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Caffarelli–Kohn–Nirenberg inequality for biharmonic equations with inhomogeneous term and Rellich potential

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Abstract. In this article, multiplicity of nontrivial solutions for an inhomogeneous singular biharmonic equation with Rellich potential are studied. Firstly, a negative energy solution of the studied equations is achieved via the Ekeland's variational principle and Caffarelli–Kohn–Nirenberg inequality. Then by applying Mountain pass theorem lack of Palais–Smale conditions, the second solution with positive energy is also obtained.

Keywords: singular biharmonic equations, nontrivial solution, inhomogeneous, Caffarelli–Kohn–Nirenberg inequality.

Mathematics Subject Classification 35J35, 35J62, 35J75, 35D30.

1 Introduction

We investigate multiplicity of solutions for the following singular biharmonic equations with inhomogeneous terms

$$\begin{cases} \Delta^2 u - \mu \frac{u}{|x|^4} = \frac{|u|^{p_\alpha - 2}u}{|x|^\alpha} + \lambda f(x), & \text{in } \mathbb{R}^N, \\ u \in H_0^2(\mathbb{R}^N), \quad u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\Delta^2 u = \Delta(\Delta u)$, $N \geq 5$, $0 < \mu < \overline{\mu} := \frac{N^2(N-4)^2}{16}$, $p_\alpha = \frac{2(N-\alpha)}{N-4}$, $0 \leqslant \alpha < 4$, $f(x) \in H_0^{-2}(\mathbb{R}^N)$ is a given function and $f(x) \not\equiv 0$, $H_0^{-2}(\mathbb{R}^N)$ denotes the dual of $H_0^2(\mathbb{R}^N)$, the singular term $\frac{u}{|x|^4}$ comes from models in physics.

In the past decades, nonlinear elliptic equations involving biharmonic operator have received much attention due to their wide application to mechanical and physical models such as clamped plates, thin-elastic plates, and in the research of the Paneitz–Branson equation and the Willmore equation (see [11]). Under the framework of nonlinear function analysis, there are many results on qualitative properties, the existence and multiplicity of solutions for biharmonic equations with singular potential (see [1,7,9,12,14–16,19–22,25,26], and the references

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therein). At the beginning, Brezis and Nirenberg [4] considered the following problems:

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{N+2}{N-2}}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, and let

$$S_\lambda = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \int_\Omega |u|^2 dx}{\int_\Omega |u|^{2^*} dx}, \qquad \lambda \in \mathbb{R},$$

where $2^* = \frac{2N}{N-2}$ as Sobolev critical exponent. They basically proved that S_λ is reachable when N and λ satisfy different conditions. Since the seminal work of Brezis and Nirenberg, the study of critical growth in semilinear and quasilinear problem have gradually become a hot subject. On the basis of (1.2), Jannelli [13] studied the following semilinear elliptic equations involving the Hardy terms and critical exponents, and obtained at least a nontrivial solution when $N \geq 3$ and

$$\lambda < \lambda_1(\mu) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega (|\nabla u|^2 dx - \mu \frac{u^2}{|x|^2}) dx}{\int_\Omega |u|^2 dx}.$$

Furthermore, Wang and Zhou [24] considered the problem of [13] with $u^{2^*-1} + \lambda u$ being replaced by $\frac{u^{2*(s)-2}}{|x|^s} + h(x)$, where $N \geq 3$, $0 \leq \mu < \frac{(N-2)^2}{4}, 2^*(s) = \frac{2(N-s)}{N-2}, 0 \leq s < 2$, $h(x) \geq 0$. By using the upper and lower solution method and Mountain pass theorem, they proved the given problem has at least two nontrivial solutions.

Tarantello [23] studied the following semilinear elliptic equations involving inhomogeneous perturbation and critical exponential terms:

$$\begin{cases} -\Delta u = u^{2^*-2} u + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

When $\|f\|$ is appropriately small, the author proved that problem (1.3) admits at least two solutions by applying the Mountain pass theorem and the Ekeland's variational principle.

