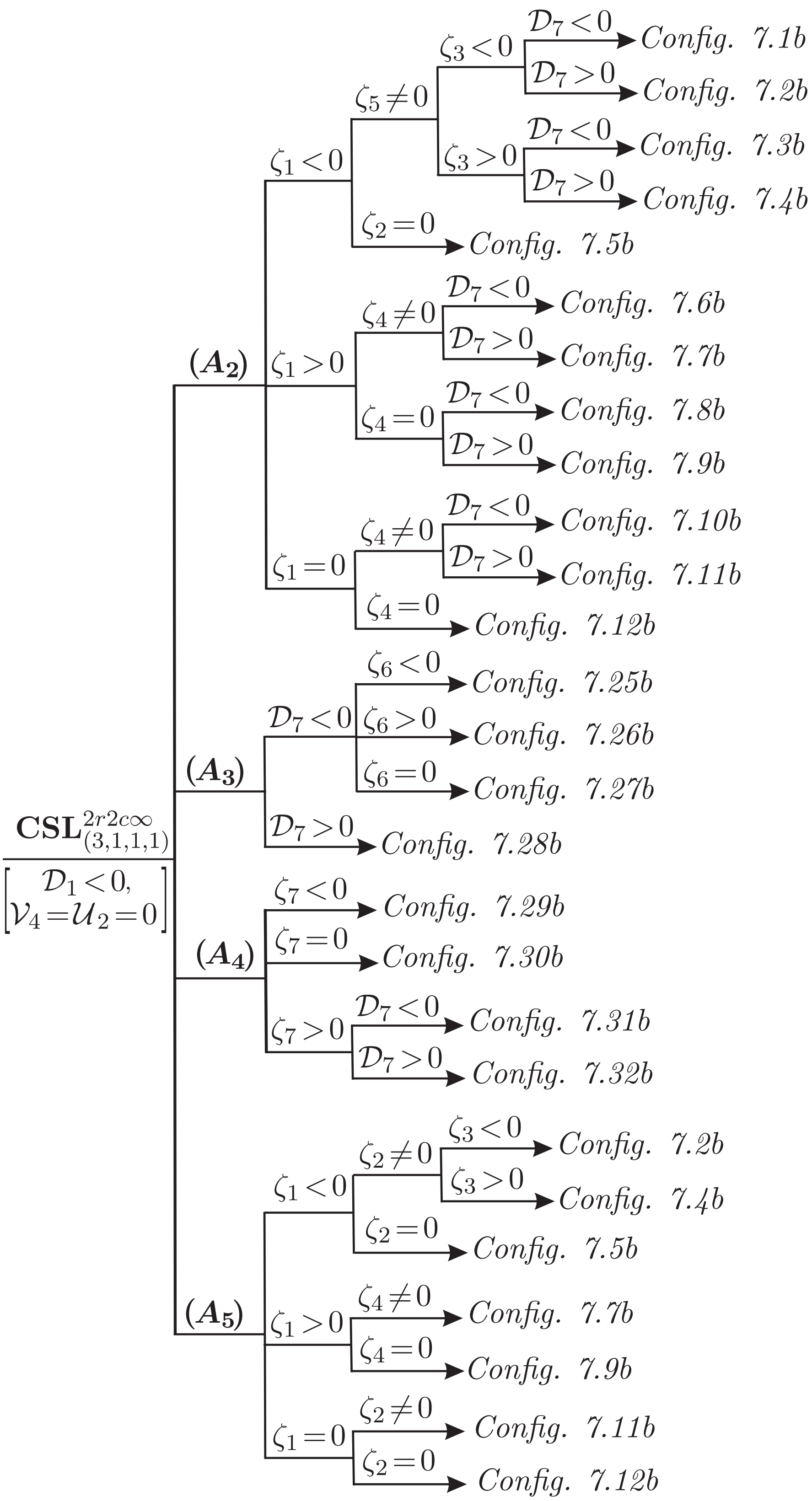

Figure 1.3: Diagram of the configurations for the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$: statements (A_2)–(A_5)

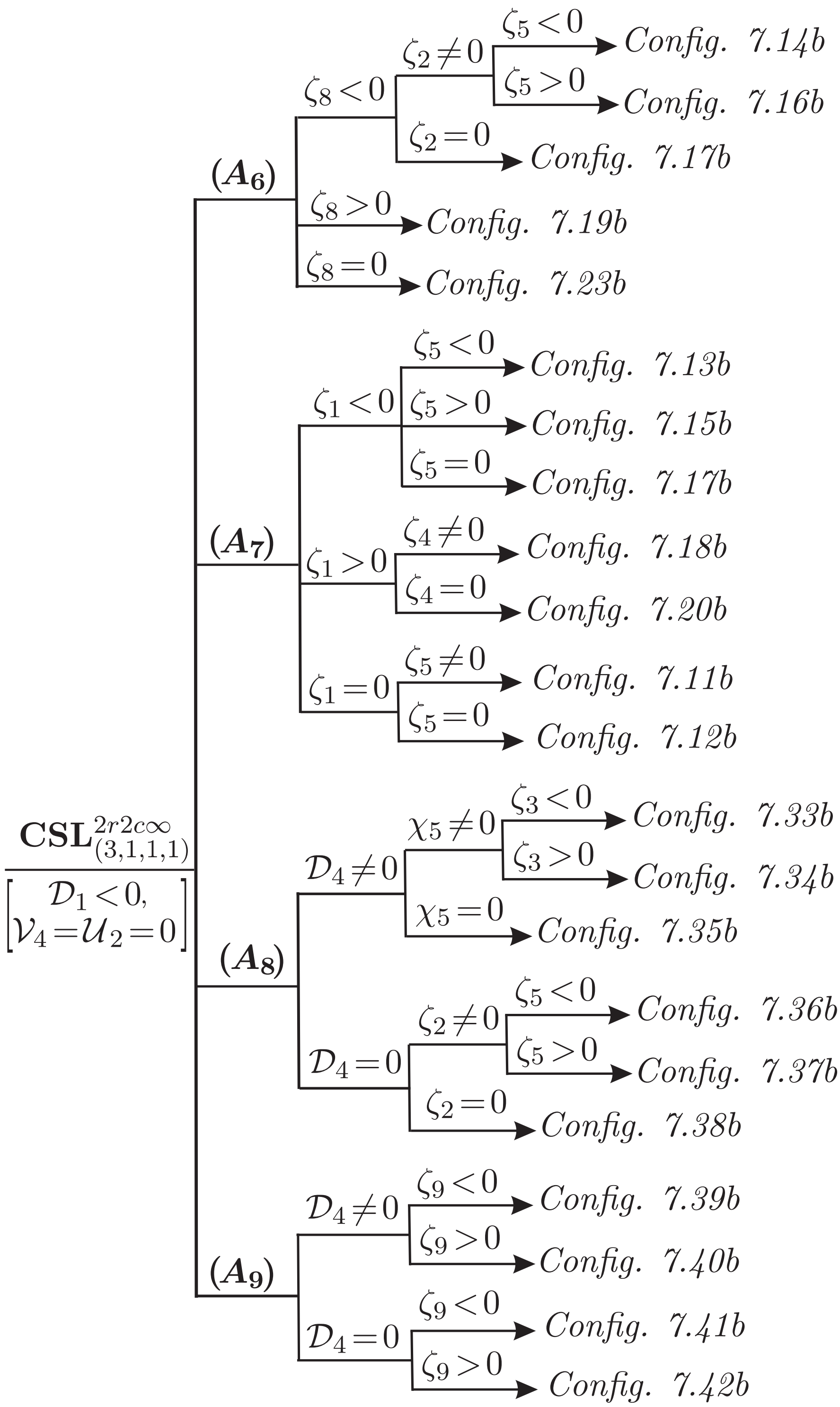

Figure 1.4: Diagram of the configurations for the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$: statements $(\boldsymbol{A_6})$–$(\boldsymbol{A_9})$

for this paper. In Section 3 we prove our Main Theorem considering the family of cubic systems possessing invariant lines in the configuration of the type $(3,1,1,1)$ and having two real and two complex distinct infinite singularities. More exactly, in Subsection 3.1 we prove the statement (A) of the Main Theorem, constructing the canonical systems and determining the corresponding configurations which these systems could possess. Moreover, the necessary and sufficient conditions for the realization of each one the obtained configurations are determined. In Subsection 3.2 we prove the statement (B) of the Main Theorem. Using the geometric invariants, we prove that all the 42 detected configurations of invariant lines for the class of cubic systems in $\mathbf{CSL}_{(3,1,1,1)}^{2r2c\infty}$ are distinct according to Definition 1.3.

2 Preliminaries

Consider real cubic systems, i.e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(a,x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(a,x,y) \end{aligned} \tag{2.1}$$

with variables x and y and real coefficients. The polynomials p_i and q_i $(i = 0,1,2,3)$ are homogeneous polynomials of degree i in x and y:

$$\begin{aligned} &p_0 = a_{00}, \quad p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3, \\ &p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ &q_0 = b_{00}, \quad q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3, \\ &q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (2.1) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03}, x, y]$.

2.1 The main invariant polynomials associated to configurations of invariant lines

It is known that on the set of polynomial systems (1.1), in particular on the set **CS** of all cubic differential systems (2.1), acts the group $Aff(2,\mathbb{R})$ of affine transformation on the plane [40]. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **CS**. We can identify the set **CS** of systems (2.1) with a subset of $\mathbb{R}^{20}$ via the map $\mathbf{CS} \longrightarrow \mathbb{R}^{20}$ which associates to each system (2.1) the 20-tuple $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ of its coefficients.

For the definitions of an affine or GL-comitant or invariant as well as for the definition of a T-comitant and CT-comitant we refer the reader to [36]. Here we shall only construct the necessary affine invariant polynomials which are needed to detect the existence of invariant lines for the class of cubic systems with four real distinct infinite singularities and with exactly seven invariant straight lines including the line at infinity and including multiplicities.

Let us consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y], \quad i = 0,1,2,3, \\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y], \quad i = 1,2,3. \end{aligned}$$

As it was shown in [42] the polynomials

$$\{C_0(a,x,y),\ C_1(a,x,y),\ C_2(a,x,y),\ C_3(a,x,y),\ D_1(a),\ D_2(a,x,y)\ D_3(a,x,y)\} \tag{2.2}$$

of degree one in the coefficients of systems (2.1) are GL-comitants of these systems.

Notation 2.1. Let $f, g \in \mathbb{R}[a, x, y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

$(f,g)^{(k)} \in \mathbb{R}[a, x, y]$ is called the transvectant of index k of (f,g) (cf. [20,28]).

Theorem 2.2 ([46]). *Any GL-comitant of systems (2.1) can be constructed from the elements of the set (2.2) by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(f,g)^{(k)}$.*

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $P(a,x,y)$ and $Q(a,x,y)$. We obtain $\tilde{P}(\tilde{a}(a,x_0,y_0),x',y') = P(a,x'+x_0,y'+y_0)$, $\tilde{Q}(\tilde{a}(a,x_0,y_0),x',y') = Q(a,x'+x_0,y'+y_0)$. We construct the following polynomials

$$\Omega_i(a,x_0,y_0) \equiv \text{Res}_{\,x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big), C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1},$$
$$\Omega_i(a,x_0,y_0) \in \mathbb{R}[a,x_0,y_0], \quad (i=1,2,3)$$

and we denote

$$\tilde{\mathcal{G}}_i(a,x,y) = \Omega_i(a,x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[a,x,y] \quad (i=1,2,3).$$

Remark 2.3. We note that the polynomials $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ and $\tilde{\mathcal{G}}_3(a,x,y)$ are affine comitants of systems (2.1) and are homogeneous polynomials in the coefficients $a_{00},\ldots,b_{03}$ and non-homogeneous in x,y and

$$\begin{array}{lll} \deg_a \tilde{\mathcal{G}}_1 = 3, & \deg_a \tilde{\mathcal{G}}_2 = 4, & \deg_a \tilde{\mathcal{G}}_3 = 5, \\ \deg_{(x,y)} \tilde{\mathcal{G}}_1 = 8, & \deg_{(x,y)} \tilde{\mathcal{G}}_2 = 10, & \deg_{(x,y)} \tilde{\mathcal{G}}_3 = 12. \end{array}$$

Notation 2.4. Let $\mathcal{G}_i(a,X,Y,Z)$ $(i=1,2,3)$ be the homogenization of $\tilde{\mathcal{G}}_i(a,x,y)$, i.e.

$$\begin{aligned} \mathcal{G}_1(a,X,Y,Z) &= Z^8 \tilde{\mathcal{G}}_1(a,X/Z,Y/Z), \\ \mathcal{G}_2(a,X,Y,Z) &= Z^{10} \tilde{\mathcal{G}}_2(a,X/Z,Y/Z), \\ \mathcal{G}_3(a,X,Y,Z) &= Z^{12} \tilde{\mathcal{G}}_3(a,X/Z,Y/Z), \end{aligned}$$

and $\mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{G}_1(a,X,Y,Z),\ \mathcal{G}_2(a,X,Y,Z),\ \mathcal{G}_3(a,X,Y,Z)\Big)$ in $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [25]):

Lemma 2.5. *The straight line $\mathcal{L}(x,y) \equiv ux + vy + w = 0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant line for a cubic system (2.1) if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\tilde{\mathcal{G}}_1(x,y)$, $\tilde{\mathcal{G}}_2(x,y)$ and $\tilde{\mathcal{G}}_3(x,y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{G}}_i(x,y) = (ux+vy+w)\widetilde{W}_i(x,y) \quad (i=1,2,3),$$

where $\widetilde{W}_i(x,y) \in \mathbb{C}[x,y]$.

Lemma 2.6. *Consider a cubic system (2.1) and let $\boldsymbol{a} \in \mathbb{R}^{20}$ be its 20-tuple of coefficients.*

1) *If $\mathcal{L}(x,y) \equiv ux+vy+w=0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant straight line of multiplicity k for the system associated to $\boldsymbol{a}$ then $[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, \tilde{\mathcal{G}}_3)$ in $\mathbb{C}[x,y]$, i.e. there exist $W_i(a,x,y) \in \mathbb{C}[x,y]$ $(i=1,2,3)$ such that*

$$\tilde{\mathcal{G}}_i(\boldsymbol{a},x,y) = (ux+vy+w)^k W_i(a,x,y), \quad i=1,2,3.$$

2) *If the line $l_\infty : Z=0$ is of multiplicity $k>1$ then $Z^{k-1} \mid \gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$, i.e. we have $Z^{k-1} \mid H(\boldsymbol{a},X,Y,Z)$.*

Consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ constructed in [3] and acting on $\mathbb{R}[a,x,y]$, where

$$\begin{aligned}
\mathbf{L}_1 = {} & 3a_{00}\frac{\partial}{\partial a_{10}} + 2a_{10}\frac{\partial}{\partial a_{20}} + a_{01}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}} + a_{20}\frac{\partial}{\partial a_{30}} \\
& + 3b_{00}\frac{\partial}{\partial b_{10}} + 2b_{10}\frac{\partial}{\partial b_{20}} + b_{01}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}} + b_{20}\frac{\partial}{\partial b_{30}}, \\
\mathbf{L}_2 = {} & 3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{10}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}} + a_{02}\frac{\partial}{\partial a_{03}} \\
& + 3b_{00}\frac{\partial}{\partial b_{01}} + 2b_{01}\frac{\partial}{\partial b_{02}} + b_{10}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}} + b_{02}\frac{\partial}{\partial b_{03}}.
\end{aligned}$$

Using this operator and the affine invariant $\mu_0 = \operatorname{Resultant}_x\big(p_3(a,x,y), q_3(a,x,y)\big)/y^9$ we construct the following polynomials

$$\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \quad i=1,\ldots,9,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (2.1) with respect to the group $GL(2,\mathbb{R})$ (see [3]). The polynomial $\mu_i(a,x,y)$, $i \in \{0,1,\ldots,9\}$ is homogeneous of degree 6 in the coefficients of systems (2.1) and homogeneous of degree i in the variables x and y. The geometrical meaning of these polynomial is revealed in the next lemma.

Lemma 2.7 ([2,3]). *Assume that a cubic system (S) with coefficients $\boldsymbol{a} \in \mathbb{R}^{20}$ belongs to the family (2.1). Then:*

(i) The total multiplicity of all finite singularities of this system equals $9-k$ if and only if for every $i \in \{0,1,\ldots,k-1\}$ we have $\mu_i(\boldsymbol{a},x,y)=0$ in the ring $\mathbb{R}[x,y]$ and $\mu_k(\boldsymbol{a},x,y) \neq 0$. In this case the factorization $\mu_k(\boldsymbol{a},x,y) = \prod_{i=1}^{k}(u_i x - v_i y) \neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of singularities at infinity which in perturbations generate finite singularities of the system (S). Moreover the number of distinct factors in this factorization is less than or equal to four (the maximum number of infinite singularities of a cubic system) and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system (S) which have coalesced with the infinite singular point $[v_i : u_i : 0]$.

(ii) The point $M_0(0,0)$ is a singular point of multiplicity k $(1 \leq k \leq 9)$ for the cubic system (S) if and only if for every i such that $0 \leq i \leq k-1$ we have $\mu_{9-i}(\boldsymbol{a},x,y) = 0$ in $\mathbb{R}[x,y]$ and $\mu_{9-k}(\boldsymbol{a},x,y) \neq 0$.

(iii) The system (S) is degenerate (i.e. $\gcd(p,q) \neq \text{const}$) if and only if $\mu_i(\boldsymbol{a},x,y) = 0$ in $\mathbb{R}[x,y]$ for every $i=0,1,\ldots,9$.

In order to define the invariant polynomials we need, we first construct the following comitants of second degree with respect to the coefficients of initial systems (2.1):

$$
\begin{array}{lll}
S_1 = (C_0, C_1)^{(1)}, & S_{10} = (C_1, C_3)^{(1)}, & S_{19} = (C_2, D_3)^{(1)}, \\
S_2 = (C_0, C_2)^{(1)}, & S_{11} = (C_1, C_3)^{(2)}, & S_{20} = (C_2, D_3)^{(2)}, \\
S_3 = (C_0, D_2)^{(1)}, & S_{12} = (C_1, D_3)^{(1)}, & S_{21} = (D_2, C_3)^{(1)}, \\
S_4 = (C_0, C_3)^{(1)}, & S_{13} = (C_1, D_3)^{(2)}, & S_{22} = (D_2, D_3)^{(1)}, \\
S_5 = (C_0, D_3)^{(1)}, & S_{14} = (C_2, C_2)^{(2)}, & S_{23} = (C_3, C_3)^{(2)}, \\
S_6 = (C_1, C_1)^{(2)}, & S_{15} = (C_2, D_2)^{(1)}, & S_{24} = (C_3, C_3)^{(4)}, \\
S_7 = (C_1, C_2)^{(1)}, & S_{16} = (C_2, C_3)^{(1)}, & S_{25} = (C_3, D_3)^{(1)}, \\
S_8 = (C_1, C_2)^{(2)}, & S_{17} = (C_2, C_3)^{(2)}, & S_{26} = (C_3, D_3)^{(2)}, \\
S_9 = (C_1, D_2)^{(1)}, & S_{18} = (C_2, C_3)^{(3)}, & S_{27} = (D_3, D_3)^{(2)}.
\end{array}
$$

We shall use here the following invariant polynomials constructed in [25] and [10]:

$$
\begin{aligned}
\mathcal{D}_1(a) &= 6S_{24}^3 - \left[(C_3, S_{23})^{(4)}\right]^2, \\
\mathcal{D}_2(a,x,y) &= -S_{23}, \\
\mathcal{D}_3(a,x,y) &= (S_{23}, S_{23})^{(2)} - 6C_3(C_3, S_{23})^{(4)}, \\
\mathcal{D}_4(a) &= (C_3, \mathcal{D}_2)^{(4)}, \\
\mathcal{D}_5(a) &= A_1 - A_2, \\
\mathcal{D}_6(a) &= 3A_1 + A_2, \\
\mathcal{D}_7(a) &= -A_1 - 3A_2, \\
\mathcal{D}_8(a) &= 2A_1^3 - 9A_6^2 + 2A_1A_{10} + A_{16}, \\
\mathcal{V}_1(a,x,y) &= S_{23} + 2D_3^2, \\
\mathcal{V}_2(a,x,y) &= S_{26}, \\
\mathcal{V}_3(a,x,y) &= 6S_{25} - 3S_{23} - 2D_3^2, \\
\mathcal{V}_4(a,x,y) &= C_3\left[(C_3, S_{23})^{(4)} + 36\,(D_3, S_{26})^{(2)}\right], \\
\mathcal{V}_5(a,x,y) &= 6T_1(9A_5-7A_6) + 2T_2(4T_{16} - T_{17}) - 3T_3(3A_1+5A_2) + 3A_2T_4 + 36T_5^2 - 3T_{44}, \\
\mathcal{U}_1(a) &= S_{24} - 4S_{27}, \\
\mathcal{U}_2(a,x,y) &= 6\,(S_{23} - 3S_{25}, S_{26})^{(1)} - 3S_{23}(S_{24} - 8S_{27}) - 24S_{26}^2 \\
&\quad + 2C_3\,(C_3, S_{23})^{(4)} + 24D_3\,(D_3, S_{26})^{(1)} + 24D_3^2S_{27},
\end{aligned}
$$

In order to characterize the cubic systems belonging to the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$ we define here the following new invariant polynomials:

$$
\begin{aligned}
\chi_1(a,x,y) &= T_{13} - 2T_{11}, \\
\chi_2(a,x,y) &= 8A_3T_2 + 22A_4T_2 + 15T_{57} + 9T_{60} - 21T_{62} + 6T_{63} + 9T_{65}, \\
\chi_3(a,x,y) &= 2T_1T_8T_{15} + 2T_5T_{74} + T_5T_{75}, \\
\chi_4(a) &= A_7 + A_8 - A_9, \\
\chi_5(a) &= A_7, \\
\chi_6(a,x,y) &= 30(6A_3T_1^2 + 9T_5T_6 - 3T_4T_9 - T_2T_{26}) - T_1(29T_2T_{14} + 32T_2T_{15} - 108T_{36} - 45T_{42}), \\
\chi_7(a,x,y) &= T_{12} - T_{13},
\end{aligned}
$$

$$\begin{aligned}
\chi_8(a,x,y) &= 10A_3T_2 + 30A_4T_2 - 6T_{59} + 15T_{60} + 15T_{57} - 31T_{62} + 17T_{63} + 5T_{64} + 5T_{65},\\
\chi_9(a,x,y) &= 6T_5(3T_{11} - 4T_{13}) + 10T_3T_{18} + 6T_4T_{18} - 3T_2T_{48} + 2T_2(T_{49} + T_{50}) + 22T_1T_{71} + T_{86},\\
\chi_{10}(a,x,y) &= 880A_3T_1(101T_2T_6 - 36T_1T_9) + 337920(T_{11} - T_{13})(T_{74} + T_{75}) - 880(5T_2^2 + 27T_3)T_9^2\\
&\quad - 528T_9(120A_4T_1^2 + 11658T_5T_6 + 50T_2T_{26} - 60T_1T_{37} + 25T_{76} - 80T_{78})\\
&\quad - 44T_{19}(21442T_2T_{15} - 259854T_{36} + 42588T_{37} + 59307T_{42} - 42888T_{38})\\
&\quad - 2640T_{26}(128T_{25} - 3T_{23} + 10T_{24} + 24T_{26}) + 24T_4T_6(344426T_{14} - 921997T_{15})\\
&\quad - 3T_6(345752T_3T_{15} - 1006720T_{80} - 1019038T_{81} + 969523T_{82} + 2177623T_{83} - 11264T_{84}),\\
\chi_{11}(a,x,y) &= 360A_7T_2 + 3066T_{110} - 270T_{111} + 148T_{113} - 1895T_{114} + 2675T_{115} - 1176T_{116} + 3090T_{117}\\
&\quad - 540T_{118} - 680T_{119} + 155T_{120} + 1375T_{121},\\
\chi_{12}(a,x,y) &= 18T_2^2T_9 - T_2(36T_{23} + 324T_{24} - 737T_{26}) - T_1(108T_2T_{15} - 6(460T_{36} - 629T_{37} - 656T_{42}))\\
&\quad - 3(29028T_5T_6 + 54T_3T_9 - 629T_4T_9 + 96T_{78}),\\
\chi_{13}(a,x,y) &= -60(2A_{14} + 47A_{15})T_2 - 12180A_4T_{17} + 30A_3(47T_{16} + 51T_{17}) - 105A_1(T_{57} + 12T_{63})\\
&\quad - A_2(1200T_{60} - 174T_{59} - 255T_{57} - 5754T_{62} + 2403T_{63} - 4435T_{64} + 7820T_{65}),\\
\chi_{14}(a,x,y) &= 3T_1T_8T_{15} - 3T_5T_{75},\\
\chi_{15}(a,x,y) &= T_8,\\
\chi_{16}(a,x,y) &= 96T_6T_8 + 12T_{133} + 9T_{135} + 28T_2T_{74} + 21T_2T_{75},\\
\chi_{17}(a,x,y) &= 9T_6T_9(174T_1T_9 + 193T_{19}) + T_6^2(77T_2T_9 + 1164T_1T_{14} - 69T_{23} - 57T_{24}) - 696T_{74}^2,\\
\widetilde{\chi}_1(a,x,y) &= T_{13} - 2T_{12},\\
\widetilde{\chi}_2(a,x,y) &= 3T_2T_6 + 2T_1T_9 + T_{19},\\
\zeta_1(a,x,y) &= (A_1 - A_2)(5A_3T_2 + 25A_4T_2 - 9T_{59} + 15T_{57} - 39T_{62} + 33T_{63}),\\
\zeta_1'(a,x,y) &= 972T_1(A_8T_2 + 6T_{107}) - 5832T_5(5T_{36} + T_{38}) + 27(14904T_{11}^2 + 216T_{10}T_{15} - 16344T_8T_{18}\\
&\quad - 7T_2^2T_{59} - 81T_3T_{59} + 18T_4T_{59}),\\
\zeta_2(a) &= 432A_2A_4 - 162A_{12} - 81A_{13} - 27A_{14} - 648A_{15},\\
\zeta_3(a) &= A_7(2A_1A_9 - 3A_4A_6),\\
\zeta_4(a,x,y) &= T_{59},\\
\zeta_5(a) &= 36A_1^2A_4^2 - (A_{12} - 4A_{13} - A_{14} - 2A_{15})^2,\\
\zeta_6(a) &= 8A_1^2A_2 + 58A_2^3 - 29A_6^2 + 82A_2A_{10} + 245A_{16},\\
\zeta_7(a) &= -(5A_1 + 3A_2),\\
\zeta_8(a) &= A_4(2A_3 + 3A_4),\\
\zeta_9(a) &= -T_9T_{17},
\end{aligned}$$

where

$$\begin{aligned}
&A_1 = S_{24}/288, \quad A_2 = S_{27}/72, \quad A_3 = \big(72D_1A_2 + (S_{22}, D_2)^{(1)}\big)/24,\\
&A_4 = \big[9D_1S_{24} - 2592D_1A_2 + 36(S_{11}, D_3)^{(2)} + 24(S_{18}, D_2)^{(1)} - 8(S_{14}, D_3)^{(2)} - 8(S_{20}, D_2)^{(1)}\\
&\qquad - 32(S_{22}, D_2)^{(1)}\big]/2^7/3^3, \quad A_6 = \big(S_{26}, D_3\big)^{(2)}/2^5/3^3, \quad A_7 = (T_9, C_3)^{(4)}/2^5/3^2,\\
&A_8 = (T_{14}, D_3)^{(2)}/12, \quad A_9 = (T_{15}, D_3)^{(2)}/12, \quad A_{10} = [\![S_{23}, D_3)^{(2)}, D_3)^{(2)}/2^9/3^4,\\
&A_{12} = [\![T_9, C_3)^{(3)}, D_3)^{(2)}/2^6/3^3, \quad A_{13} = [\![T_9, C_3)^{(2)}, C_3)^{(4)}/2^7/3^3,\\
&A_{14} = [\![T_9, D_3)^{(2)}, D_3)^{(2)}/2^5/3^2, \quad A_{15} = [\![T_{14}, C_3)^{(2)}, D_3)^{(2)}/2^5/3^2,\\
&A_{16} = [\![S_{23}, C_3)^{(1)}, D_3)^{(2)}, D_3)^{(2)}, D_3)^{(2)}/5/2^{13}/3^7
\end{aligned}$$

are affine invariants, whereas the polynomials

$$T_1 = C_3, \quad T_2 = D_3, \quad T_3 = S_{23}/18, \quad T_4 = S_{25}/6, \quad T_5 = S_{26}/72,$$
$$T_6 = \big[3C_1(D_3^2 - 9T_3 + 18T_4) - 2C_2(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21}) + $$
$$+ 2C_3(2D_2^2 - S_{14} + 8S_{15})\big]/2^4/3^2, \quad T_7 = (S_{23}, C_3)^{(1)}/72,$$
$$T_8 = \big[5D_2(D_3^2 + 27T_3 - 18T_4) + 20D_3S_{19} + 12\big(S_{16}, D_3\big)^{(1)} - 8D_3S_{17}\big]/5/2^5/3^3,$$
$$T_9 = \big[9D_1(9T_3 - 18T_4 - D_3^2) + 2D_2(D_2D_3 - 3S_{17} - S_{19} - 9S_{21}) + 18\big(S_{15}, C_3\big)^{(1)} -$$
$$- 6C_2(2S_{20} - 3S_{22}) + 18C_1S_{26} + 2D_3S_{14}\big]/2^4/3^3, \quad T_{10} = (S_{23}, D_3)^{(1)}/2^5/3^3,$$
$$T_{11} = \big[\big(D_3^2 - 9T_3 + 18T_4, C_2\big)^{(2)} - 6\big(D_3^2 - 9T_3 + 18T_4, D_2\big)^{(1)} - 12\big(S_{26}, C_2\big)^{(1)} +$$
$$+ 12D_2S_{26} + 432C_2(A_1 - 5A_2)\big]/2^7/3^4,$$
$$T_{12} = \big[\big(D_3^2 + 15T_3 - 6T_4, C_2\big)^{(2)} - 6\big(D_3^2 - 3T_3 + 12T_4, D_2\big)^{(1)} - 4\big(S_{26}, C_2\big)^{(1)} +$$
$$+ 10D_2S_{26} - 720(A_1 + 3A_2)C_2\big]/2^7/3^3,$$
$$T_{13} = \big[\big(D_3^2 + 27T_3 - 18T_4, C_2\big)^{(2)} - 216\big(T_4, D_2\big)^{(1)} + 48D_3S_{22} + 36D_2S_{26} - 432C_2(3A_1 + 17A_2)\big]/2^7/3^4,$$
$$T_{14} = \big[\big(8S_{19} + 9S_{21}, D_2\big)^{(1)} - D_2(8S_{20} + 3S_{22}) + 18D_1S_{26} + 1296C_1A_2\big]/2^4/3^3,$$
$$T_{15} = \big[72(S_{19}, D_2)^{(1)} - (S_{17}, C_2)^{(2)} + 16(S_{21}, D_2)^{(1)} - 3(D_3^2, C_1)^{(2)} + 27(T_3, C_1)^{(2)} - 54(T_4, C_1)^{(2)}$$
$$+ 36D_1S_{26} + 2160C_1A_1 + 4752C_1A_2 - 16D_2S_{22} + 4(S_{14}, D_3)^{(1)} - 68(S_{15}, D_3)^{(1)} - 8(S_{14}, C_3)^{(2)}$$
$$- 8(S_{15}, C_3)^{(2)} - 4D_2S_{18}\big]/2^6/3^3, \quad T_{16} = (S_{23}, D_2)^{(2)}/2^6/3^3, \quad T_{17} = (S_{26}, D_3)^{(1)}/2^5/3^3,$$
$$T_{18} = \big[4\big(D_3^2 + 6T_4, C_2\big)^{(3)} + 2(C_2D_3, C_3)^{(4)} - 9D_2(96A_2 + S_{24}),$$
$$T_{19} = \big(T_6, C_3\big)^{(1)}/2, \quad T_{23} = \big(T_6, C_3\big)^{(2)}/6, \quad T_{24} = \big(T_6, D_3\big)^{(1)}/6,$$
$$T_{25} = \big[16\big[(C_2, D_3)^{(1)}\big]^2 + 5184C_1(3A_2C_3 - T_5D_3) + D_2^2(D_3^2 - 81T_3 - 54T_4) + 4D_2(648T_5C_2$$
$$+ 3D_3S_{17} + 2D_3S_{19} - 18C_3S_{20})\big]/2^6/3^4, \quad T_{26} = \big(T_9, C_3\big)^{(1)}/4, \quad T_{36} = \big(T_6, D_3\big)^{(2)}/12,$$
$$T_{37} = \big(T_9, C_3\big)^{(2)}/12, \quad T_{38} = \big(T_9, D_3\big)^{(1)}/6, \quad T_{42} = \big(T_{14}, C_3\big)^{(1)}/2,$$
$$T_{44} = \big((S_{23}, C_3)^{(1)}, D_3\big)^{(2)}/5/2^6/3^3, \quad T_{50} = \big(T_{12}, D_3\big)^{(1)}/6,$$
$$T_{57} = \big(T_9, D_3\big)^{(2)}/12, \quad T_{59} = \big(T_6, C_3\big)^{(4)}/2^4/3^2, \quad T_{60} = \big(T_9, C_3\big)^{(3)}/72,$$
$$T_{62} = \big(T_{14}, C_3\big)^{(2)}/6, \quad T_{63} = \big(T_{15}, C_3\big)^{(2)}/6, \quad T_{64} = \big(T_{15}, D_3\big)^{(1)}/6, \quad T_{65} = \big(T_{14}, D_3\big)^{(1)}/6,$$
$$T_{74} = \big[18(27T_3C_2D_1 - 54T_4C_2D_1 - 64T_6D_2 - 3T_3C_1D_2 + 6T_4C_1D_2)D_3 - 6(9C_2D_1 - C_1D_2)D_3^3$$
$$+ 27C_0D_3^4 + D_3^2(-486T_3C_0 + 972T_4C_0 + 108C_3D_1D_2 - 8C_2D_2^2 - 54C_3S_8 + 108C_3S_9$$
$$+ 27C_2S_{11} - 27C_2S_{12} + 4C_2S_{14} - 32C_2S_{15} + 54D_1S_{16} - 3C_1S_{17} + 6C_1S_{19} - 18C_1S_{21})$$
$$- 972(T_3 - 2T_4)C_3D_1D_2 + 72(T_3 - 2T_4)C_2D_2^2 - 486C_3(128T_{11}C_1 - T_3S_8 + 2T_4S_8$$
$$+ 2T_3S_9 - 4T_4S_9) + 20736T_{11}C_2^2 - 9C_2(T_3 - 2T_4)(27S_{11} - 27S_{12} + 4S_{14} - 32S_{15})$$
$$+ 2187(T_3 - 2T_4)^2C_0 + 576T_6(S_{17} - 2S_{19} + 6S_{21}) - 27T_3(18D_1S_{16} - C_1S_{17} + 2C_1S_{19}$$
$$- 6C_1S_{21}) + 54T_4(18D_1S_{16} - C_1S_{17} + 2C_1S_{19} - 6C_1S_{21})\big]/2^8/3^4,$$
$$T_{75} = \big[-18(40C_3D_2 + 137S_{16})\big(D_3^2, C_1\big)^{(1)} - 48(4C_3D_2 - 3C_2D_3 - S_{16})\big(S_{14}, C_3\big)^{(1)}$$
$$- 768C_3D_2\big(S_{15}, C_3\big)^{(1)} + 9(16C_3D_2 - 9C_2D_3 + 5S_{16})\big(S_{23}, C_1\big)^{(1)} - 648C_0D_3^4$$
$$+ 162(C_2D_3 + 3S_{16})\big(S_{25}, C_1\big)^{(1)} + 144(9C_2D_1 + 2C_1D_2)D_3^3 - 12D_3^2(32C_2D_2^2 - 18C_3S_8$$
$$+ 9C_2S_{11} - 54C_2S_{12} + 24C_2S_{14} - 96C_2S_{15} - 324D_1S_{16} - 6C_1S_{17} + 12C_1S_{19} - 18C_0S_{23}$$

$$
\begin{aligned}
&+216C_0S_{25})+8D_3(64C_3D_2^3+64C_3D_2S_{14}+16D_2^2S_{16}+12S_{14}S_{16}-96S_{15}S_{16}-36C_2^2S_{18}\\
&-96C_2D_2S_{19}+108C_2^2S_{20}+240C_2D_2S_{21}-297C_2D_1S_{23}-24C_1D_2S_{23}+1134C_2D_1S_{25})\\
&+62208C_3(3T_{13}C_1-16T_8D_1)+2(1728C_3D_1D_2+32C_2D_2^2+18C_3S_8+4176C_3S_9-9C_2S_{11}\\
&-1395C_2S_{12}-16C_2S_{14}+96C_2S_{15}-108D_1S_{16}-18C_1S_{17}-60C_1S_{19}+2160C_1S_{21})S_{23}\\
&+54C_0S_{23}^2+32(5832T_{13}C_1C_3-31104T_8C_3D_1-34992T_8S_{10}-3C_3S_{14}S_{17}-4D_2S_{16}S_{17}\\
&+3C_2S_{17}^2+12C_2C_3D_2S_{18}-3C_2S_{16}S_{18}+16C_3D_2^2S_{19}-2C_3S_{14}S_{19}+16C_3S_{15}S_{19}\\
&+24D_2S_{16}S_{19}-12C_2S_{19}^2-36C_2C_3D_2S_{20}+9C_2S_{16}S_{20}-48C_3D_2^2S_{21}-12C_3S_{14}S_{21}\\
&-24D_2S_{16}S_{21}+12C_2S_{17}S_{21})-36(288C_3D_1D_2+474C_3S_8+528C_3S_9-237C_2S_{11}\\
&-255C_2S_{12}-180D_1S_{16}-86C_1S_{17}+156C_1S_{19}+276C_1S_{21})S_{25}-1944C_0S_{25}^2]/2^{11}/3^4,
\end{aligned}
$$

$$
\begin{aligned}
&T_{76}=[\![T_6,C_3)^{(2)},C_3)^{(1)}/36,\ T_{78}=\big(T_{25},C_3\big)^{(1)}/2,\ T_{80}=[\![T_9,C_3)^{(2)},C_3)^{(1)}/144,\\
&T_{81}=[\![T_6,C_3)^{(3)},C_3)^{(1)}/2^6/3^2,\ T_{82}=[\![T_6,C_3)^{(2)},D_3)^{(1)}/2^3/3^3,\ T_{83}=[\![T_6,C_3)^{(1)},D_3)^{(2)}/24,\\
&T_{84}=\big(T_{25},C_3\big)^{(2)}/6,\ T_{86}=[\![T_{11},C_3)^{(2)},C_3)^{(1)}/36,\ T_{107}=[\![T_9,D_3)^{(2)},D_3)^{(1)}/432,\\
&T_{110}=[\![T_6,C_3)^{(4)},C_3)^{(2)}/2^7/3^3,\ T_{111}=[\![T_6,C_3)^{(3)},D_3)^{(2)}/2^7/3^3,\ T_{113}=[\![T_{14},C_3)^{(2)},C_3)^{(2)}/72,\\
&T_{114}=[\![T_{14},C_3)^{(2)},D_3)^{(1)}/72,\ T_{115}=[\![T_{14},C_3)^{(1)},D_3)^{(2)}/72,\ T_{116}=[\![T_{15},C_3)^{(2)},C_3)^{(2)}/72,\\
&T_{117}=[\![T_6,D_3)^{(2)},D_3)^{(2)}/2^5/3^3,\ T_{118}=[\![T_9,C_3)^{(3)},C_3)^{(2)}/2^5/3^3,\ T_{119}=\big(T_{25},C_3\big)^{(4)}/2^4/3^2,\\
&T_{120}=[\![T_9,C_3)^{(3)},D_3)^{(1)}/2^5/3^3,\ T_{121}=[\![T_9,C_3)^{(2)},D_3)^{(2)}/2^4/3^3,\ T_{133}=\big(T_{74},C_3\big)^{(1)},\\
&T_{135}=\big(T_{75},C_3\big)^{(1)},
\end{aligned}
$$

are T-comitants of cubic systems (2.1) (see for details [36]). In the above list the bracket "$[\![$" means a succession of two or up to four parentheses "(" depending on the row in which it appears.

We note that these invariant polynomials are the elements of the polynomial basis of T-comitants up to degree six constructed by Iu. Calin [15].

2.2 Preliminary results

In order to determine the degree of the common factor of the polynomials $\tilde{\mathcal{G}}_i(a,x,y)$ for $i=1,2,3$, we shall use the notion of the k^{th} *subresultant* of two polynomials with respect to a given indeterminate (see for instance, [22,28]).

Following [25] we consider two polynomials

$$f(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n,\qquad g(z)=b_0z^m+b_1z^{m-1}+\cdots+b_m,$$

in the variable z of degree n and m, respectively.

We say that the k–th *subresultant* (see for example, [28]) with respect to variable z of the two polynomials $f(z)$ and $g(z)$ is the $(m+n-2k)\times(m+n-2k)$ determinant

$$
R_z^{(k)}(f,g)=\left|\begin{array}{cccccc}
a_0 & a_1 & a_2 & \dots & \dots & a_{m+n-2k-1}\\
0 & a_0 & a_1 & \dots & \dots & a_{m+n-2k-2}\\
0 & 0 & a_0 & \dots & \dots & a_{m+n-2k-3}\\
\dots & \dots & \dots & \dots & \dots & \dots\dots\dots\\
0 & 0 & b_0 & \dots & \dots & b_{m+n-2k-3}\\
0 & b_0 & b_1 & \dots & \dots & b_{m+n-2k-2}\\
b_0 & b_1 & b_2 & \dots & \dots & b_{m+n-2k-1}
\end{array}\right|
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\}(m-k)\text{ times}\\
\\
\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\}(n-k)\text{ times}
\end{array}
\tag{2.3}
$$

in which there are $m-k$ rows of a's and $n-k$ rows of b's, and $a_i = 0$ for $i > n$, and $b_j = 0$ for $j > m$.

For $k = 0$ we obtain the standard resultant of two polynomials. In other words we can say that the k–th subresultant with respect to the variable z of the two polynomials $f(z)$ and $g(z)$ can be obtained by deleting the first and the last k rows and the first and the last k columns from its resultant written in the form (2.3) when $k = 0$.

The geometrical meaning of the subresultants is based on the following lemma.

Lemma 2.8 (see [22,28]). *Polynomials $f(z)$ and $g(z)$ have precisely k roots in common (considering their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f,g) = R_z^{(1)}(f,g) = R_z^{(2)}(f,g) = \cdots = R_z^{(k-1)}(f,g) = 0 \neq R_z^{(k)}(f,g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 2.8 the following result.

Lemma 2.9. *Two polynomials $\tilde{f}(x_1, x_2, \ldots, x_n)$ and $\tilde{g}(x_1, x_2, \ldots, x_n)$ have a common factor of degree k with respect to the variable x_j if and only if the following conditions are satisfied:*

$$R_{x_j}^{(0)}(\tilde{f},\tilde{g}) = R_{x_j}^{(1)}(\tilde{f},\tilde{g}) = R_{x_j}^{(2)}(\tilde{f},\tilde{g}) = \cdots = R_{x_j}^{(k-1)}(\tilde{f},\tilde{g}) = 0 \neq R_{x_j}^{(k)}(\tilde{f},\tilde{g}),$$

where $R_{x_j}^{(i)}(\tilde{f},\tilde{g}) = 0$ in $\mathbb{R}[x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_n]$.

In paper [25] all the possible configurations of invariant lines are determined in the case, when the total multiplicity of these lines (including the line at infinity) equals nine. All possible configurations of invariant lines in the case when the total multiplicity of these lines (including the line at infinity) equals eight, are determined in [5,9–12].

In the above mentioned articles, several lemmas are proved concerning the number of triplets and/or couples of parallel invariant straight lines which could have a cubic system. Taking together these lemmas produce the following theorem.

Theorem 2.10. *If a cubic system (2.1) possesses a given number of triplets or/and couples of invariant parallel lines real or/and complex, then the following conditions are satisfied, respectively:*

(i)	*two triplets*	$\Rightarrow$	$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$;
(ii)	*one triplet and one couple*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$;
(iii)	*one triplet*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{U}_2 = 0$;
(iv)	*3 couples*	$\Rightarrow$	$\mathcal{V}_3 = 0$;
(v)	*2 couples*	$\Rightarrow$	$\mathcal{V}_5 = 0$.

Remark 2.11. The above conditions depend only on the coefficients of the cubic homogeneous parts of the systems (2.1).

We rewrite the systems (2.1) using a different notation for the coefficients::

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3 \equiv p(x,y), \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3 \equiv q(x,y). \end{aligned} \tag{2.4}$$

Let $L(x,y) = Ux + Vy + W = 0$ be an invariant straight line of this family of cubic systems. Then, we have

$$Up(x,y) + Vq(x,y) = (Ux + Vy + W)(Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F),$$

and this identity provides the following 10 relations:

$$\begin{array}{ll} Eq_1 = (p-A)U + tV = 0, & Eq_6 = (2h-E)U+(2m-D)V-2BW=0, \\ Eq_2 = (3q-2B)U + (3u-A)V = 0, & Eq_7 = kU + (n-E)V - CW = 0, \\ Eq_3 = (3r-C)U + (3v-2B)V = 0, & Eq_8 = (c-F)U + eV - DW = 0 \\ Eq_4 = (s-C)U + Vw = 0, & Eq_9 = dU + (f-F)V - EW = 0, \\ Eq_5 = (g-D)U + lV - AW = 0, & Eq_{10} = aU + bV - FW = 0. \end{array} \tag{2.5}$$

It is well known that in the case of the non-singular infinite invariant line the infinite singularities (real or complex) of systems (2.4) are determined by the linear factors of the polynomial

$$C_3 = yp_3(x,y) - xq_3(x,y).$$

Remark 2.12. Let $C_3 = \prod_{i=1}^{4}(\alpha_i x + \beta_i y)$, $i = 1,2,3,4$. Then $[\beta_i : \alpha_i : 0]$ are the singular points at infinity. Hence the invariant affine lines must be of the form $Ux + Vy + W = 0$ with (U,V) among (α_i, β_i). In this case, for any fixed (α_i, β_i), for a specific value of W, six equations among (2.5) become linear with respect to the parameters $\{A, B, C, D, E, F\}$ (with the corresponding non-zero determinant) and we can determine their values, which annihilate some of the equations (2.5). So in what follows, for each direction given by (α_i, β_i), we will examine only the non-zero equations containing the last parameter W.

For the proof of the Main Theorem it is useful to consider the following homogeneous cubic systems associated to systems (2.4):

$$x' = p_3(x,y), \quad y' = q_3(x,y). \tag{2.6}$$

Clearly in the case of two real and two complex distinct infinite singularities the polynomial $C_3(x,y)$ has four distinct linear factors over $\mathbb{C}$: two of them being real and two complex. The following remark concerning the associated homogeneous cubic systems (2.6) is useful.

Remark 2.13. Assume that a cubic system (2.4) in $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$ possesses invariant lines of total multiplicity three in a real direction. Then the corresponding associated homogeneous cubic systems (2.6) has one invariant line of multiplicity three in the same direction.

Indeed, if a system (2.4) possesses a triplet of parallel invariant lines (distinct or coinciding) in a real direction then via an affine transformation this system could be brought to the form

$$\dot{x} = x[(x+b)^2 + u], \quad \dot{y} = q(a,x,y).$$

It is clear that if $u < 0$ (respectively $u > 0$) then we have three real (respectively one real and two complex) all distinct invariant lines. In the case $u = 0$ we either have one simple and one double invariant lines if $b \neq 0$, or one triple invariant line if $b = 0$. It remains to observe that in all four cases the corresponding associated homogeneous cubic systems possess the invariant line $x = 0$ of multiplicity at least three.

According to [9,25] (see also [30]) we have the following result.

Lemma 2.14. *A cubic system (2.4) has 2 real and two complex all distinct infinite singularities if and only if the condition $\mathcal{D}_1 < 0$ holds. Moreover its associated homogeneous cubic systems (2.6) could be brought via a linear transformation to the canonical form*

$$(S_{II}) \quad \begin{cases} x' = (1+u)x^3 + (s+v)x^2y + rxy^2, \\ y' = -sx^3 + ux^2y + vxy^2 + (r-1)y^3. \end{cases} \quad C_3 = x(sx+y)(x^2+y^2), \tag{2.7}$$

3 The proof of the Main Theorem

Considering Lemma 2.14 we deduce that for the systems in the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$ the condition $\mathcal{D}_1 < 0$ holds and these systems could be brought via a linear transformation to the family of systems

$$\begin{aligned}\dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + (1+u)x^3 + (s+v)x^2y + rxy^2,\\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 - sx^3 + ux^2y + vxy^2 + (r-1)y^3\end{aligned} \tag{3.1}$$

with $C_3 = x(sx+y)(x^2+y^2)$. In what follows we examine cubic systems possessing configurations of invariant lines of the type $\mathfrak{T} = (3,1,1,1)$.

3.1 The proof of the statement (A)

The configurations of the type $\mathcal{T} = (3,1,1,1)$ could only have one triplet of parallel invariant lines and clearly in the case of two real and two complex infinite singularities such triplet could be only in a real direction.

3.1.1 Construction of the associated homogeneous systems

Since systems with the configuration of the type $\mathcal{T} = (3,1,1,1)$ could only possess one triplet of parallel invariant lines, according to Theorem 2.10 the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$ are necessary for systems (3.1). Taking the corresponding associated homogeneous systems (2.7) we force the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$.

We observe that the invariant polynomial $\mathcal{U}_2$ is a homogeneous polynomial of degree four in x and y. So we shall use the following notation:

$$\mathcal{U}_2 = \sum_{j=0}^{4} \mathcal{U}_{2j} x^{4-j} y^j.$$

On the other hand a straightforward computation of the value of $\mathcal{V}_4$ for systems (2.7) yields

$$\begin{aligned}&\mathcal{V}_4 = 9216\,\widehat{\mathcal{V}}_4\, C_3(x,y), \quad \text{where}\\ &\widehat{\mathcal{V}}_4 = 6r^2 s + r(2su - 9s - 3v) + (s+v)(sv - 3u).\end{aligned}$$

As for systems (2.7) we have $C_3 = x(sx+y)(x^2+y^2) \neq 0$, we conclude that the condition $\mathcal{V}_4 = 0$ for these systems is equivalent to $\widehat{\mathcal{V}}_4 = 0$.

For systems (2.7) we evaluate

$$\mathcal{U}_2 = 3 \cdot 2^{12} \sum_{j=0}^{4} \widehat{\mathcal{U}}_{2j} x^{4-j} y^j,$$

where $\widehat{\mathcal{U}}_{2j}$ are polynomials in the parameters r, s, u and v. We have

$$\widehat{\mathcal{U}}_{24} = r(9u - 12ru + 4r^2u - 3sv + 2rsv - rv^2) = 0$$

and we consider two cases: $r \neq 0$ and $r = 0$.

1: *The case $r \neq 0$.* Then we must have

$$9u - 12ru + 4r^2u - 3sv + 2rsv - rv^2 = (3-2r)^2 u + v(-3s + 2rs - rv) = 0$$

and we examine two subcases: $3-2r\neq 0$ and $3-2r=0$.

1.1: *The subcase* $3-2r\neq 0$. Then the condition $\widehat{\mathcal{U}}_{24}=0$ gives $u=\frac{3sv-2rsv+rv^2}{(3-2r)^2}$ and we obtain:

$$\widehat{\mathcal{U}}_{23}=\frac{3r(3s-2rs+v)\big[(3-2r)^2+v^2\big)}{2r-3}=(3-2r)\widehat{\mathcal{V}}_4=0.$$

Since $r(3-2r)\neq 0$ the above condition gives $v=(2r-3)s$ and this implies $\mathcal{U}_2=\widehat{\mathcal{V}}_4=0$. Therefore we get the family of systems

$$\begin{aligned}\dot{x}&=(1-s^2+rs^2)x^3+2(-1+r)sx^2y+rxy^2,\\ \dot{y}&=-sx^3+(-1+r)s^2x^2y+(-3+2r)sxy^2+(-1+r)y^3.\end{aligned}\tag{3.2}$$

1.2: *The subcase* $3-2r=0$. We get $r=3/2$ and therefore the condition $\widehat{\mathcal{U}}_{24}=0$ gives $v=0$. Then we obtain $\widehat{\mathcal{V}}_4=0$ and

$$\widehat{\mathcal{U}}_{20}=-3(s^2-2u)\big[4s^2+(3+2u)^2\big]/4=0$$

and we discuss two possibilities: $s^2-2u=0$ or $s=3+2u=0$.

1.2.1: *The possibility* $s^2-2u=0$. We have $u=s^2/2$ and then $\mathcal{U}_2=\widehat{\mathcal{V}}_4=0$. In this case we arrive at the family of systems

$$\begin{aligned}\dot{x}&=(1+s^2/2)x^3+sx^2y+3xy^2/2,\\ \dot{y}&=-sx^3+1/2s^2x^2y+y^3/2.\end{aligned}\tag{3.3}$$

We observe that the above family of systems is a subfamily of (3.2) defined by the value $r=3/2$.

1.2.2: *The possibility* $s=3+2u=0$. In this case we get again $\mathcal{U}_2=\widehat{\mathcal{V}}_4=0$ and we obtain the system

$$\dot{x}=-x^3/2+3xy^2/2,\quad \dot{y}=-3x^2y/2+y^3/2.\tag{3.4}$$

However for this system we calculate (see the definition of the polynomial $H(X,Y,Z)$ on the page 14, Notation 2.4):

$$H(X,Y,Z)=\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)=3XY(X^2+Y^2)^3/4.$$

So system (3.4) possesses two triple invariant lines $x\pm iy=0$ and by Remark 2.13, systems (3.1) could have triplets of parallel invariant lines only in these two directions. However since these lines will be complex, we deduce that systems (3.1) with the associated homogeneous cubic system (3.4) could not possess invariant lines with the configuration of the type $\mathfrak{T}=(3,1,1,1)$.

2: *The case* $r=0$. Then we calculate

$$\widehat{\mathcal{U}}_{23}=3(s+v)(3u-sv),\quad \widehat{\mathcal{V}}_4=-(s+v)(3u-sv)$$

and we examine two subcases: $s+v=0$ or $s+v\neq 0$ and $3u-sv=0$.

2.1: *The subcase* $s+v=0$. Then $v=-s$ and this implies $\mathcal{U}_2=\widehat{\mathcal{V}}_4=0$. Therefore we get the family of systems (we set new parameters and variables: $s=s_1$, $u=u_1$, $x=x_1$, $y=y_1$)

$$\dot{x}_1=(1+u_1)x_1^3,\quad \dot{y}_1=-s_1x_1^3+u_1x_1^2y_1-s_1x_1y_1^2-y_1^3.\tag{3.5}$$

In this case we observe that the systems (3.2) via the transformation

$$x_1 = -(sx+y), \quad y_1 = -x+sy, \quad t_1 = -t/(s^2+1)$$

can be transformed to systems (3.5) after additional change of the parameters: $s = s_1$ and $s^2 - r(1+s^2) = u_1$.

2.2: *The subcase* $3u - sv = 0$ *and* $s+v \neq 0$. Then $u = sv/3$ and we calculate

$$\widehat{\mathcal{U}}_{22} = -(s+v)(3s+v)(9+v^2).$$

Since $s+v \neq 0$ we get $v = -3s$ and this implies $\mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$. In this case we obtain the family of systems

$$\dot{x} =(1-s^2)x^3 - 2sx^2y, \quad \dot{y} = -sx^3 - s^2x^2y - 3sxy^2 - y^3. \tag{3.6}$$

We observe that the above family of systems is a subfamily of (3.2) defined by the value $r = 0$.

So we have proved the next lemma.

Lemma 3.1. *If for a homogeneous cubic system (2.7) the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$ hold then this system could be brought via a linear transformation and time rescaling to the form (3.5) with one exception: when the conditions $s = v = 0$ and $r = -u = 3/2$ (which imply $\mathcal{V}_4 = \mathcal{U}_2 = 0$) then we get the system (3.4) that has two triple complex invariant lines $x \pm iy = 0$.*

Thus according to this lemma forcing the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$ to be satisfied for systems (3.1) we obtain two families of systems. The first one with the associated homogeneous cubic systems of the form (3.5) and due to an additional translation having the parameter $n = 0$ in the quadratic parts of systems (3.1):

$$\begin{aligned} &\dot{x} = a + cx + dy + gx^2 + 2hxy + ky^2 + (1+u)x^3, \\ &\dot{y} = b + ex + fy + lx^2 + 2mxy - sx^3 + ux^2y - sxy^2 - y^3, \end{aligned} \tag{3.7}$$

The second family has the associated homogeneous cubic systems of the form (3.4) and applying an additional translation we can assume that two parameters vanish: $m = 0$ and $n = 0$. As a result we arrive at the following family of systems:

$$\begin{aligned} &\dot{x} = a + cx + dy + gx^2 + 2hxy + ky^2 - x^3/2 + 3xy^2/2, \\ &\dot{y} = b + ex + fy + lx^2 - 3x^2y/2 + y^3/2. \end{aligned} \tag{3.8}$$

As it was mentioned above, by Remark 2.13 systems (3.8) could not possess invariant lines with the configuration of the type $\mathfrak{T} = (3,1,1,1)$. And later (see Lemma 3.25) will be proved that none of the sets of conditions provided by the statement A) of Main Theorem could be satisfied for systems (3.8).

Next we prove the following lemma which is the first step in the classification of the configuration of systems in the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$.

Lemma 3.2. *Assume that for a non-degenerate cubic system (2.4) the conditions $\mathcal{D}_1 < 0$ and $\mathcal{V}_4 = \mathcal{U}_2 = 0$ hold. Then the infinite invariant line $Z = 0$ of this system has the multiplicity indicated below if and only if the corresponding conditions are satisfied, respectively:*

(i) *one* $\Leftrightarrow \mathcal{D}_7 \neq 0$;

(*ii*) *two* $\Leftrightarrow \mathcal{D}_7 = 0$ *and* $\widetilde{\chi}_1 \neq 0$;

(*iii*) *three* $\Leftrightarrow \mathcal{D}_7 = \widetilde{\chi}_1 = 0$ *and* $\widetilde{\chi}_2 \neq 0$;

(*iv*) *four* $\Leftrightarrow \mathcal{D}_7 = \widetilde{\chi}_1 = \widetilde{\chi}_2 = 0$.

Moreover the maximum multiplicity which could have the line at infinity of a non-degenerate system with $\mathcal{D}_1 < 0$ *and* $\mathcal{V}_4 = \mathcal{U}_2 = 0$ *is four.*

Proof. First of all we mention that by Lemma 2.14 the condition $\mathcal{D}_1 < 0$ implies the existence of 2 real and 2 complex infinite singularities.

On the other hand as it was mentioned earlier, according to Lemma 3.1 the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$ lead either to the family of systems (3.7) or to (3.8).

According to Lemma 2.6 if the invariant line $Z = 0$ is of multiplicity $k > 1$ then Z^{k-1} is a common factor of the invariant polynomials $\mathcal{G}_i(a, X, Y, Z)$, $i = 1, 2, 3$ defined in Notation 2.4 of the manuscript. So the existence of such a common factor of the above mentioned three polynomials is a necessary condition for the invariant line $Z = 0$ of systems (3.7) to be of the multiplicity k.

For systems (3.7) calculations yield:

$$\begin{aligned}
\mathcal{G}_1(X, Y, Z) &= (1+u)X^3(sX+Y)(X^2+Y^2)(uX^2-2sXY-3Y^2) + Z\big[\Psi_1(X, Y, Z)\big],\\
\mathcal{G}_2(X, Y, Z) &= (1+u)X^3(sX+Y)(X^2+Y^2)\big[(s^2+2u+2u^2)X^4 - 4suX^3Y\\
&\qquad + (s^2-3-6u)X^2Y^2 + 4sXY^3 + 3Y^4\big] + Z\big[\Psi_2(X, Y, Z)\big],\\
\mathcal{G}_3(X, Y, Z) &= 24(1+u)X^3(sX+Y)(uX^2-Y^2)(X^2+Y^2)\big[(1+s^2+2u+u^2)X^4\\
&\qquad - 2suX^3Y + (s^2-1-2u)X^2Y^2 + 2sXY^3 + Y^4\big] + Z\big[\Psi_3(X, Y, Z)\big],
\end{aligned} \tag{3.9}$$

where $\Psi_j(X, Y, Z)$ $(j = 1, 2, 3)$ are some polynomials in X, Y and Z.

Evidently Z will be a common factor of the polynomials $\mathcal{G}_i(X, Y, Z)$ (i.e. $\mathcal{G}_i(X, Y, 0) = 0$ for each $i = 1, 2, 3$) if and only if $u + 1 = 0$. Since the condition $\mathcal{D}_7 \neq 0$ implies $u + 1 \neq 0$ we deduce that Z could not be the needed common factor and hence the infinite invariant line $Z = 0$ for systems (3.7) is of multiplicity one.

On the other hand for systems (3.8) we calculate

$$\mathcal{G}_1(X, Y, Z) = \ 6XY(X^2+Y^2)^3 + Z\big[\Phi(X, Y, Z)\big], \quad \mathcal{D}_7 = 4 \neq 0,$$

where $\Phi(X, Y, Z)$ is a polynomials in X, Y and Z. So we can see that the polynomial $\mathcal{G}_1(X, Y, Z)$ could not have as a factor Z and hence all three polynomials $\mathcal{G}_i(X, Y, Z)$ $i = 1, 2, 3$ could not have the common factor Z. So we arrive at the following remark.

Remark 3.3. The family of systems (3.8) could not have the infinite invariant line $Z = 0$ of multiplicity greater than one.

Thus we conclude that in the case $\mathcal{D}_7 \neq 0$ a non-degenerate cubic system with $\mathcal{D}_1 < 0$ and $\mathcal{V}_4 = \mathcal{U}_2 = 0$ has the line at infinity of multiplicity one. This completes the proof of the statement (i) of the lemma.

(ii) Assume now that the condition $\mathcal{D}_7 = 0$ holds and taking into account Remark 3.3 we consider the family of systems (3.7). In this case the condition $\mathcal{D}_7 = 0$ gives us $u = -1$ and considering (3.9) we deduce that Z is a common factor of the polynomials $\mathcal{G}_i(X, Y, Z)$, $i = 1, 2, 3$. We claim that the invariant line $Z = 0$ of systems (3.7) has multiplicity at least two. For this it is sufficient to apply the following perturbation to systems (3.7) with $u = -1$ (remaining in the class of cubic systems):

$$\dot{x} = (a + cx + dy + gx^2 + 2hxy + ky^2)(1 + \varepsilon x), \quad \dot{y} = q(x, y), \quad |\varepsilon| \ll 1.$$

It is clear that the perturbed systems possess the invariant line $\varepsilon x + 1 = 0$ which coalesces with infinite one when ε tends to zero. So we deduce that the invariant line $Z = 0$ is of multiplicity at least 2 and in order to determine exactly its multiplicity we calculate:

$$\begin{aligned}
\mathcal{G}_1(X,Y,Z)/Z = {}& -(sX+Y)(X^2+Y^2)\big[(g-2hs)X^4 + 2(g-k)sX^3Y\\
&+(3g-k+2hs)X^2Y^2 + 4hXY^3 + kY^4\big] + Z\big[\Psi'_1(X,Y,Z)\big],\\
\mathcal{G}_2(X,Y,Z)/Z = {}& (sX+Y)^2(X^2+Y^2)^2\big[(g-k)sX^3 + (3g-k+2hs)X^2Y\\
&+6hXY^2 + 2kY^3\big] + Z\big[\Psi'_2(X,Y,Z)\big],\\
\mathcal{G}_3(X,Y,Z)/Z = {}& -24(sX+Y)^3(X^2+Y^2)^3(gX^2+2hXY+kY^2) + Z\big[\Psi'_3(X,Y,Z)\big],
\end{aligned} \tag{3.10}$$

where $\Psi'_j(X,Y,Z)$ $(j = 1,2,3)$ are some polynomials in X, Y and Z.

We observe that each one of the polynomials $\mathcal{G}_i(X,Y,Z)/Z$, $i = 1,2,3$ has the factor Z if and only if $k = h = g = 0$. This condition is governed by the invariant polynomials $\widetilde{\chi}_1$ because for systems (3.7) with $u = -1$ we have

$$\text{Coefficient}[\widetilde{\chi}_1, xy^2] = -8ks/3, \quad \text{Coefficient}[\widetilde{\chi}_1, y^3] = 2k(s^2-3)/9$$

and clearly the condition $\widetilde{\chi}_1 = 0$ implies $k = 0$. Then we calculate

$$\widetilde{\chi}_1 = 2x^2\big[2(h+2gs-3hs^2)x + (3g-8hs-gs^2)y\big]/9 = 0,$$

and we determine that for $s = 0$ we get $h = g = 0$. If $s \neq 0$ we obtain:

$$h + 2gs - 3hs^2 = 0 \Rightarrow g = \frac{h(3s^2-1)}{2s} \Rightarrow 3g - 8hs - gs^2 = -\frac{3h(1+s^2)^2}{2s} = 0 \Rightarrow h = g = 0.$$

So in the case $\widetilde{\chi}_1 \neq 0$ we have $k^2 + h^2 + g^2 \neq 0$ and therefore the invariant line $Z = 0$ has the multiplicity exactly two.

(iii) Admit now that the conditions $\mathcal{D}_7 = 0$ and $\widetilde{\chi}_1 = 0$ are satisfied. This implies $u = -1$ and $k = h = g = 0$ and considering (3.10) we deduce that Z^2 is a common factor of the polynomials $\mathcal{G}_i(X,Y,Z)$, $i = 1,2,3$. We claim that the invariant line $Z = 0$ of systems (3.7) has the multiplicity at least three. For this it is sufficient to apply to (3.7) with $u = -1$ and $k = h = g = 0$ the following perturbation (remaining in the class of cubic systems):

$$\dot{x} = (a + cx + dy)(1 + \varepsilon_1 x + \varepsilon_2 x^2), \quad \dot{y} = q(x, y)$$

with $|\varepsilon_i| \ll 1$ $(i = 1,2)$. Clearly the perturbed systems possess the two invariant lines defined by the equation $1 + \varepsilon_1 x + \varepsilon_2 x^2 = 0$ which coalesces with infinite one when ε_1 and ε_2 tend to zero. So we deduce that the invariant line $Z = 0$ is of multiplicity at least 3 and in order to determine precisely its multiplicity we calculate:

$$\begin{aligned}
\mathcal{G}_1(X,Y,Z)/Z^2 = {}& -(sX+Y)(X^2+Y^2)\big[(c-ds)X^3 + 2csX^2Y + (3c+ds)XY^2\\
&+2dY^3\big] + Z\big[\Psi''_1(X,Y,Z)\big],\\
\mathcal{G}_2(X,Y,Z)/Z^2 = {}& (sX+Y)^2(sX+3Y)(cX+dY)(X^2+Y^2)^2 + Z\big[\Psi''_2(X,Y,Z)\big],\\
\mathcal{G}_3(X,Y,Z)/Z^2 = {}& -24(sX+Y)^3(cX+dY)(X^2+Y^2)^3 + Z\big[\Psi''_3(X,Y,Z)\big],
\end{aligned} \tag{3.11}$$

where $\Psi''_j(X,Y,Z)$ $(j=1,2,3)$ are some polynomials in X,Y and Z. We observe that each of the polynomials $\mathcal{G}_i(X,Y,Z)/Z^2$, $i=1,2,3$ has as a factor Z if and only if $c=d=0$. This condition is governed by the invariant polynomials $\widetilde{\chi}_2$ because for of systems (3.7) $u=-1$ and $k=h=g=0$ we have

$$\widetilde{\chi}_2 = 4x^2(sx+y)(cx+dy)(x^2+y^2)\big[(3s^2-1)x^2+8sxy+(3-s^2)y^2\big]/3.$$

Evidently the condition $\widetilde{\chi}_2=0$ is equivalent to $c=d=0$. So in the case $\widetilde{\chi}_2\neq 0$ we have $c^2+d^2\neq 0$ and therefore we deduce that the invariant line $Z=0$ has the multiplicity exactly three.

(iv) Admit now that the conditions $\mathcal{D}_7=0$ (i.e. $u=-1$) and $\widetilde{\chi}_1=\widetilde{\chi}_2=0$ which implies $k=h=g=d=c=0$. Then considering (3.11) we deduce that Z^3 is a common factor of the polynomials $\mathcal{G}_i(X,Y,Z)$, $i=1,2,3$. We claim that the invariant line $Z=0$ of systems (3.7) is of multiplicity at least four. For this it is sufficient to apply to (3.7) with $u=-1$ and $k=h=g=d=c=0$ the following perturbation (remaining in the class of cubic systems):

$$\dot{x}=a(1+\varepsilon_1x+\varepsilon_2x^2+\varepsilon_3x^3),\quad \dot{y}=q(x,y)$$

with $|\varepsilon_i|\ll 1$ $(i=1,2,3)$. Clearly the perturbed systems possess the three parallel invariant lines defined by the equation $1+\varepsilon_1x+\varepsilon_2x^2+\varepsilon_3x^3=0$ which coalesce with the infinite one when ε_i $(i=1,2,3)$ tend to zero. So we deduce that the invariant line $Z=0$ is of multiplicity at least 4 and in order to determine precisely its multiplicity we calculate:

$$\mathcal{G}_1(X,Y,Z)/Z^3=-(sX+Y)(X^2+Y^2)(X^2+2sXY+3Y^2)+Z\big[\Psi'''_1(X,Y,Z)\big],$$

where $\Psi'''_1(X,Y,Z)$ $(j=1,2,3)$ is a polynomial in X,Y and Z. As we can see the polynomial $\mathcal{G}_1(X,Y,Z)/Z^3$ could not have Z as a factor and therefore we deduce that the maximum multiplicity of the invariant line $Z=0$ for systems (3.7) equals four.

As all the cases are examined we conclude that Lemma 3.2 is proved. □

Thus considering Lemma 3.1 as well as Lemma 3.25 (which will be proved later) in what follows we consider the family of systems (3.7), i.e. the systems

$$\begin{aligned}\dot{x}&=a+cx+dy+gx^2+2hxy+ky^2+(1+u)x^3,\\ \dot{y}&=b+ex+fy+lx^2+2mxy-sx^3+ux^2y-sxy^2-y^3,\end{aligned}\tag{3.12}$$

for which we have $C_3(x,y)=x(sx+y)(x^2+y^2)$. For the corresponding associated homogeneous cubic systems we calculate (see the definition of the polynomial $H(X,Y,Z)$ on the page 14, Notation 2.4):

$$H(X,Y,Z)=\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)=(1+u)X^3(sX+Y)(X^2+Y^2).\tag{3.13}$$

So by Remark 2.13, systems (3.12) could have one triplet of parallel invariant lines in the direction $x=0$. However for some values of the parameters u and s the common divisor $\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$ could contain additional factors (see Notation 2.4 and Lemma 2.6). We prove the following lemma.

Lemma 3.4. *Systems (3.12) could possess a triplet of parallel invariant lines in the real direction $sx+y=0$ if and only if $s=u=0$.*

Proof. Consider the corresponding homogeneous cubic systems associated to (3.12):

$$\dot{x} = (1+u)x^3, \quad \dot{y} = -sx^3 + ux^2y - sxy^2 - y^3. \tag{3.14}$$

It was shown above that for these systems the value of $H(X,Y,Z)$ is given in (3.13). Since the factor $(sX+Y)$ in $\gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ depends on Y, according to Lemma 2.9 in order to increase its multiplicity up to 3 it is necessary

$$R_Y^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = R_Y^{(1)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = 0.$$

We calculate

$$R_Y^{(1)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = 6s(9+s^2)X^3 = 0$$

which implies $s = 0$. Then for systems (3.14) with $s = 0$ we obtain

$$R_Y^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = 9u^2(3+u)^2X^8 = 0.$$

Therefore this condition gives $u(3+u) = 0$. If $u = -3$ we get the homogeneous system

$$\dot{x} = -2x^3, \quad \dot{y} = -y(3x^2 + y^2),$$

for which we have $H(X,Y,Z) = 6X^3Y(X^2+Y^2)^2$, i.e. considering Remark 2.13 we could not have a triplet of parallel invariant lines in he direction $y = 0$.

Assuming $u = 0$ we get the homogeneous system

$$\dot{x} = x^3, \quad \dot{y} = -y^3, \tag{3.15}$$

for which we have $H(X,Y,Z) = 3X^3Y^3(X^2+Y^2)$ and this completes the proof of the lemma. □

3.1.2 Construction of the cubic systems possessing configuration of the type $\mathcal{T} = (3,1,1,1)$

In what follows we examine systems (3.12) considering each one of the cases provided by Lemma 3.2.

1: The case $\mathcal{D}_7 \neq 0$. Then by Lemma 3.2 the infinite invariant line $Z = 0$ of systems (3.12) is of multiplicity one and hence, we have to detect the conditions for the existence of invariant affine lines of total multiplicity six. Moreover these lines have to be in the configuration of the type $(3,1,1,1)$. Since the existence of a triplet of parallel invariant lines in the real direction $x = 0$ for systems (3.12) is a generic case we begin with the study of this case.

Considering the equations (2.5) and Remark 2.12 for systems (3.12) in the case of the direction $x = 0$ we obtain the following non-vanishing equations Eq_i:

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a - cW + gW^2 - (1+u)W^3. \tag{3.16}$$

It is clear that these three equations can have three common solutions if and only if $k = d = h = 0$ and since $\mathcal{D}_7 = (u+1) \neq 0$ we obtain that in this case the equation $Eq_{10} = 0$ has three solutions. They could be real or/and complex, distinct or coinciding. This means that systems (3.12) have in the direction $x = 0$ a triplet of parallel invariant lines.

Next we have to determine the conditions for the existence of three invariant lines in three distinct directions: one real ($sx + y = 0$) and two complex ($x \pm iy = 0$). Since the coefficients of systems (3.12) are real it is clear that for the complex directions it is sufficient to examine only one of them: $x + iy = 0$. In this case we have $U = 1$, $V = i$ and considering (2.5) and Remark 2.12 we obtain

$$\begin{aligned} Eq_7 &= 2m - g - il + (3 + u - 2is)W, \\ Eq_9 &= e + i(f - c) - 2\big[l + i(m - g)\big]\big)W - \big[3s + i(3 + 2u)\big]W^2, \\ Eq_{10} &= a + ib - (c + ie)W + (g + il)W^2 - (1 + u - is)W^3. \end{aligned} \tag{3.17}$$

Calculations yield

$$Res_W(Eq_7, Eq_9) = H_1 + iH_2, \quad Res_W(Eq_7, Eq_{10}) = H_3 + iH_4$$

where

$$\begin{aligned} H_1 =& -\big[4s^2 - (3 + u)^2\big]e + \big[4(f - c)(3 + u)s + (g^2 - 4m^2)s - l^2 s - 2l(3g - 3m + mu)\big], \\ H_2 =& \big[4s^2 - (3 + u)^2\big](c - f) + 2gm(u - 3) - 3l^2 + 3g^2 - 12es + 2lgs - 4m^2u - 4esu\big] \\ H_3 =& -a(3 + u)\big[12s^2 - (3 + u)^2\big] - 2bs\big[4s^2 - 3(3 + u)^2\big] + (cg - le - 2cm)\big[4s^2 - (3 + u)^2\big] \\ &- l^3 s - 2cl^2(3g - 3m + mu) + 2(g - 2m)(g^2 - gm - 2m^2 - 6es + gmu - 2m^2u - 2esu) \\ &- ls(12c - 3g^2 + 4gm + 4m^2 + 4cu), \\ H_4 =& 2as\big[4s^2 - 3(3+u)^2\big] - b(3+u)\big[12s^2 - (3+u)^2\big] + (cl + eg - 2em)\big[4s^2 - (3+u)^2\big] - 2l^3 \\ &- (g - 2m)(g^2 - 12c - 4m^2 - 4cu)s + 2l(3g^2 - 6gm - 6es + 2gmu - 4m^2u - 2esu) \\ &+ l^2(3g - 2m)s. \end{aligned} \tag{3.18}$$

It is clear that for the existence of a common solution of equations $Eq_7 = Eq_9 = Eq_{10} = 0$ with respect to W it is necessary and sufficient $H_1 = H_2 = H_3 = H_4 = 0$.