By applying similar methods as in Ref. [23], Deng and Wang [8] studied the following nonlinear biharmonic problems with inhomogeneous perturbation terms and critical exponential terms:

$$\begin{cases} \Delta^2 u - \lambda u = |u|^{2_*-2} u + f(x), & \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \Omega}|_{\partial\Omega} = 0, \end{cases} \tag{1.4}$$

where $N \geq 5$ and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $2_* = \frac{2N}{N-4}$. They proved that problem (1.4) has at least two solutions when $\|f\|$ is appropriately small. Furthermore, they dealt with the non-existence of solutions for the above studied equation under some assumptions on the perturbation term f.

By using the strong Maximum principle and the Comparison principle, Ref. [17] discussed the existence and nonexistence results of the following semilinear biharmonic problems with the optimal exponent p:

$$\begin{cases} \Delta^2 u - \mu \frac{u}{|x|^4} = \lambda f(x) + u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = -\Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p > 1, \mu > 0, \lambda > 0$ and $\Omega \subset \mathbb{R}^N(N > 4)$ is a smooth bounded domain and $0 \in \Omega$.

Mousomi Bhakta [2] considered the following elliptic problem with singular terms:

$$\begin{cases} \Delta^2 u - \mu \frac{u}{|x|^4} = \frac{|u|^{p_\alpha - 2}u}{|x|^\alpha}, & \text{in } \Omega, \\ u \in H_0^2(\Omega), \quad u > 0, & \text{in } \Omega, \end{cases} \tag{1.5}$$

when Ω is an open subset of $\mathbb{R}^N$ $(N \geq 5)$, some nonexistence of solutions results are obtained by applying Pohozaev identity and Nehari manifold, In addition, they further discussed the existence of positive solutions when $\alpha = 0$.

Through the analysis of the above mentioned studies, **a quite natural question to ask is whether the inhomogeneous biharmonic problem (1.1) possesses multiple nontrivial solution in $\mathbb{R}^N$?** As far as we know, when $\alpha \neq 0$ and $\Omega \neq \mathbb{R}^N$ in (1.5), the problem (1.5) does not have a solution. Thanks to lack of compactness of the functional energy, the author obtain that the non-existence result of solution in a bounded domain. Therefore, we consider adding a perturbation term to overcome this difficulty and prove that the energy function I of problem (1.1) admits at least two critical points. One is a negative energy solution obtained by using Ekeland's variational method in [10], and other is a positive energy solution achieved by applying Mountain pass theorem in [1] without Palais–Smale (PS) conditions. The main result of this paper is the following theorem.

Theorem 1.1. *Assume that $N \geq 5, 0 < \mu < \overline{\mu}, 0 \leq \alpha < 4, p_\alpha = \frac{2(N-\alpha)}{N-4}$,and $f(x) \in H_0^{-2}(\mathbb{R}^N)$ with $f(x) \not\equiv 0$. Then there exists a constant $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the problem (1.1) admits at least two nontrivial solutions which one is of negative energy and the other solution with positive energy, if*

$$\|f\|_{H_0^{-2}(\mathbb{R}^N)} < \frac{p_\alpha - 2}{2\lambda(p_\alpha - 1)} \left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2(p_\alpha - 1)} \right)^{\frac{1}{p_\alpha - 2}},$$

where S_μ will be given in (2.3).

2 Preliminaries

This section will mainly give some preparation to the proof of Theorem 1.1.

Due to the fact that the space $H_0^2(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ in regard to the following norm

$$\|u\|_{H_0^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx \right)^{1/2}.$$

Note that $\mu < \bar{\mu}$ and by the following Rellich inequality [18]

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq \bar{\mu} \int_{\mathbb{R}^N} \frac{u^2}{|x|^4} dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{2.1}$$

where $\bar{\mu} = \frac{N^2(N-4)^2}{16}$ is optimal, then we can show that the norm

$$\|u\|_\mu = \left(\int_{\mathbb{R}^N} |\Delta u|^2 - \mu \frac{u^2}{|x|^4} dx \right)^{1/2}$$

is an equivalent norm to $\|u\|_{H_0^2(\mathbb{R}^N)}$.