Solving the system of equations $H_1 = H_2 = 0$ with respect to the parameters e and f we obtain:

$$\begin{aligned} e =& \frac{1}{\big[4s^2 + (3+u)^2\big]^2}\Big[l^2 s(u^2 - 27 - 4s^2 - 6u) + 2l\big[m(3 - u)(4s^2 - (3+u)^2) + gu(4s^2 + 18 + 3u) \\ &+ 27g\big] + s(g - 2m)(27g - 18m + 4gs^2 + 8ms^2 + 6gu + 12mu - gu^2 + 6mu^2)\Big], \\ f =& c + \frac{1}{\big[4s^2 + (3+u)^2\big]^2}\Big[l^2(27 + 18u + 4s^2u + 3u^2) + 2ls(27g - 36m + 4gs^2 + 6gu - gu^2 + 4mu^2) \\ &- (g - 2m)\big[4s^2(6m + gu) + (3 + u)^2(3g + 2mu)\big]\Big] \end{aligned} \tag{3.19}$$

and evidently we could do this only in the case $4s^2 + (3 + u)^2 \neq 0$.

On the other hand for systems (3.12) we have

$$\mathcal{D}_8 = -8(s^2 - u)\big[4s^2 + (3 + u)^2\big]/27 \tag{3.20}$$

and as we will see later the condition $s^2 - u = 0$ is also essential.

So in what follows we have to consider two subcases: $\mathcal{D}_8 \neq 0$ and $\mathcal{D}_8 = 0$.

1.1: *The subcase* $\mathcal{D}_8 \neq 0$, i.e. $4s^2 + (3 + u)^2 \neq 0$. Considering this condition we examine all the needed directions.

(i) *The direction* $x+iy=0$. In this case we have the conditions (3.19) and solving the system of equations $H_3=H_4=0$ with respect to the parameters a and b we obtain:

$$
\begin{aligned}
a=&\frac{(g-2m)(3+u)-2ls}{\left[4s^2+(3+u)^2\right]^3}\Big[c(4s^2+(3+u)^2)^2+\left[4ls-2g(3+u)+4m(3+u)\right]\\
&\times\left[g(3+2s^2+u)+(1+u)(ls+3m+mu)\right]\Big],\\
b=&\frac{2(g-2m)s+l(3+u)}{\left[4s^2+(3+u)^2\right]^3}\Big[c\left[4s^2+(3+u)^2\right]^2-2(g-2m)\left[2gs^2(1+u)+m(8s^2+(3+u)^3)\right]\\
&+2l^2(3+u)(3+2s^2+u)+8lsm(1+u)(3+u)+2lgs\left[4s^2-(u-1)(3+u)\right]\Big].
\end{aligned}\tag{3.21}
$$

Thus if for systems (3.12) the conditions $k=d=h=0$, (3.19) and (3.21) are satisfied then these systems possess five invariant affine lines: three in the direction $x=0$ and two in the complex directions $x\pm iy$.

(ii) *The direction* $sx+y=0$. Then we have $U=s$, $V=1$ and considering (2.5) and the above conditions we obtain

$$
Eq_5=l+gs-2ms+(s^2-u)W \tag{3.22}
$$

and since the condition $\mathcal{D}_8\neq 0$ implies $s^2-u\neq 0$ we deduce that the above equation is linear with respect to the parameter W. Then the condition $Eq_5=0$ gives $W=(l+gs-2ms)/(u-s^2)$ and we calculate:

$$
Eq_8=\frac{2(1+s^2)\,\widehat{H}\,H_5}{(s^2-u)^2\left[4s^2+(3+u)^2\right]^2},\quad Eq_{10}=\frac{(1+s^2)\,\widehat{H}\,H_6}{(s^2-u)^3\left[4s^2+(3+u)^2\right]^3},\tag{3.23}
$$

where

$$
\begin{aligned}
\widehat{H}=&\,s(g-2m)(9+u)+l(9-2s^2+3u),\\
H_5=&\,ls(9+s^2)(1+u)+m(1+u)\left[s^2(u-9)-3u(3+u)\right]+g\left[2s^4+u^2(3+u)+s^2(9+5u)\right],\\
H_6=&\,c(s^2-u)^2\left[4s^2+(3+u)^2\right]^2+l^2(2s^2-9-3u)(1+u)(9+7s^2+2s^4+3u-s^2u)\\
&+g^2\left[6s^4(u^2-9-4u)-4s^6(3+u)-u^2(3+u)^3-s^2(81+99u+55u^2+13u^3)\right]\\
&+2m\left[4gs^6(3+u)+4ls^5(1+u)(3+u)+gu^2(3+u)^3-8ls^3(1+u)(u^2-9+2u)\right.\\
&+6ls(1+u)(3+u)(9+4u+u^2)-gs^4(u^3-81-45u+13u^2)\\
&\left.+2gs^2(81+126u+76u^2+20u^3+u^4)\right]+4m^2s^2(1+u)(9+u)(s^2u-9-3s^2-7u-2u^2)\\
&+2lgs\left[4s^6+s^4(3-10u-u^2)+2s^2(2u^2+u^3-18-23u)-(3+u)(27+39u+12u^2+2u^3)\right].
\end{aligned}\tag{3.24}
$$

We observe that the equations $Eq_8=Eq_{10}=0$ imply either $\widehat{H}=0$ or $H_5=H_6=0$ and we examine both possibilities.

First we observe that if for systems (3.12) the conditions of the existence of a triplet in the direction $x=0$ are satisfied (i.e. $k=d=h=0$) then for these systems we have $\chi_1=-\widehat{H}(g,l,m,s,u)x^3/9$. Therefore we conclude that the condition $\widehat{H}=0$ is equivalent to $\chi_1=0$ in the case under consideration.

1.1.1: *The possibility* $\chi_1=0$, i.e. $\widehat{H}=0$. We observe that the polynomial $\widehat{H}$ is linear with respect to the parameter l with the coefficient $2s^2-3(u+3)$ in front.

On the other hand for systems (3.12) we have

$$\mathcal{D}_6 = 4\big[2s^2 - 3(u+3)\big]/9$$

and therefore we have to consider two cases: $\mathcal{D}_6 \neq 0$ and $\mathcal{D}_6 = 0$.

1.1.1.1: *The case $\mathcal{D}_6 \neq 0$.* Then $9 - 2s^2 + 3u \neq 0$ and we calculate $l = \frac{(g-2m)s(9+u)}{2s^2-3u-9}$. So considering conditions (3.19) and (3.21) we arrive at the following lemma.

Lemma 3.5. *Assume that for a system (3.12) the conditions*

$$u+1 \neq 0, \quad (s^2-u)\big[4s^2+(3+u)^2\big] \neq 0, \quad \varkappa \equiv 2s^2 - 3(u+3) \neq 0. \tag{3.25}$$

hold. Then this system possesses invariant lines in the configuration $(3,1,1,1)$ if and only if the following conditions are satisfied:

$$\begin{aligned}
k = d = h = 0, \quad l &= \frac{s}{\varkappa}(u+9)(g-2m),\\
e &= \frac{s}{\varkappa^2}(g-2m)\big[g(s^2-27)+2m(s^2-3u+18)\big],\\
f &= c + \frac{3}{\varkappa^2}(g-2m)\big[3g(s^2-3)-2m(s^2+3u)\big],\\
a &= -\frac{3}{\varkappa^3}(g-2m)\big[c\,\varkappa^2+6(g-2m)(gs^2-3g-3m-3mu)\big],\\
b &= \frac{s}{\varkappa^3}(g-2m)\big[c\,\varkappa^2+2(g-2m)(4gs^2-2ms^2-9mu-27m)\big].
\end{aligned} \tag{3.26}$$

Next we construct the invariant conditions corresponding to (3.26).

Lemma 3.6. *Assume that for a cubic system (3.12) the conditions $\chi_1 = 0$ and $\mathcal{D}_6\mathcal{D}_7\mathcal{D}_8 \neq 0$ hold. Then this system has invariant lines in the configuration $(3,1,1,1)$ if and only if the conditions $\chi_3 = \chi_6 = 0$ are satisfied.*

Proof. For systems (3.12) we have $\mathcal{D}_4 = 2304s(9+s^2)$ and we examine two possibilities: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

a) *The possibility $\mathcal{D}_4 \neq 0$.* For systems (3.12) we calculate

$$\text{Coefficient}[\chi_1, y^3] = k(1+u)$$

and since $\mathcal{D}_7 = 4(1+u) \neq 0$ the condition $\chi_1 = 0$ implies $k = 0$. Then we get the conditions

$$\text{Coefficient}[\chi_1, xy^2] = \frac{2}{9}h(s^2+3u) = 0, \quad \text{Coefficient}[\chi_1, x^2y] = \frac{4}{9}hs(u-3) = 0$$

and since $s \neq 0$ (due to $\mathcal{D}_4 = 2304s(9+s^2) \neq 0$) we obtain $h = 0$. In this case we calculate

$$\chi_1 = \frac{1}{9}\big[l(-9+2s^2-3u) - (g-2m)s(9+u)\big]x^3 = 0$$

which implies $l = \frac{s(u+9)(g-2m)}{2s^2-3(u+3)}$. Thus the condition $\chi_1 = 0$ for systems (3.12) gives us the conditions on the parameters k, h and l from (3.26).

Next assuming that these conditions are satisfied we examine the other conditions from (3.26). Evaluating the invariant polynomial χ_6 we obtain

$$\text{Coefficient}[\chi_6, xy^7] = 10d(s^2-9-6u), \quad \text{Coefficient}[\chi_6, x^2y^6] = \frac{10}{3}ds(81+23s^2-42u)$$

and since $s \neq 0$ we claim that the vanishing of these coefficients implies $d = 0$. Indeed supposing $d \neq 0$ we get $s^2 - 9 - 6u = 0$ which gives $u = (s^2 - 9)/6$. Then we obtain $81 + 23s^2 - 42u = 16(9 + s^2) \neq 0$ and the contradiction we obtained proves our claim.

Thus $d = 0$ and calculations yield

$$\begin{aligned}
\text{Coefficient}[\chi_6, x^3y^5] = 110\Big[&e(s^2 - 9 - 6u) - fs(3 + s^2 - 2u) + cs(3 + s^2 - 2u)\\
&+ \frac{4s}{\varkappa}(g - 2m)(g - m)(9 + s^2)\Big],\\
\text{Coefficient}[\chi_6, x^4y^4] = \frac{10s}{3}\Big[&e(7s^2 - 927 - 330u) - fs(21 + 39s^2 - 110u) + cs(21 + 39s^2 - 110u)\\
&+\frac{4s}{\varkappa^2}(g-2m)(9+s^2)(927m-711g+86gs^2-62ms^2-165gu+165mu)\Big]
\end{aligned}$$

and we observe that the above polynomials are linear with respect to the parameters e and f with the corresponding determinant $-32fs(9 + s^2)^2\varkappa^3 \neq 0$. So forcing these polynomials to vanish we get

$$\begin{aligned}
e &= \frac{s}{\varkappa^2}(g - 2m)\big[g(s^2 - 27) + 2m(s^2 - 3u + 18)\big],\\
f &= c + \frac{3}{\varkappa^2}(g - 2m)\big[3g(s^2 - 3) - 2m(s^2 + 3u)\big].
\end{aligned}$$

Thus provided the condition $\chi_1 = 0$ is fulfilled, the condition $\chi_6 = 0$ for systems (3.12) gives us the conditions on the parameters d, e and f from (3.26).

So it remains to determine the invariant polynomials which are responsible for the conditions on the parameters a and b given in (3.26). Evaluating the invariant polynomial χ_3 for systems (3.12) for which the conditions on the parameters k, d, h, l, e and f are given in (3.26) we have:

$$\chi_3 = \frac{2x^5}{27\varkappa^3}\psi_1\psi_2\psi_3\big[\psi_4\, x + \psi_5\, y\big],$$

where

$$\begin{aligned}
\psi_1 &= s(u - 3)x + (s^2 + 3u)y,\\
\psi_2 &= (u^2 - 3s^2)x^2 - 4s(3 + u)xy + (s^2 - 9 - 6u)y^2,\\
\psi_3 &= (6s^2 + 3u + u^2)x^2 + 2s(9 + u)xy + (9 - 2s^2 + 3u)y^2,\\
\psi_4 &= -b\varkappa^3 + s(g - 2m)\big[c\,\varkappa^2 + 2(g - 2m)(4gs^2 - 2ms^2 - 9mu - 27m)\big],\\
\psi_5 &= a\varkappa^3 + 3(g - 2m)\big[c\,\varkappa^2 + 6(g - 2m)(gs^2 - 3g - 3m - 3mu)\big].
\end{aligned}$$

It is not too difficult to see that due to $s \neq 0$ the condition $\psi_1\psi_2\psi_3 \neq 0$ holds. Therefore the condition $\chi_3 = 0$ is equivalent to $\psi_4 = \psi_5 = 0$ and solving these equations with respect to the parameters a and b we get the expressions for these parameters given in (3.26). This completes the proof of Lemma 3.6 as well as the statement (A_1) of the Main Theorem in the case $\mathcal{D}_4 \neq 0$. □

b) *The possibility* $\mathcal{D}_4 = 0$. Then $s = 0$ and we observe that the conditions (3.26) become of the form:

$$\begin{aligned}
k = d = h = l = e = b = 0,\quad f &= c - \frac{1}{(3 + u)^2}(g - 2m)(3g + 2mu),\\
a &= \frac{1}{(3 + u)^3}(g - 2m)\big[c(3 + u)^2 - 2(g - 2m)(g + m + mu)\big],
\end{aligned}\tag{3.27}$$

For systems (3.12) with $s = 0$ we calculate

$$\chi_1 = -\frac{1}{9}\left[3l(u+3) + 2hu^2\right]x^3 - \frac{1}{9}k(u-3)ux^2y + \frac{2}{3}huxy^2 + k(u+1)y^3$$

and due to the condition $\mathcal{D}_7\mathcal{D}_8 = 32u(1+u)(3+u)^2/27 \neq 0$ we deduce that the condition $\chi_1 = 0$ is equivalent to $k = h = l = 0$.

On the other hand for systems (3.12) with $s = k = h = l = 0$ we calculate

$$\text{Coefficient}[\chi_6, xy^7] = -30d(3+2u), \quad \text{Coefficient}[\chi_6, x^3y^5] = 180d - 10(3+2u)(42d+33e+40du)$$

and evidently the condition $\chi_6 = 0$ implies $d = 0$. Then we calculate again

$$\text{Coefficient}[\chi_6, x^3y^5] = -330e(3+2u), \quad \text{Coefficient}[\chi_6, x^5y^3] = 5e\left[81 + (3+2u)(10u-99)\right]$$

and we observe that in this case the condition $\chi_6 = 0$ implies $e = 0$. We finally calculate

$$\chi_6 = 60u\left[(c-f)(3+u)^2 - (g-2m)(3g+2mu)\right]x^6y^2$$

and since $u(3+u) \neq 0$ the condition $\chi_6 = 0$ yields

$$f = c - \frac{1}{(3+u)^2}(g-2m)(3g+2mu),$$

i.e. we get the condition for the parameter f given in (3.27).

Next assuming the above mentioned conditions are fulfilled for systems (3.12) we calculate

$$\begin{aligned}\chi_3 = {} & \frac{2u}{9(3+u)^2}x^5y(ux^2+3y^2)(u^2x^2-9y^2-6uy^2)\\ & \times\Big\{b(3+u)^3x - \left[a(3+u)^3 - (g-2m)(c(3+u)^2 - 2(g-2m)(g+m+mu)\right]y\Big\}].\end{aligned}$$

Therefore due to $u(3+u) \neq 0$ the condition $\chi_3 = 0$ implies

$$b = 0, \quad a = \frac{1}{(3+u)^3}(g-2m)\left[c(3+u)^2 - 2(g-2m)(g+m+mu)\right],$$

i.e. we get the two conditions for the parameters b and a given in (3.27). This completes the proof of our claim and hence the statement (A_1) the Main Theorem is valid also in the case $\mathcal{D}_4 = 0$.

1.1.1.2: *The case* $\mathcal{D}_6 = 0$. This implies $9 - 2s^2 + 3u = 0$ and we have $u = (2s^2-9)/3$. Then we obtain:

$$\widehat{H} \equiv s(g-2m)(9+u) + l(9-2s^2+3u) = 2(g-2m)s(9+s^2)/3 = 0$$

i.e. we get $s(g-2m) = 0$. On the other hand for $u = (2s^2-9)/3$ we calculate (see (3.20))

$$\mathcal{D}_8 = -8(s^2-u)\left[4s^2 + (3+u)^2\right]/27 = -\frac{32}{729}s^2\left(s^2+9\right)^2 \neq 0.$$

So $s \neq 0$ and this implies $g = 2m$. Considering (3.19) and (3.21) we arrive at the next lemma.

Lemma 3.7. *Assume that for a system (3.12) the conditions $\chi_1 = 0$, $\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\mathcal{D}_6 = 0$ hold. Then this system possesses invariant lines in the configuration $(3,1,1,1)$ if and only if the following conditions are satisfied:*

$$\begin{aligned}
&k = d = h = 0, \quad u = (2s^2-9)/3, \quad g = 2m,\\
&e = \frac{3l\left[3l(s^2-27)+4ms(9+s^2)\right]}{4s(9+s^2)^2},\\
&f = \frac{81l^2(s^2-3)+36lms(9+s^2)+4cs^2(9+s^2)^2}{4s^2(9+s^2)^2},\\
&a = -\frac{9l\left[27l^2(s^2-3)+18lms(9+s^2)+2cs^2(9+s^2)^2\right]}{4s^3(9+s^2)^3},\\
&b = \frac{3l\left[18l^2s+9lm(9+s^2)+cs(9+s^2)^2\right]}{2s(9+s^2)^3}.
\end{aligned} \tag{3.28}$$

Next we determine the invariant conditions equivalent to those provided in the above lemma. More exactly we prove the following lemma.

Lemma 3.8. *Assume that for a system (3.12) the conditions $\chi_1 = 0$, $\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\mathcal{D}_6 = 0$ hold. Then this system has invariant lines in the configuration $(3,1,1,1)$ if and only if the conditions $\chi_2 = \chi_3 = 0$ are satisfied.*

Proof. For systems (3.12) with $u = (2s^2-9)/3$ we calculate:

$$\mathcal{D}_7 = \frac{8}{3}(s^2-3), \quad \text{Coefficient}[\chi_1, y^3] = 2k(s^2-3)/3, \quad \text{Coefficient}[\chi_1, xy^2]\big|_{k=0} = 2h(s^2-3)/3.$$

So it is clear that due to $\mathcal{D}_8 \neq 0$ (i.e. $s \neq 0$) the condition $\chi_1 = 0$ implies $k = h = 0$ and then calculations yield:

$$\chi_1 = -2(g-2m)s(9+s^2)x^3/27, \quad \mathcal{D}_8 = -\frac{32}{729}s^2(9+s^2)^2 \neq 0.$$

So we conclude that the condition $\chi_1 = 0$ for systems (3.12) with $\mathcal{D}_6 = 0$ (i.e. $u = (2s^2-9)/3$) is equivalent to $k = h = 0$ and $g = 2m$. Assuming that these conditions are fulfilled for systems (3.12) we obtain:

$$\text{Coefficient}[\chi_2, y^2] = 56ds(9+s^2)/3 = 0 \quad \Leftrightarrow \quad d = 0$$

and then we calculate

$$\chi_2 = -\frac{8}{9}\varphi_1' x^2 + \frac{16}{3}\varphi_2' xy,$$

where

$$\begin{aligned}
\varphi_1' &= 36es(3+s^2) - 8fs^4 + 81l^2 - 36lms + 8cs^4,\\
\varphi_2' &= 9e(s^2-3) - fs(-27+s^2) - 18clm - 27cs + cs^3.
\end{aligned}$$

We observe that the polynomials φ_1' and φ_2' are linear with respect to the parameters e and f with the corresponding determinant $36s^2(9+s^2)^2 \neq 0$ and therefore the equations $\varphi_1' = \varphi_2' = 0$ give us

$$\begin{aligned}
e &= \frac{3l}{4s(9+s^2)^2}\left[3l(s^2-27)+4ms(9+s^2)\right],\\
f &= \frac{1}{4s^2(9+s^2)^2}\left[81l^2(s^2-3)+36lms(9+s^2)+4cs^2(9+s^2)^2\right].
\end{aligned}$$

Thus provided $\chi_1 = 0$ is fulfilled, the condition $\chi_2 = 0$ for systems (3.12) gives us the conditions on the parameters d, e and f from (3.28).

Next evaluating the invariant polynomial χ_3 for systems (3.12) for which the conditions on the parameters k, d, h, u, g, e and f are given in (3.28) we have:

$$\chi_3 = \frac{2x^6(sx+3y)}{6561s^2(9+s^2)^2}\hat{\psi}_1\hat{\psi}_2\big[-2s^2\hat{\psi}_3\,x + \hat{\psi}_4\,y\big],$$

where

$$\begin{aligned}
\hat{\psi}_1 &= 2s(s^2-9)x + 9(s^2-3)y,\\
\hat{\psi}_2 &= (81-63s^2+4s^4)x^2 - 24s^3xy - 27(s^2-3)y^2,\\
\hat{\psi}_3 &= -2bs(9+s^2)^3 + 3l\big[18l^2s + 9lm(9+s^2) + cs(9+s^2)^2\big],\\
\hat{\psi}_4 &= -4as^3(9+s^2)^3 - 9l\big[27l^2(s^2-3) + 18lms(9+s^2) + 2cs^2(9+s^2)^2\big].
\end{aligned}$$

It is not too difficult to see that due to $s(s^2-3) \neq 0$ the condition $\hat{\psi}_1\hat{\psi}_2 \neq 0$ holds. Therefore the condition $\chi_3 = 0$ is equivalent to $\hat{\psi}_3 = \hat{\psi}_4 = 0$ and solving these equations with respect to the parameters a and b we get the expressions for these parameters given in (3.28). This completes the proof of Lemma 3.8 as well as the statement (A_2) of the Main Theorem. □

1.1.2: *The possibility* $\chi_1 \neq 0$. Then considering (3.23) in order to have invariant lines of total multiplicity seven we must force $H_5 = H_6 = 0$. Taking into account (3.24) we consider two cases: $s \neq 0$ and $s = 0$ and this condition is governed by the invariant polynomial $\mathcal{D}_4 = 2304s(9+s^2)$.

1.1.2.1: *The case* $\mathcal{D}_4 \neq 0$. Then $s \neq 0$ and solving the equations $H_5 = H_6 = 0$ with respect to the parameters c and l we obtain:

$$\begin{aligned}
c = {}& \frac{1}{s^2(9+s^2)^2(1+u)}\Big[-27m^2(s^2-3)(1+u)^2 + 6gm(s^2-3)(1+u)(s^2+3u)\\
&-g^2(2s^4u - 27s^2 - 7s^4 - 6s^2u - 9u^2 + 3s^2u^2)\Big],\\
l = {}& \frac{1}{s(9+s^2)(1+u)}\Big[m(1+u)(9s^2+9u-s^2u+3u^2) - g(9s^2+2s^4+5s^2u+3u^2+u^3)\Big].
\end{aligned}$$

Thus considering the conditions $k = d = h = 0$ and the conditions for the parameters e and f from (3.19) as well as for the parameters a and b from (3.21) and the above conditions we conclude that altogether these conditions guarantee the existence of common solutions of the equations (2.5) for each one of the four directions for invariant lines of systems (3.12). So we arrive at the following lemma.

Lemma 3.9. *Assume that for a system* (3.12) *the conditions* $\chi_1\mathcal{D}_7\mathcal{D}_8 \neq 0$ *and* $\mathcal{D}_4 \neq 0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the following conditions* *are satisfied:*

$$
\begin{aligned}
&k=d=h=0,\\
&f=\frac{1}{s^2\,(s^2+9)^2\,(u+1)^2}\big[3m^2(1+u)^2(27-9s^2+s^4+27u-3s^2u+9u^2)\\
&\quad+18gm(1+u)(2s^2+s^4-3u-3u^2-u^3)-g^2(s^2+3u)(2s^2+s^4-3u-3u^2-u^3)\big],\\
&e=\frac{1}{s\,(s^2+9)^2\,(u+1)^2}\big[m^2(3u-18-s^2)(1+u)^2(s^2+3u)\\
&\quad+6gm(1+u)(6s^2+s^4+9u+4s^2u+9u^2-u^3)\\
&\quad+g^2(-27s^2-11s^4-s^6-24s^2u-4s^4u-18u^2-8s^2u^2-12u^3+u^4)\big],\\
&a=\frac{g(3+s^2+2u)-6m(1+u)}{s^2\,(s^2+9)^2\,(u+1)^2}\big[9m^2(1+u)^2-6gmu(1+u)+g^2(s^2+u^2)\big],\\
&b=-\frac{m(s^2-3u)(1+u)+g(s^2+u^2)}{s^3\,(s^2+9)^3\,(u+1)^3}\big[m^2(1+u)^2(81+81s^2+2s^4+81u-3s^2u+18u^2)\\
&\quad+g^2(3+s^2+2u)(9s^2+2s^4+5s^2u+3u^2+u^3)\\
&\quad-2gm(1+u)(36s^2+10s^4+27u+33s^2u+27u^2+s^2u^2+6u^3)\big],\\
&c=\frac{1}{s^2(9+s^2)^2(1+u)}\big[27m^2(3-s^2)(1+u)^2+6gm(s^2-3)(1+u)(s^2+3u)\\
&\quad+g^2(27s^2+7s^4+6s^2u-2s^4u+9u^2-3s^2u^2)\big],\\
&l=-\frac{1}{s(9+s^2)(1+u)}\big[m(1+u)(s^2u-9s^2-9u-3u^2)+g(9s^2+2s^4+5s^2u+3u^2+u^3)\big]
\end{aligned}
\tag{3.29}
$$

Next we determine the invariant conditions equivalent to those provided by the above lemma. More exactly we prove the following lemma.

Lemma 3.10. *Assume that for a system (3.12) the conditions $\chi_1\mathcal{D}_7\mathcal{D}_8\neq 0$ and $\mathcal{D}_4\neq 0$ hold. Then this system has invariant lines in the configuration $(3,1,1,1)$ if and only if the conditions $\chi_7=\chi_8=\chi_9=\chi_{10}=0$ and either $\mathcal{D}_5\neq 0$ and $\chi_{11}=0$ or $\mathcal{D}_5=\chi_{12}=0$ are satisfied.*

Proof. For systems (3.12) we calculate:

$$\chi_7=\frac{1}{9}(hx+ky)\big[(3s^2+3u+2u^2)x^2-2s(u-3)xy-(s^2+3u)y^2\big].$$

We claim that the condition $\chi_7=0$ is equivalent to $k=h=0$. Indeed assume that $\chi_7=0$ and $k^2+h^2\neq 0$. Then we must have $3s^2+3u+2u^2=s(u-3)=s^2+3u=0$. However since $s\neq 0$ (due to $\mathcal{D}_4\neq 0$) we obtain $u=3$ and this leads to a contradiction $s^2+9=0$. So our claim is proved and we conclude that the condition $\chi_7=0$ gives $k=h=0$ from (3.29).

Assuming that for systems (3.12) the conditions $k=h=0$ hold we calculate

$$\chi_8=\frac{160}{9}d\big[s(3s^2-9+4u^2)x^2+2(6s^2+9u+s^2u+6u^2)xy+s(9+s^2)y^2\big]$$

and due to $s\neq 0$ we deduce that the condition $\chi_8=0$ is equivalent to $d=0$.

Next we evaluate the invariant polynomial χ_9 for systems (3.12) with the conditions $k=h=d=0$:

$$
\begin{aligned}
\chi_9=&-\frac{16}{9}\big[ls(9+s^2)(1+u)+m(1+u)(s^2u-9s^2-9u-3u^2)\\
&+g(9s^2+2s^4+5s^2u+3u^2+u^3)\big]x^5.
\end{aligned}
$$

Evidently forcing the condition $\chi_9 = 0$ to be fulfilled we get the condition for the parameter l given in (3.29).

Assuming that for systems (3.12) the conditions under the parameters k, h, d and l provided by Lemma 3.9 are fulfilled we calculate

$$\chi_{11} = -\frac{40}{9s^2(9+s^2)^2(1+u)^2}\Big[\big(u_{11}c + u_{12}e - u_{13}f + \widetilde{U}(g,m,s,u)\big)x^2 + \big(u_{21}c + u_{22}e - u_{23}f + \widetilde{V}(g,m,s,u)\big)xy + \big(u_{31}c + u_{32}e - u_{33}f + \widetilde{W}(g,m,s,u)\big)y^2\Big],$$

where

$$\begin{aligned}
u_{11} &= s^3(9+s^2)^2(1+u)^2(427s^2u + 912s^2 - 905u^2 - 237u + 2277),\\
u_{12} &= s^2(9+s^2)^2(1+u)^2(529s^2u + 1830s^2 - 2103u^2 - 7479u - 6831),\\
u_{13} &= s^3(9+s^2)^2(1+u)^2(529s^2u + 606s^2 - 599u^2 - 1155u + 2277),\\
u_{21} &= 2s^2(9+s^2)^2(1+u)^2(332s^4 - 613s^2u + 996s^2 - 306u^2 - 459u),\\
u_{22} &= 2s^3(9+s^2)^2(1+u)^2(383s^2 - 1686u - 1611),\\
u_{23} &= 2s^2(9+s^2)^2(1+u)^2(383s^4 - 307s^2u + 996s^2 + 153u^2 - 459u),\\
u_{31} &= 843s^3(9+s^2)^2(1+u)^2(3 + s^2 - 2u),\\
u_{32} &= 843s^2(9+s^2)^2(1+u)^2(s^2 - 6u - 9),\\
u_{33} &= 843s^3(9+s^2)^2(1+u)^2(3 + s^2 - 2u)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{U} = 2s\Big[&m^2(1+u)^2(s^2+3u)(1058s^4u + 1824s^4 - 1740s^2u^2 + 2025s^2u + 20601s^2 + 459u^3\\
&- 8775u^2 - 55080u - 73872) - 3gm(1+u)(427s^6u^2 + 281s^6u + 924s^6 + 376s^4u^3 + 3404s^4u^2\\
&+ 4065s^4u - 6777s^4 - 1657s^2u^4 - 3456s^2u^3 - 9999s^2u^2 - 50841s^2u - 61479s^2 + 306u^5\\
&- 12159u^4 - 78084u^3 - 137052u^2 - 61479u) + g^2(612s^8 + 427s^6u^3 - 643s^6u^2 - 1805s^6u\\
&+ 1713s^6 + 2658s^4u^3 - 12786s^4u^2 - 54540s^4u - 35424s^4 - 1128s^2u^5 - 5259s^2u^4 - 21321s^2u^3\\
&- 112509s^2u^2 - 220725s^2u - 122958s^2 + 153u^6 - 9234u^5 - 59724u^4 - 112428u^3 - 61479u^2)\Big],\\
\widetilde{V} = 2\Big[&m^2(1+u)^2(s^2+3u)(1532s^6 - 1686s^4u + 6894s^4 - 1377s^2u^2 - 17928s^2u - 24867s^2\\
&+ 1377u^3 + 8262u^2 + 12393u) - 6gm(1+u)(332s^8u - 434s^8 + 383s^6u^2 + 1150s^6u - 4599s^6\\
&- 1379s^4u^3 - 1074s^4u^2 - 8055s^4u - 18630s^4 - 153s^2u^4 - 9198s^2u^3 - 26919s^2u^2 - 10368s^2u\\
&+ 459u^5 + 2754u^4 + 4131u^3) + g^2(664s^8u^2 - 2656s^8u - 1484s^8 + 153s^6u^3 + 2298s^6u^2\\
&- 30507s^6u - 25308s^6 - 1992s^4u^4 - 3378s^4u^3 - 19341s^4u^2 - 99360s^4u - 70389s^4 - 13338s^2u^4\\
&- 37287s^2u^3 - 4212s^2u^2 + 12393s^2u + 459u^6 + 2754u^5 + 4131u^4)\Big],\\
\widetilde{W} = 1686&(g-m)s(9+s^2)(1+u)\big[g(s^4u - 5s^4 - 6s^2u - 18s^2 - 3u^3 - 9u^2)\\
&- m(2s^2 - 9 - 3u)(1+u)(s^2+3u)\big].
\end{aligned}$$

We observe that the condition $\chi_{11} = 0$ yields the equations

$$\text{Coefficient}[\chi_{11}, x^2] = \text{Coefficient}[\chi_{11}, xy] = \text{Coefficient}[\chi_{11}, y^2] = 0 \tag{3.30}$$

which are linear with respect to the parameters c, e and f. Calculating the corresponding determinant $\det ||u_{ij}||$ $(i,j = 1,2,3)$ we obtain

$$\det ||u_{ij}|| = 26311716s^7(s^2+3u)(9+s^2)^7(1+u)^6(s^2-u)\big[4s^2 + (u+3)^2\big].$$

On the other hand for systems (3.12) we have

$$\mathcal{D}_5 = \frac{4}{9}(s^2+3u),\ \ \mathcal{D}_4 = 2304s(9+s^2),\ \ \mathcal{D}_7 = 4(1+u),\ \ \mathcal{D}_8 = -8(s^2-u)\big[4s^2+(3+u)^2\big]/27$$

and since $\mathcal{D}_4\mathcal{D}_7\mathcal{D}_8 \neq 0$ we conclude that in the case $\mathcal{D}_5 \neq 0$ we get $\det||u_{ij}|| \neq 0$.

So assuming $\mathcal{D}_5 \neq 0$ and solving the system of equations (3.30) with respect to the parameters c, e and f we get exactly the conditions provided by Lemma 3.9 for these three parameters.

We examine now the case $\mathcal{D}_5 = 0$ when the invariant polynomial χ_{11} could not be used for the determining the conditions under parameters c, e and f.

So assume that for systems (3.12) the conditions on the parameters k, h, d and l provided by Lemma 3.9 are fulfilled and in addition the condition $\mathcal{D}_5 = 0$ holds. This implies $s^2+3u = 0$ (i.e. $u = -s^2/3$) and we calculate

$$\chi_{12} = \frac{4x^2}{729(s^2-3)^2}(\phi_1' x^6+\phi_2' x^5 y+\phi_3' y^4 y^2+\phi_4' x^3 y^3+\phi_5' x^2 y^4+\phi_6' x y^5+\phi_7' y^6),\quad \mathcal{D}_7 = -\frac{4}{3}(s^2-3),$$

where

$$\phi_7' = -2751246(s^2-3)^2(3f-3c+g^2-2gm+cs^2-fs^2).$$

Since $\mathcal{D}_7 \neq 0$ (i.e. $s^2-3 \neq 0$) the condition $\phi_7' = 0$ gives us

$$f = \frac{1}{s^2-3}(cs^2-3c+g^2-2gm) \tag{3.31}$$

and then we calculate

$$\phi_6' = 655371(s^2-3)\big[9e(s^2-3)^2 - gs(18m-27g+gs^2-6ms^2)\big].$$

Therefore due to $s^2-3 \neq 0$ the condition $\phi_6' = 0$ implies

$$e = \frac{gs}{7(s^2-3)^2}(gs^2-27g-6ms^2+18m)$$

and for these values of the parameters f and e we obtain

$$\chi_{12} = -432(s^2-3)x^8(sx+3y)^2\big[cs^2(s^2-3)(9+s^2)^2 - 9m^2(s^2-3)^3 + g^2s^2(9+s^2)^2\big].$$

Again since $s^2-3 \neq 0$ as well as $s \neq 0$ (due to $\mathcal{D}_4 \neq 0$) the condition $\chi_{12} = 0$ yields

$$c = \frac{1}{s^2(s^2-3)(9+s^2)^2}\big[9m^2(s^2-3)^3 - g^2s^2(9+s^2)^2\big].$$

So considering (3.31) we obtain

$$f = \frac{m}{s^2(s^2-3)(9+s^2)^2}\big[9m(s^2-3)^3 - 2gs^2(9+s^2)^2\big].$$

Comparing the conditions obtained for the parameters c, e and f above with (3.29) for $u = -s^2/3$ we conclude that they coincide.

Thus from the conditions (3.29) it remains to construct the invariant analog for the conditions on the parameters a and b and this will be done independently on the value of the invariant polynomial $\mathcal{D}_5$.