From [3], we have the following Caffarelli–Kohn–Nirenberg (CKN) inequality

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \leq C(N,\alpha) \left(\int_{\mathbb{R}^N} \frac{|u|^{p_\alpha}}{|x|^\alpha} dx \right)^{2/p_\alpha}, \qquad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{2.2}$$

where the constant $C(N,\alpha) > 0$. For each μ with $0 < \mu < \overline{\mu}$, the best Sobolev constant S_μ can be given by

$$S_\mu = \inf_{u \in H_0^2(\mathbb{R}^N), u(x) \neq 0} \frac{\int_{\mathbb{R}^N} (|\Delta u|^2 - \mu \frac{u^2}{|x|^4}) dx}{(\int_{\mathbb{R}^N} |x|^{-\alpha} |u|^{p_\alpha} dx)^{\frac{2}{p_\alpha}}}, \tag{2.3}$$

where S_μ is achieved in $\mathbb{R}^N$. By applying (2.1) and (2.2), we know $S_\mu > 0$.

To obtain our results, the energy function I of problem (1.1) can be defined by

$$I(u) = \frac{1}{2}\|u\|_\mu^2 - \frac{1}{p_\alpha} \int_{\mathbb{R}^N} \frac{|u|^{p_\alpha}}{|x|^\alpha} dx - \lambda \int_{\mathbb{R}^N} f(x) u dx, \qquad u \in H_0^2(\mathbb{R}^N). \tag{2.4}$$

According to $f \in H_0^{-2}(\mathbb{R}^N)$ and (2.1)–(2.2), it is easy to obtain that the energy function $I(u)$ is a well defined C^1 function in $H_0^2(\mathbb{R}^N)$.

A function $u \in H_0^2(\mathbb{R}^N)$ is said to be a weak solution of the equations (1.1) if u satisfies

$$\int_{\mathbb{R}^N} \left(\Delta u \Delta v - \mu \frac{uv}{|x|^4} \right) dx = \int_{\mathbb{R}^N} \frac{|u|^{p_\alpha - 2} uv}{|x|^\alpha} dx + \lambda \int_{\mathbb{R}^N} f(x) uv dx \tag{2.5}$$

for any $v \in H_0^2(\mathbb{R}^N)$.

Definition 2.1. A sequence $\{u_n\}_{n=1}^\infty \subset H_0^2(\mathbb{R}^N)$ satisfy $I(u_n) \to c$ $(c \in \mathbb{R})$ and $I'(u_n) \to 0$ in $H_0^{-2}(\mathbb{R}^N)$ as $n \to \infty$. Then the sequence $\{u_n\}_{n=1}^\infty$ is called a $(PS)_c$ sequence.

Lemma 2.1. *Assume that the sequence $\{u_n\}_{n=1}^\infty$ in $H_0^2(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the energy function I of problem (1.1) at level $c \in \mathbb{R}$. Then $u_n \rightharpoonup u$ in $H_0^2(\mathbb{R}^N)$ and $I'(u) = 0$.*

Proof. For n sufficiently large, there hold

$$\frac{1}{2}\|u_n\|_\mu^2 - \frac{1}{p_\alpha} \int_{\mathbb{R}^N} \frac{|u_n|^{p_\alpha}}{|x|^\alpha} dx - \lambda \int_{\mathbb{R}^N} f(x) u_n dx = c + o_n(1),$$

and

$$\|u_n\|_\mu^2 - \int_{\mathbb{R}^N} \frac{|u_n|^{p_\alpha}}{|x|^\alpha} dx - \lambda \int_{\mathbb{R}^N} f(x) u_n dx = o_n(1),$$

where $o_n(1)$ means that for $n \to \infty$, $o_n(1) \to 0$. Thus, there holds

$$\begin{aligned} c + o_n(1) &= I(u_n) - \frac{1}{p_\alpha} \langle I'(u_n), u_n \rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{p_\alpha} \right) \|u_n\|_\mu^2 - \lambda \left(1 - \frac{1}{p_\alpha} \right) \|f\|_{H_0^{-2}(\mathbb{R}^N)} \|u_n\|_\mu, \end{aligned} \tag{2.6}$$

which means that $\{u_n\}_{n=1}^\infty$ is a bounded sequence in $H_0^2(\mathbb{R}^N)$. Up to a subsequence if necessary, there holds