So evaluating the invariant polynomial χ_{10} for systems (3.12) for which the conditions on the parameters k, d, h, l, c, e and f are given in (3.29) we have:

$$\chi_{10} = \frac{112640x^6}{9s^4(9+s^2)^4(1+u)^4}\psi_1'\psi_2'\psi_3'\big[9m(1+u) - g(s^2+3u)\big]\big[\psi_4' x + \psi_5' y\big],$$

where

$$\begin{aligned}
\psi_1' &= (2s^4 + s^2u^2 - 3s^2u - 2u^3 - 9u^2 - 9u)\,x - s(s^2+9)(u+1)y,\\
\psi_2' &= (3s^2 + 2u^2 + 3u)\,x^2 - 2s(u-3)xy - (s^2+3u)\,y^2,\\
\psi_3' &= (3s^2 - u^2)x^2 + 4s(3+u)xy - (s^2 - 9 - 6u)y^2,\\
\psi_4' &= bs^3(9+s^2)^3(1+u)^3 + \big[m(s^2-3u)(1+u) + g(s^2+u^2)\big]\big[m^2(1+u)^2(81+81s^2+2s^4+81u\\
&\quad -3s^2u+18u^2)+g^2(3+s^2+2u)(9s^2+2s^4+5s^2u+3u^2+u^3)-2gm(1+u)(36s^2+10s^4\\
&\quad +27u+33s^2u+27u^2+s^2u^2+6u^3)\big],\\
\psi_5' &= -s(9+s^2)(1+u)\Big[as^2(9+s^2)^2(1+u)^2 - (3g-6m+gs^2+2gu-6mu)\big[g^2s^2\\
&\quad +(gu-3m-3mu)^2\big]\Big].
\end{aligned}$$

We observe that the polynomial χ_{10} contains as a factor the expression $9m(1+u) - g(s^2 + 3u)$ which is different from zero due to the condition $\chi_1 \neq 0$ because in the case under consideration we have:

$$\chi_1 = \frac{(s^2-u)}{9s(9+s^2)(1+u)}\big[4s^2 + (u+3)^2\big]\big[9m(1+u) - g(s^2+3u)\big].$$

It is evidently due to $\mathcal{D}_4\mathcal{D}_7\mathcal{D}_8 \neq 0$ that the condition $\chi_1 \neq 0$ is equivalent to $9m(1+u) - g(s^2+3u) \neq 0$.

Thus due to $\chi_1 \neq 0$ we conclude that the condition $\chi_{10} = 0$ is equivalent to $\psi_4' = \psi_5' = 0$, because for $s \neq 0$ the condition $\psi_1'\psi_2'\psi_3' \neq 0$ holds. Solving the equations $\psi_4' = \psi_5' = 0$ with respect to the parameters a and b we get exactly the expressions for these parameters given in (3.29).

So Lemma 3.10 is proved and this means that the statement (A_3) of the Main Theorem is also proved. □

1.1.2.2: *The case* $\mathcal{D}_4 = 0$. Then $s = 0$ and considering (3.24) and systems (3.12) with the conditions (3.19) and (3.21) we obtain

$$\mathcal{D}_8 = 8u(3+u)^2/27, \quad \widehat{H} = 3l(3+u), \quad H_5 = -u(3+u)(3m - gu + 3mu),$$
$$H_6 = -(3+u)^2\big[9l^2(1+u) + g(g-2m)u^2(3+u) - cu^2(3+u)^2\big].$$

Since in this case the conditions $\widehat{H} \neq 0$ and $\mathcal{D}_8 \neq 0$ imply $lu(3+u) \neq 0$, solving the equations $H_5 = H_6 = 0$ with respect to the parameters c and g we obtain:

$$c = \frac{3(1+u)\big[3l^2 + m^2(3+u)^2\big]}{u^2(3+u)^2}, \quad g = \frac{3m(1+u)}{u}.$$

So we arrive at the next lemma.

Lemma 3.11. *Assume that for a system (3.12) the conditions $\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\mathcal{D}_4 = 0$ hold. Then this system possesses invariant lines in the configuration $(3,1,1,1)$ if and only if the following conditions are satisfied:*

$$\begin{aligned}
&k = d = h = s = 0, \quad e = \frac{2lm}{u}, \quad g = \frac{3m(1+u)}{u},\\
&f = \frac{1}{u^2(3+u)^2}\big[m^2u(3+u)^2 + 3l^2(3+3u+u^2)\big],\\
&c = \frac{3(1+u)}{u^2(3+u)^2}\big[3l^2 + m^2(3+u)^2\big], \quad l(3+u) \neq 0,\\
&a = \frac{m(1+u)}{u^3(3+u)^2}\big[9l^2 + m^2(3+u)^2\big],\\
&b = \frac{l}{u^2(3+u)^2}\big[m^2(3+u)^2 + l^2(3+2u)\big],
\end{aligned} \tag{3.32}$$

Next we determine the invariant conditions equivalent to those provided in the above lemma. More exactly we prove the following lemma.

Lemma 3.12. *Assume that for a cubic system (3.12) the conditions $\chi_1\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\mathcal{D}_4 = 0$ hold. Then this system possesses invariant lines in the configuration $(3,1,1,1)$ if and only if the conditions $\chi_4 = \chi_5 = \chi_7 = \chi_9 = \chi_{13} = \chi_{14} = 0$ are satisfied.*

Proof. For systems (3.12) the condition $\mathcal{D}_4 = 2304s(9+s^2) = 0$ gives $s = 0$ and we calculate:

$$\chi_7 = \frac{1}{9}u(hx+ky)\big((3x^2 + 2ux^2 - 3y^2)\big).$$

Is evidently due to $\mathcal{D}_8 \neq 0$ the condition $\chi_7 = 0$ is equivalent to $k = h = 0$. Assuming these conditions to be satisfied for systems (3.12) as well as the condition $s = 0$ we calculate

$$\begin{gathered}
\chi_4 = -2du(1+u), \quad \chi_5 = -2lm - du + eu,\\
\chi_9 = -\frac{3376}{27}u(3+u)(gu - 3m - 3mu)x^5, \quad \mathcal{D}_7 = 4(1+u).
\end{gathered}$$

Therefore due to $\mathcal{D}_7\mathcal{D}_8 \neq 0$ (i.e. $u(u+1)(u+3) \neq 0$) we conclude that the condition $\chi_4 = 0$ is equivalent to $d = 0$ and in this case the condition $\chi_5 = 0$ gives us $e = 2lm/u$. Moreover the condition $\chi_9 = 0$ implies $g = 3m(1+u)/u$. So we get the conditions for the parameters d, e and g given in (3.32).

Next provided these conditions are satisfied for systems (3.12) and evaluating the invariant polynomial χ_{13} we have:

$$\chi_{13} = \frac{120}{u}(\theta_1' x^2 + \theta_2' y^2)$$

where

$$\begin{aligned}
\theta_1' = &-cu^2(138 + 178u + 69u^2) - fu^2(-138 - 145u + 17u^2) + 3l^2u(11+17u)\\
&+ 2m^2(207 + 405u + 298u^2 + 112u^3),\\
\theta_2' = &-3cu^2(3+2u) - 3f(-3+u)u^2 + 3\big[3l^2u + m^2(9+12u+7u^2)\big].
\end{aligned}$$

We observe that the equations $\theta_1' = \theta_2' = 0$ are linear with respect to the parameters c and f and the corresponding determinant equals $105u^5(3+u)^2 \neq 0$. Solving these equations we obtain:

$$c = \frac{3(1+u)}{u^2(3+u)^2}\big[3l^2 + m^2(3+u)^2\big], \quad f = \frac{1}{u^2(3+u)^2}\big[m^2u(3+u)^2 + 3l^2(3+3u+u^2)\big],$$

i.e. we get exactly the values for the parameters c and f presented in (3.32).

Thus from the conditions (3.32) it remains to construct the invariant analog for the conditions under parameters a and b. Evaluating the invariant polynomial χ_{14} for systems (3.12) for which the conditions on the parameters k, d, h, s, e, g, c and f are given in (3.32) we have:

$$\chi_{14} = \frac{2(u+1)x^5y}{u^2(3+u)^2}(ux^2-3y^2)\big[u^2x^2-3(3+2u)y^2\big]\big[u\theta_1''\,x+\theta_2''\,y\big],$$

where

$$\begin{aligned}\theta_1'' &= -bu^2(3+u)^2 + l\big[m^2(3+u)^2 + l^2(3+2u)\big],\\ \theta_2'' &= au^3(3+u)^2 - m(1+u)(9l^2+9m^2+6m^2u+m^2u^2).\end{aligned}$$

It is clear that due to $u(u+1)(u+3)\neq 0$ the condition $\chi_{14}=0$ is equivalent to $\theta_1''=\theta_2''=0$ and this implies exactly the conditions for the parameters a and b given in (3.32).

It remains to observe that the condition $l(3+u)\neq 0$ from (3.32) is equivalent to $\chi_1 = -l(3+u)x^3/3 \neq 0$. This completes the proof of Lemma 3.12 as well as of the statement (A_4) of the Main Theorem. □

1.2: *The subcase* $\mathcal{D}_8=0$, i.e. $(s^2-u)\big[4s^2+(3+u)^2\big]=0$.

Remark 3.13. For systems (3.12) the condition $\mathcal{D}_8=0=\mathcal{D}_6$ is equivalent to $4s^2+(3+u)^2=0$.

Indeed for systems (3.12) we have $\mathcal{D}_6 = 4(2s^2-9-3u)/9$. Assume that $\mathcal{D}_8=\mathcal{D}_6=0$ but $4s^2+(3+u)^2\neq 0$. Then we get $u=s^2$ and this implies $\mathcal{D}_6=-4(9+s^2)/9\neq 0$. This contradiction proves the validity of the above remark. So in what follows we examine two possibilities: $\mathcal{D}_6\neq 0$ and $\mathcal{D}_6=0$.

1.2.1: *The possibility* $\mathcal{D}_6\neq 0$. Then the condition $\mathcal{D}_8=0$ yields $s^2-u=0$. We mention that earlier (up to **1.1:** *The subcase* $\mathcal{D}_8\neq 0$, see page 29) we have investigated the directions $x=0$ and $x\pm iy=0$. So now we examine the remaining direction for the invariant lines, i.e. $sx+y=0$.

Thus we have $u=s^2$ and considering (3.22) for the direction $sx+y=0$ the condition $Eq_5=0$ gives $l=s(2m-g)$. In this case for the equations $Eq_8=0$ and $Eq_{10}=0$ we obtain

$$\begin{aligned}Eq_8 &= \frac{2(2m-g)s+(9+s^2)W}{(9+s^2)^2}\Phi_1(g,m,s,W)=0,\\ Eq_{10} &= \frac{2(2m-g)s+(9+s^2)W}{(9+s^2)^2}\Phi_2(g,m,s,W)=0,\end{aligned}$$

where $\Phi_i(g,m,s,W)$ is a polynomial in the parameters g, m and s and it is of degree i with respect to the variable W.

As we can see the above equations have a common solution in variable W, i.e. in the direction $sx+y=0$ we have one invariant line and altogether we get six invariant affine lines.

Thus considering (3.19) and (3.21) in the case $u=s^2$ as well as the condition $l=s(2m-g)$ we arrive at the following lemma.

Lemma 3.14. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7\mathcal{D}_6\neq 0$ *and* $\mathcal{D}_8=0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the following conditions* *are satisfied:*

$$
\begin{aligned}
& k=d=h=0, \quad l=s(2m-g), \quad u=s^2,\\
& f=c+\frac{3(g-2m)}{(s^2+9)^2}\,(3gs^2-9g-8ms^2),\\
& e=\frac{s(g-2m)}{(s^2+9)^2}\,(gs^2-27g-4ms^2+36m),\\
& a=\frac{3(g-2m)}{(s^2+9)^3}\,\big[c(9+s^2)^2+6(g-2m)(gs^2-3g-3m-3ms^2)\big],\\
& b=-\frac{s(g-2m)}{(s^2+9)^3}\,\big[c(9+s^2)^2+2(g-2m)(4gs^2-27m-11ms^2)\big].
\end{aligned}
\tag{3.33}
$$

In order to detect the corresponding invariant conditions we consider two cases: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

1.2.1.1: *The case* $\mathcal{D}_4 \neq 0$. We observe that the conditions (3.33) can be obtained as a particular case from the conditions (3.26) by setting $u = s^2$ (i.e. we allow the condition $\mathcal{D}_8 = 0$ to be satisfied).

On the other hand in the proof of Lemma 3.6 we did not use the condition $s^2 - u \neq 0$ and this means that Lemma 3.6 is valid in the case $\mathcal{D}_8 = 0$ too. Therefore we deduce that the statement (A_5) of the Main Theorem is true.

1.2.1.2: *The case* $\mathcal{D}_4 = 0$. Then $s = 0$ and we have $u = 0 = s$. Then according to Lemma 3.4 we could have a triplet of invariant lines either in the direction $x = 0$ or in the direction $y = 0$. Therefore we have to construct the affine invariant conditions taking into consideration the second possibility for the existence of a triplet.

In the first case (i.e. when a triplet of invariant lines is in the direction $x = 0$) we have constructed the corresponding conditions which coincide with (3.33) for $s = 0$. Now we have to determine the conditions on the parameters of systems (3.12) in order to possess invariant lines in the configuration $(3, 1, 1, 1)$ with the triplet in the direction $y = 0$.

So we have to examine each one of the directions for the invariant lines in this case.

(i) *The direction* $y = 0$. Considering the equations (2.5) and Remark 2.12 for systems (3.12) with $s = u = 0$ in the case of the direction $y = 0$ we obtain the following non-vanishing equations containing the parameter W:

$$
Eq_5 = l, \quad Eq_8 = e - 2mW, \quad Eq_{10} = b - fW + W^3.
$$

So it is evident that for the existence of a triplet the conditions $l = e = m = 0$ have to be satisfied.

(ii) *The direction* $x + iy = 0$. In this case we have $U = 1$, $V = i$ and considering (2.5), Remark 2.12 and the conditions $u = s = l = e = m = 0$ we obtain

$$
\begin{gathered}
Eq_7 = k - g - 2ih + 3W,\\
Eq_9 = d + i(f - c) - 2(h - ig)W - 3iW^2,\\
Eq_{10} = a + Ib - cW + gW^2 - W^3.
\end{gathered}
$$

Calculations yield

$$
Res_W(Eq_7, Eq_9) = V_1 + iV_2, \quad Res_W(Eq_7, Eq_{10}) = V_3 + iV_4
$$

$$\begin{aligned} V_1 &= 3d - 2gh - 2hk, \quad V_2 = 3f - 3c + g^2 - k^2 \\ V_3 &= 27a - 9cg + 2g^3 + 9ck - 3g^2k - 12h^2k + k^3, \\ V_4 &= 27b - 18ch + 6g^2h + 8h^3 - 6hk^2. \end{aligned} \tag{3.34}$$

It is clear that for the existence of a common solution of equations $Eq_7 = Eq_9 = Eq_{10} = 0$ with respect to W it is necessary and sufficient $V_1 = V_2 = V_3 = V_4 = 0$. Solving these equations we get

$$f = \frac{1}{3}(3c - g^2 + k^2), \quad d = \frac{2h}{3}(g + k), \quad b = \frac{2h}{27}(9c - 3g^2 - 4h^2 + 3k^2),$$
$$a = \frac{1}{27}\left[9c(g - k) - 2g^3 + 3(g^2 + 4h^2)k - k^3\right]$$

(iii) *The direction $x = 0$.* In this case considering the above already detected conditions and (2.5) as well as Remark 2.12 we obtain $Eq_7 = k = 0$. Hence $k = 0$ and we calculate the remaining non-vanishing equations:

$$Eq_9 = \frac{2}{3}h(g - 3W), \quad Eq_{10} = -\frac{1}{27}(g - 3W)(-9c + 2g^2 + 6gW - 9W^2).$$

As we can see the equations $Eq_9 = 0$ and $Eq_{10} = 0$ have a common solution $W = g/3$.

Thus we conclude that the following lemma is valid.

Lemma 3.15. *Assume that for a system (3.12) the conditions $\mathcal{D}_7\mathcal{D}_6 \neq 0$ and $\mathcal{D}_8 = \mathcal{D}_4 = 0$ hold. Then this system possesses invariant lines in the configuration $(3, 1, 1, 1)$ if and only if one of the following sets of the conditions is satisfied:*

– for a triplet in the direction $x = 0$:

$$\begin{aligned} &u = s = k = d = h = l = e = b = 0, \quad f = c + \frac{g(2m - g)}{3}, \\ &a = -\frac{g - 2m}{27}\left(2g^2 - 9c - 2gm - 4m^2\right). \end{aligned} \tag{3.35}$$

– for a triplet in the direction $y = 0$:

$$\begin{aligned} &u = s = k = l = e = m = 0, \quad d = \frac{2gh}{3}, \quad f = c - \frac{g^2}{3}, \\ &a = \frac{g}{27}(9c - 2g^2), \quad b = -\frac{2h}{27}(-9c + 3g^2 + 4h^2). \end{aligned} \tag{3.36}$$

We point out that in order to construct the equivalent invariant conditions for a system (3.12) to possess invariant lines in the configuration $(3, 1, 1, 1)$ we have to take into considerations both sets of conditions: (3.35) (when the triplet is in the direction $x = 0$) and (3.36) (when the triplet is in the direction $y = 0$).

First of all we recall that for systems (3.12) the conditions $\mathcal{D}_8 = 0$ and $\mathcal{D}_6 \neq 0$ yields $s^2 - u = 0$ (see page 41) and $\mathcal{D}_4 = 0$ gives $s = 0$, i.e. we have for systems (3.12) $s = u = 0$.

Considering these conditions we evaluate the invariant polynomial χ_1 for systems (3.12):

$$\chi_1 = -lx^3 + ky^3$$

and evidently the condition $\chi_1 = 0$ implies $k = l = 0$. We observe that these conditions are included in (3.35) as well as in (3.36). Then we calculate

$$\begin{aligned} \chi_3 &= 2(mx + hy)x^3y^3(x^2 + y^2)\left[3ex^2 - 2(3c - 3f - g^2 + 2gm)xy - (3d - 2gh)y^2\right], \\ \chi_8 &= -960hmxy \end{aligned}$$

and we prove the next lemma.

Lemma 3.16. *Assume that for a system (3.12) are satisfied either the conditions (3.35) or (3.36) and in addition we have $h = m = 0$. Then this system possess invariant lines of total multiplicity 9.*

Proof. Supposing $h = m = 0$ a straightforward calculation shows us that the conditions (3.35) coincide with (3.36) and have:

$$u = s = k = d = h = l = e = b = 0, \quad f = c - g^2/3, \quad a = -\frac{g}{27}\left(2g^2 - 9c\right).$$

The above conditions lead to the family of systems

$$\dot{x} = -\frac{1}{27}(g + 3x)\left(2g^2 - 9c - 6gx - 9x^2\right),$$
$$\dot{y} = \frac{1}{3}y\left(3c - g^2 - 3y^2\right),$$

which evidently possess two triplets of parallel invariant lines: one in the direction $x = 0$ and another in the direction $y = 0$. Moreover in addition these systems possess the following two complex invariant lines: $g + 3(x \pm iy) = 0$ and this completes the proof of the lemma. □

Thus we conclude that in the case of the conditions (3.35) or (3.36) the conditions $h^2 + m^2 \neq 0$ must hold. It remains to observe that this condition is governed by the invariant polynomial χ_{15} because for systems (3.12) with $s = u = k = l = 0$ we have $\chi_{15} = x^2y^2(mx + hy)$.

So in what follows we assume that the condition $\chi_{15} \neq 0$ holds, i.e. $h^2 + m^2 \neq 0$. Then the condition $\chi_3 = 0$ implies

$$e = 0, \quad f = \frac{1}{3}(3c - g^2 + 2gm), \quad d = \frac{2gh}{3},$$

whereas the condition $\chi_8 = 0$ implies $hm = 0$.

In the case $h = 0$ we get $e = d = 0$ and $f = (3c - g^2 + 2gm)/3$ and we observe that we obtain exactly the conditions from (3.35) provided for the parameters h, e, d and f.

On the other hand if $m = 0$ we obtain $e = 0$, $d = 2gh/3$ and $f = c - g^2/3$, i.e. we obtain exactly the conditions from (3.36) provided for the parameters m, e, d and f.

We examine each one of the cases mentioned above.

***α*)** Assume first that for systems (3.12) all the conditions (3.35) are satisfied except the conditions on the parameters a and b. Then for these systems we calculate

$$\chi_{16} = -12x^5y^5\left[(27a - 9cg + 2g^3 + 18cm - 6g^2m + 8m^3)x + 27by\right]$$

and we determine that the condition $\chi_{16} = 0$ implies

$$a = \frac{1}{27}(g - 2m)(9c - 2g^2 + 2gm + 4m^2), \quad b = 0.$$

So we obtain exactly the conditions on the parameters a and b given in (3.35).

***β*)** Suppose now that for systems (3.12) all the conditions (3.36) are satisfied excepting the conditions on the parameters a and b. Then for these systems we calculate

$$\chi_{16} = -12x^5y^5\left[(27a - 9cg + 2g^3)x + (27b - 18ch + 6g^2h + 8h^3)y\right]$$

and we obtain that the condition $\chi_{16} = 0$ implies in this case

$$a = \frac{1}{27}g(9c - 2g^2), \quad b = \frac{2}{27}h(9c - 3g^2 - 4h^2).$$

So we obtain exactly the conditions on the parameters a and b given in (3.36).

Thus we have proved the following lemma.

Lemma 3.17. *Assume that for a cubic system* (3.12) *the conditions* $\mathcal{D}_6\mathcal{D}_7 \neq 0$ *and* $\mathcal{D}_8 = \mathcal{D}_4 = 0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the conditions* $\chi_1 = \chi_3 = \chi_8 = \chi_{16} = 0$ *and* $\chi_{15} \neq 0$ *are satisfied.*

From this lemma the validity of the statement (A_6) of the Main Theorem follows.

1.2.2: *The possibility* $\mathcal{D}_6 = 0$. Since $\mathcal{D}_8 = 0$, according to Remark 3.13 the condition $4s^2 + (3+u)^2 = 0$ holds. Then $s = 0$, $u = -3$ and by Lemma 3.4 we conclude that a triplet could be only in the direction $x = 0$. So considering the condition $k = d = h = 0$ which guarantees the existence of a triplet of parallel invariant lines in the direction $x = 0$ we examine the directions $sx + y = 0$ (which becomes $y = 0$) and $x + iy = 0$.

a) *The direction* $y = 0$. Considering (2.5) and Remark 2.12 we obtain

$$Eq_5 = l + 3W, \quad Eq_8 = e - 2mW, \quad Eq_{10} = b - fW + W^3.$$

Therefore the condition $Eq_5 = 0$ yields $W = -l/3$ and then we obtain

$$Eq_8 = (3e + 2lm)/3 = 0, \quad Eq_{10} = (27b - l^3 + 9lf)/27 = 0.$$

Solving these equations with respect to the parameters b and e we get

$$b = l(l^2 - 9f)/27, \quad e = -2lm/3 \tag{3.37}$$

and these conditions guarantee the existence of one invariant line in the direction $y = 0$.

b) *The direction* $x + iy = 0$. In this case taking into account the conditions $k = d = h = 0$ and (3.37) we obtain

$$\begin{gathered} Eq_7 = 2m - g - il, \quad Eq_9 = -2lm/3 + i(f-c) - 2\big[l + i(m-g)\big]\big)W + 3iW^2, \\ Eq_{10} = a + il(l^2 - 9f)/27 - (c - 2lmi/3)W + (g + il)W^2 + 2W^3. \end{gathered}$$

Clearly the condition $Eq_7 = 0$ implies $l = 0$ and $g = 2m$ and therefore we have

$$Eq_9 = i(-c + f + 2mW + 3W^2), \quad Eq_{10} = a - cW + 2mW^2 + 2W^3.$$

Calculations yield

$$\begin{aligned} &Res_W(Eq_9, Eq_{10}) = i\big[27a^2 + 2am(9c + 4m^2) - (c-f)(c^2 + 4cf + 4f^2 + 4fm^2)\big] \equiv iH', \\ &Res_W^{(2)}(Eq_9, Eq_{10}) = 3c + 6f + 4m^2. \end{aligned} \tag{3.38}$$

Thus the condition $H' = 0$ implies the existence of at least one common solution $W = W_0$ of the equations $Eq_9 = 0$ and $Eq_{10} = 0$. Moreover in this case the condition $Res_W^{(2)}(Eq_9, Eq_{10}) \neq 0$ must hold (i.e. $3c + 6f + 4m^2 \neq 0$), otherwise we get that the mentioned equations have two common solutions and therefore the corresponding systems do not belong to the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$.

We observe that the polynomial H' is quadratic with respect to the parameter a and we calculate

$$\text{Discrim}[H', a] = 4(3c - 3f + m^2)(3c + 6f + 4m^2)^2.$$

Since the condition $3c + 6f + 4m^2 \neq 0$ has to be fulfilled (see the previous paragraph) we deduce that the condition $3c - 3f + m^2 \geq 0$ must hold.

So we have proved the following lemma.

Lemma 3.18. *Assume that for a system (3.12) the conditions $\mathcal{D}_7 \neq 0$ and $\mathcal{D}_8 = \mathcal{D}_6 = 0$ hold. Then this system possesses invariant lines in the configuration $(3,1,1,1)$ if and only if the following conditions are satisfied:*

$$\begin{gathered} s=0,\ u=-3,\ k=d=h=e=l=b=0,\ g=2m,\\ 27a^2+2am(9c+4m^2)-(c-f)(c^2+4cf+4f^2+4fm^2)=0,\\ 3c-3f+m^2\geq 0,\quad 3c+6f+4m^2\neq 0. \end{gathered} \tag{3.39}$$

Next we determine the invariant conditions equivalent to those provided in the above lemma. More exactly we prove the following lemma.

Lemma 3.19. *Assume that for a cubic system (3.12) the conditions $\mathcal{D}_7 \neq 0$ and $\mathcal{D}_8 = \mathcal{D}_6 = 0$ hold. Then this system possesses invariant lines in the configuration $(3,1,1,1)$ if and only if the conditions $\chi_1=\chi_2=\chi_4=\chi_6=\chi_{17}=0$, $\chi_{11}\neq 0$ and $\zeta_4\leq 0$ are satisfied.*

Proof. As it was mentioned earlier (see Remark 3.13) the conditions $\mathcal{D}_8 = \mathcal{D}_6 = 0$ imply for systems (3.12) $s=0$ and $u=-3$. Then for these systems we calculate:

$$\chi_1 = -2(hx+ky)(x^2+y^2)=0 \quad \Leftrightarrow \quad h=k=0.$$

Herein calculations yield

$$\text{Coefficient}[\chi_6, xy^7] = 90d, \quad \text{Coefficient}[\chi_6, x^4y^4] = -720l^2$$

and evidently the condition $\chi_6=0$ implies $d=0$ and $l=0$. Then we calculate again

$$\chi_6 = 90x^3y\big[15ex^4+6(g-2m)^2x^3y+26ex^2y^2+11ey^4\big]$$

and clearly the condition $\chi_6=0$ yields $e=0$ and $g=2m$. Then considering the above detected conditions we obtain

$$\chi_{11} = 4080(3c+6f+4m^2)xy$$

and we deduce that the condition $3c+6f+4m^2\neq 0$ fixed in (3.39) is equivalent to $\chi_{11}\neq 0$.

Next we calculate

$$\zeta_4 = -(3c-3f+m^2)(13x^2+3y^2)$$

and clearly the condition $3c-3f+m^2\geq 0$ given in (3.39) is equivalent to $\zeta_4\leq 0$.

Thus all the conditions provided by Lemma 3.18 are defined by the corresponding invariant polynomials except the conditions $b=0$ and $H'=0$ (see (3.38)). These conditions are governed by the invariant polynomials χ_{17} which being evaluated for systems (3.12) under the conditions (3.39) (except for $b=0$ and $H'=0$) has the form

$$\chi_{17} = -18792x^8(x^2+y^2)^4\Big[27b^2x^2-2b(27a+9cm+4m^3)xy+H'y^2\Big].$$

The condition $\chi_{17}=0$ is evidently equivalent to $b=0=H'$ and this completes the proof of Lemma 3.19 as well as the proof of the statement $(\mathcal{A}_7)$ of the Main Theorem. □

2: The case $\mathcal{D}_7=0$. Then $u=-1$ and by Lemma 3.4 we could not have a triplet of parallel invariant lines in the direction $y=0$. Since for the direction $x=0$ we have the equations

$$Eq_7 = k,\quad Eq_9 = d-2hW,\quad Eq_{10} = a-cW+gW^2. \tag{3.40}$$

we arrive at the conditions $k = d = h = 0$ and considering $u = -1$ we have

$$\widetilde{\chi}_1 = 2gx^2\left[4sx + (3 - s^2)y\right]/9.$$

According to Lemma 3.2 we consider two subcases: $\widetilde{\chi}_1 \neq 0$ and $\widetilde{\chi}_1 = 0$.

2.1: *The subcase* $\widetilde{\chi}_1 \neq 0$. We prove the following lemma.

Lemma 3.20. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7 = 0$ *and* $\widetilde{\chi}_1 \neq 0$ *hold. Then this system possesses invariant lines in the configuration* $(3, 1, 1, 1)$ *if and only if the conditions* $\chi_1 = \chi_2 = \chi_3 = 0$ *are satisfied.*

Proof. As it was mentioned above the condition $\widetilde{\chi}_1 \neq 0$ implies $g \neq 0$ and according to Lemma 3.2 the infinite line $Z = 0$ of systems (3.12) with the conditions $u = -1$ has the multiplicity exactly 2. Moreover in the direction $x = 0$ we have two parallel invariant affine lines (due to $g \neq 0$).

Thus we have to examine the remaining three directions: $x \pm iy = 0$ and $sx + y = 0$.

(i) *The direction* $x + iy = 0$. We repeat the examinations of the corresponding equations (3.17) for this particular case (i.e. $u = -1$) and considering (3.18) we arrive at the equations

$$H_i|_{\{u=-1\}} = 0, \quad i = 1, 2, 3, 4.$$

Solving these equations with respect o the parameters a, b, e and f we obtain the values of these parameters given in (3.19) and (3.21) for this particular case $u = -1$. More precisely we get the following conditions:

$$\begin{aligned}
a &= \frac{1}{4(1+s^2)^2}(g - 2m - ls)\left[2c(1+s^2) - g(g - 2m - ls)\right],\\
b &= \frac{1}{4(1+s^2)^2}(l + gs - 2ms)\left[l^2 - 2(g - 2m)m + lgs + 2c(1+s^2)\right],\\
e &= \frac{1}{4(1+s^2)^2}\left[(g - 2m)s(5g - 6m + gs^2 + 2ms^2) - l^2s(5+s^2) + 2l(3g - 4m - gs^2 + 4ms^2)\right],\\
f &= \frac{1}{4(1+s^2)^2}\left[4c(1+s^2)^2 + l^2(3-s^2) + 2ls(5g - 8m + gs^2) + (g - 2m)(2m - 3g + gs^2 - 6ms^2)\right].
\end{aligned} \tag{3.41}$$

(ii) *The direction* $sx + y = 0$. Considering (3.22) and $u = -1$ for this direction we obtain

$$Eq_5 = l + gs - 2ms + (s^2 + 1)W = 0,$$

which yields $W = -(l + gs - 2ms)/(s^2 + 1)$. Then considering (3.23) we obtain

$$\begin{aligned}
Eq_8 &= \frac{g}{2(1+s^2)}\left[4(g - 2m)s + l(3 - s^2)\right],\\
Eq_{10} &= \frac{1}{4(1+s^2)^2}(2c - g^2 + 2gm + lgs + 2cs^2)\left[4(g - 2m)s + l(3 - s^2)\right].
\end{aligned} \tag{3.42}$$

Since $g \neq 0$ the condition $Eq_8 = 0$ gives $4(g - 2m)s + l(3 - s^2) = 0$, which implies also $Eq_{10} = 0$, and we consider two possibilities: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

2.1.1: *The possibility* $\mathcal{D}_4 \neq 0$. Then $s \neq 0$ and we obtain $m = (3l + 4gs - ls^2)/(8s)$. So taking into consideration (3.41) we obtained that a system possesses invariant lines in the configuration $(3,1,1,1)$ if and only if the following conditions are satisfied:

$$\begin{aligned}
&u=-1, \quad k=d=h=0, \quad e=-\frac{l(21l-8gs+ls^2)}{64s},\\
&f=\frac{24lgs+64cs^2+3l^2(s^2-3)}{64s^2}, \quad a=-\frac{3l(3lg+8cs)}{64s^2},\\
&b=\frac{l\left[12lgs+32cs^2+l^2(9+s^2)\right]}{256s^2}, \quad m=\frac{(3l+4gs-ls^2)}{8s}.
\end{aligned} \tag{3.43}$$

Considering the conditions provided by Lemma 3.20 for systems (3.12) with $u=-1$ we calculate:

$$\text{Coefficient}[\chi_1, xy^2]=2\left[4ks+h(s^2-3)\right]/9, \quad \text{Coefficient}[\chi_1, x^2y]=4\left[k(3s^2-1)-4hs\right]/9$$

and due to $s\neq 0$ the condition $\chi_1=0$ implies $k=h=0$. Then we obtain

$$\chi_1=2\left[l(s^2-3)-4(g-2m)s\right]/9=0$$

which gives us $m=(3l+4gs-ls^2)/(8s)$. So we deduce that forcing $\chi_1=0$ we get the conditions for the parameters k, h and m given in (3.43).

Herein calculations yield

$$\text{Coefficient}[\chi_2, y^2]=56ds(9+s^2)/3=0 \quad \Leftrightarrow \quad d=0$$

due to $s\neq 0$. Then we obtain $\chi_2=\hat{\varphi}_1x^2/6+\hat{\varphi}_2xy/s$, where

$$\begin{aligned}
\hat{\varphi}_1=&128fs^2-64es(15+s^2)-128cs^2-l(9+s^2)(33l-8gs+ls^2),\\
\hat{\varphi}_2=&16es(s^2-3)-16fs^2(5+s^2)+16cs^2(5+s^2)+l(9+s^2)(4gs-3l+ls^2).
\end{aligned}$$

We observe that the equations $\hat{\varphi}_1=0$ and $\hat{\varphi}_2=0$ are real with respect to the parameters e and f and the corresponding determinant equals $1024s^3(9+s^2)^2\neq 0$ due to $s\neq 0$. Solving these equations we get exactly the expressions for the parameters e and f given in (3.43).

Next supposing that for systems (3.12) the conditions on the parameters u,k,h,d,e,m and f given in (3.43) are satisfied, we calculate

$$\chi_3=-\frac{x^5}{1728s^2}\hat{\varphi}_3\hat{\varphi}_4^2(\hat{\varphi}_5x+\hat{\varphi}_6y),$$

where

$$\begin{aligned}
\hat{\varphi}_3=&4sx+(3-s^2)y, \quad \hat{\varphi}_4=(3s^2-1)x^2+8sxy+(3-s^2)y^2,\\
\hat{\varphi}_5=&-256bs^2+l^3(9+s^2)+4ls(3lg+8cs),\\
\hat{\varphi}_6=&4\left[64as^2+3l(3lg+8cs)\right].
\end{aligned}$$

We observe that due to $s\neq 0$ we have $\hat{\varphi}_3\hat{\varphi}_4\neq 0$ and therefore the condition $\chi_3=0$ is equivalent to $\hat{\varphi}_5=\hat{\varphi}_6=0$. Solving these equations with respect to the parameters a and b we get exactly the expressions given in (3.43) for these parameters. This completes the proof of Lemma 3.20 in the case $\mathcal{D}_4\neq 0$.

2.1.2: *The possibility* $\mathcal{D}_4=0$. Then $s=0$ and the condition (see (3.42))

$$Eq_8=\frac{g}{2(1+s^2)}\left[4(g-2m)s+l(3-s^2)\right]=0$$

implies $l = 0$. Then considering (3.41) in the case $s = 0$ we arrive at the following conditions:

$$u = -1, \quad k = d = h = e = b = s = l = 0,$$
$$f = \frac{1}{4}(4c - 3g^2 + 8gm - 4m^2), \quad a = -\frac{1}{4}(g - 2m)(-2c + g^2 - 2gm). \tag{3.44}$$

Next for systems (3.12) with $u = -1$ and $s = 0$ we calculate

$$\chi_1 = -2x(3lx^2 + hx^2 + 2kxy + 3hy^2)/9$$

and evidently the condition $\chi_1 = 0$ is equivalent to $k = h = l = 0$. Then calculations yield

$$\chi_2 = -16(2d + 3e)xy, \quad \text{Coefficient}[\chi_3, x^4y^7] = -4dg/3,$$
$$\text{Coefficient}[\chi_3, x^6y^5] = 2(6b - 2dg - 3eg + 6em)/3$$

and clearly the conditions $\chi_2 = \chi_3 = 0$ implies to $d = e = b = 0$. Herein we obtain

$$\chi_3 = 2x^5y(x^2 - 3y^2)(\widehat{\psi}_1x^2 + \widehat{\psi}_2y^2)/9,$$

where

$$\widehat{\psi}_1 = -2a + 3cg - 2fg - 2g^3 - 2cm + 6g^2m - 4gm^2,$$
$$\widehat{\psi}_2 = 6a - cg - 2fg + 6cm - 2g^2m + 4gm^2.$$

So the condition $\chi_3 = 0$ implies $\widehat{\psi}_1 = \widehat{\psi}_2 = 0$ and solving these two equations with respect to the parameters a and f we obtain exactly the expressions given in (3.44) for these parameters. This complete the proof of Lemma 3.20 as well as the proof of the statement (A_8) of the Main Theorem. □

2.2: *The subcase* $\widetilde{\chi}_1 = 0$. Then $g = 0$ and according to Lemma 3.2 we examine two possibilities: $\widetilde{\chi}_2 \neq 0$ and $\widetilde{\chi}_2 = 0$. Since for systems (3.12) with the conditions $u = -1$ and $k = d = h = g = 0$ we have

$$\widetilde{\chi}_2 = 4cx^3(sx + y)(x^2 + y^2)\big[(3s^2 - 1)x^2 + 8sxy + (3 - s^2)y^2\big]/3$$

we deduce that the condition $\widetilde{\chi}_2 = 0$ is equivalent to $c = 0$.

2.2.1: *The possibility* $\widetilde{\chi}_2 \neq 0$. Then by Lemma 3.2 systems (3.12) possess a triple infinite invariant line $Z = 0$ and since $\widetilde{\chi}_2 \neq 0$ implies $c \neq 0$ we deduce that in the direction $x = 0$ systems (3.12) possess only one invariant line, which is real.

So we have to examine the remaining three directions: $x \pm iy = 0$ and $sx + y = 0$.

(i) *The direction* $x + iy = 0$. We repeat the examinations of the corresponding equations (3.17) for this particular case $u = -1$ and $g = 0$ and considering (3.18) we arrive at the equations

$$H_i|_{\{u=-1,g=0\}} = 0, \quad i = 1, 2, 3, 4.$$

Solving these equations with respect o the parameters a, b, e and f we get the values of these parameters given in (3.19) and (3.21) for this particular case $u = -1$ and $g = 0$. More precisely we get the following conditions:

$$a = -\frac{c(2m + ls)}{2(1 + s^2)}, \quad b = \frac{(l - 2ms)}{4(1 + s^2)^2}\Big[l^2 + 4m^2 + 2c(1 + s^2)\Big],$$
$$e = -\frac{1}{4(1 + s^2)^2}\Big[4m^2s(-3 + s^2) - 8lm(-1 + s^2) + l^2s(5 + s^2)\Big], \tag{3.45}$$
$$f = c - \frac{1}{4(1+s^2)^2}\Big[16lms + 4m^2(1 - 3s^2) + l^2(-3 + s^2)\Big].$$

(ii) *The direction* $sx + y = 0$. Considering (3.22) and the conditions $u = -1$ and $g = 0$ for this direction we obtain

$$Eq_5 = l - 2ms + (s^2 + 1)W = 0,$$

i.e. $W = -(l - 2ms)/(s^2 + 1)$. Then considering (3.23) we obtain

$$Eq_8 = 0, \quad Eq_{10} = \frac{c}{2(1 + s^2)}\left[l(3 - s^2) - 8ms\right]. \tag{3.46}$$

Since $c \neq 0$ the condition $Eq_{10} = 0$ gives $l(3 - s^2) - 8ms = 0$ and we consider two cases: $s \neq 0$ and $s = 0$. As it was mentioned earlier these conditions are governed by the invariant polynomial $\mathcal{D}_4 = 2304s(9 + s^2)$.

2.2.1.1: *The case* $\mathcal{D}_4 \neq 0$. Then $s \neq 0$ and we obtain $m = l(3 - s^2)/(8s)$ and considering the conditions $u = -1$, $k = d = h = g = 0$ and (3.45) we arrive at the following lemma.

Lemma 3.21. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7 = \widetilde{\chi}_1 = 0$, $\widetilde{\chi}_2 \neq 0$ *and* $\mathcal{D}_4 \neq 0$ *hold. Then this system possesses invariant lines in the configuration* $(3, 1, 1, 1)$ *if and only if the following conditions are satisfied:*

$$\begin{aligned}
&u = -1,\ k = d = h = g = 0, \quad e = -\frac{l^2(21 + s^2)}{64s},\\
&f = \frac{64cs^2 + 3l^2(s^2 - 3)}{64s^2}, \quad a = -\frac{3c\,l}{8s},\\
&b = \frac{l\left[32cs^2 + l^2(9 + s^2)\right]}{256s^2}, \quad m = \frac{l(3 - s^2)}{8s}.
\end{aligned} \tag{3.47}$$

Next we determine the invariant conditions equivalent to those provided by the above lemma. More exactly we prove the following lemma.

Lemma 3.22. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7 = \widetilde{\chi}_1 = 0$, $\widetilde{\chi}_2 \neq 0$ *and* $\mathcal{D}_4 \neq 0$ *hold. Then this system possesses invariant lines in the configuration* $(3, 1, 1, 1)$ *if and only if the conditions* $\chi_1 = \chi_3 = \chi_6 = 0$ *are satisfied.*

Proof. Clearly the condition $\mathcal{D}_7 = 0$ imply $u = -1$. Then for systems (3.12) we calculate

$$\text{Coefficient}[\widetilde{\chi}_1, xy^2] = -(8ks)/3$$

and clearly due to $\mathcal{D}_4 \neq 0$ (i.e. $s \neq 0$) the condition $\widetilde{\chi}_1 = 0$ implies $k = 0$. Then we calculate

$$\widetilde{\chi}_1 = 2x^2\left[2(h + 2gs - 3hs^2)x + (3g - 8hs - gs^2)y\right]/9 = 0$$

and we claim that the condition $\widetilde{\chi}_1 = 0$ implies $g = h = 0$. Indeed assuming $h + 2gs - 3hs^2 = 0$ we get $g = \frac{h(3s^2-1)}{2s}$ and then

$$\widetilde{\chi}_1 = -\frac{h(s^2 + 1)^2}{2s} = 0 \ \Rightarrow\ h = 0 \ \Rightarrow\ g = 0$$

and this completes the proof of our claim.

Thus the condition $\widetilde{\chi}_1 = 0$ for systems (3.12) with $u = -1$ gives us $k = h = g = 0$. Then calculations yield

$$\text{Coefficient}[\chi_6, x^2y^6] = 10ds(123 + 23s^2)/3 = 0 \ \Rightarrow\ d = 0$$

due to $s \neq 0$. Herein we obtain

$$\begin{aligned}
\text{Coefficient}[\chi_6, x^3y^5] &= 110\big[e(s^2-3) - fs(5+s^2) - (6lm - 5cs - 4m^2s - cs^3)\big]\\
\text{Coefficient}[\chi_6, x^4y^4] &= \frac{10}{3}\big[es(7s^2-597) - fs^2(131+39s^2) - 216l^2 - 42lms\\
&\qquad + 131cs^2 + 124m^2s^2 + 39cs^4\big]
\end{aligned}$$

and we observe that the above polynomials are linear with respect to the parameters e and f with the corresponding determinant $-35200s^2(9+s^2)^2/3 \neq 0$. So forcing these polynomials to vanish we get

$$\begin{aligned}
e &= \frac{1}{4s(9+s^2)^2}(2ms-3l)(45l+6ms+9ls^2+2ms^3),\\
f &= \frac{1}{4s^2(9+s^2)^2}\big[4cs^2(9+s^2)^2 - 3(3l-2ms)(42ms-9l+3ls^2+2ms^3)\big],
\end{aligned} \tag{3.48}$$

and then calculations yield

$$\begin{aligned}
\chi_6 = &-\frac{20}{s^2(9+s^2)^2}(8ms-3l+ls^2)^2x^5\big[16s^2(6+s^2)x^3 + s(63+30s^2-s^4)x^2y\\
&+9(14s^2-3+s^4)xy^2 + 12s(9+s^2)y^3\big].
\end{aligned}$$

Due to $s \neq 0$ the condition $\chi_6 = 0$ evidently implies $8ms - 3l + ls^2 = 0$, and considering (3.48) we determine:

$$e = -\frac{l^2(21+s^2)}{64s}, \quad f = \frac{64cs^2 + 3l^2(s^2-3)}{64s^2}, \quad m = \frac{l(3-s^2)}{8s}.$$

So we obtain exactly the expressions for the parameters e, f and m given in (3.47).

Next considering that all the conditions from (3.47) are satisfied except the conditions for the parameters a and b we calculate:

$$\begin{aligned}
\chi_3 = &-\frac{1}{1728s^2}x^5(4sx+3y-s^2y)\big[(3s^2-1)x^2 + 8sxy + (3-s^2)y^2\big]^2\\
&\times\big[(9l^3 - 256bs^2 + 32cls^2 + l^3s^2)x + 32s(3cl+8as)y\big].
\end{aligned}$$

Therefore due to $s \neq 0$ it is simple to determine that the condition $\chi_3 = 0$ gives us exactly the expressions for the parameters a and b given in (3.47). This completes the proof of Lemma 3.22. □

2.2.1.2: *The case $\mathcal{D}_4 = 0$.* Then $s = 0$ and the condition $Eq_{10} = 0$ (see (3.46)) gives us $Eq_{10} = 3cl/2 = 0$ and due to $c \neq 0$ this implies $l = 0$. In this case considering the conditions $u = -1, k = d = h = g = s = 0$ and (3.45) we arrive at the following lemma.

Lemma 3.23. *Assume that for a system (3.12) the conditions $\mathcal{D}_7 = \widetilde{\chi}_1 = 0, \widetilde{\chi}_2 \neq 0$ and $\mathcal{D}_4 = 0$ hold. Then this system possesses invariant lines in the configuration $(3,1,1,1)$ if and only if the following conditions are satisfied:*

$$u = -1,\ s = 0, \quad k = d = h = g = l = b = e = 0, \quad a = -cm, \quad f = c - m^2. \tag{3.49}$$

Now we determine the invariant conditions equivalent to those provided by the above lemma. We claim that Lemma 3.22 which was proved for $\mathcal{D}_4 \neq 0$ is also true in the case $\mathcal{D}_4 = 0$.

Indeed it is clear that the conditions $\mathcal{D}_7 = \mathcal{D}_4 = 0$ imply $u = -1$ and $s = 0$. Then for systems (3.12) we calculate

$$\widetilde{\chi}_1 = 2\big[2hx^3 + (3g+2k)x^2y - 3ky^3\big]/9$$

and evidently the condition $\widetilde{\chi}_1 = 0$ implies $k = h = g = 0$. Then calculations yield

$$\chi_1 = -2lx^3/3 = 0 \quad \Rightarrow \quad l = 0$$

and we obtain

$$\chi_6 = -10xy\big[(10d-129e)x^6+72(c-f-m^2)x^5y+(25d+42e)x^4y^2-3(16d-33e)x^2y^4+9dy^6\big]/3.$$

It is clear that the condition $\chi_6 = 0$ implies $d = e = 0$ and $f = c - m^2$ and we observe that all the conditions given in (3.49) are obtained except for the parameters a and b.

Finally we calculate

$$\chi_3 = 4\big[bx - (a+cm)y\big]x^5y(x^2-3y^2)^2/9$$

and evidently the condition $\chi_3 = 0$ gives us $b = 0$ and $a = -cm$. This complete the proof of the statement (A_9) of the Main Theorem.

2.2.2: *The possibility* $\widetilde{\chi}_2 = 0$. We prove the following lemma.

Lemma 3.24. *If for a system* (3.12) *the conditions* $\mathcal{D}_7 = \widetilde{\chi}_1 = \widetilde{\chi}_2 = 0$ *hold then this system could hot have a configuration of the type* $(3,1,1,1)$.

Proof. Assume that for a system (3.12) the conditions provided by this lemma are fulfilled. As we already know the condition $\mathcal{D}_7 = 0$ implies $u = -1$ and by Lemma 3.4 we could not have a triplet of parallel invariant lines in the direction $y = 0$. Since for the direction $x = 0$ we have the equations (see (3.40))

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a - cW + gW^2.$$

we arrive at the conditions $k = d = h = 0$ and considering $u = -1$ we have

$$\widetilde{\chi}_1 = 2gx^2\big[4sx + (3-s^2)y\big]/9.$$

Therefore the condition $\widetilde{\chi}_1 = 0$ implies $g = 0$ and evaluating the invariant polynomial $\widetilde{\chi}_2$ for systems (3.12) with $u = -1$ and $k = d = h = g = 0$ we get

$$\widetilde{\chi}_2 = 4cx^3(sx+y)(x^2+y^2)\big[(3s^2-1)x^2 + 8sxy + (3-s^2)y^2\big]/3.$$

It is clear that the condition $\widetilde{\chi}_2 = 0$ implies $c = 0$ and hence we arrive at the family of systems

$$\dot{x} = a, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy - sx^3 - x^2y - sxy^2 - y^3. \tag{3.50}$$

Suppose the contrary, that these systems possess the configuration invariant lines of the type $(3,1,1,1)$. Therefore we must have two complex invariant lines: one in the direction $x + iy = 0$ and another in the direction $x - iy = 0$.

Thus considering the equations (2.5) and Remark 2.12 for the direction $x + iy = 0$ we obtain $U = 1$, $V = i$ and

$$Eq_7 = 2m - il + 2(1 - is)W, \quad Eq_9 = e + if - 2(l + im)W - (i + 3s)W^2,$$
$$Eq_{10} = a + ib - ieW + ilW^2 + isW^3.$$

We calculate $Res_W(Eq_7, Eq_9) = \widehat{H}_1 + i\widehat{H}_2$ where

$$\widehat{H}_1 = 8lm - s(l^2 - 8f + 4m^2) - 4e(s^2 - 1),$$
$$\widehat{H}_2 = 4m^2 - 3l^2 - 8es - 4f(s^2 - 1).$$

So solving the system of equations $\widehat{H}_1 = 0$ and $\widehat{H}_2 = 0$ which are linear with respect to the parameters e and f we obtain:

$$e = -\frac{4m^2s(s^2 - 3) - 8lm(s^2 - 1) + l^2s(5 + s^2)}{4(1 + s^2)^2},$$
$$f = -\frac{16lms + 4m^2(1 - 3s^2) + l^2(s^2 - 3)}{4(1 + s^2)^2}.$$

Then calculations yield

$$Res_W(Eq_7, Eq_{10}) = \frac{(s + i)^3}{(1 + s^2)^2}\Big[(l^2 + 4m^2)(l - 2ms) - 4b(1 + s^2)^2 + 4ia(1 + s^2)^2\Big] = 0$$

and since the parameters of the systems are real this condition implies $a = 0$. However in this case systems (3.50) become degenerate and this completes the proof of the Lemma 3.24. □

Since all the possibilities for cases provided by the statement (A) of the Main Theorem are examined we conclude that this statement is proved.

As we mentioned earlier (see page 24) we have to prove the following lemma.

Lemma 3.25. *None of the sets of the conditions (A_1)–(A_9) could be satisfied for systems (3.8).*

Proof. For systems (3.8) calculations yield:

$$\mathcal{D}_4 = 0, \quad \mathcal{D}_6 = -4, \quad \mathcal{D}_7 = 4, \quad \mathcal{D}_8 = 0$$

and comparing with the sets of the conditions provided by the statement (A) of the Main Theorem we conclude that all the sets of the conditions (A_i) for $i = 1, 2, \ldots, 5, 7, \ldots, 9$ could not be satisfied for systems (3.8). It remains to prove that set of of the conditions (A_6) could also not be satisfied for this family of systems.

Indeed for systems (3.8) we have

$$\chi_1 = \frac{1}{4}\big[-(l + 2h)x^3 + 3(g - k)x^2y + 3(cl + 2h)xy^2 + (k - g)y^3\big]$$

and therefore the condition $\chi_1 = 0$ provided by the statement (A_6) gives $l = -2h$ and $g = k$. Then we calculate

$$\chi_8 = -240(h^2 + k^2)(x^2 + y^2), \qquad \chi_{15} = -(kx - hy)(x^2 + y^2)^2/4$$

and since according to the statement (A_6) we must have $\chi_8 = 0$ and $\chi_{15} \neq 0$ we evidently get a contradiction and this completes the proof of the lemma. □

3.2 The proof of the statement (B) of the Main Theorem

In this section we examine step by step each one of the statements (A_i) $(i = 1, \dots, 9)$ of Main Theorem and determine the possible configurations of invariant lines, correspondingly. Moreover we find out necessary and sufficient affine invariant conditions for the realization of each one of the configurations found.

3.2.1 The statement (A_1)

It was shown in the proof of the statement (A) of the Main Theorem that the affine invariant conditions provided by the statement (A_1) for the family of systems (3.12) lead to the conditions (3.26).

It is not too difficult to determine that in this cases we arrive at the family of systems

$$\begin{aligned}
\dot{x} &= \Big[x - \frac{3}{\varkappa}(g-2m)\Big]\Big[c + \frac{6}{\varkappa^2}(g-2m)\left(gs^2 - 3g - 3mu - 3m\right)\\
&\quad + \frac{2}{\varkappa}(gs^2 - 3g - 3m - 3mu)\,x + (1+u)x^2\Big] \equiv\ L_1(x)L_{2,3}(x),\\
\dot{y} &= \frac{s}{\varkappa^2}(g-2m)\Big[c\varkappa^2 + 2(g-2m)\big[4gs^2 - m(27 + 2s^2 + 9u)\big]\Big]\\
&\quad + \frac{s}{\varkappa^2}(g-2m)\big[g\left(s^2 - 27\right) + 2m\left(s^2 - 3u + 18\right)\big]\,x + \frac{s}{\varkappa}(g-2m)(9+u)\,x^2\\
&\quad + \Big[c + \frac{3}{\varkappa^2}(g-2m)\big[3g\left(s^2-3\right) - 2m\left(s^2+3u\right)\big]\Big]\,y + 2mxy - sx^3 + ux^2y - sxy^2 - y^3,
\end{aligned} \tag{3.51}$$

where $\varkappa = 2s^2 - 3(u+3) \neq 0$ and $s \neq 0$ or $s = 0$ depending on the value of the invariant polynomial $\mathcal{D}_4$.

We need to determine if the two lines defined by the equation $L_{2,3} = 0$ are real or complex and in the case when they are real, if one of them coincides with the invariant line $L_1 = 0$. So we calculate

$$\begin{aligned}
\text{Discrim}\,[L_{2,3}, x] &= -\frac{4}{\varkappa^2}\lambda(c, g, m, s, u),\\
Res_x(L_1, L_{2,3}) &= \frac{1}{\varkappa^2}\mu(c, g, m, s, u)
\end{aligned}$$

where

$$\begin{aligned}
\lambda &= c(1+u)\varkappa^2 - \big[g(s^2-3) - 3m(1+u)\big]\big[g(s^2 - 9 - 6u) + 9m(1+u)\big],\\
\mu &= c\varkappa^2 + 3(g-2m)\big[g(4s^2 - 9 + 3u) - 18m(1+u)\big].
\end{aligned} \tag{3.52}$$

We observe that

$$\text{sign}\left(\text{Discrim}\,[L_{2,3}, x]\right) = -\text{sign}\,(\lambda),$$

i.e. the invariant lines $L_{2,3} = 0$ are real (respectively complex; coinciding) if $\lambda < 0$ (respectively $\lambda > 0$; $\lambda = 0$). The invariant line $L_1 = 0$ coincides with one of the lines $L_{2,3} = 0$ if and only if $\mu = 0$.

Evaluating for systems (3.51) the invariant polynomials ζ_1 and ζ_2 we obtain:

$$\zeta_1 = \frac{80}{3\varkappa^2}(s^2+3u)^2x^2\lambda, \quad \zeta_2 = 8\mu, \quad \mathcal{D}_5 = 4(s^2+3u)/9$$

and therefore the condition $\zeta_2 = 0$ is equivalent to $\mu = 0$. On the other hand we have $\text{sign}\,(\lambda) = \text{sign}\,(\zeta_1)$ only if $\mathcal{D}_5 \neq 0$. So in what follows we examine two possibilities: $\mathcal{D}_5 \neq 0$ and $\mathcal{D}_5 = 0$.

1: The possibility $\mathcal{D}_5 \neq 0$. Then the sign of λ is governed by the invariant polynomial ζ_1 and we prove the next proposition.

Proposition 3.26. *Assume that for a system (3.51) the conditions $\mathcal{D}_6\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\mathcal{D}_5 \neq 0$ hold. Then this system possesses one of the configurations of invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$
\begin{array}{lcl}
\mathcal{D}_4 \neq 0,\, \zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_3 < 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.1b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_3 < 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.2b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_3 > 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.3b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_3 > 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.4b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 < 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.5b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 > 0,\, \zeta_4 \neq 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.6b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 > 0,\, \zeta_4 \neq 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.7b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 > 0,\, \zeta_4 = 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.8b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 > 0,\, \zeta_4 = 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.9b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 = 0,\, \zeta_2 \neq 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.10b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 = 0,\, \zeta_2 \neq 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.11b;}\\
\mathcal{D}_4 \neq 0,\, \zeta_1 = 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.12b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_5 < 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.13b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_5 < 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.14b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_5 > 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.15b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_5 > 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.16b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 < 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.17b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 > 0,\, \zeta_4 \neq 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.18b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 > 0,\, \zeta_4 \neq 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.19b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 > 0,\, \zeta_4 = 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.20b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 > 0,\, \zeta_4 = 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.21b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 = 0,\, \zeta_2 \neq 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.22b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 = 0,\, \zeta_2 \neq 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.23b;}\\
\mathcal{D}_4 = 0,\, \zeta_1 = 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.24b.}
\end{array}
$$

Proof. Following the conditions provided by the above proposition we consider two cases: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

1.1: The case $\mathcal{D}_4 \neq 0$. We examine three subcases: $\zeta_1 < 0$, $\zeta_1 > 0$ and $\zeta_1 = 0$.

a) *The subcase $\zeta_1 < 0$ ($\mathcal{D}_5 \neq 0$).* Then $\lambda < 0$ and we may use a new parameter v setting $\lambda = -v^2 < 0$. Since we have $\mathcal{D}_6\mathcal{D}_7 \neq 0$ (i.e. $(1+u)\varkappa \neq 0$) considering (3.52) we obtain

$$c = \frac{1}{(1+u)\varkappa^2}\left[g^2(s^2-3)(s^2-6u-9) - 27m^2(1+u)^2 + 6gm(1+u)(s^2+3u) - v^2\right]. \tag{3.53}$$

This leads to the following family of systems

$$\dot{x} = (1+u)\left[x - \frac{3(g-2m)}{\varkappa}\right]\left[x + \frac{gs^2-3g-3m-3mu-v}{\varkappa(1+u)}\right]\times \left[x + \frac{gs^2-3g-3m-3mu+v}{(1+u)}\right],$$

$$\begin{aligned}\dot{y} = &-\frac{(g-2m)s}{(1+u)\varkappa^3}\Big[(2gm(27+7s^2)(1+u)-m^2(1+u)(81+8s^2+9u)\\ &-g^2\big[s^4+2s^2(u-2)+9(3+2u)\big]+v^2)\Big]+\frac{(g-2m)s}{\varkappa^2}(36m-27g+gs^2+2ms^2-6mu)\,x\\ &+\frac{1}{\varkappa^2(1+u)}\big[g^2(s^2-3)(s^2+3u)-18gm(s^2-3)(1+u)+3m^2(1+u)(4s^2-9+3u)-v^2\big]\,y\\ &+\frac{s}{\varkappa}(g-2m)(9+u)\,x^2+2mxy-sx^3+ux^2y-sxy^2-y^3. \qquad (3.54)\end{aligned}$$

On the other hand for the value of c given in (3.53) we calculate

$$\mu=\frac{1}{1+u}(\gamma^2-v^2),\quad \gamma=g(s^2+3u)-9m(1+u) \qquad (3.55)$$

and since the condition $\mu=0$ leads to the coalescence of two invariant lines of the triplet, we examine two possibilities: $\zeta_2\neq 0$ and $\zeta_2=0$.

a.1) *The possibility* $\zeta_2\neq 0$. Then $\mu\neq 0$, i.e. $(\gamma-v)(\gamma+v)\neq 0$ and setting a new parameter $a=\frac{\gamma+v}{\gamma-v}$ we observe that $a-1\neq 0$. Indeed calculation yields: $a-1=\frac{2v}{\gamma-v}\neq 0$ due to $v\neq 0$. So from the relation $a=\frac{\gamma+v}{\gamma-v}$ we can determine the parameter m as follows:

$$m=\frac{g(a-1)(s^2+3u)-(a+1)v}{9(a-1)(1+u)}.$$

Then we can apply to systems (3.54) the following transformation (we recall that $\varkappa=2s^2-3(u+3)\neq 0$):

$$x_1=\alpha x-\frac{\nu}{6v},\quad y_1=\alpha y+\frac{s\nu}{18v},\quad t_1=\frac{t}{\alpha^2},$$
$$\alpha=-\frac{(a-1)(1+u)\varkappa}{2v},\quad \nu=(a-1)g\varkappa-2(1+a)v.$$

This transformation brings these systems to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned}\dot{x}&=(1+u)x(x-1)(x-a),\\ \dot{y}&=(1+u)\big[a-x(a+1)\big]y-sx^3+ux^2y-sxy^2-y^3,\end{aligned} \qquad (3.56)$$

for which the parameters s and u satisfy the conditions (3.25).

We detect that systems (3.56) possess six distinct invariant affine straight lines

$$L_1: x=0,\; L_2: x=1,\; L_3: x=a,\; L_4: y=-sx,\quad L_{5,6}: y=\pm ix \qquad (3.57)$$

and the following nine finite singularities:

$$M_1(0,0),\; M_{2,3}\big(0,\pm\sqrt{a(1+u)}\,\big),\; M_4(1,-s),\; M_{5,6}(1,\pm i),\; M_7(a,-as),\; M_{8,9}(a,\pm ia). \qquad (3.58)$$

For systems (3.56) calculations yield

$$\zeta_1=-\frac{20}{3}(a-1)^2(u+1)^2x^2\left(s^2+3u\right)^2,\quad \zeta_2=8a\varkappa^2(1+u),$$
$$\mathcal{D}_5=\frac{4}{9}\left(s^2+3u\right),\quad \mathcal{D}_6=\frac{4}{9}\big[2s^2-3(u+3)\big]\equiv\frac{4}{9}\varkappa,\quad \mathcal{D}_7=4(1+u).$$

Since the conditions $\mathcal{D}_5\neq 0$ and $\mathcal{D}_6\mathcal{D}_7\neq 0$ hold we obtain that the conditions $\zeta_1<0$ and $\zeta_2\neq 0$ imply for the parameter a of systems (3.56) the condition $a(a-1)\neq 0$.

We observe that the singular points M_2 and M_3 could be real (if $a(1+u) > 0$) or complex (if $a(1+u) < 0$). On the other hand due to the condition $a(a-1) \neq 0$ we conclude that the line $L_3 : x = a$ could neither coincide with L_1 (for $a = 0$) nor with L_2 (for $a = 1$). So considering the condition $a(a-1) \neq 0$ we examine the following two cases: $a(1+u) > 0$ and $a(1+u) < 0$.

a.1.1) *The case* $a(1+u) > 0$. So the singular points $M_{2,3}(0, \pm\sqrt{a(1+u)}\,)$ are real and we observe that all the singularities (3.58) are located at the intersection of the invariant lines, except for $M_{2,3}(0, \pm\sqrt{a(1+u)}\,)$ which lie on the line $x = 0$ and are symmetric with respect to the origin of coordinates. Moreover fixing the position of all invariant lines and moving only the singularities $M_{2,3}$ we could not obtain new configurations. So the distinct configurations depend on the position of the invariant lines.

We deduce that only two lines are not fixed, and namely: $L_3 : x = a$, $L_4 : y = -sx$. Moreover four of them (i.e. L_1, L_4 and $L_{5,6}$) intersect at the same point $(0, 0)$. Since this point lies on the line L_1, considering the triplet of parallel invariant lines (L_1, L_2 and L_3) we deduce that we could get different configurations depending on the position of the line $L_3 = a$. More precisely, if $a < 0$ then L_3 is located on the left of L_1 and if $a > 0$ then L_3 is located on the right of L_1.

Regarding the invariant line $L_4 : y = -sx$ we make the following remark.

Remark 3.27. Considering our Convention on page 8 we deduce that the invariant line $y = -sx$ coincides with the projection of the complex invariant lines $y = \pm ix$ on the plan (x, y) if and only if $s = 0$.

Since in the case under consideration we have $s \neq 0$ it is not too difficult to determine that systems (3.56) possess the configuration of invariant lines *Config. 7.1b* if $a < 0$ and *Config. 7.2b* if $a > 0$.

a.1.2) *The case* $a(1+u) < 0$. Then the singular points $M_{2,3}(0, \pm\sqrt{a(1+u)}\,)$ are complex and on the invariant line L_1 there are no real singularities except $M_1(0, 0)$. So applying the same argument as in the previous case above we obtain the following two configurations of invariant lines for systems (3.56): *Config. 7.3b* if $a > 0$ and *Config. 7.4b* if $a < 0$.

Thus we obtain that in the case $\zeta_1 < 0$ and $\zeta_2 \neq 0$ systems (3.56) could possess only four distinct configurations *Config. 7.1b - Config. 7.4b*.

Next we determine the corresponding invariant conditions for distinguishing these configurations of invariant lines. We evaluate for systems (3.56) the next invariant polynomials:

$$\zeta_3 = -2(a-1)^2 a s^2 (9+s^2)^2 (1+u)^3/81, \quad \mathcal{D}_7 = 4(1+u) \neq 0, \quad \mathcal{D}_4 = 2304 s (9+s^2) \neq 0$$

and due to $a(a-1)(u+1)s \neq 0$ we have $\operatorname{sign}(\zeta_3) = -\operatorname{sign}(a(u+1))$ and $\operatorname{sign}(\mathcal{D}_7) = \operatorname{sign}(u+1)$.

Considering the conditions on the parameters a, u determined above which define the configurations *Config. 7.1b - Config. 7.4b* for systems (3.56) in the case $\zeta_1 < 0$ and $\zeta_2 \neq 0$ as well as the expressions for the invariant polynomials given above we obtain the following affine invariant conditions for distinguishing these configurations (as well as the corresponding examples of their realization):

$$\begin{array}{lcll}
\zeta_3 < 0, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.1b} & (a = -1,\, u = -2,\, s = 1); \\
\zeta_3 < 0, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.2b} & (a = 2,\, u = 2,\, s = 1); \\
\zeta_3 > 0, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.3b} & (a = 2,\, u = -2,\, s = 1); \\
\zeta_3 > 0, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.4b} & (a = -1,\, u = 1,\, s = 1).
\end{array}$$

a.2) The possibility $\zeta_2 = 0$. Then $\mu = 0$ and considering (3.55) we get $(\gamma - v)(\gamma + v) = 0$. We may assume $\gamma - v = 0$ due to change $v \to -v$ and setting $v = \gamma \neq 0$ in systems (3.54) we arrive at the family of systems

$$
\begin{aligned}
\dot{x} =& (1+u)\Big[x - \frac{3(g-2m)}{\varkappa}\Big]^2\Big[x - \frac{12m(1+u) - g(2s^2+3u-3)}{\varkappa(1+u)}\Big],\\
\dot{y} =& -\frac{(g-2m)s}{(1+u)\varkappa^3}\Big[(2gm(27+7s^2)(1+u) - m^2(1+u)(81+8s^2+9u)\\
&- g^2\big[s^4+2s^2(u-2)+9(3+2u)\big]+v^2)\Big] + \frac{(g-2m)s}{\varkappa^2}(36m-27g+gs^2+2ms^2-6mu)\,x\\
&+\frac{1}{\varkappa^2(1+u)}\big[g^2(s^2-3)(s^2+3u) - 18gm(s^2-3)(1+u)+3m^2(1+u)(4s^2-9+3u)-v^2\big]\,y\\
&+\frac{s}{\varkappa}(g-2m)(9+u)\,x^2 + 2mxy - sx^3 + ux^2y - sxy^2 - y^3,
\end{aligned}
$$

where $\varkappa = 2s^2 - 3(u+3) \neq 0$. Since $\gamma = g(s^2+3u) - 9m(1+u) \neq 0$ then applying the transformation

$$
x_1 = \alpha x + \frac{3(g-2m)(1+u)}{2\gamma}, \quad y_1 = \alpha y - \frac{s(g-2m)(1+u)}{2\gamma},
$$
$$
t_1 = \frac{t}{\alpha^2}, \quad \alpha = \frac{\varkappa(1+u)}{2\gamma}
$$

we obtain the following 2-parameter family of systems:

$$
\begin{aligned}
\dot{x} =& (1+u)x^2(x-1),\\
\dot{y} =& -(1+u)xy - sx^3 + ux^2y - sxy^2 - y^3.
\end{aligned} \tag{3.59}
$$

We observe that this family of systems is a subfamily of (3.56) defined by the condition $a = 0$. So the above systems possess invariant lines (3.57) among which only five are distinct, because the line $L_4 \equiv L_1 : x = 0$ is double. These systems have only 4 distinct finite singularities because setting $a = 0$ in (3.58) we obtain that the five singularities $M_{2,3}(0, \pm\sqrt{a(1+u)}\ )$, $M_7(a, -as)$ and $M_{8,9}(a, \pm ia)$ coalesce with the singular point $M_1(0,0)$. As a result we get a singular point of multiplicity 6 which is a point of intersection of four invariant lines: L_1, L_4 and $L_{5,6}$. Therefore considering Remark 3.27 due to the condition $s \neq 0$ we obtain the unique configuration of singularities given by *Config. 7.5b*.

***b)** The subcase $\zeta_1 > 0$ ($\mathcal{D}_5 \neq 0$).* Then $\lambda > 0$ and we set $\lambda = v^2 > 0$. Since $(1+u)(9-2s^2+3u) \neq 0$ we obtain

$$
c = \frac{1}{(1+u)(9-2s^2+3u)^2}\big[g^2(s^2-3)(s^2-6u-9) - 27m^2(1+u)^2 + 6gm(1+u)(s^2+3u) + v^2\big].
$$

This leads to the following family of systems

$$
\begin{aligned}
\dot{x} =& (1+u)\left[x - \frac{3(g-2m)}{\varkappa}\right]\left[\frac{\big[g(s^2-3) - 3m(1+u)\big]^2 + v^2}{\varkappa^2(1+u)^2}\right.\\
&\left. + \frac{2(gs^2-3g-3m-3mu)}{\varkappa(1+u)}\,x + x^2\right],
\end{aligned}
$$

$$\begin{aligned}\dot y=&\frac{(g-2m)s}{\varkappa^3(1+u)}\big[m^2(1+u)(81+8s^2+9u)-2gm(27+7s^2)(1+u)+g^2(s^4+2s^2(u-2)\\&+9(3+2u))+v^2\big]+\frac{(g-2m)s}{\varkappa^2}(36m-27g+36m+gs^2+2ms^2-6mu)\,x\\&+\frac{1}{\varkappa^2(1+u)}\big[g^2(s^2-3)(s^2+3u)+3m^2(1+u)(4s^2+3u-9)\\&-18gm(s^2-3)(1+u)+v^2\big]\,y+\frac{s}{\varkappa}(g-2m)(9+u)\,x^2+2mxy-sx^3+ux^2y-sxy^2-y^3.\end{aligned}\tag{3.60}$$

In order to simplify these systems we need to use a transformation which depends on the condition: either $\gamma\neq 0$ or $\gamma=0$ (we recall that $\gamma=g(s^2+3u)-9m(1+u)$).

On the other hand for systems (3.60) we calculate:

$$\zeta_4=\frac{\gamma^2}{\varkappa^2}\big[(4s^2-13)x^2-2sxy-3y^2\big]$$

and therefore the condition $\gamma=0$ is equivalent to $\zeta_4=0$.

b.1) *The possibility* $\zeta_4\neq 0$. Then $\gamma\neq 0$ and setting a new parameter $a=\frac{v}{\gamma}\neq 0$ we can determine the parameter m as follows:

$$m=\frac{ags^2+3agu-v}{9a(1+u)}.$$

Then we can apply the following transformation

$$x_1=\alpha x-\frac{\nu}{3v},\quad y_1=\alpha y+\frac{s\nu}{9v},\quad t_1=\frac{t}{\alpha^2},\quad \alpha=-\frac{a(1+u)\varkappa}{v},\quad \nu=ag\varkappa-2v,$$

which brings systems (3.60) to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned}\dot x&=(1+u)x\big[(x-1)^2+a^2\big],\\ \dot y&=(1+a^2)(1+u)y-2(1+u)xy-sx^3+ux^2y-sxy^2-y^3.\end{aligned}\tag{3.61}$$

For the above systems calculations yield

$$\zeta_1=\frac{80}{3}a^2(1+u)^2(s^2+3u)^2x^2,\quad \mathcal{D}_7=4(1+u)$$

and clearly the condition $\zeta_1>0$ implies $a\neq 0$ and we must have $u+1\neq 0$ (i.e. $\mathcal{D}_7\neq 0$), otherwise we get degenerate systems.

We determine that systems (3.61) possess six distinct invariant affine straight lines

$$L_1: x=0,\ L_{2,3}: x=1\pm ia,\ L_4: y=-sx,\quad L_{5,6}: y=\pm ix$$

and the following nine finite singularities:

$$\begin{aligned}&M_1(0,0),\ M_{2,3}\big(0,\pm\sqrt{(1+a^2)(1+u)}\,\big),\ M_{4,5}\big(1+ia,\pm(i-a)\big),\\&M_{6,7}\big(1-ia,\pm(i+a)\big),\ M_{8,9}\big(1\pm ia,-s\mp is\big).\end{aligned}\tag{3.62}$$

We observe that the singular points M_2 and M_3 could be real (if $1+u>0$) or complex (if $1+u<0$), but they could not coincide due to $1+u\neq 0$. So we consider two cases: $1+u<0$ and $1+u>0$, taking into account that sign $(1+u)=$ sign $(\mathcal{D}_7)$.

b.1.1) *The case* $\mathcal{D}_7 < 0$. Then $1 + u < 0$ and therefore the singular points $M_{2,3}$ are complex and the unique real finite singular point of systems (3.61) is $M_1(0,0)$ which is the point of intersection of four invariant lines: L_1, L_4 and $L_{5,6}$. As a result taking into consideration Remark 3.27 due to the condition $s \neq 0$ we obtain the unique configuration *Config. 7.6b*.

b.1.2) *The case* $\mathcal{D}_7 > 0$. Then $1 + u > 0$ and hence the singular points $M_{2,3}$ are real. We observe that all the singularities (3.62) are located at the intersection of the invariant lines, except for $M_{2,3}$ which lie on the line $x = 0$ and are symmetric with respect to the origin of coordinates. Therefore considering Remark 3.27 and the condition $s \neq 0$ we arrive at the configuration of invariant lines given by *Config. 7.7b*.