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H_0^2(\mathbb{R}^N), \\ u_n \rightharpoonup u, & \text{in } L_{p_\alpha}(\mathbb{R}^N, |x|^{-\alpha}), \\ u_n \to u, & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{2.7}$$

Thus, it is easy to obtain that

$$\int_{\mathbb{R}^N} (\Delta u \Delta v - \mu \frac{uv}{|x|^4} - \frac{|u|^{p_\alpha - 2} uv}{|x|^\alpha} - \lambda f(x) uv) dx = 0$$

for all $v \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, which means $I'(u) = 0$. □

Lemma 2.2. *For some $c \in \mathbb{R}$, let $\{u_n\}_{n=1}^\infty$ in $H_0^2(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the energy functional I, that is to say $I(u_n) \to c, I'(u_n) \to 0$ in $H_0^{-2}(\mathbb{R}^N)$ as $n \to \infty$. Then there is a $u_0 \in H_0^2(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_0$ in $H_0^2(\mathbb{R}^N)$ holds and*

$$\textit{either} \quad u_n \to u_0 \quad \textit{or} \quad c \geq I(u_0) + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) S_\mu^{\frac{p_\alpha}{p_\alpha - 2}}.$$

Proof. It follows from (2.6) that $\{u_n\}_{n=1}^\infty$ is bounded in $H_0^2(\mathbb{R}^N)$. Due to boundedness of $\{u_n\}_{n=1}^\infty$, we know that the sequence $\{u_n\}_{n=1}^\infty$ possesses a weak convergent subsequence, still denoted by $\{u_n\}_{n=1}^\infty$, then we can get that $u_n \rightharpoonup u_0$ in $H_0^2(\mathbb{R}^N)$, and $u_n \to u_0$ a.e. in $\mathbb{R}^N$, as $n \to \infty$. Denote $w_n = u_n - u_0$, then we have $w_n \rightharpoonup 0$, as $n \to +\infty$.

On the basis of Brezis–Lieb Lemma (see [5]), we could obtain that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \left(\frac{|u_n|^{p_\alpha}}{|x|^\alpha} - \frac{|u_n - u_0|^{p_\alpha}}{|x|^\alpha} \right) dx = \int_{\mathbb{R}^N} \frac{|u_0|^{p_\alpha}}{|x|^\alpha} dx.$$

Therefore, there holds

$$I(u_n) - I(u_0) = \frac{1}{2} \|w_n\|_\mu^2 - \frac{1}{p_\alpha} \int_{\mathbb{R}^N} \frac{|w_n|^{p_\alpha}}{|x|^\alpha} dx + o_n(1). \tag{2.8}$$

And It follows from Lemma 2.1 that $I'(u_0) = 0$, combining with (2.8) we can infer that

$$\langle I'(u_n), u_n \rangle = \langle I'(u_n) - I'(u_0), w_n + u_0 \rangle = \|w_n\|_\mu^2 - \int_{\mathbb{R}^N} \frac{|w_n|^{p_\alpha}}{|x|^\alpha} dx + o_n(1).$$

In this situation, we may assume that

$$\lim_{n \to \infty} \|w_n\|_\mu^2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|w_n|^{p_\alpha}}{|x|^\alpha} dx = \xi \geq 0.$$

Suppose $\xi > 0$, together with the definition of S_μ, we have $\xi \geq S_\mu^{\frac{p_\alpha}{p_\alpha - 2}}$. Furthermore, by (2.8), we obtain

$$c = I(u_0) + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) \xi \geq I(u_0) + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) S_\mu^{\frac{p_\alpha}{p_\alpha - 2}}.$$

This ends the proof of Lemma 2.2. □

3 Proof of Theorem 1.1

In this section, we first take advantage of some analytical skills and functional idea to prove that the functional I can admit a local minimizer, which is a nontrivial negative energy solution. After that we show the existence of a nontrivial solution with positive energy via using Mountain pass theorem without (PS) condition.