So we have proved that if $\zeta_1 > 0$, $\zeta_4 \neq 0$ and $\mathcal{D}_4 \neq 0$ systems (3.54) possess the configuration *Config. 7.6b* $(a = 1, u = -2, s = 1)$ if $\mathcal{D}_7 < 0$ and *Config. 7.7b* $(a = 1, u = 1, s = 1)$ if $\mathcal{D}_7 > 0$.

b.2) *The possibility* $\zeta_4 = 0$. This implies $\gamma = 0$ and considering (3.55) the condition $\gamma = 0$ gives us

$$m = \frac{g(s^2 + 3u)}{9(1 + u)}$$

Then we can apply the transformation

$$x_1 = \alpha x + \frac{g\varkappa}{3v}, \quad y_1 = \alpha y - \frac{sg\varkappa}{9v}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = \frac{(1 + u)\varkappa}{v}.$$

which brings systems (3.60) to the following family of systems (we keep the old notations for the variables):

$$\dot{x} = (1 + u)x(x^2 + 1), \quad \dot{y} = (1 + u)y - sx^3 + ux^2y - sxy^2 - y^3 \tag{3.63}$$

with $1 + u \neq 0$. We determine that systems (3.63) possess six distinct invariant affine straight lines

$$L_1 : x = 0,\ L_{2,3} : x = \pm i,\ L_4 : y = -sx, \quad L_{5,6} : y = \pm ix$$

and the following nine finite singularities:

$$M_1(0,0),\ M_{2,3}(0, \pm\sqrt{1 + u}\ ),\ M_{4,5}(\pm i, 1),\ M_{6,7}(\pm i, -1),\ M_{8,9}(\pm i, \mp is).$$

We observe that the singular points M_2 and M_3 could be real (if $1 + u > 0$) or complex (if $1 + u < 0$), but they could not coincide due to $1 + u \neq 0$.

So, similarly as in the case of systems (3.61) we have two real and four complex invariant lines. However in this case considering our Convention (see page 8) we determine that the real invariant line $x = 0$ coincides with the projection of the complex invariant lines $L_{2,3} : x = \pm i$ on the plane (x, y). As it was mentioned earlier the invariant line $L_4 : y = -sx$ coincides with the projection of the complex invariant lines $L_{5,6} : y = \pm ix$ on the plane (x, y) (see our Convention on page 8) if and only if $s = 0$. Therefore due to the condition $s \neq 0$ we arrive at the configuration *Config. 7.8b* if $u < -1$ and at *Config. 7.9b* if $u > -1$.

Since sign $(u + 1) =$ sign $(\mathcal{D}_7)$ we deduce that for $\zeta_1 > 0$, $\zeta_4 \neq 0$ and $\mathcal{D}_4 \neq 0$ systems (3.54) possess the configuration *Config. 7.8b* $(u = -2, s = 1)$ if $\mathcal{D}_7 < 0$ and *Config. 7.9b* $(u = 1, s = 1)$ if $\mathcal{D}_7 > 0$.

c) *The subcase* $\zeta_1 = 0$ *(*$\mathcal{D}_5 \neq 0$*)*. This implies $\lambda = 0$ and considering (3.52) and solving the equation $\lambda = 0$ with respect to the parameter c, it is clear that we get (3.53) for $v = 0$. This leads to the systems (3.54) with $v = 0$, which we denote by $(3.54)_{\{v=0\}}$.

In this case for systems (3.54)$_{\{v=0\}}$ we have

$$\zeta_2 = \frac{8\gamma^2}{(1+u)}$$

and we again consider two possibilities: $\zeta_2 \neq 0$ and $\zeta_2 = 0$.

c.1) *The possibility* $\zeta_2 \neq 0$. This implies $\gamma \neq 0$ and via the transformation

$$x_1 = \alpha x + \frac{3\nu}{\gamma}, \quad y_1 = \alpha y - \frac{s\nu}{\gamma}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = -\frac{(1+u)\varkappa}{\gamma}, \quad \nu = (g-2m)(1+u)$$

systems (3.54)$_{\{v=0\}}$ can be brought to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned} \dot{x} &= (1+u)x(x-1)^2, \\ \dot{y} &= (1+u)(1-2x)y - sx^3 + ux^2y - sxy^2 - y^3. \end{aligned} \tag{3.64}$$

We observe that this family of systems is a subfamily of (3.56) defined by the condition $a = 1$. So systems (3.64) possess invariant lines (3.57) among which only five are distinct. More exactly the line $L_3 \equiv L_2 : x = 1$ is double.

These systems have 6 distinct finite singularities because setting $a = 1$ in (3.58) we obtain that the real singularity M_7 coalesces with the real singularity M_4, whereas the complex singularity M_8 (respectively M_9) coalesces with the complex singularity M_5 (respectively M_6). Moreover we observe that the simple singular point $M_1(0,0)$ is the point of intersection of four invariant lines: L_1, L_4 and $L_{5,6}$. Therefore considering Remark 3.27 and the condition $s \neq 0$ we arrive at the configuration *Config. 7.10b* if $u+1 < 0$ and at configuration *Config. 7.11b* if $u+1 > 0$.

So since sign$\,(u+1) = $ sign$\,(\mathcal{D}_7)$ we conclude that in the case $\zeta_1 = 0$, $\zeta_2 \neq 0$ and $\mathcal{D}_4 \neq 0$ systems (3.54) possess the configuration *Config. 7.10b* $(u = -2,\ s = 1)$ if $\mathcal{D}_7 < 0$ and *Config. 7.11b* $(u = 1,\ s = 1)$ if $\mathcal{D}_7 > 0$.

c.2) *The possibility* $\zeta_2 = 0$. This implies $\gamma = 0$ and considering (3.55) we determine $m = \frac{g(s^2+3u)}{9(1+u)}$. In this case systems (3.54) with $v = 0$ for this value of the parameter m become the systems

$$\begin{aligned} \dot{x} =& \frac{(g+3x+3ux)^3}{27(1+u)^2}, \\ \dot{y} =& -\frac{g^3 s(27+2s^2+9u)}{729(1+u)^3} - \frac{g^2 s(27+s^2+6u)}{81(1+u)^2}\,x + \frac{g^2(s^2+3u)}{27(1+u)^2}\,y \\ & -\frac{gs(9+u)}{9(1+u)}\,x^2 + \frac{2g(s^2+3u)}{9(1+u)}\,xy - sx^3 + ux^2y - sxy^2 - y^3, \end{aligned}$$

and after the transformation

$$x_1 = x + \frac{g}{3(1+u)}, \quad y_1 = y - \frac{gs}{9(1+u)}, \quad t_1 = t$$

we arrive at the homogeneous systems

$$\dot{x} = (1+u)x^3, \quad \dot{y} = -sx^3 + ux^2y - sxy^2 - y^3, \quad 1+u \neq 0. \tag{3.65}$$

These systems possess the invariant lines

$$L_{1,2,3} : x = 0,\ L_4 : y = -sx, \quad L_{5,6} : y = \pm ix$$

and the unique finite singularities $M_1(0,0)$ of the multiplicity nine. As a result, taking into consideration Remark 3.27 and the condition $s \neq 0$ we obtain the unique configuration *Config. 7.12b*.

1.2: The case $\mathcal{D}_4 = 0$. Then we arrive at the family of systems (3.51) with $s = 0$.

So we could follow step by step the investigations given earlier for systems (3.51) but now considering the condition $s = 0$. This condition is essential because considering Remark 3.27 we could obtain new configurations of invariant lines. More exactly we have the following remark.

Remark 3.28. We observe that in the case $s \neq 0$ we have constructed 6 canonical forms of systems (3.51) depending on the the values of the invariant polynomials ζ_1, ζ_2 and ζ_4. And the algorithm of the construction does not depends on the value of parameter s. More precisely we have the following canonical systems and their corresponding form in the case $s = 0$:

$$\begin{array}{lclcl}
\zeta_1 < 0, \zeta_2 \neq 0 & \Rightarrow & (3.56) & \overset{s=0}{\Rightarrow} & (3.56)_{\{s=0\}}; \\
\zeta_1 < 0, \zeta_2 = 0 & \Rightarrow & (3.59) & \overset{s=0}{\Rightarrow} & (3.59)_{\{s=0\}}; \\
\zeta_1 > 0, \zeta_4 \neq 0 & \Rightarrow & (3.61) & \overset{s=0}{\Rightarrow} & (3.61)_{\{s=0\}}; \\
\zeta_1 > 0, \zeta_4 \neq 0 & \Rightarrow & (3.63) & \overset{s=0}{\Rightarrow} & (3.63)_{\{s=0\}}; \\
\zeta_1 = 0, \zeta_2 \neq 0 & \Rightarrow & (3.64) & \overset{s=0}{\Rightarrow} & (3.64)_{\{s=0\}}; \\
\zeta_1 = 0, \zeta_2 = 0 & \Rightarrow & (3.65) & \overset{s=0}{\Rightarrow} & (3.65)_{\{s=0\}}.
\end{array}$$

As it was shown in the case $s \neq 0$ all the canonical systems enumerated above possess among their invariant lines the following three ones: $L_4 : y = -sx$ (or $y = 0$ if $s = 0$) and $L_{5,6} : y = \pm ix$. Considering Remark 3.27 the positions of these three invariant lines in configurations in the case $s = 0$ are different from that in the case $s \neq 0$.

So considering the above remark we conclude that in the case $s = 0$ systems (3.51) possess also 12 configurations of invariant lines which are distinct from those in the case $s \neq 0$. In order to determine the corresponding affine invariant conditions we evaluate for systems $(3.51)_{\{s=0\}}$ the invariant polynomials which distinguished the configurations *Config. 7.1b* – *Config. 7.12b*.

Considering Remark 3.28 we observe that the invariant polynomials ζ_1, ζ_2 and ζ_4 were used for constructing the canonical forms mentioned in this remark. On the other hand the invariant polynomials $\mathcal{D}_7$ and ζ_3 were applied for distinguishing the configurations *Config. 7.1b* – *Config. 7.12b* (see the statement of Proposition 3.26, case $\mathcal{D}_4 \neq 0$). Evaluating these two polynomials for systems $(3.51)_{\{s=0\}}$ we have

$$\mathcal{D}_7 = 4(1+u), \quad \zeta_3 = 0$$

and hence the invariant polynomial ζ_3 could not be used for systems $(3.51)_{\{s=0\}}$.

On the other hand we observe that this invariant polynomial is applied only in the case of systems (3.56) and in the case $s \neq 0$ it is responsible for the sign of the expression $a(u+1)$ because for systems (3.56) we have

$$\zeta_3 = -2(a-1)^2 as^2(9+s^2)^2(1+u)^3/81.$$

Therefore for these systems in the case $s = 0$ we need another invariant polynomial and we define the invariant ζ_5 which for systems $(3.56)_{\{s=0\}}$ has the value

$$\zeta_5 = -144(a-1)^2 a(1+u)^3$$

and clearly if $\zeta_5 \neq 0$ then $\operatorname{sign}(\zeta_5) = -\operatorname{sign}(a(u+1))$.

Thus considering Remark 3.28 and the first part of the statement of Proposition 3.26 corresponding to the case $D_4 \neq 0$ as well as our Convention on page 8 and Remark 3.27, in the case $D_4 = 0$ we arrive at the configurations *Config. 7.13b* – *Config. 7.24b*. For the realization of each one of these configurations it is sufficient to take the corresponding examples presented in the proof of the case $\mathcal{D}_4 \neq 0$ and substitute $s = 1$ by $s = 0$. Thus we conclude that Proposition 3.26 is completely proved. $\square$

2: The possibility $\mathcal{D}_5 = 0$. In this case we get $u = -s^2/3$ and then $\zeta_1 = 0$. So we have to detect another invariant polynomial which governs the sign of the polynomial λ. We observe that in this particular case we have

$$\lambda = 3(s^2-3)^2(3c - g^2 + m^2 - cs^2),$$

where $s^2 - 3 \neq 0$ due to $\mathcal{D}_7 = -4(s^2-3)/3 \neq 0$.

On the other hand for systems (3.51) with $u = -s^2/3$ we calculate

$$\zeta_1' = 64s^2(9+s^2)^2(3c - g^2 + m^2 - cs^2)x^6, \quad \mathcal{D}_8 = -32s^2(9+s^2)^2/729. \tag{3.66}$$

Therefore due to $\mathcal{D}_8 \neq 0$ we have $s \neq 0$ and we conclude that in the case $\mathcal{D}_5 = 0$ we have $\operatorname{sign}(\lambda) = \operatorname{sign}(\zeta_1')$.

We prove the following proposition.

Proposition 3.29. *Assume that for a system (3.51) the conditions $\mathcal{D}_6\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\mathcal{D}_5 = 0$ hold. Then this system possesses one of the configurations of invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_1' < 0,\ \zeta_2 \neq 0,\ \zeta_3 < 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.1b;} \\
\zeta_1' < 0,\ \zeta_2 \neq 0,\ \zeta_3 < 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.2b;} \\
\zeta_1' < 0,\ \zeta_2 \neq 0,\ \zeta_3 > 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.3b;} \\
\zeta_1' < 0,\ \zeta_2 \neq 0,\ \zeta_3 > 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.4b;} \\
\zeta_1' < 0,\ \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.5b;} \\
\zeta_1' > 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.6b;} \\
\zeta_1' > 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.7b;} \\
\zeta_1' > 0,\ \zeta_4 = 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.8b;} \\
\zeta_1' > 0,\ \zeta_4 = 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.9b;} \\
\zeta_1' = 0,\ \zeta_2 \neq 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.10b;} \\
\zeta_1' = 0,\ \zeta_2 \neq 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.11b;} \\
\zeta_1' = 0,\ \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.12b.}
\end{array}$$

Proof. First of all we observe that for systems (3.51) with $u = -s^2/3$ according to (3.66) the condition $\mathcal{D}_8 \neq 0$ implies $s \neq 0$ (i.e. $\mathcal{D}_4 \neq 0$).

On the other hand, the proof of Proposition 3.26 for the case $\mathcal{D}_4 \neq 0$ was performed for the condition $u = -s^2/3$ inclusively, because this condition is not essential for the proof. Therefore a system (3.51) with $u = -s^2/3$ could possess only one of the configurations *Config. 7.1b*–*Config. 7.12b* provided by Proposition 3.26 in the case $\mathcal{D}_4 \neq 0$. We claim that each one of these 12 configurations is realizable in the case $u = -s^2/3$.

Indeed for systems (3.51) with $u = -s^2/3$ we have

$$\zeta_1 = 0, \quad \zeta_2 = -72(s^2-3)(3c-g^2+m^2-cs^2),$$
$$\zeta_3 = -\frac{8s^2(9+s^2)^2}{243(-3+s^2)}(3c-g^2+m^2-cs^2)(3c-g^2+4m^2-cs^2),$$
$$\zeta_4 = m^2(4s^2x^2-13x^2-2sxy-3y^2), \quad \mathcal{D}_7 = -4(s^2-3)/3, \quad \mathcal{D}_8 = -32s^2(9+s^2)^2/729$$

and to prove the compatibility of the conditions provided by Proposition 3.29 it is sufficient to present the examples of the realizations of the corresponding configurations for systems (3.54) with $u = -s^2/3$ in terms of the parameters $(c,g,m,s) = (c_0,g_0,m_0,s_0)$ with $s_0 \neq 0$. So we have

Config. 7.1b: $(c_0,g_0,m_0,s_0) = (1,1,0,-2)$;
Config. 7.2b: $(c_0,g_0,m_0,s_0) = (-1,0,1,-1)$;
Config. 7.3b: $(c_0,g_0,m_0,s_0) = (2,0,1,-2)$;
Config. 7.4b: $(c_0,g_0,m_0,s_0) = (0,1,0,-1)$;
Config. 7.5b: $(c_0,g_0,m_0,s_0) = (4,0,1,2)$;
Config. 7.6b: $(c_0,g_0,m_0,s_0) = (-1,0,-2,-2)$;
Config. 7.7b: $(c_0,g_0,m_0,s_0) = (-1,0,-2,-1)$;
Config. 7.8b: $(c_0,g_0,m_0,s_0) = (-1,0,0,-2)$;
Config. 7.9b: $(c_0,g_0,m_0,s_0) = (1,0,0,-1)$;
Config. 7.10b: $(c_0,g_0,m_0,s_0) = (0,1,1,-2)$;
Config. 7.11b: $(c_0,g_0,m_0,s_0) = (0,1,1,-1)$;
Config. 7.12b: $(c_0,g_0,m_0,s_0) = (0,0,0,1)$.

This completes the proof of Proposition 3.29. □

3.2.2 The statement (A_2)

As it was shown in the proof of statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_2) for the family of systems (3.12) lead to the conditions (3.28).

Assuming these conditions to be fulfilled for systems (3.12) we arrive at the family of systems

$$\begin{aligned}
\dot{x} = {}& \left[x - \frac{9l}{2s(9+s^2)}\right]\left[\frac{27l^2(s^2-3)+18lms(9+s^2)+2cs^2(9+s^2)^2}{2s^2(9+s^2)^2}\right.\\
&\left.+\frac{3l(s^2-3)+2ms(9+s^2)}{s(9+s^2)}\,x + 2(s^2-3)x^2/3\right] \equiv L_1^{(1)}(x)L_{2,3}^{(1)}(x),\\
\dot{y} = {}& \frac{3l\left[18l^2s+9lm(9+s^2)+cs(9+s^2)^2\right]}{2s(9+s^2)^3} + \frac{3l\left[3l(s^2-27)+4ms(9+s^2)\right]}{4s(9+s^2)^2}\,x\\
&+\frac{81l^2(s^2-3)+36lms(9+s^2)+4cs^2(9+s^2)^2}{4s^2(9+s^2)^2}\,y + lx^2 + 2mxy\\
&- sx^3 + (2s^2-9)x^2y/3 - sxy^2 - y^3,
\end{aligned} \tag{3.67}$$

for which we have

$$\mathcal{D}_7 = \frac{8}{3}(s^2-3) \neq 0, \quad \mathcal{D}_8 = -\frac{32}{729}s^2(9+s^2)^2 \neq 0.$$

We need to determine if the two lines defined by the equation $L_{2,3}^{(1)} = 0$ are real or complex and in the case when they are real, if one of them coincides with the invariant line $L_1^{(1)} = 0$. So we calculate

$$\text{Discrim}\,[L_{2,3}^{(1)}, x] = -\frac{1}{3s^2(9+s^2)^2}\lambda^{(1)}(c,l,m,s),$$
$$Res_x(L_1^{(1)}, L_{2,3}^{(1)}) = \frac{1}{2s^2(9+s^2)^2}\mu^{(1)}(c,l,m,s)$$

where

$$\begin{aligned}\lambda^{(1)} &= 81l^2(s^2-3)^2 + 36lms(s^2-3)(9+s^2) + 4s^2(9+s^2)^2(2cs^2-6c-3m^2),\\ \mu^{(1)} &= 81l^2(s^2-3) + 36lms(9+s^2) + 2cs^2(9+s^2)^2.\end{aligned} \tag{3.68}$$

We observe that

$$\text{sign}\left(\text{Discrim}\,[L_{2,3}^{(1)}, x]\right) = -\text{sign}\,(\lambda^{(1)}),$$

i.e. the invariant lines $L_{2,3}^{(1)} = 0$ are real (respectively complex; coinciding) if $\lambda^{(1)} < 0$ (respectively $\lambda^{(1)} > 0$; $\lambda^{(1)} = 0$). Moreover, the invariant line $L_1^{(1)} = 0$ coincides with one of the lines $L_{2,3}^{(1)} = 0$ if and only if $\mu^{(1)} = 0$.

On the other hand for systems (3.67) calculations yield:

$$\zeta_1 = \frac{20\lambda^{(1)}\left(s^2-3\right)^2 x^2}{s^2\left(s^2+9\right)^2}, \quad \chi_5 = -\frac{\mu^{(1)}}{9s\left(s^2+9\right)}$$

and hence due to $\mathcal{D}_7 \neq 0$ we have $\text{sign}\,(\lambda^{(1)}) = \text{sign}\,(\zeta_1)$. Moreover we observe that the condition $\mu^{(1)} = 0$ is equivalent to $\chi_5 = 0$.

Proposition 3.30. *Assume that for a system* (3.67) *the condition $\mathcal{D}_7\mathcal{D}_8 \neq 0$ holds. Then this system possesses one of the configurations of invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_1 < 0,\ \zeta_5 \neq 0,\ \zeta_3 < 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.1b ;}\\
\zeta_1 < 0,\ \zeta_5 \neq 0,\ \zeta_3 < 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.2b;}\\
\zeta_1 < 0,\ \zeta_5 \neq 0,\ \zeta_3 > 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.3b ;}\\
\zeta_1 < 0,\ \zeta_5 \neq 0,\ \zeta_3 > 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.4b;}\\
\zeta_1 < 0,\ \zeta_5 = 0 & \Leftrightarrow & \textit{Config. 7.5b;}\\
\zeta_1 > 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.6b;}\\
\zeta_1 > 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.7b;}\\
\zeta_1 > 0,\ \zeta_4 = 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.8b;}\\
\zeta_1 > 0,\ \zeta_4 = 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.9b;}\\
\zeta_1 = 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.10b;}\\
\zeta_1 = 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.11b;}\\
\zeta_1 = 0,\ \zeta_4 = 0 & \Leftrightarrow & \textit{Config. 7.12b.}
\end{array}$$

Proof. We examine three cases: $\zeta_1 < 0$, $\zeta_1 > 0$ and $\zeta_1 = 0$.

a) *The case* $\zeta_1 < 0$. This implies $\lambda^{(1)} < 0$ and we may set $\lambda^{(1)} = -3v^2 < 0$. We observe that the polynomial $\lambda^{(1)}$ is linear with respect to the parameter c with the coefficient $8s^2(s^2 - 3)(9+s^2)^2 \neq 0$ (due to $\mathcal{D}_7\mathcal{D}_8 \neq 0$).

Thus solving the equation $\lambda^{(1)} = -3v^2$ we obtain

$$c = -\frac{3}{8s^2(s^2-3)(9+s^2)^2}\Big[\big[9l(s^2-3)-2ms(9+s^2)\big]\big[3l(s^2-3)+2ms(9+s^2)\big]+v^2\Big]. \tag{3.69}$$

This leads to the following family of systems

$$\begin{aligned}
\dot{x} =&\frac{2(s^2-3)}{3}\Big[x-\frac{9l}{2s(9+s^2)}\Big]\Big[x+\frac{3(18ms-9l+3ls^2+2ms^3-v)}{4s(s^2-3)(9+s^2)}\Big]\times\\
&\Big[x+\frac{3(18ms-9l+3ls^2+2ms^3+v)}{4s(s^2-3)(9+s^2)}\Big],\\
\dot{y} =&\frac{9l}{16s^2(s^2-3)(9+s^2)^3}\big[12lms(s^2-3)(9+s^2)+4m^2(9s+s^3)^2\\
&+3l^2(6s^2-81+7s^4)-v^2\big]+\frac{3l}{4s(9+s^2)^2}(36ms-81l+3ls^2+4ms^3)\,x\\
&+\frac{3}{8s^2(s^2-3)(9+s^2)^2}\big[27l^2(s^2-3)^2+12lms(s^2-3)(9+s^2)\\
&+4m^2(9s+s^3)^2-v^2\big]\,y+lx^2+2mxy-sx^3+(2s^2-9)x^2y/3-sxy^2-y^3.
\end{aligned} \tag{3.70}$$

On the other hand for the value of c given in (3.69) we calculate

$$\mu^{(1)} = \frac{3\big[(\gamma^{(1)})^2-v^2\big]}{4(s^2-3)}, \quad \gamma^{(1)} = 9l(s^2-3)+2ms(9+s^2) \tag{3.71}$$

and since the condition $\mu^{(1)} = 0$ (i.e. $\chi_5 = 0$) leads to the coalescence of two invariant lines of the triplet, we examine two possibilities: $\chi_5 \neq 0$ and $\chi_5 = 0$.

a.1) *The possibility* $\chi_5 \neq 0$. Then $\mu^{(1)} \neq 0$, i.e. $(\gamma^{(1)}-v)(\gamma^{(1)}+v) \neq 0$ and setting a new parameter $a = \frac{\gamma^{(1)}+v}{\gamma^{(1)}-v}$ we observe that $a-1 \neq 0$. Indeed calculation yields: $a-1 = \frac{2v}{\gamma^{(1)}-v} \neq 0$ due to $v \neq 0$. So from the relation $a = \frac{\gamma^{(1)}+v}{\gamma^{(1)}-v}$ we can determine the value of the parameter m:

$$m = \frac{-9l(a-1)(s^2-3)+(a+1)v}{2s(a-1)(9+s^2)}.$$

Then we can apply the following transformation

$$\begin{gathered}
x_1 = \alpha x+\frac{3\nu}{v}, \quad y_1 = \alpha y-\frac{s\nu}{v}, \quad t_1 = \frac{t}{\alpha^2},\\
\alpha = \frac{2(1-a)s(s^2-3)(9+s^2)}{3v}, \quad \nu = l(a-1)(s^2-3),
\end{gathered}$$

which brings systems (3.70) to the family of systems (we keep the old notations for the variables)

$$\begin{aligned}
\dot{x} &= \frac{2}{3}(s^2-3)x(x-1)(x-a),\\
\dot{y} &=\frac{2}{3}(s^2-3)\big[a-x(a+1)\big]y-sx^3+\frac{1}{3}(2s^2-9)x^2y-sxy^2-y^3.
\end{aligned} \tag{3.72}$$

It remains to observe that this family of systems is a subfamily of systems (3.56) defined by the condition $u = (2s^2-9)/3$. This family was investigated earlier and since $u = (2s^2-9)/3$ is not a point of bifurcation, we deduce that there are no new configurations. However we have to determine the conditions for the realization of the corresponding configurations of invariant lines in this case.

For systems (3.72) we calculate:

$$\zeta_1 = -\frac{80}{3}(a-1)^2\left(s^2-3\right)^4x^2, \quad \chi_5 = -\frac{4}{27}as\left(s^2-3\right)\left(s^2+9\right),$$
$$\zeta_3 = -\frac{16}{2187}(a-1)^2a\left(s^2-3\right)^3s^2\left(s^2+9\right)^2.$$

Since the condition $\mathcal{D}_7\mathcal{D}_8 \neq 0$ is satisfied we conclude that the condition $\zeta_1 < 0$ gives us $a-1 \neq 0$ and the condition $\chi_5 \neq 0$ implies $a \neq 0$.

As it was shown earlier systems (3.56) in the case $a(a-1) \neq 0$ and $s \neq 0$ could possess only 4 configurations *Config. 7.1b* – *Config. 7.4b*. More precisely for systems (3.56) we have obtained the following configurations when the corresponding conditions are satisfied, respectively:

$$\begin{array}{lcl} \textit{Config. 7.1b} & \Leftrightarrow & a(a-1)>0,\ a<0; \\ \textit{Config. 7.2b} & \Leftrightarrow & a(a-1)>0,\ a>0; \\ \textit{Config. 7.3b} & \Leftrightarrow & a(a-1)<0,\ a>0; \\ \textit{Config. 7.4b} & \Leftrightarrow & a(a-1)<0,\ a<0. \end{array}$$

On the other hand for systems (3.72) we have $s \neq 0$ due to $\mathcal{D}_8 \neq 0$ and furthermore we have

$$\operatorname{sign}(a(u+1)) = \operatorname{sign}(a(s^2-3)) = \operatorname{sign}(\zeta_3), \quad \operatorname{sign}(u+1) = \operatorname{sign}(s^2-3) = \operatorname{sign}(\mathcal{D}_7).$$

Therefore we conclude that in the case $\zeta_1 < 0$ and $\chi_5 \neq 0$ the statement of Proposition 3.30 is valid.

a.2) *The possibility* $\chi_5 = 0$. Then $\mu^{(1)} = 0$ and considering (3.71) we get $(\gamma^{(1)} - v)(\gamma^{(1)} + v) = 0$. So we may assume $\gamma^{(1)} - v = 0$ due to change $v \to -v$ and setting $v = \gamma^{(1)} \neq 0$ in systems (3.70) we arrive at the family of systems

$$\begin{aligned} \dot{x} =& \frac{2(s^2-3)}{3}\left[x - \frac{9l}{2s(9+s^2)}\right]^2\left[x + \frac{3(9ms-9l+3ls^2+ms^3)}{s(s^2-3)(9+s^2)}\right], \\ \dot{y} =& -\frac{27l^2(18ms-27l+5ls^2+2ms^3)}{4s^2(9+s^2)^3} + \frac{3l}{4s(9+s^2)^2}(36ms-81l+3ls^2+4ms^3)\,x \\ & - \frac{9l}{4s^2(9+s^2)^2}\left[36ms-27l+9ls^2+4ms^3\right]y + lx^2 + 2mxy \\ & - sx^3 + (2s^2-9)x^2y/3 - sxy^2 - y^3 \end{aligned}$$

So since $\gamma^{(1)} = 9l(s^2-3) + 2ms(9+s^2) \neq 0$ then applying the transformation

$$x_1 = \alpha x + \frac{3l(s^2-3)}{\gamma^{(1)}}, \quad y_1 = \alpha y - \frac{ls(s^2-3)}{\gamma^{(1)}}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = -\frac{2s(s^2-3)(9+s^2)}{3\gamma^{(1)}},$$

we obtain the following 1-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{2}{3}(s^2-3)x^2(x-1), \\ \dot{y} &= -\frac{2}{3}(s^2-3)xy - sx^3 + \frac{1}{3}(2s^2-9)x^2y - sxy^2 - y^3. \end{aligned}$$

We again observe that the above family of systems is a subfamily of systems (3.59) defined by the condition $u = (2s^2 - 9)/3$. This family was investigated earlier and it was shown that it possesses the unique configuration *Config. 7.5b* including the case $u = (2s^2 - 9)/3$.

b) *The case* $\zeta_1 > 0$. Then $\lambda^{(1)} > 0$ and we set $\lambda^{(1)} = 3v^2 > 0$ and since $s(s^2-3) \neq 0$ (due to $\mathcal{D}_7\mathcal{D}_8 \neq 0$) we obtain

$$c = -\frac{3}{8s^2(s^2-3)(9+s^2)^2}\Big[\big[9l(s^2-3) - 2ms(9+s^2)\big]\big[3l(s^2-3) + 2ms(9+s^2)\big] - v^2\Big].$$

This leads to the following family of systems

$$\begin{aligned}
\dot{x} =& \Big[x - \frac{9l}{2s(9+s^2)}\Big]\Big[\frac{9v^2 + \big[9l(s^2-3) + 2s(9+s^2)(3m + 2(s^2-3)x)\big]^2}{24s^2(s^2-3)(9+s^2)^2}\Big],\\
\dot{y} =& \frac{9l}{16s^2(s^2-3)(9+s^2)^3}\big[12lms(s^2-3)(9+s^2) + 4m^2(9s+s^3)^2\\
&+ 3l^2(6s^2 - 81 + 7s^4) + v^2\big] + \frac{3l}{4s(9+s^2)^2}(36ms - 81l + 3ls^2 + 4ms^3)\,x\\
&+ \frac{3}{8s^2(s^2-3)(9+s^2)^2}\big[27l^2(s^2-3)^2 + 12lms(s^2-3)(9+s^2)\\
&+ 4m^2(9s+s^3)^2 + v^2\big]\,y + lx^2 + 2mxy - sx^3 + (2s^2-9)x^2y/3 - sxy^2 - y^3.
\end{aligned} \tag{3.73}$$

In order to simplify these systems we need to use a transformation which depends on the condition: either $\gamma^{(1)} \neq 0$ or $\gamma^{(1)} = 0$. Since for the above systems we have

$$\zeta_4 = \frac{\big(\gamma^{(1)}\big)^2}{4s^2(9+s^2)^2}\big[(4s^2-13)x^2 - 2sxy - 3y^2\big]$$

we conclude that the condition $\gamma^{(1)} = 0$ is equivalent to $\zeta_4 = 0$. So we discuss two possibilities: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

b.1) *The possibility* $\zeta_4 \neq 0$. This implies $\gamma^{(1)} \neq 0$ and setting a new parameter $a = \frac{v}{\gamma^{(1)}} \neq 0$ we have $v = a\gamma^{(1)}$. Then we can apply to systems (3.73) the following transformation

$$x_1 = \alpha x + \frac{6l(s^2-3)}{\gamma^{(1)}}, \quad y_1 = \alpha y - \frac{2ls(s^2-3)}{\gamma^{(1)}}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = -\frac{4s(s^2-3)(9+s^2)}{3\gamma^{(1)}},$$

which brings these systems to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned}
\dot{x} &= \frac{2}{3}(s^2-3)x\big[(x-1)^2 + a^2\big],\\
\dot{y} &= \frac{2}{3}(s^2-3)(1+a^2)y - \frac{4}{3}(s^2-3)xy - sx^3 + \frac{1}{3}(2s^2-9)x^2y - sxy^2 - y^3.
\end{aligned} \tag{3.74}$$

It remains to observe that this family of systems is a subfamily of systems (3.61) defined by the condition $u = (2s^2-9)/3$. The family (3.61) was investigated earlier and it was proved the existence of only two configurations of the invariant lines: *Config. 7.6b* if $\mathcal{D}_7 < 0$ and *Config. 7.7b* if $\mathcal{D}_7 > 0$.

Since for systems (3.74) we have $\mathcal{D}_7 = 8\left(s^2-3\right)/3$ we deduce that both configurations are also realizable in the case under consideration.

b.2) *The possibility* $\zeta_4 = 0$. This implies $\gamma^{(1)} = 0$ and considering (3.71) the condition $\gamma^{(1)} = 0$ gives

$$m = -\frac{9l(s^2-3)}{2s(9+s^2)}$$

Then we can apply to systems (3.73) the transformation

$$x_1 = \alpha x - \frac{6l(s^2-3)}{v}, \quad y_1 = \alpha y + \frac{2ls(s^2-3)}{v}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = \frac{4s(s^2-3)(9+s^2)}{3v},$$

which brings these systems to the following family of systems (we keep the old notations for the variables):

$$\dot{x} = \frac{2}{3}(s^2-3)x(x^2+1), \quad \dot{y} = \frac{2}{3}(s^2-3)y - sx^3 + \frac{1}{3}(2s^2-9)x^2y - sxy^2 - y^3. \tag{3.75}$$

It is easy to observe that this family is a subfamily of systems (3.63) defined by the condition $u = (2s^2-9)/3$. The family (3.63) was investigated earlier and we have proved the existence of two configurations: *Config. 7.8b* if $\mathcal{D}_7 < 0$ and *Config. 7.9b* if $\mathcal{D}_7 > 0$. So by the same reasons as in the possibility ***b.1)*** above we conclude that in the case $\zeta_1 > 0$ and $\zeta_4 = 0$ the statement of Proposition 3.30 is valid.

c) *The case* $\zeta_1 = 0$. This implies $\lambda^{(1)} = 0$ and considering (3.68) and solving the equation $\lambda^{(1)} = 0$ with respect to the parameter c, it is clear that we get (3.69) for $v = 0$. This leads to the systems (3.70) with $v = 0$, which we denote by $(3.70)_{\{v=0\}}$ and for these systems we have

$$\zeta_4 = \frac{\left(\gamma^{(1)}\right)^2}{4s^2(9+s^2)^2}\left[(4s^2-13)x^2 - 2sxy - 3y^2\right]$$

and we again consider two possibilities: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

c.1) *The possibility* $\zeta_4 \neq 0$. Then $\gamma^{(1)} \neq 0$ and via the transformation

$$x_1 = \alpha x + \frac{6l(s^2-3)}{\gamma^{(1)}}, \quad y_1 = \alpha y - \frac{2ls(s^2-3)}{\gamma^{(1)}}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = -\frac{4s(s^2-3)(9+s^2)}{3\gamma^{(1)}},$$

systems $(3.70)_{\{v=0\}}$ can be brought to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned} \dot{x} &= \frac{2}{3}(s^2-3)x(x-1)^2, \\ \dot{y} &= \frac{2}{3}(s^2-3)y - \frac{4}{3}(s^2-3)xy - sx^3 + \frac{1}{3}(2s^2-9)x^2y - sxy^2 - y^3. \end{aligned} \tag{3.76}$$

We observe that this family of systems is a subfamily of systems (3.64) defined by the condition $u = (2s^2-9)/3$. This family was investigated earlier and we have detected *Config. 7.10b* if $\mathcal{D}_7 < 0$ and *Config. 7.11b* if $\mathcal{D}_7 > 0$. Clearly we get the same configurations in the case $u = (2s^2-9)/3$, i.e. when $\mathcal{D}_6 = 0$.

c.2) *The possibility* $\zeta_4 = 0$. Then $\gamma^{(1)} = 0$ and considering (3.71) we determine $m = -\frac{9l(s^2-3)}{2s(9+s^2)}$. In this case systems (3.70) with $v = 0$ for this value of the parameter m after the transformation

$$x_1 = x - \frac{9l}{2s(9+s^2)}, \quad y_1 = y + \frac{3ls}{2(9+s^2)}, \quad t_1 = t$$

we will be brought to the homogeneous systems (3.65) with $u = (2s^2-9)/3$. However these systems are already examined and we found only the configuration *Config. 7.12b*.

Since all the cases are examined we conclude that Proposition 3.30 is proved. □

3.2.3 The statement (A_3)

According to the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by this statement for the family of systems (3.12) lead to the conditions (3.29).

Next we determine the canonical form of the systems (3.12) subject to the conditions (3.29). Assuming these conditions to be fulfilled for systems (3.12) we arrive at the following family of systems

$$\begin{aligned}\dot{x} = &(1+u)\Big[x + \frac{3g-6m+gs^2+2gu-6mu}{(9+s^2)(1+u)}\Big]\Big[\frac{g^2s^2+[gu-3m(1+u)]^2}{s^2(9+s^2)(1+u)^2}\\ &-\frac{2(gu-3g-3m-3mu)}{(9+s^2)(1+u)}\,x + x^2\Big] \equiv\ (1+u)L_1^{(2)}(x)L_{2,3}^{(2)}(x),\\ \dot{y} =&\widetilde{Q}(x,y),\end{aligned} \tag{3.77}$$

where the polynomial $\widetilde{Q}(x,y)$ depends on the parameters g, m, s and u and it is determined by the conditions (3.29). According to the statement (A_3) of the Main Theorem for the above systems the conditions $\mathcal{D}_7\mathcal{D}_8\mathcal{D}_4 \neq 0$ and $\chi_1 \neq 0$ must hold. So calculations yield:

$$\begin{aligned}&\mathcal{D}_7 = 4(1+u) \neq 0, \quad \mathcal{D}_8 = -8(s^2-u)\big[4s^2+(3+u)^2\big]/27 \neq 0, \quad \mathcal{D}_4 = 2304s(9+s^2) \neq 0,\\ &\chi_1 = \frac{1}{9s(9+s^2)(1+u)}(s^2-u)\big[9m(1+u)-g(s^2+3u)\big]\big[4s^2+(u+3)^2\big] \neq 0.\end{aligned}$$

Considering the first equation of systems (3.77) we observe that

$$\text{Discrim}\,[L_{2,3}^{(2)}, x] = -\frac{4(\gamma^{(2)})^2}{s^2(9+s^2)^2}, \quad Res_x(L_1^{(2)}, L_{2,3}^{(2)}) = \frac{(s^2+1)(\gamma^{(2)})^2}{s^2(9+s^2)^2}$$

where

$$\gamma^{(2)} = 9m(1+u) - g(s^2+3u) \neq 0$$

due to the condition $\chi_1 \neq 0$.

Thus we deduce that the invariant lines $L_{2,3}^{(2)} = 0$ are complex and they could not coalesce. Moreover all three invariant lines are distinct.

Since $\gamma^{(2)} \neq 0$ applying the transformation

$$\begin{aligned}&x_1 = \alpha x - \frac{3g-6m+gs^2+2gu-6mu}{\gamma^{(2)}}, \quad \alpha = -\frac{(u+1)(9+s^2)}{\gamma^{(2)}},\\ &y_1 = \alpha y + \frac{m(1+u)(9+s^2+6u)-gu(3+s^2+2u)}{s\gamma^{(2)}}, \quad t_1 = \frac{t}{\alpha^2},\end{aligned}$$

to systems (3.77) we arrive at the family of systems

$$\begin{aligned}&\dot{x} = (1+u)x\big[(x-1)^2+\frac{1}{s^2}\big], \quad s(1+u)\neq 0,\\ &\dot{y} = (1+u)^2x/s - (1+u)(2+s^2+u)y/s^2 + (s^2-2u-u^2)x^2/s\\ &\qquad + (3+s^2+2u)y^2/s - sx^3 + ux^2y - sxy^2 - y^3.\end{aligned} \tag{3.78}$$

We determine that systems (3.78) possess six distinct invariant affine straight lines

$$L_1 : x = 0,\ L_{2,3} : x = 1 \pm i/s,\ L_4 : y = -sx + (s^2+u+2)/s, \quad L_{5,6} : y = \pm ix + (1+u)/s$$

and the following nine finite singularities:

$$M_1(0,0),\ M_2\big(0,(1+u)/s\big),\ M_3\big(0,(2+s^2+u)/s\big),\ M_{4,5}\big(1\pm i/s,\ 1/s\mp i\big),\\ M_{6,7}\big(1\pm i/s,\ u/s\pm i\big),\ M_{8,9}\big(1\pm i/s,\ (2+u)/s\mp i\big), \tag{3.79}$$

which due to $s(1+u)\neq 0$ are all distinct except for the case $2+s^2+u=0$ which implies the coalescence of the real singular point M_3 with M_1.

We observe that all the singularities (3.79) are located at the intersections of the invariant lines, except for the real singularity $M_1(0,0)$ and the complex singularities $M_{4,5}\big(1\pm i/s,\ 1/s\mp i\big)$. Moreover we have exactly three real singularities, which are all located on the invariant line $x=0$. We note that the real singularity M_2 (respectively M_3) is the point of intersection of the invariant line L_1 with the two complex lines $L_{5,6}$ (respectively with the real line L_4).

On the other hand the complex singularity M_6 (respectively M_7) is a point of intersection of two invariant lines L_2 and L_5 (respectively L_3 and L_6), whereas the complex singularity M_8 (respectively M_9) is a point of intersection of three invariant lines L_2, L_4 and L_6 (respectively L_3, L_4 and L_5).

So, considering the fact that we have exatly three real finite singularities M_1, M_2 and M_3 and all of them are located on the invariant line $x=0$ we conclude that we could obtain three distinct configurations of invariant lines defined by the distinct positions of the free point M_1 with respect to the other two real singularities (M_2 and M_3).

In order to describe the positions of the finite real singularities located on the same invariant line we use the following notations.

Notation 3.31. Assume that two finite real singular points $\widetilde{M}_1(x_1,y_1)$ and $\widetilde{M}_2(x_2,y_2)$ of a cubic system are located on the real invariant line $ax+by+c=0$ of this system. Then:

(α) in the case $a\neq 0$ we say that the singular point $\widetilde{M}_1$ is located *below* (respectively *above*) or coincides with, the singularity $\widetilde{M}_2$ if $y_1\leq y_2$ (respectively $y_2<y_1$) and we denote this position by $\widetilde{M}_1\preceq\widetilde{M}_2$ (respectively $\widetilde{M}_2\prec\widetilde{M}_1$);

(β) in the case $a=0$ (then $y_1=y_2$) we say that the singular point $\widetilde{M}_1$ is located *on the left* (respectively *on the right*) or coincides with, the singularity $\widetilde{M}_2$ if $x_1\leq x_2$ (respectively $x_2<x_1$) and we again denote this position by $\widetilde{M}_1\preceq\widetilde{M}_2$ (respectively $\widetilde{M}_2\prec\widetilde{M}_1$).

Since $y_3-y_2=(1+s^2)/s$ it is easy to determine that the positions of the real singularities on the line $x=0$ are determined by the following conditions:

$$M_2\prec M_1\ \Leftrightarrow\ (1+u)s<0;\quad M_3\preceq M_1\ \Leftrightarrow\ (2+u+s^2)s\leq 0;\quad M_3\preceq M_2\ \Leftrightarrow\ s<0.$$

Therefore considering these conditions we obtain the following conditions for the realization of the corresponding configurations of invariant lines:

$$\begin{array}{llllll}
1+u<0,\ 2+u+s^2<0 & \Rightarrow & M_2\prec M_3\prec M_1 & \Rightarrow & \textit{Config. 7.25b}; & (s=1,\ u=-7/2)\\
1+u<0,\ 2+u+s^2>0 & \Rightarrow & M_2\prec M_1\prec M_3 & \Rightarrow & \textit{Config. 7.26b}; & (s=1,\ u=-3/2)\\
1+u<0,\ 2+u+s^2=0 & \Rightarrow & M_2\prec M_3\equiv M_1 & \Rightarrow & \textit{Config. 7.27b}; & (s=1,\ u=-3)\\
1+u>0 & \Rightarrow & M_1\prec M_2\prec M_3 & \Rightarrow & \textit{Config. 7.28b}. & (s=1,\ u=0)
\end{array}$$

In order to determine the corresponding invariant conditions we evaluate for systems (3.78) the following invariant polynomials:

$$\mathcal{D}_4=2304s(9+s^2),\quad \mathcal{D}_7=4(1+u),\quad \zeta_6=8(2+u+s^2)\big[4s^2+(u-1)^2\big].$$

So due to the condition $s \neq 0$ (as $\mathcal{D}_4 \neq 0$) we have sign $(\zeta_6) =$ sign $(2+u+s^2)$ and sign $(\mathcal{D}_7) =$ sign $(1+u)$.

Considering the conditions for the configurations of invariant lines presented above we arrive at the following proposition.

Proposition 3.32. *Assume that for a system (3.77) the condition $\mathcal{D}_7\mathcal{D}_8\chi_1 \neq 0$ and $\mathcal{D}_4 \neq 0$ holds. Then this system possesses one of following four configurations of invariant lines if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\mathcal{D}_7 < 0,\, \zeta_6 < 0 & \Leftrightarrow & \textit{Config. 7.25b;}\\
\mathcal{D}_7 < 0,\, \zeta_6 > 0 & \Leftrightarrow & \textit{Config. 7.26b;}\\
\mathcal{D}_7 < 0,\, \zeta_6 = 0 & \Leftrightarrow & \textit{Config. 7.27b;}\\
\mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.28b.}
\end{array}$$

3.2.4 The statement (A_4)

As it was shown in the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_4) for the family of systems (3.12) lead to the conditions (3.32).

Assuming these conditions to be fulfilled for systems (3.12) we arrive at the family of systems

$$\begin{aligned}
\dot{x} &= (1+u)\Big(x+\frac{m}{u}\Big)\Big[\frac{9l^2}{u^2(3+u)^2}+\Big(x+\frac{m}{u}\Big)^2\Big],\\
\dot{y} &= \frac{(l+uy)}{u^2}\Big[\frac{l^2(3+2u)}{(3+u)^2}+(ux+m)^2+ly-uy^2\Big].
\end{aligned} \tag{3.80}$$

According to the statement (A_4) of the Main Theorem for the above systems the conditions $\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\chi_1 \neq 0$ must hold. So calculations yield:

$$\mathcal{D}_7 = 4(1+u) \neq 0, \quad \mathcal{D}_8 = 8u(3+u)^2/27 \neq 0, \quad \chi_1 = -l(3+u)x^3/3 \neq 0$$

and hence for systems (3.80) the condition $lu(3+u) \neq 0$ holds. Then via the transformation

$$x_1 = \alpha x + \frac{m(3+u)}{3l}, \quad y_1 = \alpha y + \frac{3+u}{3}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = \frac{u(3+u)}{3l}$$

systems (3.80) can be brought to the systems

$$\dot{x} = (1+u)x(x^2+1), \quad \dot{y} = y\big[-2-u+(3+u)y+ux^2-y^2\big], \tag{3.81}$$

for which we have $\mathcal{D}_7 = 4(1+u) \neq 0$ and $\mathcal{D}_8 = 8u(3+u)^2/27 \neq 0$. Therefore for the above systems the condition $u(1+u)(3+u) \neq 0$ is satisfied.

We determine that systems (3.81) possess six distinct invariant affine straight lines

$$L_1 : x = 0,\ L_{2,3} : x = \pm i,\ L_4 : y = 0, \quad L_{5,6} : y = 1 \pm ix$$

and the following nine finite singularities:

$$M_1(0,0),\ M_2(0,1),\ M_3(0,2+u),\ M_{4,5}(\pm i,0),\ M_{6,7}(\pm i,2),\ M_{8,9}(\pm i,1+u).$$

We observe that we could have multiple singularities for some values of the parameter u. More exactly, in the case $u = -2$ the singular point M_3 coalesces with M_1 and we obtain a double singular point $(0,0)$. On the other hand we determine that for $u=1$ the complex singular point $M_8(i,1+u)$ (respectively $M_9(-i,1+u)$ coalesces with the complex singular point $M_6(i,2)$ (respectively $M_7(-i,2)$. As a result we get two double complex singular points, however according to Definition 1.2 this fact is irrelevant for a configuration because we take into consideration only real singularities located on the invariant lines.

We remark that we have only three real singularities and all of them are located on the invariant line $x=0$. Two among these real singularities are fixed: $M_1(0,0)$ (which is a point of the intersection of the invariant lines L_1 and L_4) and $M_2(0,1)$ (which is a point of the intersection of the invariant lines L_1, L_5 and L_6). The singular point $M_3(0,u+2)$ depends on the parameter u and hence could change its position with respect to the singularities M_1 and M_2.

Thus, since we have $M_1 \prec M_2$, taking into consideration our Convention (see page 8) we conclude that the position of $M_3(0,u+2)$ leads to the following four distinct configurations of invariant lines:

$$\begin{array}{rcccl}
u<-2 & \Rightarrow & M_3 \prec M_1 \prec M_2 & \Rightarrow & \textit{Config. 7.29b;}\\
u=-2 & \Rightarrow & M_3 = M_1 \prec M_2 & \Rightarrow & \textit{Config. 7.30b;}\\
-2<u<-1 & \Rightarrow & M_1 \prec M_3 \prec M_2 & \Rightarrow & \textit{Config. 7.31b;}\\
u>-1 & \Rightarrow & M_1 \prec M_2 \prec M_3 & \Rightarrow & \textit{Config. 7.32b.}
\end{array}$$

On the other hand for systems (3.81) we have

$$\mathcal{D}_7 = 4(1+u), \quad \zeta_7 = 4(2+u)$$

and evidently we arrive at the following proposition.

Proposition 3.33. *Assume that for a system (3.80) the conditions $\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\chi_1 \neq 0$ hold. Then this system possesses one of the following four configurations of the invariant lines if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_7<0 & \Leftrightarrow & \textit{Config. 7.29b;}\\
\zeta_7=0 & \Leftrightarrow & \textit{Config. 7.30b;}\\
\zeta_7>0,\ \mathcal{D}_7<0 & \Leftrightarrow & \textit{Config. 7.31b;}\\
\zeta_7>0,\ \mathcal{D}_7>0 & \Leftrightarrow & \textit{Config. 7.32b.}
\end{array}$$

3.2.5 The statement (A_5)

According to the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_5) for the family of systems (3.12) lead to the conditions (3.33).

Remark 3.34. We observe that the conditions (3.33) can be obtained as a particular case from the conditions (3.26) by setting $u=s^2$ (i.e. we allow the condition $\mathcal{D}_8=0$ to be satisfied). This means that we could follow all the steps we have done in the case of the conditions (3.26) if these steps do not depend on the condition $u=s^2$.

Thus applying the conditions (3.33) to systems (3.12) we arrive at the family of systems $(3.51)_{\{u=s^2\}}$ which is a subfamily of (3.51) defined by the condition $u=s^2$.

We remark that all the configurations of the family (3.51) were investigated and Proposition 3.26 provides the necessary and sufficient affine invariant conditions for the realization of each one of the possible 12 possible configurations in the case $\mathcal{D}_4 \neq 0$.

Thus we have to determine which sets of the conditions provided by Proposition 3.26 (for $\mathcal{D}_4 \neq 0$) are compatible in the case $u = s^2 \neq 0$. We prove the following proposition.

Proposition 3.35. *Assume that for a system* $(3.51)_{\{u=s^2\}}$ *the conditions* $\mathcal{D}_7\mathcal{D}_6 \neq 0$ *and* $\mathcal{D}_4 \neq 0$ *hold. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$
\begin{array}{lcl}
\zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_3 < 0 & \Leftrightarrow & \textit{Config. 7.2b;}\\
\zeta_1 < 0,\, \zeta_2 \neq 0,\, \zeta_3 > 0 & \Leftrightarrow & \textit{Config. 7.4b;}\\
\zeta_1 < 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.5b;}\\
\zeta_1 > 0,\, \zeta_4 \neq 0 & \Leftrightarrow & \textit{Config. 7.7b;}\\
\zeta_1 > 0,\, \zeta_4 = 0 & \Leftrightarrow & \textit{Config. 7.9b;}\\
\zeta_1 = 0,\, \zeta_2 \neq 0 & \Leftrightarrow & \textit{Config. 7.11b;}\\
\zeta_1 = 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.12b.}
\end{array}
$$

Proof. Evaluating for systems $(3.54)_{\{u=s^2\}}$ the invariant polynomials ζ_1, ζ_2, ζ_3, ζ_4, $\mathcal{D}_4$ and $\mathcal{D}_7$ which are involved in Proposition 3.26 (for $\mathcal{D}_4 \neq 0$) we obtain:

$$
\zeta_1 = \frac{1280s^4}{3(s^2+9)^2}\kappa_1 x^2, \quad \zeta_2 = 8\kappa_2, \quad \zeta_3 = \frac{8s^2}{81(s^2+9)^2}\kappa_1\kappa_2, \quad \mathcal{D}_4 = 2304s(9+s^2)
$$

$$
\zeta_4 = \frac{1}{(s^2+9)^2}\kappa_3^2\left[(4s^2-13)x^2 - 2sxy - 3y^2\right], \quad \mathcal{D}_7 = 4(1+s^2), \quad \mathcal{D}_6 = -4(9+s^2)/9,
$$

where

$$
\begin{aligned}
\kappa_1 &= c(1+s^2)(9+s^2)^2 + (9g-9m+5gs^2-9ms^2)(gs^2-3g-3m-3ms^2),\\
\kappa_2 &= c(9+s^2)^2 + 3(g-2m)(7gs^2-9g-18m-18ms^2),\\
\kappa_3 &= 9m-4gs^2+9ms^2.
\end{aligned}
$$

As we can see the condition $\mathcal{D}_7 > 0$ holds. Therefore we conclude that the configurations *Configs. 7.1b, 7.3b, 7.6b, 7.8b, 7.10b* which correspond to the case $\mathcal{D}_7 < 0$ and are realizable for systems (3.54) (see Proposition 3.26), could not be realizable for systems $(3.54)_{\{u=s^2\}}$.

To prove the compatibility of other conditions provided by Proposition 3.26 it is sufficient to present the examples of the realizations of the corresponding configurations for systems $(3.54)_{\{u=s^2\}}$ in terms of the parameters $(c, g, m, s) = (c_0, g_0, m_0, s_0)$ with $s_0 \neq 0$. So we have

$$
\begin{array}{ll}
\textit{Config. 7.2b}: & (c_0, g_0, m_0, s_0) = (-1, 1, 1, -1);\\
\textit{Config. 7.4b}: & (c_0, g_0, m_0, s_0) = (-2, 1, 1, -1);\\
\textit{Config. 7.5b}: & (c_0, g_0, m_0, s_0) = (-57/50, 1, 1, -1);\\
\textit{Config. 7.7b}: & (c_0, g_0, m_0, s_0) = (0, 1, 1, -1);\\
\textit{Config. 7.9b}: & (c_0, g_0, m_0, s_0) = (1, 0, 0, -1);\\
\textit{Config. 7.11b}: & (c_0, g_0, m_0, s_0) = (0, -3, 1, 1);\\
\textit{Config. 7.12b}: & (c_0, g_0, m_0, s_0) = (0, 0, 0, 1)
\end{array}
$$

This completes the proof of Proposition 3.35. □

3.2.6 The statement (A_6)

As it was shown in the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_6) for the family of systems (3.12) according to Lemma 3.15 lead either to the conditions

$$\begin{aligned} &u = s = k = d = h = l = e = b = 0, \quad f = c + \frac{g(2m-g)}{3}, \\ &a = -\frac{g-2m}{27}\left(2g^2 - 9c - 2gm - 4m^2\right). \end{aligned} \tag{3.82}$$

(for a triplet in the direction $x = 0$), or to the conditions

$$\begin{aligned} &u = s = k = l = e = m = 0, \quad d = \frac{2gh}{3}, \quad f = c - \frac{g^2}{3}, \\ &a = \frac{g}{27}\left(9c - 2g^2\right), \quad b = -\frac{2h}{27}(-9c + 3g^2 + 4h^2) \end{aligned} \tag{3.83}$$

(for a triplet in the direction $y = 0$).

It is not too difficult to detect that when conditions (3.82) are satisfied then (3.12) become the systems

$$\begin{aligned} \dot{x} &= \frac{1}{27}(g - 2m + 3x)(9c - 2g^2 + 2gm + 4m^2 + 6gx + 6mx + 9x^2), \\ \dot{y} &= \frac{1}{3}y\left(3c - g^2 + 2gm + 6mx - 3y^2\right). \end{aligned} \tag{3.84}$$

On the other hand, if conditions (3.83) are satisfied then we arrive at the family of systems

$$\begin{aligned} \dot{x} &= -\frac{1}{27}(g + 3x)\left(-9c + 2g^2 - 6gx - 18hy - 9x^2\right), \\ \dot{y} &= -\frac{1}{27}(2h + 3y)\left(-9c + 3g^2 + 4h^2 - 6hy + 9y^2\right). \end{aligned} \tag{3.85}$$

We claim that systems (3.84) and (3.85) are affinely equivalent. Indeed since some parameters of the two systems coincide we set for systems (3.85) free parameters $\tilde{c} = c$, $\tilde{g} = g$ and $\tilde{h} = h$. Then the transformation

$$x_1 = y - \tilde{g}/3, \quad y_1 = -x - \tilde{g}/3, \quad t_1 = -t$$

leads to the systems

$$\begin{aligned} \dot{x}_1 &= \frac{1}{27}(g_1 - 2m_1 + 3x)\left(9c_1 - 2g_1^2 + 2g_1m_1 + 4m_1^2 + 6g_1x_1 + 6m_1x_1 + 9x_1^2\right), \\ \dot{y}_1 &= \frac{1}{3}y_1\left(3c_1 - g_1^2 + 2g_1m_1 + 6m_1x - 3y_1^2\right) \end{aligned}$$

with $g_1 = \tilde{g}$, $m_1 = -\tilde{h}$ and $c_1 = (-3\tilde{c} + 2\tilde{g}^2)/3$. In other words we have obtained exactly systems (3.84) with new parameters c_1, g_1, m_1. This completes the proof of our claim.

Thus in this case either the conditions (3.82) or (3.83) are satisfied in both cases using an affine transformation and time rescaling we arrive at the same family of systems (3.84).

We observe that the family of systems (3.84) is a subfamily of (3.51) defined by the condition $u = s = 0$. We have shown that systems (3.51) possess three parallel invariant lines in the direction $x = 0$ and the kind of these lines (real, complex, distinct or coinciding) are determined by the polynomials λ and μ given in (3.52). For the particular case $u = s = 0$ (i.e. for systems (3.84)) these polynomials become

$$\lambda\big|_{\{u=s=0\}} = 27(3c - g^2 + m^2), \quad \mu\big|_{\{u=s=0\}} = 27(3c - g^2 + 4m^2).$$

On the other hand we observe that the sign of the polynomial λ as well as the the value of the polynomial μ are governed by the invariant polynomials ζ_1 and ζ_2 which for systems (3.51) have the form (see page 54)

$$\zeta_1 = \frac{80}{3\varkappa^2}(s^2 + 3u)^2 x^2 \lambda, \quad \zeta_2 = 8\mu.$$

As we can see for $u = s = 0$ the invariant ζ_1 vanishes, i.e. it could not be used to define the sign of $\lambda\big|_{\{u=s=0\}}$, i.e. the sign of the polynomial $3c - g^2 + m^2$.

Thus we have to define another invariant polynomial which captures the sign of $3c - g^2 + m^2$. Such a polynomial could be ζ_8 which for systems (3.84) has the value

$$\zeta_8 = 8m^2(3c - g^2 + m^2).$$

On the other hand according to the conditions provided by the statement (A_6) of the Main Theorem the condition $\chi_{15} \neq 0$ must hold. For systems (3.84) we calculate $\chi_{15} = mx^3y^2 \neq 0$, i.e. $m \neq 0$ and we have

$$\mathrm{sign}\,(\zeta_8) = \mathrm{sign}\,(3c - g^2 + m^2) = \mathrm{sign}\,(\lambda\big|_{\{u=s=0\}}).$$

Thus substituting the invariant polynomial ζ_1 (which vanishes) by ζ_8 we could determine which sets of the conditions provided by Proposition 3.26 are compatible in the case $\mathcal{D}_8 = \mathcal{D}_4 = 0$ (i.e. $u = s = 0$).

Proposition 3.36. *Assume that for a system* (3.84) *the condition* $\chi_{15} \neq 0$ *(i.e.* $m \neq 0$*) holds. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_8 < 0,\, \zeta_2 \neq 0,\, \zeta_5 < 0 & \Leftrightarrow & \textit{Config. 7.14b;} \\
\zeta_8 < 0,\, \zeta_2 \neq 0,\, \zeta_5 > 0 & \Leftrightarrow & \textit{Config. 7.16b;} \\
\zeta_8 < 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.17b;} \\
\zeta_8 > 0 & \Leftrightarrow & \textit{Config. 7.19b;} \\
\zeta_8 = 0 & \Leftrightarrow & \textit{Config. 7.23b.}
\end{array}$$

Proof. Considering Proposition 3.26 we evaluate for systems (3.84) the invariant polynomials ζ_8 (instead of ζ_1), ζ_2, ζ_4, ζ_5 and $\mathcal{D}_7$ which are involved in Proposition 3.26 in the case $\mathcal{D}_4 = 0$. The calculations yield:

$$\zeta_8 = 8m^2(3c - g^2 + m^2), \quad \zeta_2 = 216(3c - g^2 + 4m^2), \quad \zeta_4 = -m^2(13x^2 + 3y^2),$$
$$\zeta_5 = -64(3c - g^2 + 4m^2)(3c - g^2 + m^2), \quad \mathcal{D}_4 = 0, \quad \mathcal{D}_7 = 4, \quad \mathcal{D}_6 = -4.$$

As we can see the conditions $\mathcal{D}_7 > 0$ and $\zeta_4 \neq 0$ (due to $\chi_{15} \neq 0$, i.e. $m \neq 0$) hold. Therefore we conclude that the configurations *Configs. 7.13b, 7.15b, 7.18b, 7.20b, 7.21b, 7.22b* which correspond to the case $\mathcal{D}_7 < 0$ (or $\zeta_4 = 0$) and are realizable for systems (3.54) (see Proposition 3.26) could not be realizable for systems (3.84). Moreover the configuration *Configs. 7.24b* is defined by the conditions $\zeta_8 = \zeta_2 = 0$, however these conditions are incompatible with $\chi_{15} \neq 0$. Indeed, assuming $\zeta_8 = 0$ we get $c = (g^2 - m^2)/3$ and then $\zeta_2 = 648m^2 \neq 0$ (due to $\chi_{15} \neq 0$). Hence *Configs. 7.24b* could also not be realizable for systems (3.84).

To prove the compatibility of other conditions provided by Proposition 3.36 it is sufficient to present the examples of the realization of the corresponding configurations for systems (3.84) in terms of the parameters $(c, g, m) = (c_0, g_0, m_0)$. So we have

$$\begin{array}{ll} \textit{Config. 7.14b}: & (c_0, g_0, m_0) = (-3/2, 1, -1); \\ \textit{Config. 7.16b}: & (c_0, g_0, m_0) = (-1/2, 1, -1); \\ \textit{Config. 7.17b}: & (c_0, g_0, m_0) = (-1, 1, -1); \\ \textit{Config. 7.19b}: & (c_0, g_0, m_0) = (1, 1, -1); \\ \textit{Config. 7.23b}: & (c_0, g_0, m_0) = (0, 1, 1). \end{array}$$

This completes the proof of Proposition 3.36. □

3.2.7 The statement (A_7)

According to the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_7) for the family of systems (3.12) lead to the conditions (3.39). We observe that these conditions contain the equality $H' = 0$ where the polynomial

$$H' = 27a^2 + 2am(9c + 4m^2) - (c - f)(c^2 + 4cf + 4f^2 + 4fm^2)$$

is quadratic with respect to parameter a. So in order to construct the canonical form of systems (3.12) subject to conditions (3.39) we have to examine this polynomial. We observe that

$$\text{Discrim}[H', a] = 4(3c - 3f + m^2)(3c + 6f + 4m^2)^2$$

and since according to the conditions (3.39) we must have $3c + 6f + 4m^2 \neq 0$ and $3c - 3f + m^2 \geq 0$ we set a new parameter v as follows: $3c - 3f + m^2 = v^2 \geq 0$. Then we obtain $f = (3c + m^2 - v^2)/3$ and this implies

$$H' = \left[27a + 9cm + 4m^3 + 3(3c + 2m^2)v - 2v^3\right]\left[27a + 9cm + 4m^3 - 3(3c + 2m^2)v + 2v^3\right]/27 = 0.$$

Due to the change $v \to -v$ we may assume that the first factor vanishes and we obtain

$$a = -(m + v)(9c + 4m^2 + 2mv - 2v^2)/27.$$

This leads to the family of systems

$$\begin{aligned} \dot{x} &= (3x - m - v)(9c + 4m^2 + 2mv - 2v^2 + 12mx - 6vx - 18x^2)/27 \\ &\equiv \frac{1}{27}\tilde{L}_1(x)\tilde{L}_{2,3}(x), \\ \dot{y} &= y(3c + m^2 - v^2 + 6mx - 9x^2 - 3y^2)/3. \end{aligned} \tag{3.86}$$

We need to determine if the two lines defined by the equation $\tilde{L}_{2,3} = 0$ are real or complex and in the case when they are real, if one of them coincides with the invariant line $\tilde{L}_1 = 0$ or not. So we calculate

$$\text{Discrim}\,[\tilde{L}_{2,3}, x] = 108(6c + 4m^2 - v^2) \equiv 108\tilde{\lambda}, \quad Res_x(\tilde{L}_1, \tilde{L}_{2,3}) = 27(3c + 2m^2 - 2v^2) \equiv 27\tilde{\mu} \tag{3.87}$$

and clearly the invariant lines $\tilde{L}_{2,3} = 0$ are real (respectively complex; coinciding) if $\tilde{\lambda} > 0$ (respectively $\tilde{\lambda} < 0$; $\tilde{\lambda} = 0$). Moreover the invariant line $\tilde{L}_1 = 0$ coincides with one of the lines $\tilde{L}_{2,3} = 0$ if and only if $\tilde{\mu} = 0$.

On the other hand for systems (3.86) we calculate

$$\zeta_1 = -720\tilde{\lambda}x^2, \quad \zeta_5 = 64\tilde{\lambda}\tilde{\mu}$$

and evidently we have $\text{sign}\,(\zeta_1) = -\text{sign}\,(\tilde{\lambda})$ and in the case $\zeta_1 \neq 0$ the condition $\tilde{\mu} = 0$ is equivalent to $\zeta_5 = 0$.

Proposition 3.37. *Assume that for a system* (3.86) *the condition* $\chi_{11} \neq 0$ *holds. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_1 < 0,\, \zeta_5 < 0 & \Leftrightarrow & \textit{Config. 7.13b;} \\
\zeta_1 < 0,\, \zeta_5 > 0 & \Leftrightarrow & \textit{Config. 7.15b;} \\
\zeta_1 < 0,\, \zeta_5 = 0 & \Leftrightarrow & \textit{Config. 7.17b;} \\
\zeta_1 > 0,\, \zeta_4 \neq 0 & \Leftrightarrow & \textit{Config. 7.18b;} \\
\zeta_1 > 0,\, \zeta_4 = 0 & \Leftrightarrow & \textit{Config. 7.20b;} \\
\zeta_1 = 0,\, \zeta_5 \neq 0 & \Leftrightarrow & \textit{Config. 7.22b;} \\
\zeta_1 = 0,\, \zeta_5 = 0 & \Leftrightarrow & \textit{Config. 7.24b.}
\end{array}$$

Proof. Considering the above proposition we consider three cases: $\zeta_1 < 0$, $\zeta_1 > 0$ and $\zeta_1 = 0$.

a) *The case* $\zeta_1 < 0$. This implies $\tilde{\lambda} > 0$ and we may set $\tilde{\lambda} = 3w^2 > 0$. Then we obtain $c = (v^2 + 3w^2 - 4m^2)/6$ and this leads to the factorization

$$\tilde{L}_{2,3} = -(2m - v - 3w - 6x)(2m - v + 3w - 6x)/2$$

and since the condition $\tilde{\mu} = 0$ implies the coalescence of two invariant lines from the triplet we examine two subcases: $\zeta_5 \neq 0$ and $\zeta_5 = 0$.

a.1) *The subcase* $\zeta_5 \neq 0$. Then $\tilde{\mu} \neq 0$ and for the value of the parameter c given above we calculate: $\tilde{\mu} = 3(w - v)(w + v)/2 \neq 0$ and we can apply to systems (3.86) the transformation

$$x_1 = \frac{2}{(w - v)}\,x - \frac{2(m + v)}{3(w - v)}, \quad y_1 = \frac{2}{(w - v)}\,y, \quad t_1 = t(w - v)^2/4.$$

Then setting an additional parameter $a = (v + w)/(v - w) \neq 0, 1$ (because $\tilde{\mu} \neq 0$ and $a - 1 = 2w/(v - w) \neq 0$), we arrive at the following family of systems (we keep the old notations for the variables):

$$\begin{aligned} \dot{x} &= -2x(x - 1)(x - a), \\ \dot{y} &= y(-2a + 2x + 2ax - 3x^2 - y^2). \end{aligned} \tag{3.88}$$

with $a(a - 1) \neq 0$. It remains to observe that this family of systems is a subfamily of systems (3.56) defined by the conditions $u = -3$ and $s = 0$. The canonical form (3.56) was obtained from (3.51) via an affine transformation and time rescaling in the case $\zeta_1 < 0$ and $\zeta_2 \neq 0$ (which imply $\lambda > 0$ and $\mu \neq 0$, respectively) and therefore all the invariant lines from the triplet are real and distinct.

In the proof of Proposition 3.26 it was shown that systems (3.56) with $s = 0$ and $a(a - 1)(u + 1) \neq 0$ possess the following configurations of invariant lines if and only if the corresponding conditions are satisfied, respectively:

$$\begin{array}{lcl} a(1+u)>0,\, 1+u<0 & \Leftrightarrow & \textit{Config. 7.13b}\,; \\ a(1+u)>0,\, 1+u>0 & \Leftrightarrow & \textit{Config. 7.14b}; \\ a(1+u)<0,\, 1+u<0 & \Leftrightarrow & \textit{Config. 7.15b}\,; \\ a(1+u)<0,\, 1+u>0 & \Leftrightarrow & \textit{Config. 7.16b}. \end{array}$$

Since for the systems (3.88) is a subfamily of systems (3.56) defined by the conditions $u = -3$ and $s = 0$ we have $1 + u = -2 < 0$. Therefore we conclude that systems (3.88) could not possess configurations *Config. 7.14b* and *Config. 7.16b*.

On the other hand for these systems we have

$$\zeta_5 = -1152a(a-1)^2 \quad \Rightarrow \quad \text{sign}\,(\zeta_5) = -\text{sign}\,(a)$$

and hence we arrive at *Config. 7.13b* if $\zeta_5 < 0$ and at *Config. 7.15b* if $\zeta_5 > 0$. So we deduce that in the case $\zeta_1 < 0$ and $\zeta_5 \neq 0$ systems Proposition 3.37 is true.

a.2) *The subcase* $\zeta_5 = 0$. Then $\tilde{\mu} = 0$ and we get $(w-v)(w+v) = 0$. We may assume $w - v = 0$ due to change $w \to -w$. So setting $v = w \neq 0$ we obtain $c = -2(m-w)(m+w)/3$ and therefore systems (3.86) become as systems

$$\begin{aligned} \dot{x} &= 2(m-2w-3x)(m+w-3x)^2/27, \\ \dot{y} &= y\big[-(m-w)(m+w)y/3 + 2mxy - 3x^2y - y^3\big]. \end{aligned}$$

We observe that the above systems via the transformation

$$x_1 = -\frac{1}{w}x + \frac{m+w}{3w}, \quad y_1 = -\frac{1}{w}y, \quad t_1 = tw^2$$

can be brought to the system

$$\dot{x} = -2x^2(x-1), \quad \dot{y} = y(2x - 3x^2 - y^2).$$

This system is contained in the family (3.59) for $u = -3$ and $s = 0$. Since systems (3.59) in the case $s = 0$ possess the unique configuration of invariant line given by *Config. 7.17b* we conclude that Proposition 3.37 is true also in the case $\zeta_1 < 0$ and $\zeta_5 = 0$.

b) *The case* $\zeta_1 > 0$. This implies $\tilde{\lambda} < 0$ and we may set $\lambda = -3w^2 < 0$. So we obtain $c = (v^2 - 3w^2 - 4m^2)/6$ and this leads to the family of systems

$$\begin{aligned} \dot{x} &= (m+v-3x)\big[9w^2 + (-2m+v+6x)^2\big]/54, \\ \dot{y} &= -y(2m^2 + v^2 + 3w^2 - 12mx + 18x^2 + 6y^2)/6, \end{aligned} \tag{3.89}$$

for which we examine two subcases: $v \neq 0$ and $v = 0$. These conditions are governed by the invariant polynomial $\zeta_4 = -v^2(13x^2 + 3y^2)$.

b.1) *The subcase* $\zeta_4 \neq 0$. Then $v \neq 0$ and via the transformation

$$x_1 = -\frac{2}{v}x + \frac{2(m+w)}{3v}, \quad y_1 = -\frac{2}{v}y, \quad t_1 = tv^2/4$$

after the additional setting of the parameter $a = w/v \neq 0$ systems (3.89) can be brought to the systems

$$\dot{x} = -2x\big[(x-1)^2 + a^2\big], \quad \dot{y} = y(-2 - 2a^2 + 4x - 3x^2 - y^2). \tag{3.90}$$

So we get a subfamily of systems (3.61) defined by the conditions $u = -3$ and $s = 0$. We observe that systems (3.61) in the case $s = 0$ possess 2 configurations: *Config. 7.18b* if $u + 1 < 0$

and *Config. 7.19b* if $1+u>0$. However for systems (3.90) we have $1+u=-2<0$ and therefore we obtain the unique configuration *Config. 7.18b*.

b.2) *The subcase* $\zeta_4=0$. Then $v=0$ and since $w\neq 0$ in this case we apply to systems (3.89) the transformation

$$x_1=-\frac{2}{w}\,x-\frac{2m}{3w},\quad y_1=-\frac{2}{w}\,y,\quad t_1=tw^2/4$$

obtaining the following system

$$\dot{x}=\,-2x(1+x^2),\quad \dot{y}=-y(2+3x^2+y^2).$$

which is contained in the family (3.63) for $u=-3$ and $s=0$. Since for this system we have $\mathcal{D}_4=0$, $\zeta_1>0$, $\zeta_4=0$ and $\mathcal{D}_7=-8<0$, according to Proposition 3.26 we deduce that the above system possesses the unique configuration given by *Config. 7.20b*.

c) *The case* $\zeta_1=0$. This implies $\tilde{\lambda}=0$ and considering (3.87) we obtain $c=(v^2-4m^2)/6$ and this leads to the systems

$$\begin{aligned}\dot{x}&=(2m-v-6x)^2(m+v-3x)/54,\\ \dot{y}&=-\,y(2m^2+v^2-12mx+18x^2+6y^2)/6,\end{aligned}$$

for which we calculate $\zeta_4=-v^2(13x^2+3y^2)$.

c.1) *The subcase* $\zeta_4\neq 0$. Then $v\neq 0$ and via the transformation

$$x_1=-\frac{2}{v}\,x+\frac{2(m+v)}{3v},\quad y_1=-\frac{2}{v}\,y,\quad t_1=tv^2/4$$

we arrive at the following system

$$\dot{x}=\,-2(x-1)^2x,\quad \dot{y}=y(-2+4x-3x^2-y^2)$$

which belongs to the family (3.63) for $u=-3$ and $s=0$ already examined. We observe that systems (3.63) in the case $s=0$ possess 2 configurations: *Config. 7.22b* if $u+1<0$ and *Config. 7.23b* if $u+1>0$. However for the above system we have $1+u=-2<0$ and therefore we obtain the unique configuration *Config. 7.22b*.

c.1) *The subcase* $\zeta_4=0$. Then $v=0$ and we get the systems

$$\dot{x}=2(m-3x)^3/27,\quad \dot{y}=-y(m^2-6mx+9x^2+3y^2)/3$$

which via the transformation $x_1=x-m/3$, $y_1=y$, $t_1=t$ will be brought to the homogeneous systems

$$\dot{x}=\,-2x^3,\quad \dot{y}=-y(3x^2+y^2).$$

This system belongs to the family (3.65) for $u=-3$ and $s=0$ already examined and in the case $s=0$ it was determined that we have the unique configuration *Config. 7.24b*.