Lemma 3.1. *Suppose that* $0 < \mu < \overline{\mu} = \frac{N^2(N-4)^2}{16})$, $N \geq 5$, $0 \leq \alpha < 4$. *Then there exist constants* $\Lambda_1, \eta_0, \xi > 0$ *such that for every* $\lambda \in (0, \Lambda_1)$, *there holds*

$$I(u) \geq \xi > 0 \quad \text{for } \|u\|_\mu = \eta_0. \tag{3.1}$$

Proof. From (2.4), Young inequality, and the definition of S_μ, we get

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_\mu^2 - \frac{1}{p_\alpha}\int_{\mathbb{R}^N} \frac{|u|^{p_\alpha}}{|x|^\alpha}dx - \lambda \int_{\mathbb{R}^N} f(x)udx \\ &\geq \frac{1}{2}\|u\|_\mu^2 - \frac{1}{p_\alpha} S_\mu^{-\frac{p_\alpha}{2}} \|u\|_\mu^{p_\alpha} - \lambda \|f\|_{H_0^{-2}(\mathbb{R}^N)}\|u\|_\mu \\ &= \|u\|_\mu \left(\frac{1}{2}\|u\|_\mu - \frac{1}{p_\alpha} S_\mu^{-\frac{p_\alpha}{2}} \|u\|_\mu^{p_\alpha - 1} - \lambda \|f\|_{H_0^{-2}(\mathbb{R}^N)} \right). \end{aligned} \tag{3.2}$$

Set

$$h(z) = \frac{1}{2}z - \frac{1}{p_\alpha} S_\mu^{-\frac{p_\alpha}{2}} z^{p_\alpha - 1}, \quad z \geq 0.$$

Then from $h'(z_0) = 0$, there holds

$$z_0 = \left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2} \right)^{\frac{1}{p_\alpha - 2}},$$

which indicates that

$$\begin{aligned} h(z_0) &= \frac{1}{2}\left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2} \right)^{\frac{1}{p_\alpha - 2}} - \frac{1}{p_\alpha} S_\mu^{-\frac{p_\alpha}{2}} \left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2} \right)^{\frac{p_\alpha - 2 + 1}{p_\alpha - 2}} \\ &= \frac{p_\alpha - 2}{2p_\alpha - 2}\left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2} \right)^{\frac{1}{p_\alpha - 2}} > 0. \end{aligned}$$

In order to obtain $h(z_0) > \lambda \|f\|_{H_0^{-2}(\mathbb{R}^N)}$, we could choose

$$\Lambda_1 := \frac{p_\alpha - 2}{2p_\alpha - 2}\left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2} \right)^{\frac{1}{p_\alpha - 2}} / \|f\|_{H_0^{-2}(\mathbb{R}^N)}. \tag{3.3}$$

Due to $0 \leq \alpha < 4$, then $p_\alpha > 2$. Choosing $\eta_0 = z_0$ and $\xi = z_0(h(z_0) - \lambda\|f\|_{H_0^{-2}(\mathbb{R}^N)})$, it follows from (3.2) that there exists $\Lambda_1 > 0$ (be given in (3.3)) such that

$$I(u) \geq \xi > 0 \quad \text{for any } \|u\|_\mu = \eta_0, \text{ and } \lambda \in (0, \Lambda_1),$$

and the conclusion is achieved.

We now show that there exists a nontrivial solution with negative solution.

On account of the continuity of f on $\mathbb{R}^N$ and combining with $f \not\equiv 0$, we can choose $\phi \in C_0(\mathbb{R}^N \setminus \{0\})$ such that $\int_{\mathbb{R}^N} f(x)\phi dx > 0$. Then for $t > 0$ sufficiently small with $\|t\phi\|_\mu < \eta_0$, there holds

$$I(t\phi) = \frac{t^2}{2}\|\phi\|_\mu^2 - \frac{t^{p_\alpha}}{p_\alpha}\int_{\mathbb{R}^N} \frac{|\phi|^{p_\alpha}}{|x|^\alpha}dx - \lambda t \int_{\mathbb{R}^N} f(x)\phi dx < 0.$$

Therefore, we have

$$c_1 = \inf\{I(u) : u \in \overline{B}_{\eta_0}\} < 0, \quad \text{where} \quad \overline{B}_{\eta_0} = \{u \in H_0^2(\mathbb{R}^N), \|u\|_\mu < \eta_0\}.$$

According to the complete metric space $\overline{B}_{\eta_0}$ with respect to the norm of $H_0^2(\mathbb{R}^N)$, then applying the Ekeland's variational principle to $I(u)$ in $\overline{B}_{\eta_0}$ yields that there exist a $(PS)_{c_1}$ sequence $\{u_n\}_{n=1}^\infty$ in $\overline{B}_{\eta_0}$ and a $u_* \in H_0^2(\mathbb{R}^N)$ with $\|u_*\|_\mu < \eta_0$, such that $u_n \rightharpoonup u_*$.

We now turn to show that $u_n \to u_*$ in $H_0^2(\mathbb{R}^N)$ as $n \to \infty$. Otherwise, it follows from Lemma 2.2 that

$$c_1 \geq I(u_*) + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} \geq c_1 + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) S^{\frac{p_\alpha}{p_\alpha - 2}} > c_1,$$

which is a contradiction.

Then the above proof yields that u_* is a critical point of the functional I satisfying $c_1 = I(u_*) < 0$. Furthermore, it follows from (2.3) and (3.2) that

$$\begin{aligned}
c_1 &= \frac{p_\alpha - 2}{2p_\alpha}\|u_*\|_\mu^2 - \frac{p_\alpha - 1}{p_\alpha}\int_{\mathbb{R}^N} \lambda f(x) u_*(x) dx \\
&\geq \frac{p_\alpha - 2}{2p_\alpha}\|u_*\|_\mu^2 - \frac{\lambda(p_\alpha - 1)}{p_\alpha}\|f\|_{H_0^{-2}(\mathbb{R}^N)}\|u_*\|_\mu \\
&\geq \frac{(p_\alpha - 2)(p_\alpha - 1)^2\|\lambda f\|^2_{H_0^{-2}(\mathbb{R}^N)}}{2p_\alpha(p_\alpha - 2)^2} - \frac{\lambda(p_\alpha - 1)}{p_\alpha}\|f\|_{H_0^{-2}(\mathbb{R}^N)}\frac{p_\alpha - 1}{p_\alpha - 2}\|\lambda f\|_{H_0^{-2}(\mathbb{R}^N)} \\
&= -\frac{(p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)}\lambda^2\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}.
\end{aligned}$$

Thus, we can deduce that the problem (1.1) possesses a nontrivial solution u_* with negative energy. □

Lemma 3.2. *Let constant $\Lambda_2 > 0$ such that*

$$(p_\alpha - 2)^2 S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \lambda^2(p_\alpha - 1)^2\|f\|^2_{H_0^{-2}(\mathbb{R}^N)} > 0, \quad \text{for any } \lambda \in (0, \Lambda_2). \tag{3.4}$$

Then there are a $\widetilde{u}(x) \in H_0^2(\mathbb{R}^N)$ and constant Λ_3 with $0 < \Lambda_3 \leq \Lambda_2$ such that

$$\sup_{t \geq 0} I(t\widetilde{u}) < \frac{p_\alpha - 2}{2p_\alpha} S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \frac{\lambda^2(p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}, \quad \text{for all } \lambda \in (0, \Lambda_3). \tag{3.5}$$