As all the cases are examined we deduce that Proposition 3.37 is proved. □

3.2.8 The statement (A_8)

We prove the following proposition.

Proposition 3.38. *Assume that for a system (3.12) the conditions provided by the statement (A_8) of the Main Theorem are satisfied. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\mathcal{D}_4 \neq 0,\, \chi_5 \neq 0,\, \zeta_3 < 0 & \Leftrightarrow & \textit{Config. 7.33b;}\\
\mathcal{D}_4 \neq 0,\, \chi_5 \neq 0,\, \zeta_3 > 0 & \Leftrightarrow & \textit{Config. 7.34b;}\\
\mathcal{D}_4 \neq 0,\, \chi_5 = 0 & \Leftrightarrow & \textit{Config. 7.35b;}\\
\mathcal{D}_4 = 0,\, \zeta_2 \neq 0,\, \zeta_5 < 0 & \Leftrightarrow & \textit{Config. 7.36b;}\\
\mathcal{D}_4 = 0,\, \zeta_2 \neq 0,\, \zeta_5 > 0 & \Leftrightarrow & \textit{Config. 7.37b;}\\
\mathcal{D}_4 = 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.38b.}
\end{array}$$

Proof. As it was proved in the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_8) for the family of systems (3.12) lead to the conditions (3.43) in the case $\mathcal{D}_4 \neq 0$ and to the conditions (3.44) in the case $\mathcal{D}_4 = 0$. So we consider two cases: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

1: The case $\mathcal{D}_4 \neq 0$. Then for the family of systems (3.12) the conditions (3.43) are satisfied and we arrive at the systems

$$\begin{aligned}
\dot{x} &= \frac{1}{64s^2}(8sx-3l)(8gsx+3lg+8cs) \equiv \frac{1}{64s^2}L_1^{(1)}L_2^{(1)},\\
\dot{y} &= \frac{l}{256s^2}(9l^2+12lgs+32cs^2+l^2s^2) - \frac{l}{64s}(21l-8gs+ls^2)x + lx^2\\
&\quad + \frac{1}{64s^2}(3l^2s^2-9l^2+24lgs+64cs^2)y - \frac{1}{4s}(ls^2-3l-4gs)xy - sx^3 - x^2y - sxy^2 - y^3.
\end{aligned} \tag{3.91}$$

Next we investigate if the invariant lines $L_1^{(1)} = 0$ and $L_2^{(1)} = 0$ could coincide. So we calculate

$$Res_x(L_1^{(1)}, L_2^{(1)}) = 16s(3lg+4cs) \equiv 16s\mu^{(1)}$$

and since $s \neq 0$ we conclude that these two parallel invariant lines could coincide if and only if $\mu^{(1)} = 0$. We determine that this condition is governed by the invariant polynomial χ_5 because for systems (3.91) we have

$$\chi_5 = -(3lg+4cs)(9+s^2)/18.$$

a) *The case* $\chi_5 \neq 0$. Then $\mu^{(1)} \neq 0$ and due to $gs \neq 0$ via the transformation

$$x_1 = -\frac{4gs}{\mu^{(1)}}x + \frac{3lg}{2\mu^{(1)}}, \quad y_1 = -\frac{4gs}{\mu^{(1)}}y - \frac{lgs}{2\mu^{(1)}}, \quad t_1 = \frac{[\mu^{(1)}]^2}{16g^2s^2}t,$$

after the additional setting of a new parameter $a = -\frac{4g^2s}{\mu^{(1)}}$ we arrive at the systems

$$\dot{x} = ax(x-1), \quad \dot{y} = -ay + axy - sx^3 - x^2y - sxy^2 - y^3 \tag{3.92}$$

for which we have $\chi_5 = 2as(9+s^2)/9 \neq 0$, i.e. $as \neq 0$.

We determine that systems (3.92) possess five distinct invariant affine straight lines

$$L_1 : x = 0, \quad L_2 : x = 1, \quad L_3 : y = -sx, \quad L_{4,5} : y = \pm ix$$

and by Lemma 3.2 the line at infinity is of multiplicity 2. On the other hand these systems possess the following six finite singularities:

$$M_1(0,0),\ M_{2,3}(0, \pm\sqrt{-a}\,),\ M_{4,5}(1, \pm i), \quad M_6(1, -s).$$

We observe that the singular points $M_{2,3}$ could be real (if $a<0$) or complex (if $a>0$), but they could not coincide due to $a\neq 0$. We draw attention to the fact that all these finite singularities are simple, because three finite singular points coalesced with infinite singularities.

Indeed considering Lemma 2.7 for systems (3.92) we calculate

$$\mu_0=\mu_1=\mu_2=0,\quad \mu_3=a^3(sx+y)(x^2+y^2)\neq 0.$$

So by Lemma 2.7 (see statement (i)) considering the factorization of the invariant polynomial μ_3 we deduce that one real finite singular point coalesced with the real infinite singularity $N[1:-s:0]$ which becomes of the multiplicity $(1,1)$ (see Remark 1.4). And simultaneously two complex finite singularities coalesced with the complex infinite singularities located at the intersection of the complex lines $y=\pm ix$ with the line at infinity $Z=0$ (however according to Definition 1.2 of a configuration, we do not consider the complex singularities).

On the other hand all the invariant lines of systems (3.92) are fixed, except for the invariant line $L_3: y=-sx$. Moreover we will determine according to our Convention (see page 8) the position of this line with respect to the complex lines $L_{4,5}: y=\pm ix$. Since $s\neq 0$, according to Remark 3.27 the invariant line $y=-sx$ does not coincide with the projection of the complex invariant lines $y=\pm ix$ on the plane (x,y).

We remark that the singular point $M_1(0,0)$ is a point of intersection of four invariant lines: L_1, L_3, L_4 and L_5 and that in the case $a<0$ the real singular points $M_{2,3}(0,\pm\sqrt{-a}\ )$, located on the invariant line $x=0$, are symmetric with respect to the origin of coordinates. As a result we arrive at the following two distinct configurations of invariant lines for systems (3.92) with $as\neq 0$: *Config. 7.33b* if $a<0$ and *Config. 7.34b* if $a>0$.

On the other hand for systems (3.92) we calculate $\zeta_3 = 2a^3s^2(9+s^2)^2/81$ and hence $\operatorname{sign}(a)=\operatorname{sign}(\zeta_3)$. So we deduce that systems (3.92) possess the configuration *Config. 7.33b* if $\zeta_3<0$ and *Config. 7.34b* if $\zeta_3>0$.

b) *The case* $\chi_5=0$. This implies $\mu^{(1)}=0$ and this means that the invariant line $L_1^{(1)}$ coalesces with $L_2^{(1)}$ and we have a double invariant line in the direction $x=0$. The condition $\mu^{(1)}=0$ yields $3lg+4cs=0$, i.e. $c=-3lg/(4s)$. In this case systems (3.91) can be brought via the transformation

$$x_1=\frac{1}{g}\,x-\frac{3l}{8gs},\quad y_1=\frac{1}{g}\,y+\frac{l}{8g},\quad t_1=g^2\,t,$$

to the family of systems

$$\dot{x}=x^2,\quad \dot{y}=xy-sx^3-x^2y-sxy^2-y^3 \tag{3.93}$$

with $s\neq 0$ (due to $\mathcal{D}_4\neq 0$). We determine that the above systems possess four distinct invariant affine straight lines

$$L_{1,2}: x=0,\quad L_3: y=-sx,\quad L_{4,5}: y=\pm ix.$$

We observe that the line $x=0$ as well as the line at infinity are of multiplicity 2 (see Lemma 3.2). On the other hand these systems possess the unique singularity $M_1(0,0)$ which is of the multiplicity six. Indeed considering Lemma 2.7 for systems (3.93) we calculate

$$\mu_0=\mu_1=\mu_2=0,\quad \mu_3=(sx+y)(x^2+y^2)\neq 0,\quad \mu_4=\mu_5=\mu_6=\mu_7=\mu_8=\mu_9=0.$$

Therefore by Lemma 2.7 (see statement (ii)) the above finite singularity has multiplicity six. On the other hand by the same arguments which we provided for systems (3.92) we deduce that the infinite singularity $N[1:-s:0]$ is of the multiplicity $(1,1)$.

So taking into account the condition $s \neq 0$ and Remark 3.27 as well as the fact that all the invariant affine lines of systems (3.93) intersect at the same singular point $M_1(0,0)$ (of multiplicity 6) we arrive at the unique configuration *Config. 7.35b*.

2: The case $\mathcal{D}_4 = 0$. Then for the family of systems (3.12) the conditions (3.44) are satisfied and we arrive at the systems

$$\begin{aligned} \dot{x} &= \frac{1}{4}(g - 2m + 2x)(2c - g^2 + 2gm + 2gx) \equiv \frac{1}{4} L_1^{(2)} L_2^{(2)}, \\ \dot{y} &= \frac{1}{4}(4c - 3g^2 + 8gm - 4m^2)y + 2mxy - x^2y - y^3. \end{aligned} \tag{3.94}$$

We calculate

$$Res_x(L_1^{(2)}, L_2^{(2)}) = 4(c - g^2 + 2gm) \equiv 4\mu^{(2)}$$

and clearly the parallel invariant lines $L_1^{(1)} = 0$ and $L_2^{(1)} = 0$ could coincide if and only if $\mu^{(2)} = 0$.

On the other hand for systems (3.94) we have $\zeta_2 = 288\mu^{(2)}$ and therefore the condition $\mu^{(2)} = 0$ is equivalent to $\zeta_2 = 0$.

a) *The case* $\zeta_2 \neq 0$. Then since $g \neq 0$ (due to $\widetilde{\chi}_1 = 2gx^2y/3 \neq 0$) via the transformation

$$x_1 = -\frac{g}{\mu^{(2)}}\, x - \frac{g(g-2m)}{2\mu^{(2)}}, \quad y_1 = -\frac{g}{\mu^{(2)}}\, y, \quad t_1 = \frac{\left[\mu^{(2)}\right]^2}{g^2}\, t,$$

after the additional setting of a new parameter $a = -\frac{g^2}{\mu^{(2)}}$ we arrive at the systems

$$\dot{x} = ax(x-1), \quad \dot{y} = -ay + axy - x^2y - y^3.$$

So we get a subfamily of systems (3.92) defined by the condition $s = 0$ and considering the investigation of systems (3.92) and Remark 3.27 we deduce that the above systems possess the configuration *Config. 7.36b* if $a < 0$ and *Config. 7.37b* if $a > 0$.

We observe that in the case $s = 0$ the invariant polynomial ζ_3 vanishes because it contains as a factor s^2. In this case for determining the sign of the parameter a we apply the invariant ζ_5 that for the above systems has the value $\zeta_5 = -144a^3$. Hence we have sign $(a) = -$sign (ζ_5) and consequently we get the configuration *Config. 7.36b* if $\zeta_5 > 0$ and *Config. 7.37b* if $\zeta_5 < 0$.

b) *The case* $\zeta_2 = 0$. Then $\mu^{(2)} = 0$ and this means that the invariant line $L_1^{(2)}$ coalesces with $L_2^{(2)}$ and we have a double invariant line in the direction $x = 0$. The condition $\mu^{(2)} = 0$ yields $c = g(g - 2m)$ and then systems(3.94) via the transformation

$$x_1 = \frac{x}{g} + \frac{g - 2m}{2g}, \quad y_1 = \frac{y}{g}\, y, \quad t_1 = g^2\, t,$$

can be brought to the system

$$\dot{x} = x^2, \quad \dot{y} = xy - x^2y - y^3,$$

which belongs to the family (3.93) defined by the condition $s = 0$. Considering Remark 3.27 we deduce that the above system possesses the configuration *Config. 7.38b*.

Since all the cases are examined we conclude that Proposition 3.38 is proved. □

3.2.9 The statement (A_9)

We prove the following proposition.

Proposition 3.39. *Assume that for a system (3.12) the conditions provided by the statement (A_9) of the Main Theorem are satisfied. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}\mathcal{D}_4 \neq 0,\, \zeta_9 < 0 & \Leftrightarrow & \textit{Config. 7.39b;}\\ \mathcal{D}_4 \neq 0,\, \zeta_9 > 0 & \Leftrightarrow & \textit{Config. 7.40b;}\\ \mathcal{D}_4 = 0,\, \zeta_9 < 0 & \Leftrightarrow & \textit{Config. 7.41b;}\\ \mathcal{D}_4 = 0,\, \zeta_9 > 0 & \Leftrightarrow & \textit{Config. 7.42b;}\end{array}$$

Proof. According to the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_9) for the family of systems (3.12) lead either to the conditions (3.47) in the case $\mathcal{D}_4 \neq 0$ or to the conditions (3.49). So we examine these two cases.

1: The case $\mathcal{D}_4 \neq 0$. Then we have the conditions (3.47) and in this case we arrive at the systems

$$\begin{aligned}\dot{x} &= cx - \frac{3cl}{8s},\\ \dot{y} &= \frac{l}{256s^2}(9l^2 + 32cs^2 + l^2s^2) - \frac{l^2}{64s}(21 + s^2)\,x + \frac{1}{64s^2}(3l^2s^2 - 9l^2 + 64cs^2)\,y\\ &\quad + lx^2 - \frac{l}{4s}(s^2 - 3)\,xy - sx^3 - x^2y - sxy^2 - y^3.\end{aligned}$$

For these systems we have

$$\widetilde{\chi}_2 = 4cx^3(sx + y)(x^2 + y^2)\big[(3s^2 - 1)x^2 + 8sxy + (3 - s^2)y^2\big]/3, \quad \mathcal{D}_4 = 2304s(9 + s^2)$$

and therefore the condition $\widetilde{\chi}_2\mathcal{D}_4 \neq 0$ implies $cs \neq 0$. Then the above systems could be brought via the transformation

$$x_1 = x - \frac{3l}{8s}, \quad y_1 = y + \frac{l}{8}, \quad t_1 = t$$

to the following family of systems

$$\dot{x} = cx, \quad \dot{y} = cy - sx^3 - x^2y - sxy^2 - y^3 \tag{3.95}$$

with $cs \neq 0$. We determine that systems (3.95) possess four distinct invariant affine straight lines

$$L_1 : x = 0, \quad L_2 : y = -sx, \quad L_{3,4} : y = \pm ix.$$

Moreover the line at infinity has multiplicity 3 (see Lemma 3.2, statement (iii)). On the other hand these systems possess the following three singularities:

$$M_1(0,0),\ M_{2,3}(0, \pm\sqrt{c}\,)$$

and the singular points M_2 and M_3 could be real (if $c > 0$) or complex (if $c < 0$). We draw attention to the fact that all these finite singularities are simple, because six finite singularities coalesced with infinite singularities.

Indeed considering Lemma 2.7 for systems (3.95) we calculate

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = -c^3(sx+y)^2(x^2+y^2)^2 \neq 0.$$

So by Lemma 2.7 (see statement (i)) considering the factorization of the invariant polynomial μ_6 we deduce that two real finite singular point coalesced with the real infinite singularity $N_1[1 : -s : 0]$ and this infinite singularity becomes of the multiplicity $(2,1)$ (see Remark 1.4), whereas four complex finite singularities coalesced with complex singularities at infinity. More exactly, two of them with $N[1 : +i : 0]$ and other two with $\bar{N}[1 : -i : 0]$. However according to Definition 1.2 this fact is irrelevant for a configuration.

So taking into account our Convention (see page 8) and the fact that all the invariant affine lines of systems (3.95) intersect at the same singular point $M_1(0,0)$ (of multiplicity 6) we arrive at the following two configurations:

$$\textit{Config. 7.39b} \Leftrightarrow \quad c > 0; \quad \textit{Config. 7.40b} \Leftrightarrow \quad (c < 0).$$

On the other hand for systems (3.95) we calculate

$$\zeta_9 = -2cx^2\left[(3s^2-1)x^2 + 8sxy + (3-s^2)y^2\right]^2/27$$

and hence sign $(\zeta_9) = -$sign (c). Therefore we get *Config. 7.39b* if $\zeta_9 < 0$ and *Config. 7.40b* if $\zeta_9 > 0$.

2: The case $\mathcal{D}_4 = 0$. Then $s = 0$ and in this case the conditions (3.49) hold for systems (3.12). In this case we arrive at the systems

$$\dot{x} = -cm + cx, \quad \dot{y} = (c - m^2)y + 2mxy - x^2y - y^3$$

applying the transformation $(x, y, t) \mapsto (x + m, y, t)$ we arrive at the systems (3.95) with $s = 0$.

Thus considering our Convention (see page 8) and the sign of the invariant polynomial ζ_9 we arrive at the configuration of invariant lines given by *Config. 7.41b* if $\zeta_9 < 0$ and by *Config. 7.42b* if $\zeta_9 > 0$. This completes the proof of Proposition 3.39. □

Since all the cases provided by the statement (A) are examined we conclude that the statement (B) of the Main Theorem is proved completely.

3.3 Geometric invariants and the proof of the statement (C)

In this subsection we complete the proof of the Main Theorem by showing that all 42 configurations of invariant lines we constructed are non-equivalent according to Definition 1.3. For this we define the invariants that split the configurations of this family into the 42 *distinct* ones. We would like these invariants to be among those best suited for describing the geometric phenomena that are specific to this class.

The basic algebraic-geometric definitions of use here are the notion of an integer valued r-cycle and its type i.e. we take $G = Z$ in the Definitions 1.5 and 1.6 and we have:

Definition 3.40. Let V be an irreducible algebraic variety of dimension n over a field K. A cycle of dimension r or r-cycle on V is a formal sum $\sum_W m(W)W$ where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $m(W) \in \mathbb{Z}$, and only a finite number of $m(W)$'s are non-zero. We call degree of an r-cycle the sum $\sum_W$. An $(n-1)$-cycle is called a divisor.

Definition 3.41. We call type of an r-cycle the set of all ordered couples (n_1, n_2) where n_1 is a coefficient, $n_1 = m(W)$ appearing in the $r - cyle$ and n_2 is the number of W's in the cycle whose coefficient is $m(W)$.

We denote the type of an r-cycle C by $\mathcal{T}(C)$. We use the following notations:

$$\mathbf{CS} = \left\{ (S) \,\middle|\, \begin{array}{l} (S) \text{ is a system (2.1) such that } \gcd(P(x,y),Q(x,y)) = 1 \\ \text{and } \max\big(\deg(P(x,y)), \deg(Q(x,y))\big) = 3 \end{array} \right\};$$

$$\mathbf{CSL} = \left\{ (S) \in \mathbf{CS} \,\middle|\, \begin{array}{l} (S) \text{ possesses at least one invariant affine line or} \\ \text{the line at infinity with multiplicity at least two} \end{array} \right\}.$$

Notation 3.42. Let

$$\begin{aligned}
\widetilde{P}(X,Y,Z) &= p_0(\boldsymbol{a})Z^2 + p_1(\boldsymbol{a},X,Y)Z + p_2(\boldsymbol{a},X,Y);\\
\widetilde{Q}(X,Y,Z) &= q_0(\boldsymbol{a})Z^2 + q_1(\boldsymbol{a},X,Y)Z + q_2(\boldsymbol{a},X,Y);\\
\widetilde{C}(X,Y,Z) &= Y\widetilde{P}(X,Y,Z) - X\widetilde{Q}(X,Y,Z);\\
\sigma(p,q) &= \{w \in \mathbb{R}^2)|\; p(w) = q(w) = 0\};\\
\mathbf{D}_S(\widetilde{P},\widetilde{Q}) &= \sum_{w\in\sigma(\widetilde{P},\widetilde{Q})} I_w(\widetilde{P},\widetilde{Q})w;\\
\mathbf{D}_S(\widetilde{C},Z) &= \sum_{w\in\{Z=0\}} I_w(\widetilde{C},Z)w \quad \text{if} \quad Z \nmid \widetilde{C}(X,Y,Z);\\
\mathbf{D}_S(\widetilde{P},\widetilde{Q};Z) &= \sum_{w\in\{Z=0\}} I_w(\widetilde{P},\widetilde{Q})w;\\
\widehat{\mathbf{D}}_S(\widetilde{P},\widetilde{Q},Z) &= \sum_{w\in\{Z=0\}} \Big(I_w(\widetilde{C},Z),\, I_w(\widetilde{P},\widetilde{Q})\Big)w,
\end{aligned}$$

where $I_w(F,G)$ is the intersection number (see [19]) of the curves defined by homogeneous polynomials $F,\ G \in \mathbb{C}[X,Y,Z]$ and $\deg(F), \deg(G) \geq 1$.

The set $\sigma(p,q)$ is thus formed by the finite (or affine) singularities of a polynomial system defined by $p(x,y), q(x,y)$. The multiplicity of a finite singular point w is the number $I_w(p,q)$ which is the intersection number of the affine curves defined by p and q. The total multiplicity of a point at infinity, i.e. located on $Z = 0$ is $I_w(\tilde{P},\tilde{Q})$ and it is the sum $I_w(\widetilde{C},Z) + I_w(\widetilde{P},\widetilde{Q})$ of the two multiplicities appearing in the last divisor above. A complex projective line $uX + vY + wZ = 0$ in $\mathbf{P}_2(\mathbb{C})$ is invariant for a system (S) if it either coincides with $Z = 0$ or it is the projective completion of an invariant affine line $ux + vy + w = 0$.

Notation 3.43. Let $(S) \in \mathbf{CSL}$. Let us denote

$$\mathbf{IL}(S) = \left\{ \; l \;\middle|\; \begin{array}{l} l \text{ is a line in } \mathbf{P}_2(\mathbb{C}) \text{ such} \\ \text{that } l \text{ is invariant for } (S) \end{array} \right\};$$

$$M(l) = \text{the multiplicity of the invariant line } l \text{ of } (S).$$

In defining $M(l)$ we assume, of course, that (S) has a finite number of invariant lines.

Remark 3.44. We note that the line $L_\infty : Z = 0$ is included in $\mathbf{IL}(S)$ for any $(S) \in \mathbf{CSL}$.

Assuming we have a finite number of invariant lines, let $l_i\ :\ f_i(x,y) = ax + by + c = 0$, $i = 1,\ldots,k$, be all the distinct invariant affine lines (real or complex) of a system $(S) \in \mathbf{CSL}$. Let $L_i\ :\ \mathcal{F}_i(X,Y,Z) = aX + bY + cZ = 0$ be the complex projective completion of l_i. Let M_i be the multiplicity of the line L_i and let M be the multiplicity of the line at infinity $Z = 0$.

Notation 3.45.

$$\mathcal{G} \,:\, \prod_i \mathcal{F}_i(X,Y,Z)^{M_i} Z^M = 0; \quad Sing\,\mathcal{G} = \{w \in \mathcal{G} \mid w \,\text{is a singular point of}\ \mathcal{G}\}\,;$$

$$m(w) = \text{the multiplicity of the point}\ w,\ \text{as a point of}\ \mathcal{G}.$$

We call $\mathcal{G}$ *the total curve*.

Suppose that a system (2.1) possesses a finite number of invariant lines $L_1, \ldots L_k$, including the line at infinity. Sometimes it is convenient to consider in our discussion a number of these invariant lines say $L_{i_1}, \ldots L_{i_l}$ of a system (S). We call *marked* system (S) by the lines $L_{i_1}, \ldots L_{i_l}$ the object denoted by $(S, L_{i_1}, \ldots L_{i_l})$ of the system (S) in which we singled out the lines $L_{i_1}, \ldots L_{i_l}$. We shall consider invariants attached to such marked systems.

Because in this paper we are concerned with triplets of parallel lines, *the affine plane* clearly plays an important role. This needs to be reflected in our choice of invariants. We now define an invariant that captures the most basic geometric distinctions of the configurations in this family:

Definition 3.46. Let $\mathcal{M}$ be the ordered couple $(M_{Aff}, M(l_\infty))$, where M_{Aff} is the maximum multiplicity of the invariant affine lines of the system and $M(l_\infty)$ is the multiplicity of the line at infinity. Clearly $\mathcal{M}$ is an invariant.

Using $\mathcal{M}$ we split the 42 configurations in 6 classes: three with $M(l_\infty) = 1$ and three with $M(l_\infty) > 1$.

We describe now the way the invariant $\mathcal{M}$ captures the geometry of the configurations related to the parallel lines by letting $\mathcal{M}$ run through all its six possible values: the generic case and five limiting cases:

$\mathcal{M} = (1,1)$ This is ***the generic case*** with 3 (distinct) parallel lines;

$\mathcal{M} = (2,1)$ is ***a first limit case*** of the preceding one, where two of the three parallel lines coalesced yielding just two parallel lines, one of them double;

$\mathcal{M} = (3,1)$ is ***a second limit case*** where the three parallel lines coalesced yielding a triple line;

$\mathcal{M} = (1,2)$ is ***a third limit case*** where a line of the triplet coalesced with the line at infinity yielding a double line at infinity;

$\mathcal{M} = (1,3)$ is ***a fourth limit case*** where two lines of the triplet disappeared at infinity yielding a triple line at infinity;

$\mathcal{M} = (2,2)$ is ***a fifth limit case*** when one one line of the triplet went to infinity and the other two lines of the triplet coalesced.

It is clear that we also need to define invariants that relate to the real singularities of the systems located on the configurations. We first observe that all the real singularities of the systems are located on the invariant lines of the configurations, occasionally even on a single line.

We encapsulate in two zero-cycles $C^{\mathbb{R}}_{Sing} = \sum_w \nu(w)w$ and $C^{R}_{\mathcal{G}} = \sum_w m(w)w$ the multiplicity properties of the real singularities of the systems located on the configurations. In the first cycle we denoted by $\nu(w)$ the multiplicity of the real singular point w and in the second cycle we denoted by $m(w)$ the multiplicity of the real singular point w this time regarded as a simple or multiple point of the total curve $\mathcal{G}$. We denote their respective types by $\mathcal{T}^{\mathbb{R}}_{Sing}$ and $\mathcal{T}^{\mathbb{R}}_{\mathcal{G}}$. In view of the geometry of the systems we actually only need to consider the restriction of these two invariants on the affine plane and we denote them by $\mathcal{T}^{\mathbb{R},aff}_{Sing}$ and $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}$. If anyone of these two invariants, say $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}$ yields the same value for two or more configurations, to be

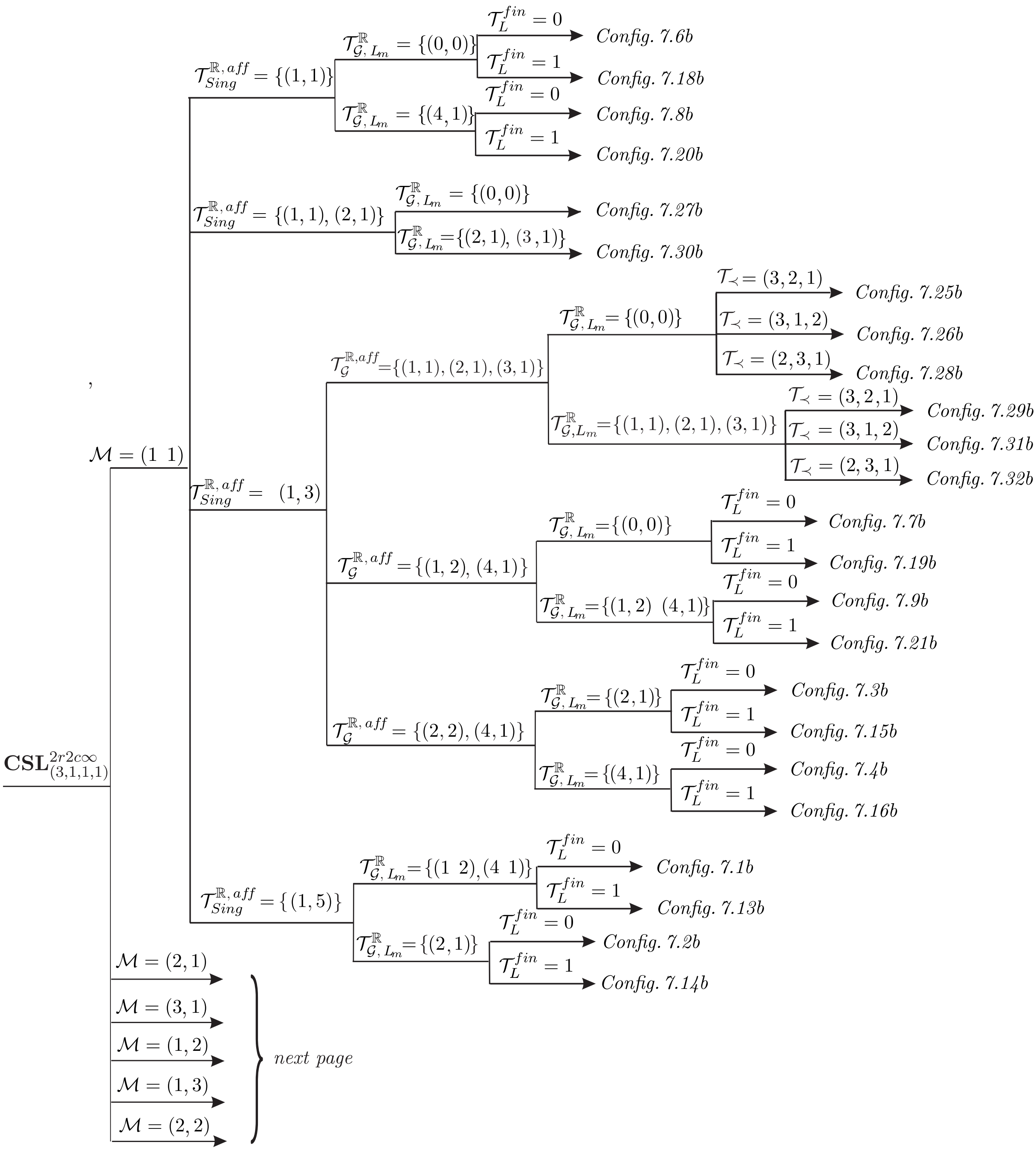

Figure 3.1: Diagram of non-equivalent configurations

able to distinguish we shall need to restrict its value to a single affine line L and in this case the resulting invariant will be denoted by $\mathcal{T}^{\mathbb{R}}_{\mathcal{G},L}$.

Assume that for a marked system $(S, L_r, L_c, \bar{L}_c)$ with a real invariant line L_r and a complex invariant line L_c together with its conjugate line $\bar{L}_c$ these three invariant lines intersect at the same real point which could be finite or infinite.

Considering our Convention (see page 8) we define an invariant $\mathcal{T}^{fin}_{L}$ for such marked systems $(S, L_r, L_c, \bar{L}_c)$ in the case when the intersection point is finite:

$\mathcal{T}^{fin}_{L} = 1$ if and only if the real invariant line L_r coincides with the line $\mathcal{R}(L_c, \bar{L}_c) : y = ax + c$

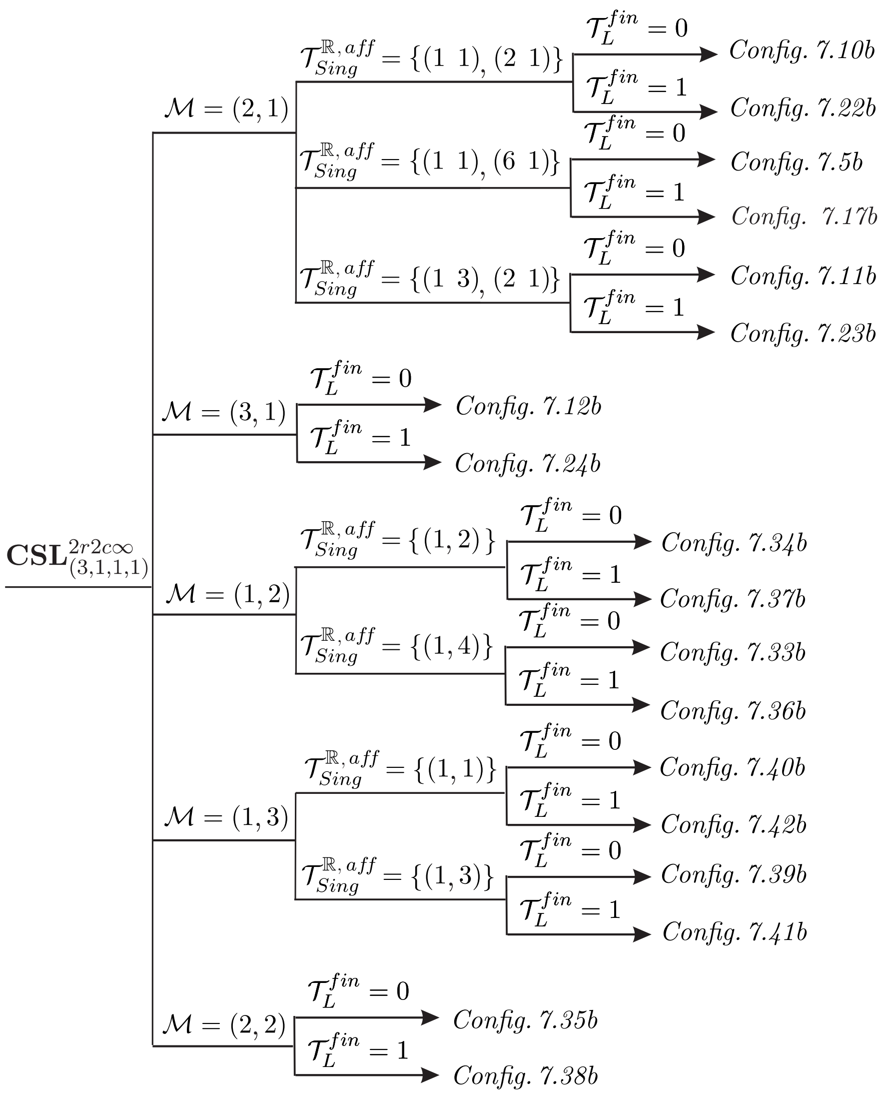

Figure 3.1 (*cont.*): Diagram of non-equivalent configurations

defined in our Convention on page 6;

$\mathcal{T}^{fin}_{L}=0$ if and only if the the real invariant line L_r does not coincide with $\mathcal{R}(L_c,\bar{L}_c)$.

Let us now consider the generic case $\mathcal{M}=(1,1)$ which is the more complex one. This class contains 30 configurations i.e. all *Config. 7.jb* with $j\leq 32$ with two exceptions: *Config. 7.12b* and *Config. 7.22b*. To distinguish the corresponding configurations the first one of the invariants we use is $\mathcal{T}^{\mathbb{R},aff}_{Sing}$ and its values for this class are: $\mathcal{T}^{\mathbb{R},aff}_{Sing}$: $\{(1,1)\},\{(1,1),(2,1)\},\{(1,3)\},\ \{(1,5)\}$. For the second case we then only need to apply $\mathcal{T}^{\mathbb{R}}_{\mathcal{G},L_m}$ while for the first and last case to distinguish further the configurations we need to apply first $\mathcal{T}^{\mathbb{R}}_{\mathcal{G},L_m}$, where L_m is the middle line in the triplet of parallel lines and secondly the invariant $\mathcal{T}^{fin}_{L}$. In the third case, i.e. $\mathcal{T}^{\mathbb{R},aff}_{Sing}=\{(1,3)\}$ we first use $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}$ which has three values and for two of them $\mathcal{T}^{\mathbb{R}}_{\mathcal{G},L_m}$ together with $\mathcal{T}^{fin}_{L}$ distinguish the configurations. For the value $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}=\{(1,1),(2,1),(3,1)\}$ we need a new invariant which we denote by $\mathcal{T}_{\prec}$ and define as follows:

We first observe that for all six configurations occurring for $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}=\{(1,1),(2,1),(3,1)\}$ all real affine singularities are located on a single real affine line and they are three in number determining a closed interval on this line. Based on this observation we introduce this new invariant. We consider these three real singular points and their associated multiplicities as simple or multiple points of the curve $\mathcal{G}$. We first note that the maximum multiplicity of the three points in all six cases is either 3 or 4 and this maximum multiplicity corresponds to a uniquely determined point. We then list the multiplicities $m(w)$ in an ordered sequence in the following way. If we have an end point of the segment determined by the three points which is of maximum multiplicity, we initiate the sequence with its multiplicity and we folow with the multiplicity of the middle point and end with the multiplicity of the other end point of the segment. If none of the end points has maximum multiplicity then we start with the multiplicity of the end point of maximum multiplicity among the two and follow with the multiplicity of the middle point and finally with the multiplicity of the other end point. In case the two end points have equal multiplicity we start with the common multiplicity followed by the multiplicity of the middle point and end with the common multiplicity of the end points. This order is clearly preserved as the multiplicities are preserved. So this is an invariant which we denote by $\mathcal{T}_{\prec}$. The case $\mathcal{T}_{\mathcal{G}}^{\mathbb{R},aff} = \{(1,1),(2,1),(3,1)\}$ is the only one where this invariant occurs. For the remaining values of $\mathcal{M}$ to distinguish the configurations the two invariants $\mathcal{T}_{Sing}^{\mathbb{R},aff}$ and $\mathcal{T}_{L}^{fin}$ do the job as we see in the bifurcation diagram for the configurations which gives all the explicit calculations of the invariants (see Figure 5).

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