Proof. From Theorem 2.1 of [2], we know that there is a nontrivial nonnegative solution as $\lambda = 0$ for problem(1.1), and then denote it as $\widetilde{z}(x)$. Next, we may choose $\widetilde{u}(x) = \widetilde{z}(x)$ if the function $f(x) \geq 0$ for each $x \in \mathbb{R}^N$, $\widetilde{u}(x) = -\widetilde{z}(x)$ if the function $f(x) \leq 0$ for each $x \in \mathbb{R}^N$, $\widetilde{u}(x) = \widetilde{z}(x - x_0)$ if there exists a point $x_0 \in \mathbb{R}^N$ satisfying $f(x_0) > 0$. We now claim that there holds

$$\int_{\mathbb{R}^N} f(x)\widetilde{u}(x) dx > 0. \tag{3.6}$$

Indeed, the inequality (3.6) holds obviously if the function $f(x) \geq 0$ or $f(x) \leq 0$ for each $x \in \mathbb{R}^N$. Now if there is a point $x_0 \in \mathbb{R}^N$ satisfying $f(x_0) > 0$, then by the continuity of the function f, we can deduce that there exists an open neighborhood $B(x_0, \tau) \subset \mathbb{R}^N$ of $x_0, \tau > 0$, such that the function $f(x) > 0$ for all $x \in B(x_0, \tau)$. Therefore, one can deduce from the definition of $\widetilde{z}(x - x_0)$, that

$$\int_{\mathbb{R}^N} f(x)\widetilde{z}(x - x_0) dx > 0.$$

To prove the inequality (3.5), we discuss the functions g and $\widetilde{g}$ defined by

$$g(t) := I(t\widetilde{u}) = \frac{t^2}{2}\|\widetilde{u}\|_\mu^2 - \frac{t^{p_\alpha}}{p_\alpha}\int_{\mathbb{R}^N}\frac{|\widetilde{u}|^{p_\alpha}}{|x|^\alpha}dx - \lambda t\int_{\mathbb{R}^N} f(x)\widetilde{u}dx,\ t \geq 0,$$

and

$$\widetilde{g}(t) := \frac{t^2}{2}\|\widetilde{u}\|_\mu^2 - \frac{t^{p_\alpha}}{p_\alpha}\int_{\mathbb{R}^N}\frac{|\widetilde{u}|^{p_\alpha}}{|x|^\alpha}dx,\ t \geq 0.$$

Obviously, there holds

$$g(0) = 0 < \frac{p_\alpha - 2}{2p_\alpha}S_\mu^{\frac{p_\alpha}{p_\alpha-2}} - \frac{\lambda^2(p_\alpha-1)^2}{2p_\alpha(p_\alpha-2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}$$

for every $\lambda \in (0, \Lambda_2)$. Thus from the continuity of function g, there exists some $t_1 > 0$ sufficiently small, such that

$$\frac{p_\alpha - 2}{2p_\alpha}S_\mu^{\frac{p_\alpha}{p_\alpha-2}} - \frac{\lambda^2(p_\alpha-1)^2}{2p_\alpha(p_\alpha-2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)} > g(t)$$

for all $t \in (0, t_1)$.

For another thing, by the definition of $\widetilde{g}$ there holds

$$\max_{t\geq 0}\widetilde{g}(t) = \frac{p_\alpha - 2}{2p_\alpha}S_\mu^{\frac{p_\alpha}{p_\alpha-2}}.$$

This together with the definition of g, we have

$$\sup_{t\geq 0} I(t\widetilde{u}) < \left(\frac{1}{2} - \frac{1}{p_\alpha}\right)S_\mu^{\frac{p_\alpha}{p_\alpha-2}} - \lambda t_1\int_{\mathbb{R}^N} f(x)\widetilde{u}dx.$$

Choose $\lambda > 0$ satisfying that

$$\lambda t_1\int_{\mathbb{R}^N} f(x)\widetilde{u}dx > \frac{\lambda^2(p_\alpha-1)^2}{2p_\alpha(p_\alpha-2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}.$$

Then from (3.6), one has

$$0 < \lambda < \frac{2p_\alpha(p_\alpha-2)t_1\int_{\mathbb{R}^N} f(x)\widetilde{u}dx}{(p_\alpha-1)^2\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}}.$$

Set

$$\Lambda_3 := \min\left\{\frac{2p_\alpha(p_\alpha-2)t_1\int_{\mathbb{R}^N} f(x)\widetilde{u}dx}{(p_\alpha-1)^2\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}},\ \Lambda_2\right\}.$$

For all $\lambda \in (0, \Lambda_3)$, we conclude that

$$\sup_{t\geq 0} I(t\widetilde{u}) < \frac{p_\alpha - 2}{2p_\alpha}S_\mu^{\frac{p_\alpha}{p_\alpha-2}} - \frac{\lambda^2(p_\alpha-1)^2}{2p_\alpha(p_\alpha-2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)},$$

and this ends the proof. □

Next we will show another critical point with positive energy of problem (1.1).

Since $I(t\widetilde{u}) \to -\infty$ as $t \to \infty$, then one may take $t^* > 0$ sufficiently large if necessary, such that $I(t^*\widetilde{z}) < 0$. Taking $\eta_0 > 0$, then Lemma 3.1 can show that $I|_{\partial B_{\eta_0}} \geq \xi > 0$ for every $\lambda \in (0, \Lambda_1)$. Set

$$\Gamma = \{\gamma \in C([0,1], H_0^2(\mathbb{R}^N)),\ \gamma(0) = 0,\ \gamma(1) = t^*\widetilde{u}\},$$

and

$$c_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)).$$

Then it follows from Mountain pass theorem without (PS) condition that there exists a $(PS)_{c_2}$ sequence $\{u_n\}_{n=1}^{\infty}$ in $H_0^2(\mathbb{R}^N)$ satisfying

$$I(u_n) \to c_2, \quad I'(u_n) \to 0, \quad \text{in } H_0^{-2}(\mathbb{R}^N)$$

as $n \to \infty$.

Furthermore, it follows from Lemma 2.1 that there exists a subsequence of $\{u_n\}_{n=1}^{\infty}$, still denoted by $\{u_n\}_{n=1}^{\infty}$, and a $u^* \in H_0^2(\mathbb{R}^N)$, such that $u_n \rightharpoonup u^*$, as $n \to \infty$. If $u_n \nrightarrow u^*$ as $n \to \infty$, then from Lemma 2.2 we can deduce that

$$c_2 \geq I(u^*) + \frac{p_\alpha - 2}{2p_\alpha} S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} \geq \frac{p_\alpha - 2}{2p_\alpha} S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \frac{\lambda^2 (p_\alpha - 1)^2}{2p_\alpha (p_\alpha - 2)} \|f\|^2_{H_0^{-2}(\mathbb{R}^N)}. \tag{3.7}$$

But Lemma 3.2 shows that

$$\sup_{t \geq 0} I(t\widetilde{u}) < \frac{p_\alpha - 2}{2p_\alpha} S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \frac{\lambda^2 (p_\alpha - 1)^2}{2p_\alpha (p_\alpha - 2)} \|f\|^2_{H_0^{-2}(\mathbb{R}^N)}, \quad \text{for any } \lambda \in (0, \Lambda_3).$$

This together with (3.7) means that $u_n \to u^*$ in $H_0^2(\mathbb{R}^N)$, as $n \to \infty$. Taking $\lambda^* := \min\{\Lambda_1, \Lambda_3\}$, it is easy to show that for any $\lambda \in (0, \lambda^*)$, the functional I has the second critical point u^* satisfying $I(u^*) > 0$. Therefore the proof of Theorem 1.1 is finished.

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